1 Luenberger Observer

Let the following system be given,

$$\dot{x} = Ax + B_w w,\ y = Cx + w,\ \dot{w} = 0$$

The classical Luenberger observer is given by,

$$\dot{\hat{x}} = A\hat{x} + B_w\hat{w} + L(y - \hat{y}),\ \dot{\hat{w}} = L_w(y - \hat{y}),\ \hat{y} = C\hat{x} + \hat{w}$$

The error dynamics are obtained as,

$$\begin{aligned} \dot{e} &= Ax + B_w w - A\hat{x} - L(Cx + w - C\hat{x} - \hat{w}) \\ \dot{e} &= (A - LC)e + (B_w - L)e_w \end{aligned}$$

and

$$\begin{aligned} \dot{e}_w &= -L_w(Cx + w - C\hat{x}) \\ \dot{e}_w &= -L_w Ce \end{aligned}$$

or put together the following is obtained,

$$\begin{bmatrix} \dot{e} \\ \dot{e}_w \end{bmatrix} = \begin{bmatrix} A - LC & B_w - L \\ -L_w C & 0 \end{bmatrix} \begin{bmatrix} e \\ e_w \end{bmatrix}$$

Using the projection

$$\Pi = I - \frac{CC^T}{C^TC}$$

the signal w is decomposed as

$$w = C\alpha + C^{\perp}\beta$$

and the system is re-stated as

$$\begin{aligned} \dot{x} