A Distributed Observer for a Time-Invariant Linear System

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***Abstract*—A time-invariant, linear, distributed observer is described for estimating the state of an $m > 0$ channel, n-dimensional continuous-time linear system of the form $\dot{x} = Ax,\ y_i = C_i x,\ i \in \{1, 2, \ldots, m\}$. The state x is simultaneously estimated by m agents assuming each agent i senses y_i and receives the state z_j of each of its neighbors' estimators. Neighbor relations are characterized by a constant directed graph $\mathbb{N}$ whose vertices correspond to agents and whose arcs depict neighbor relations. For the case when the neighbor graph is strongly connected, the overall distributed observer consists of m linear estimators, one for each agent; $m - 1$ of the estimators are of dimension n and one estimator is of dimension $n + m - 1$. Using results from the classical decentralized control theory, it is shown that subject to the assumptions that none of the C_i are zero, the neighbor graph $\mathbb{N}$ is strongly connected, the system whose state to be estimated is jointly observable, and nothing more, it is possible to freely assign the spectrum of the overall distributed observer. For the more general case, when $\mathbb{N}$ has $q > 1$ strongly connected components, it is explained how to construct a family of q distributed observers, one for each component, which can estimate x at a preassigned convergence rate.**

***Index Terms*—Distributed observer, time-invariant system, decentralized control.**

I. Introduction

STATE estimators such as Kalman filters and observers have had a huge impact on the entire field of estimation and control. This paper deals with observers for time-invariant linear systems. An observer for a process modeled by a continuous-time, time-invariant linear system with state x, measured output $y = Cx$, and state-dynamics $\dot{x} = Ax$, is a time-invariant linear system with input y, which is capable of generating an asymptotically correct estimate of x exponentially fast at a preassigned but arbitrarily fast convergence rate. As is well known, the only requirement on the system $y = Cx, \dot{x} = Ax$ for such an estimator to exist is that the matrix pair (C, A) be observable. In this paper, we will be interested in the natural generalization of this concept appropriate to a network of m agents. We now make precise what we mean by this.

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A. Problem

We are interested in a fixed network of $m > 0$ autonomous agents labeled $1, 2, \ldots, m$, which are able to receive information from their neighbors where by the *neighbor* of agent i is meant any other agent in agent i's reception range. We write $\mathcal{N}_i$ for the set of labels of agent i's neighbors and we take agent i to be a neighbor of itself. Neighbor relations between distinct pairs of agents are characterized by a directed graph $\mathbb{N}$ with m vertices and a set of arcs defined so that there is an arc from vertex j to vertex i whenever agent j is a distinct neighbor of agent i; thus $\mathbb{N}$ has no self-arcs. Each agent i can sense a signal $y_i \in \mathbb{R}^{s_i},\ i \in \mathbf{m} = \{1, 2, \ldots, m\}$, where

$$y_i = C_i x, \quad i \in \mathbf{m} \tag{1}$$

$$\dot{x} = Ax \tag{2}$$

and $x \in \mathbb{R}^n$.

Agent i estimates x using an n_i-dimensional linear system with state vector z_i and we assume the information agent i can receive from neighbor $j \in \mathcal{N}_i$ is $z_j(t)$ and $y_j(t)$. The problem of interest is to construct a suitably defined family of linear systems

$$\dot{z}_i = \sum_{j \in \mathcal{N}_i} (H_{ij} z_j + K_{ij} y_j), \quad i \in \mathbf{m} \tag{3}$$

$$x_i = \sum_{j \in \mathcal{N}_i} (M_{ij} z_j + N_{ij} y_j), \quad i \in \mathbf{m} \tag{4}$$

in such a way so that no matter what the initializations of (2) and (3) are, each signal $x_i(t)$ is an asymptotically correct estimate of $x(t)$ in the sense that each *estimation error* $e_i = x_i(t) - x(t)$ converges to zero as $t \to \infty$ at a preassigned, but arbitrarily fast convergence rate. We call such a family a *distributed (state) observer*.

We assume throughout that $C_i \neq 0,\ i \in \mathbf{m}$, and that the system defined by (1) and (2) is *jointly observable*; i.e., with $C = [C_1' \quad C_2' \quad \cdots \quad C_m']'$, the matrix pair (C, A) is observable. Generalizing the results that follow to the case when (C, A) is only detectable is quite straightforward and can be accomplished using well-known ideas. However, the commonly made assumption that each pair $(C_i, A),\ i \in \mathbf{m}$, is observable, or even just detectable, is very restrictive, grossly simplifies the problem and is unnecessary. It is precisely the exclusion of this assumption that distinguishes the problem posed here from almost all of the distributed estimator problems addressed in the literature. The main exceptions to this we are aware of are the state estimation problems considered in [1]–[4]. It should be noted, however, that none of these publications claim to provide techniques for directly controlling convergence rate, whereas this paper does.

B. Background

There is a huge literature that seeks to deal with distributed Kalman filters or distributed observers; see, for example, [1]–[9] and the many references cited therein. Many results are only partial and most problem formulations are different in detail than the problem posed here. The problem we have posed was prompted specifically by the work in [1] that seeks to devise a time-invariant distributed observer for the discrete-time analog of (1) and (2). Two particularly important contributions are made in [1]. First it is recognized that the problem of crafting a "stable" distributed observer is more or less equivalent to devising a stabilizing decentralized control as in [10] and [11]. Second, it is demonstrated that under suitable conditions, it is only necessary for the dimension of one of the agent subsystems in [1] to be larger than n, and that larger dimension need not exceed $n+m-1$.

The work reported in this paper clarifies and expands on the results of [1] in several ways. First, we outline a construction for systems with strongly connected neighbor graphs that enables one to freely adjust the observer's spectrum. Second, the results obtained here apply whether A is singular or not; the implication of this generalization is that the construction proposed can be used to craft observers for continuous time processes, whereas the construction proposed in [1] cannot unless A is nonsingular.

II. Observer Design Equations

We now develop the interrelationships between the matrices appearing in (3) and (4), which must hold for each x_i to be an asymptotically correct estimate of x. Note first that because (4) must hold even when all estimates are correct, for each $i \in \mathbf{m}$, it is necessary that the equation $x = \sum_{j\in\mathcal{N}_i}(M_{ij}z_j + N_{ij}C_j x),\ \ i \in \mathbf{m}$, has a solution $z_i^x,\ i \in \mathbf{m}$, for each possible $x \in \mathbb{R}^n$. Thus, if we define $V_i = [z_i^{u_1} \quad z_i^{u_2} \quad \cdots \quad z_i^{u_n}]_{n_i\times n},\ i \in \mathbf{m}$, where u_k is the kth unit vector in $\mathbb{R}^n$, then

$$\boxed{I = \sum_{j\in\mathcal{N}_i}(M_{ij}V_j + N_{ij}C_j), \quad i \in \mathbf{m}} \tag{5}$$

This and (4) imply that the m estimation errors satisfy

$$x_i - x = \sum_{j\in\mathcal{N}_i} M_{ij}\epsilon_j, \quad i \in \mathbf{m} \tag{6}$$

where

$$\epsilon_i = z_i - V_i x,\ i \in \mathbf{m}. \tag{7}$$

Moreover, as a direct consequence of (1)–(3)

$$\dot\epsilon_i = \sum_{j\in\mathcal{N}_i} H_{ij}\epsilon_j + \left(\sum_{j\in\mathcal{N}_i}(H_{ij}V_j + K_{ij}C_j) - V_iA\right)x, \quad i \in \mathbf{m}.$$

Thus, if we stipulate that

$$\boxed{V_iA = \sum_{j\in\mathcal{N}_i}(H_{ij}V_j + K_{ij}C_j),\ i \in \mathbf{m}} \tag{8}$$

then

$$\dot\epsilon_i = \sum_{j\in\mathcal{N}_i} H_{ij}\epsilon_j,\ i \in \mathbf{m}. \tag{9}$$

We shall refer to (5) and (8) as the *observer design equations*. These equations are quite general. They apply to all time-invariant continuous and discrete time state observers whether they are distributed or not.

It is clear from (6) that if the $V_i, H_{ij}, M_{ij}, N_{ij}$, and K_{ij} can be chosen so that the observer design equations (5) and (8) hold and the system defined by (9) is exponentially stable, then each x_i will be an asymptotically correct estimate of x. The *distributed observer design problem* is to develop constructive conditions that ensure that the $V_i, H_{ij}, M_{ij}, N_{ij}$, and K_{ij} can be so chosen and that the convergence of the ϵ_i will occur at a preassigned but arbitrarily fast rate.

III. Centralized Observers

The purpose of this section is to review the well-known concept of an (centralized) observer with the aim summarizing certain less well-known ideas that will play a role in the construction of a distributed observer. In the centralized case, $m=1$ and a state observer is an n_1-dimensional linear system with input $y = Cx$, state $z \in \mathbb{R}^{n_1}$, and output x_1 of the form $\dot z_1 = Hz_1 + Ky$, $x_1 = Mz_1 + Ny$. In this case, the observer design equations are $I = MV + NC$ and $VA = HV + KC$ and the observer design problem is to determine matrices H, K, M, N, and V so that the observer design equations hold and H is a stability matrix with a preassigned spectrum. Observers fall into three broad categories depending on the dimension n_1: full-state observers, minimal-state observers, and extended-state observers. Each type is briefly reviewed in the following.

A. Full-State Observers

Just about the easiest solution to the observer design problem that one can think of, is the one for which $M = I$, $N = 0$, $V = I_{n\times n}$, and $H = A - KC$. Any observer of this type is called a *full-state observer* because in this case z_1 is an asymptotic estimate of x. Of course, it is necessary that K be chosen so that $A - KC$ is a stable matrix with a preassigned spectrum. One way to accomplish this is to exploit duality and use spectrum assignment, as is well known. No matter how one goes about defining K, the definitions of H, M, N, and V given previously show that a full-state observer is modeled by equations of the form $\dot z_1 = (A-KC)z_1 + Ky$, $x_1 = z_1$.

B. Reduced-State Observers

By a *minimal-state observer* is meant an observer of least dimension that can generate an asymptotic estimate of x. Minimal dimensional observers are obtained by exploiting the fact that $y = Cx$ is a "partial" measurement of x. Note that the observer design equation $I = MV + NC$ implies that the number of linearly independent rows of $V_{n_1 \times n}$ must be at least equal to the dimension of $\ker C$. Thus, the dimension of any observer must be at least equal to the dimension of $\ker C$. Techniques for constructing minimal-state observers are well known [12], [13].

C. Extended-State Observers

Much less well known are what might be called "extended-state observers" or "dynamic observers" [14]. An observer of this type would be of dimension $n_1 = n + \bar{n}$, where $\bar{n}$ is a nonnegative integer chosen by the designer. With $\bar{n}$ fixed, an extended observer can be obtained by first picking $M = [I \quad 0]_{n \times (n+\bar{n})}, V' = [I \quad 0]_{n \times (n+\bar{n})}$, and $N = 0$, thereby, ensuring that observer design equation $I = MV + NC$ is satisfied. With V so chosen, z_1 must be of the corresponding form $z_1 = [x_1' \quad \bar{z}_1']'$. Accordingly, the partitioned matrices

$$H = \begin{bmatrix} A + \bar{D}C & \bar{C} \\ \bar{B}C & \bar{A} \end{bmatrix}_{(n+\bar{n}) \times (n+\bar{n})} \qquad K = -\begin{bmatrix} \bar{D} \\ \bar{B} \end{bmatrix}$$

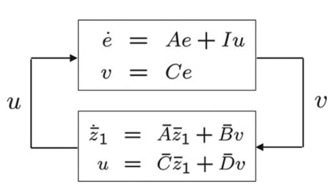
satisfy the observer design equation $VA = HV + KC$ for any values of the matrices $\bar{A}, \bar{B}, \bar{C}, \bar{D}$, and

$$\begin{aligned} \dot{x}_1 &= (A + \bar{D}C)x_1 + \bar{C}\bar{z}_1 - \bar{D}y \\ \dot{\bar{z}}_1 &= \bar{B}Cx_1 + \bar{A}\bar{z}_1 - \bar{B}y. \end{aligned}$$

Moreover, the estimation error $e = x_1 - x$ satisfies

$$\begin{aligned} \dot{e} &= (A + \bar{D}C)e + \bar{C}\bar{z}_1 \\ \dot{\bar{z}}_1 &= \bar{B}Ce + \bar{A}\bar{z}_1. \end{aligned}$$

These equations suggest the following feedback diagram.



Thus, the design of an extended state observer amounts to picking the coefficient matrices $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ of the lower subsystem in the block diagram to at least stabilize the loop and provides the overall system with a preassigned stability margin. Of course, if $\bar{n} = 0$, this subsystem is just the constant matrix $\bar{D}$ and one has again a classical full-state observer of dimension n. Exactly what might be gained by picking $\bar{n}$ greater than zero is not clear in the case of a centralized observer. However, for the decentralized observer, we describe next, the flexibility of a dynamic lower loop will become self-evident.

IV. Distributed Observers

The primary goal of the distributed observer design is to choose the matrices $V_i, H_{ij}, M_{ij}, N_{ij}$, and K_{ij} so that the observer design equations (5) and (8) hold and the system defined by (9) is exponentially stable with a preassigned convergence rate. Another goal might be to choose these matrices to reduce the information that needs to be transmitted between neighboring agents. Still another goal might be to choose these matrices so that the dimensions of the individual estimators are as small as possible. In this paper, we will consider the case when the only information transmitted between neighboring agents are estimator states z_i and we will make no attempt to construct estimators of least dimension. This means that we will set all $K_{ij} = 0$ except for K_{ii} in (3) and all $N_{ij} = 0$ in (4). Thus, in this paper, we will focus exclusively on observers of the form

$$\dot{z}_i = \sum_{j \in \mathcal{N}_i} H_{ij} z_j + K_i y_i, \quad i \in \mathbf{m} \tag{10}$$

$$x_i = \sum_{j \in \mathcal{N}_i} M_{ij} z_j, \quad i \in \mathbf{m} \tag{11}$$

where we have adopted the notation $K_i = K_{ii}$.

The easiest way to satisfy the observer design equations under these conditions is to set $V_i = I_{n \times n}$ for $i \in \mathbf{m}$ and to pick the M_{ij} so that $I = \sum_{j \in \mathcal{N}_i} M_{ij}, \ i \in \mathbf{m}$. However, as will become apparent soon, in order to have the flexibility to be able to freely control convergence rate, it will be necessary to change the definition of one of the V_i—say V_p—to $V_p = [I \quad 0]_{(n+m-1) \times n}$. This will be illustrated toward the end of Section IV-A. For the present, it is assumed that all V_i are equal to $I_{n \times n}$. With the V_i so defined, observer design equation (8) simplifies to

$$A - K_i C_i = \sum_{j \in \mathcal{N}_i} H_{ij}, \ i \in \mathbf{m}. \tag{12}$$

In view of (6) and (9), the observer design problem for this type of an observer is to try to choose the K_i and H_{ij} so that (12) holds and in addition so that $H = [H_{ij}]$ is a stability matrix with a prescribed spectrum, where $H_{ij} = 0$ if $j \notin \mathcal{N}_i$. It is possible to express H in a more explicit form, which takes into account the constraints on the H_{ij} imposed by (12). For this let $\tilde{A}$ denote the block diagonal matrix $\tilde{A} = I_{m \times m} \otimes A$, where $\otimes$ is the Kronecker product. Set $B_i = b_i \otimes I_{n \times n}, \ i \in \mathbf{m}$, where b_i is the ith unit vector in $\mathbb{R}^m$; in addition, let $C_{ii} = C_i B_i', \ i \in \mathbf{m}$, and $C_{ij} = c_{ij} \otimes I_{n \times n}, \ j \in \mathcal{N}_i, j \neq i, \ i \in \mathbf{m}$, where c_{ij} is the row in the transpose of the incidence matrix of $\mathbb{N}$ corresponding to the arc from j to i.[1] It is then possible to express H in the compact form

$$H = \tilde{A} + \sum_{i \in \mathbf{m}} \sum_{j \in \mathcal{N}_i} B_i F_{ij} C_{ij} \tag{13}$$

where $F_{ii} = -K_i, i \in \mathbf{m}$ and $F_{ij} = H_{ij}, \ j \in \mathcal{N}_i, j \neq i$, $i \in \mathbf{m}$. Note that there are no constraints on the F_{ij}. In this form, it is clear that H is what results when output feedback

[1] We have adopted this double subscript notion for the sake of conciseness; thus, for example, c_{ij} is a row vector, *not* the ijth entry of a matrix of numbers.

laws $u_{ij} = F_{ij} y_{ij}$ are applied to the system

$$\dot{\epsilon} = \tilde{A}\epsilon + \sum_{i\in\mathbf{m}} \sum_{j\in\mathcal{N}_i} B_i u_{ij} \tag{14}$$

$$y_{ij} = C_{ij}\epsilon, \quad ij \in \mathcal{I} \tag{15}$$

where $\mathcal{I} \subset \mathbf{m}\times\mathbf{m}$ is the set of double indices $\mathcal{I} = \{ij : i \in \mathbf{m},\ j \in \mathcal{N}_i\}$. The problem of constructing a distributed observer of this type, thus, reduces to trying to choose the F_{ij} to at least stabilize H if such matrices exist. Of course, one also wants control over rate of convergence, so stabilization of H alone is not all that is of interest. Whether the goal is just stabilization of H or control over convergence rate, choosing the F_{ij} to accomplish this will typically not be possible except under special conditions. In fact, the problem trying to stabilize H by appropriately choosing the F_{ij} is mathematically the same as the classical decentralized stabilization problem for which there is a substantial literature [10], [11].

A. Strongly Connected Neighbor Graph $\mathbb{N}$

One approach to decentralized stabilization problem is to try to choose the F_{ij} so that for given $p \in \mathbf{m}$ and $q \in \mathcal{N}_p$, the matrix pairs (H, B_p) and (C_{pq}, H) are controllable and observable, respectively. Having accomplished this, stabilization can then be achieved by applying standard centralized feedback techniques such as those in [15] to the resulting controllable observable system. This is the approach taken in this paper. The following proposition provides the key technical result that we need.

Proposition 1: Suppose that the neighbor graph $\mathbb{N}$ is strongly connected. There exist gain matrices $F_{ij},\ ij \in \mathcal{I}$, such that the matrix pairs (H, B_p) and (C_{pq}, H) are controllable and observable, respectively, for all $p \in \mathbf{m}$ and all $q \in \mathcal{N}_p$. Moreover, for any such pair, m is the controllability index of (H, B_p).

The proof of this proposition will be given is Section VI.

In the light of Proposition 1, the way to construct a distributed observer is clear. As a first step, choose matrices M_{ij}, $i \in \mathbf{m},\ j \in \mathcal{N}_i$ so that $I = \sum_{j\in\mathcal{N}_i} M_{ij},\ i \in \mathbf{m}$. Next choose the F_{pq} so that the conclusions of the Proposition 1 hold. Having so chosen the F_{ij} or equivalently the H_{ij} and the K_i, fix values of $p \in \mathbf{m}$ and $q \in \mathcal{N}_p$. Next set $\bar{n} = m - 1$ and use a standard construction technique such as that given in [15] to pick matrices $\bar{A}_{\bar{n}\times\bar{n}}, \bar{B}_{\bar{n}\times\omega}, \bar{C}_{n\times\bar{n}}$, and $\bar{D}_{n\times\omega}$ to assign a desirable spectrum to the matrix

$$\hat{H} = \begin{bmatrix} H + B_p\bar{D}C_{pq} & B_p\bar{C} \\ \bar{B}C_{pq} & \bar{A} \end{bmatrix}_{(nm+\bar{n})\times(nm+\bar{n})} \tag{16}$$

where $\omega = s_p$, the dimension of y_p, if $q = p$ or $\omega = n$ if $p \neq q$. This can be done because (C_{pq}, H) is an observable pair and because (H, B_p) is a controllable pair with controllability index m.

It is possible to verify that the observer design equations hold provided the matrices $M_{kp}, H_{kp}, H_{pk},\ k \in \mathbf{m}, V_p$, and K_p are appropriately redefined to take into account the dynamics added to agent p's estimator. For simplicity, assume that $p = m$ and redefine V_m to be $[I \quad 0]'_{(n+\bar{n})\times n}$. For $k \in \mathbf{m}$, redefine M_{km} to be $[M_{km} \quad 0]_{\omega\times(n+\bar{n})}$, thereby, ensuring that observer design equation (5) holds. To ensure that observer design equation (8) holds, first replace H_{km} with $[H_{km} \quad 0]_{n\times(n+\bar{n})}$ for $k \in \{1, 2, \ldots, m-1\}$. If $q = m$ replace $H_{mi},\ i \in \mathbf{m},\ i \neq m$, and H_{mm} with the matrices

$$\begin{bmatrix} H_{mi} \\ 0 \end{bmatrix}, i \in \{1, 2, \ldots, m-1\}$$

and

$$\begin{bmatrix} H_{mm} + \bar{D}C_m & \bar{C} \\ \bar{B}C_m & \bar{A} \end{bmatrix}$$

respectively; in addition, replace K_m with the matrix $[(K_m - \bar{D})' \quad -\bar{B}']'_{(n+\bar{n})\times s_m}$. If on the other hand, $m \neq q$, replace $H_{mi},\ i \in \mathbf{m},\ i \neq q, m, H_{mq}$, and H_{mm} with the matrices

$$\begin{bmatrix} H_{mi} \\ 0 \end{bmatrix}, i \in \{1, 2, \ldots, m-1\}, i \neq q$$

$$\begin{bmatrix} H_{mq} - \bar{D} \\ -\bar{B} \end{bmatrix}, \text{and} \begin{bmatrix} H_{mm} + \bar{D} & \bar{C} \\ \bar{B} & \bar{A} \end{bmatrix}$$

respectively; in addition, replace K_m with matrix $[K'_m \quad 0]'_{(n+\bar{n})\times s_m}$. In either case, observer design equation (8) holds. With this redefined notation, the overall observer is described by (10) and (11), where z_m is now the $n + \bar{n}$-dimensional state of agent m's estimator. In addition, since observer design equation (8) holds, error $\bar{\epsilon} = [\epsilon'_1 \quad \epsilon'_2 \quad \ldots \quad \epsilon'_m \quad \bar{z}']'$ satisfies $\dot{\bar{\epsilon}} = \hat{H}\bar{\epsilon}$, where $\hat{H}$ is defined as in (16) and $x_i - x = \sum_{k\in\mathcal{N}_i} M_{ik}\epsilon_i,\ i \in \mathbf{m}$.

We are led to the main result of this paper.

Theorem 1: Suppose that (1) and (2) is a jointly observable system and that $C_i \neq 0,\ i \in \mathbf{m}$. If the neighbor graph $\mathbb{N}$ is strongly connected, then for each symmetric set of $mn + m - 1$ complex numbers Λ, there is a distributed observer (10), (11) for which the spectrum of the $(mn + m - 1) \times (mn + m - 1)$ matrix $\hat{H} \stackrel{\Delta}{=} [H_{ij}]$ is Λ. Moreover, if the numbers in Λ are all points in the open left half of the complex plane, the observer's m outputs $x_i(t),\ i \in \mathbf{m}$, all asymptotically correctly estimate $x(t)$ in the sense that each estimation error $e_i = x_i(t) - x(t)$ converges to zero as $t \to \infty$ as fast as $e^{\hat{H}t}$ converges to the zero matrix, no matter what the initializations of (2) and (3) are.

B. Nonstrongly Connected Neighbor Graph $\mathbb{N}$

We now turn briefly to the problem of developing a distributed observer for the case when $\mathbb{N}$ is not strongly connected. We will assume for simplicity and without loss of generality that $\mathbb{N}$ is weakly connected. For if it is not, the ideas that follow can be applied to each maximally weakly connected subgraph of $\mathbb{N}$, since each such subgraph is isolated from the rest. As before, the goal is to devise m estimators whose estimates converge to x exponentially fast at arbitrary, preassigned rates. We suppose that $\mathbb{N}$ has q strongly connected components $\mathbb{N}_1, \mathbb{N}_2, \ldots, \mathbb{N}_q$, and for each $i \in \mathbf{q}$, we write Σ_i for the m_i channel *component subsystem* $\dot{x} = Ax\ y_j = C_j x,\ j \in \mathcal{V}_i$, where $\mathcal{V}_i$ is the set of labels of the vertices of $\mathbb{N}_i$ and m_i is the number of labels in $\mathcal{V}_i$. We say that there is a directed path {resp., arc} from strongly connected component $\mathbb{N}_i$ to strongly connected component $\mathbb{N}_j$ if there is a directed path {resp., arc} in $\mathbb{N}$ from at least one vertex in $\mathbb{N}_i$ to at least one vertex in $\mathbb{N}_j$. Following [1], we say that $\mathbb{N}_j$ is a *source component* of $\mathbb{N}$ if $\mathbb{N}_j$ has no incoming arcs from any other strongly connected component of $\mathbb{N}$. It is clear that $\mathbb{N}$ must contain at at least one source component. Moreover, since $\mathbb{N}$ is weakly connected, it is also clear that for any strongly connected component of $\mathbb{N}_i$, which is not a source, there must be at least one directed path from at least one source $\mathbb{N}_j$ to $\mathbb{N}_i$.

Let $\mathbb{N}_j$ be a source component and Σ_j be its associated component subsystem. Note that there cannot be any signal flow to any channel in Σ_j from any channel of any other component subsystem. It follows that for there to exist estimators for each channel in Σ_j, which are capable of estimating x at a preassigned convergence rate, it is necessary that Σ_j be a jointly observable subsystem. In view of Theorem 1, joint observability of Σ_j is also sufficient for such a distributed observer to exist because $\mathbb{N}_j$ is strongly connected. Suppose, therefore, that for each source component $\mathbb{N}_j$, the associated component subsystem Σ_j is jointly observable and that a distributed observer has been constructed with preassigned converge rate for each such Σ_j. If all strongly connected components of $\mathbb{N}$ are sources, then these observers solve the distribute observer design problem. Suppose, therefore, that there is at least one strongly connected component, which is not a source. Then, there must be at least one strongly connected component $\mathbb{N}_i$, which is not a source for which there is a source $\mathbb{N}_j$ with an arc to $\mathbb{N}_i$. This implies that there must be a channel $k \in \mathcal{V}_j$ of Σ_j whose estimator state z_k is available to at least one channel—say channel l of component subsystem Σ_i. But $\epsilon_k = z_k - V_k x$. Moreover, for the full-state observers, we are considering, V_k' is a left inverse of V_k so $V_k' z_k = \bar{C}_l x + V_k' \epsilon_k$, where $\bar{C}_l = I_{n \times n}$. Therefore, $V_k' z_k$ can be regarded as a measurement of x with exponentially decaying additive measurement noise $V_k' \epsilon_k$. Thus, if the readout equation $y_l = C_l x$ in the definition of Σ_i, is replaced with the augmented readout equation

$$y_l = \begin{bmatrix} C_l \\ \bar{C}_k \end{bmatrix} x + \begin{bmatrix} 0 \\ V_k' \epsilon_k \end{bmatrix}$$

then the resulting subsystem, denoted by $\bar{\Sigma}_i$, will be jointly observable with unmeasurable but exponentially decaying measurement noise. Since $\mathbb{N}_i$ is strongly connected, a distributed observer with the same convergent rate as that of ϵ_k, can, therefore, be constructed for Σ_i. If $\mathbb{N}_i$ is the only strongly connected component of $\mathbb{N}$, which is not a source, then construction is complete. If, on the other hand, $\mathbb{N}$ has other strongly connected components, which are not sources, the same ideas as just described, can be applied to each corresponding component subsystem in a sequential manner. We are led to the following.

Corollary 1: Suppose that $C_i \neq 0,\ i \in \mathbf{m}$ and that neighbor graph $\mathbb{N}$ has q strongly connected components $\mathbb{N}_i,\ i \in \mathbf{q}$. Let Σ_i be the component subsystem of (1) and (2) corresponding to strongly connected component i. In order for there to exist distributed observers for each of the component subsystems that are capable of estimating x at an arbitrary but preassigned convergence rate, it is necessary and sufficient that each of the component subsystems whose graphs are sources, is jointly observable.

V. Decentralized Control Theory

The aim of this section is to summarize the concepts and results from [11] and [16], which we will make use of to justify Proposition 1. We do this for a k channel, n-dimensional linear system of the form

$$\dot{x} = Ax + \sum_{i \in \mathcal{I}} B_i u_i \qquad y_i = C_i x,\ \ i \in \mathcal{I} \tag{17}$$

where $\mathcal{I} = \{1, 2, \ldots, k\}^2$ and $C_i \neq 0,\ i \in \mathcal{I}$. Application of decentralized feedback laws of the form $u_i = F_i y_i,\ i \in \mathcal{I}$ to this system yields the equation $\dot{x} = Hx$, where $H = A + \sum_{i \in \mathcal{I}} B_i F_i C_i$. For given $p \in \mathcal{I}$, explicit necessary and sufficient conditions under which there exist F_i which make (C_p, H, B_p) controllable and observable are given in [11] and [16]. There are two conditions. First, (17) must be jointly controllable and jointly observable. Second, each "complementary subsystem" of (17) must be "complete" (cf., [11, Th. 3]). There are as many complementary subsystems of (17) as there are strictly proper subsets of $\mathcal{I}$. By the *complementary subsystem* of (17) corresponding to a nonempty proper subset $\mathcal{C} \subset \mathcal{I}$, is meant a subsystem with input matrix $\mathbf{B}(\mathcal{C}) = \text{block row}\{B_i : i \in \mathcal{C}\}$, state matrix A and readout matrix $\mathbf{C}(\bar{\mathcal{C}}) = \text{block column}\{C_i : i \in \bar{\mathcal{C}}\}$, where $\bar{\mathcal{C}}$ is the complement of $\mathcal{C}$ in $\mathcal{I}$ [11]. The complementary subsystem determined by $\mathcal{C}$ is uniquely determined up to the orderings of the block rows and block columns of $\mathbf{B}(\mathcal{C})$ and $\mathbf{C}(\bar{\mathcal{C}})$, respectively; as will become clear in a moment, the properties that characterize completeness do not depend on these orderings.

For a given complementary subsystem $(\mathbf{C}(\bar{\mathcal{C}}), A, \mathbf{B}(\mathcal{C}))$ to be complete, its transfer matrix $\mathbf{C}(\bar{\mathcal{C}})(sI - A)^{-1}\mathbf{B}(\mathcal{C})$ must be nonzero and the *matrix pencil*

$$\pi(\mathcal{C}) = \begin{bmatrix} \lambda I - A & \mathbf{B}(\mathcal{C}) \\ \mathbf{C}(\bar{\mathcal{C}}) & 0 \end{bmatrix} \tag{18}$$

must have rank no less than n for all real and complex λ (see [16, Corollary 4] and[17]). The requirement that the transfer matrix of each complementary subsystem be nonzero, can be established in terms of the connectivity of the "graph" of (17). By the *graph* of (17), written $\mathbb{G}$, is meant that the k-vertex directed graph with vertex labels in $\mathcal{I}$, and arcs defined so that for each two labels $i, j \in \mathcal{I}$, there is an arc from vertex j to i if $C_i(sI - A)^{-1}B_j \neq 0$. For the transfer matrices of all complementary subsystems of (17) to be nonzero, it is necessary and sufficient that $\mathbb{G}$ be a strongly connected graph (see [11, Lemma 8]).

VI. Analysis

The aim of this section is to prove Proposition 1. To do this, it is useful to first establish certain properties of the subsystem of (14) and (15) defined by the equations

$$\dot{\epsilon} = \tilde{A}\epsilon + \sum_{i \in \mathbf{m}} \sum_{j \in \bar{\mathcal{N}}_i} B_i u_{ij} \tag{19}$$

$$y_{ij} = C_{ij}\epsilon, \quad ij \in \mathcal{J} \tag{20}$$

where $\mathcal{J}$ is the complement of the set $\{ii : i \in \mathbf{m}\}$ in $\mathcal{I}$ and for $i \in \mathbf{m}$, $\bar{\mathcal{N}}_i$ is the complement of the set $\{i\}$ in $\mathcal{N}_i$. This subsystem is what results when outputs $y_{ii},\ i \in \mathbf{m}$, are deleted

[2] The symbols used in this section such as x, C_i, A, and $\mathcal{I}$ are generic and do not have the same meanings as the same symbols do when used elsewhere in this paper.

from (15). Our goal here is to show that with suitable scalars f_{ij}, the matrix pairs $(\bar{H}, B_p),\ p \in \mathbf{m}$, are all controllable with controllability index m, where

$$\bar{H} = \tilde{A} + \sum_{i\in\mathbf{m}} \sum_{j\in\bar{\mathcal{N}}_i} B_i F_{ij} C_{ij} \tag{21}$$

and $F_{ij} = f_{ij} I_n$. I_n is the $n \times n$ identity matrix. Note that for any f_{ij} and any $p \in \mathbf{m}$, the submatrix $[B_p \quad \bar{H}B_p \quad \cdots \quad \bar{H}^{m-1}B_p]$ has exactly nm columns. Since nm is the dimension of the system (19), (20), m is the smallest possible controllability index which the pair $(\bar{H}, B_p)$ might attain as the f_{ij} range over all possible values. From this, it is obvious that if for each $p \in \mathbf{m}$, there exist f_{ij} for which $(\bar{H}, B_p)$ has controllability index m, then there must be f_{ij} for which $(\bar{H}, B_p)$ has controllability index m for all $p \in \mathbf{m}$, and moreover, the set of f_{ij} for which this is true is the complement of a proper algebraic set in the linear space in which the vector of f_{ij} takes values.

To proceed we will first show that with the f_{ij} chosen properly, the matrix pair (F, b_m) is controllable, where F is the $m \times m$ matrix

$$F = \sum_{i\in\mathbf{m}} \sum_{j\in\bar{\mathcal{N}}_i} b_i f_{ij} c_{ij} \tag{22}$$

and for $i \in \mathbf{m}$, b_i is the ith unit vector in $\mathbb{R}^m$. Note that F is what results when the feedback laws $v_{ij} = f_{ij} w_{ij}$ are applied to the system

$$\dot{z} = \sum_{i\in\mathbf{m}} \sum_{j\in\bar{\mathcal{N}}_i} b_i v_{ij} \tag{23}$$

$$w_{ij} = c_{ij} z, \quad ij \in \mathcal{J} \tag{24}$$

where as before, c_{ij} is the row in the transpose of the incidence matrix of $\mathbb{N}$ corresponding to the arc from j to i. Note that (23) and (24) can be viewed as a m^* channel system, where m^* is the number of labels in $\mathcal{J}$. In view of the fact that $\operatorname{span}\{b_1, b_2, \ldots, b_m\} = \mathbb{R}^m$, it is obvious that (23) is jointly controllable. Let $\mathbb{G}$ denote that m^*-vertex directed graph with vertex labels in $\mathcal{J}$ and arcs defined so that there is an arc from vertex ij to kq if $c_{kq}(sI)^{-1}b_i \neq 0$, for $j \in \bar{\mathcal{N}}_i$.

Lemma 1: If the neighbor graph $\mathbb{N}$ is strongly connected, then $\mathbb{G}$ is strongly connected.

Proof of Lemma 1: Note that for each $j \in \bar{\mathcal{N}}_i$, $c_{ij}(s)^{-1}b_i = -\frac{1}{s}$ and $c_{ij}(s)^{-1}b_j = \frac{1}{s}$. From these expressions, it follows that $c_{ij}(sI)^{-1}b_i \neq 0$ and $c_{ij}(sI)^{-1}b_j \neq 0$, for $i \in \mathbf{m},\ j \in \bar{\mathcal{N}}_i$. Therefore, for each $i \in \mathbf{m}$, the subgraph $\mathbb{G}_i$ induced by vertices $ij,\ j \in \bar{\mathcal{N}}_i$ is complete. By the *quotient graph* of $\mathbb{G}$, written $\mathbb{Q}$, is meant that the directed graph with m vertices labeled $1, 2, \ldots, m$ and an arc from i to k if there is an arc in $\mathbb{G}$ from a vertex in the set $\{ij : j \in \bar{\mathcal{N}}_i\}$ to a vertex in the set $\{kq : q \in \bar{\mathcal{N}}_k\}$. Because each of the subgraphs $\mathbb{G}_i$ is complete, $\mathbb{G}$ will be strongly connected if $\mathbb{Q}$ is strongly connected. But $\mathbb{Q} = \mathbb{N}$ so $\mathbb{Q}$ is strongly connected. Therefore, $\mathbb{G}$ is strongly connected. ■

Lemma 2: If the neighbor graph $\mathbb{N}$ is strongly connected, then each complementary subsystem of (23) and (24) is complete.

Proof of Lemma 2: Let $\mathcal{C} \subset \mathcal{J}$ be a nonempty subset and let $(\mathbf{C}, 0_{m\times m}, \mathbf{B})$ be the coefficient matrices of the complementary subsystem determined by $\mathcal{C}$. Thus, $\mathbf{B} = \text{block row}\{b_i : ij \in \mathcal{C}\}$, and $\mathbf{C} = \text{block column}\{c_{ij} : ij \in \bar{\mathcal{C}}\}$, where $\bar{\mathcal{C}}$ is the complement of $\mathcal{C}$ in $\mathcal{J}$. To prove the lemma, it is enough to show that the coefficient matrix triple $(\mathbf{C}, 0_{m\times m}, \mathbf{B})$ is complete. To establish completeness, the transfer matrix $\mathbf{C}(sI)^{-1}\mathbf{B}$ must be nonzero and the matrix pencil

$$\pi(\mathcal{C}) = \begin{bmatrix} \lambda I & \mathbf{B} \\ \mathbf{C} & 0 \end{bmatrix} \tag{25}$$

must have rank no less than m for all real and complex λ (cf., [16, Corollary 4]). In view of Lemma 1 and the assumption that $\mathbb{N}$ is strongly connected, $\mathbb{G}$ is strongly connected. Therefore, by [11, Lemma 8], $\mathbf{C}(sI)^{-1}\mathbf{B} \neq 0$.

To complete the proof, it is enough to show that for all complex numbers λ, $\operatorname{rank} \pi(\mathcal{C}) \geq m$. In view of the structure of $\pi(\mathcal{C})$ in (25), it is clear that for all such λ, $\operatorname{rank} \pi(\mathcal{C}) \geq \operatorname{rank} \mathbf{C} + \operatorname{rank} \mathbf{B}$. To establish completeness, it is, therefore, sufficient to show that

$$\operatorname{rank} \mathbf{C} + \operatorname{rank} \mathbf{B} \geq m. \tag{26}$$

Let $q \in \mathbf{m}$ denote the number of distinct integers i such that $ij \in \mathcal{C}$. In view of the definition of $\mathbf{B}$, $\operatorname{rank} \mathbf{B} = q$. If $q = m$, $\operatorname{rank} \mathbf{B} = m$ and (26) holds. Suppose next that $q < m$. Let $\mathbf{C}^*$ denote the submatrix of $\mathbf{C}$, which results when all rows c_{ij} in $\mathbf{C}$ for which $ik \in \mathcal{C}$ for some k, are deleted. Since $\operatorname{rank} \mathbf{C} \geq \operatorname{rank} \mathbf{C}^*$ and $\operatorname{rank} B = q$, (26) will hold if

$$\operatorname{rank} \mathbf{C}^* \geq (m - q). \tag{27}$$

Corresponding to the definition of $\mathbf{C}^*$, let $\mathbb{N}^*$ denote the spanning subgraph of $\mathbb{N}$, which results when any arc in $\mathbb{N}$ from i to j for which there is a k such that $ik \in \mathcal{C}$ is removed. There are exactly q distinct values of i for which $ik \in \mathcal{C}$ for some k. Moreover, for any such i, the corresponding vertex in $\mathbb{N}^*$ cannot have any outgoing arcs. Since $\mathbb{N}$ is strongly connected, any other vertex k in $\mathbb{N}^*$ must have at least one outgoing arc not incident on vertex k. This means that the unoriented version of $\mathbb{N}^*$ must have at most q connected components. Thus, if $M_{\mathbb{N}^*}$ is the incidence matrix of $\mathbb{N}^*$, then as a consequence of [18, Th. 8.3.1]

$$\operatorname{rank} M_{\mathbb{N}^*} \geq m - q. \tag{28}$$

But for any $ij \in \mathcal{J}$ such that $ik \notin \mathcal{C}$ for some k, c_{ij} is the row in the transpose of the incidence matrix of $\mathbb{N}^*$ corresponding to the arc from j to i. Therefore, up to a possible reordering of rows, $\mathbf{C}^* = M'_{\mathbb{N}^*}$. From this and (28), it follows that (27) holds. Therefore, the lemma is true. ■

Lemma 3: Let $A_{n\times n}$, $F_{m\times m}$, and $g_{m\times 1}$ be any given real-values matrices. There is an $mn \times mn$ nonsingular matrix T such that

$$\begin{aligned}&[G\ HG\ \cdots\ H^{m-1}G] \\ &= [g \otimes I_n\ \ (Fg) \otimes I_n\ \ \cdots\ \ (F^{m-1}g) \otimes I_n]T\end{aligned} \tag{29}$$

where $G = g \otimes I_n$ and $H = I_m \otimes A + F \otimes I_n$.

Proof of Lemma 3: Since $(I_m \otimes A)(F \otimes I_n) = (F \otimes I_n)(I_m \otimes A) = F \otimes A$, for $k \geq 1$

$$H^k = (I_m \otimes A + F \otimes I_n)^k = \sum_{i=0}^{k} \binom{k}{i} F^i \otimes A^{k-i}$$

where $\binom{k}{i}$ is the binomial coefficient. Thus

$$H^k G = (I_m \otimes A + F \otimes I_n)^k (g \otimes I_n) = \sum_{i=0}^{k} \binom{k}{i} F^i g \otimes A^{k-i}, \quad k \geq 1. \tag{30}$$

Define $T_1 = I_{mn}$ and for $k \in \{2, 3, \ldots, m\}$ let T_k be that $mn \times mn$ matrix composed of m^2 $n \times n$ submatrices $T_{ij}(k)$ defined so that $T_{ii}(k) = I_n, i \in \mathbf{m}$, $T_{(i+1),(k)}(k) = \binom{k-1}{i} A^{k-i-1}$, $i \in \{0, 1, \ldots, k-1\}$, and all remaining $T_{ij}(k) = 0$.

Let $X(k) = [g \otimes I_n \ (Fg) \otimes I_n \ \cdots \ (F^{k-1}g) \otimes I_n \ H^k G \ H^{k+1} G \cdots \ H^{m-1} G]$ for $k \in \mathbf{m}$. Obviously, $X(1) = [G \ HG \ \cdots \ H^{m-1} G]$, and $X(m) = [g \otimes I_n \quad (Fg) \otimes I_n \quad \cdots \ (F^{m-1} g) \otimes I_n]$.

The definition of T_k and (30) imply that

$$X(k) T_k = X(k-1), \ k \geq 1. \tag{31}$$

We claim that $T \triangleq T_m T_{m-1} \ldots T_1$ has the required properties. Note first that each of the T_i is an upper triangular matrix with ones on the main diagonal. Thus, each T_i is nonsingular that implies that T is nonsingular. According to (31)

$$\begin{aligned} [g \otimes I_n \ (Fg) \otimes I_n \ \cdots \ (F^{m-1} g) \otimes I_n] T &= X(m) T_m T_{m-1} \cdots T_1 \\ &= X(m-1) T_{m-1} T_{m-2} \cdots T_1 \\ &\ \ \vdots \\ &= X(1) T_1. \end{aligned}$$

Since $T_1 = I_{mn}$, (29) is true. ■

Lemma 4: Suppose $\mathbb{N}$ is strongly connected. The $m^* + m$ channel system (14), (15) is jointly controllable and jointly observable.

Proof of Lemma 4: In view of the definitions of the B_i, it is clear that $\mathcal{B}_1 + \mathcal{B}_2 + \cdots \mathcal{B}_m = \mathbb{R}^{nm}$, where $\mathcal{B}_i$ is the column span of B_i. It follows at once that (14), (15) is jointly controllable. To establish joint observability, it is enough to show that 0 is the only vector $x \in \mathbb{R}^{nm}$ for which $C_{ij} x = 0, \ ij \in \mathcal{I}$ and $\tilde{A} x = \lambda x$ for some complex number λ. Suppose $\tilde{A} x = \lambda x$ in which case $A x_i = \lambda x_i$, where $x = [x_1' \quad x_2' \quad \cdots \quad x_m']'$ and $x_i \in \mathbb{R}^n, \ i \in \mathbf{m}$. Moreover, if $C_{ij} x = 0, \ ij \in \mathcal{I}$, then $C_i x_i = 0, \ i \in \mathbf{m}$ and $(M_I \otimes I_n) x = 0$ where M_I is the transpose of the incidence matrix of $\mathbb{N}$. Since $\mathbb{N}$ is strongly connected, $(M_I \otimes I_n) x = 0$ implies that $x_i = x_1, \ i \in \mathbf{m}$. Thus, $C_i x_1 = 0, \ i \in \mathbf{m}$. But (C, A) is observable by assumption where $C = [C_1' \quad C_2' \quad \cdots \quad C_m']'$. Therefore, $x_1 = 0$. This implies that $x = 0$, and thus, that (14) and (15) is jointly observable. ■

Proof of Proposition 1: Since span $\{b_1, b_2, \ldots, b_m\} = \mathbb{R}^m$, the subsystem defined by (23) and (24) is jointly controllable. From this, Lemma 2, and [11, Th. 1], it follows that for each $p \in \mathbf{m}$, there exist f_{ij} such that (F, b_p) is a controllable pair where F is as defined (22). Since the set of f_{ij} for which this is true, is the complement of a proper algebraic set in the space in which the f_{ij} takes values, there also exist f_{ij} for which (F, b_p) is a controllable pair for all $p \in \mathbf{m}$. Fix such a set of f_{ij}.

By definition $B_i = b_i \otimes I_n \ i \in \mathbf{m}$, $C_{ij} = c_{ij} \otimes I_n, \ ij \in \mathcal{J}$, and $\tilde{A} = I_m \otimes A$. In view of the definition of $\bar{H}$ in (21), $\bar{H} = I_m \otimes A + F \otimes I_n$. From this and Lemma 3, it follows that for each $p \in \mathbf{m}$, there is a nonsingular matrix T_p such that $[B_p \quad \bar{H} B_p \quad \cdots \quad \bar{H}^{m-1} B_p] = [b_p \quad F b_p \quad \cdots \quad F^{m-1} b_p] \otimes I_n) T_p$. Since each T_p is nonsingular and each (F, b_p) is a controllable pair

$$\text{rank}\,[B_p \quad \bar{H} B_p \quad \cdots \quad \bar{H}^{m-1} B_p] = nm.$$

Therefore, for each $p \in \mathbf{m}$, $(\bar{H}, B_p)$ is a controllable pair with controllability index m. Note that if we define $F_{ii} = 0, \ i \in \mathbf{m}$, then in view of (13), $H = \bar{H}$. Therefore, for each $p \in \mathbf{m}$, (H, B_p) is a controllable pair with controllability index m. Clearly this must be true generically, for almost all $F_{ij}, ij \in \mathcal{I}$.

In view of [11, Th. 1], the complementary subsystems of (14) and (15) must all be complete. But by Lemma 4, (14) and (15) is a jointly controllable, jointly observable system. From this and [11, Corollary 1], it follows that there exist $F_{ij}, \ ij \in \mathcal{I}$ such that for all $p \in \mathbf{m}$ and all $q \in \mathcal{N}_p$, the matrix pairs (H, B_p) and (C_{pq}, H) controllable and observable, respectively. Since this also must be true generically for almost all F_{ij} the proposition is true. ■

VII. Concluding Remarks

In this paper, we have explained how to construct a family of distributed observers for a given neighbor graph $\mathbb{N}$ that are capable of estimating the state of the system (1) and (2) at a preassigned but arbitrarily fast convergence rate. There are many additional issues to be addressed. For example, how might one construct distributed observers of least dimension, which can estimate x? Accomplishing this will almost certainly require the transmission to each agent i from each neighbors j, the signal y_j, which agent j measures. This of course comes at a price, so there is a tradeoff to be studied between required observer dimension on one hand and the amount of information to be transferred across the network on the other. Another issue of importance would be to try to construct a distributed observer for the case when $\mathbb{N}$ changes over time; of course this problem will call for a different type of mathematics since the equations involved will be time-varying systems. Finally, it would be useful to try to determine how to construct distributed observers when in place of (2), one has $\dot{x} = Ax + \sum_{i=1}^{m} B_i u_i$, where u_i is an input signal that can be measured by agent i. Some of these problems will be addressed in the future.

One of the shortcoming of the distributed observer discussed in this paper is that it is an inherently "nonresilient" algorithm. By a *resilient algorithm* for a distributed process is meant an algorithm that, by exploiting built-in network and data redundancies, is able to continue to function correctly in the face of abrupt changes in the number of vertices and arcs in the neighbor graph upon which the algorithm depends. Such changes might arise as a result of a network communication failure, a component failure, a sensor temporarily being put to sleep to conserve energy, or even possibly a malicious attack. We conjecture that this lack of resilience is common to all distributed estimators, which are linear time-invariant systems. A "hybrid observer" has recently been proposed that can overcome this limitation [19].

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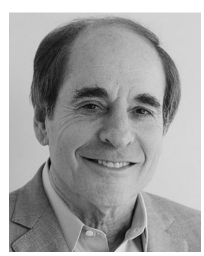

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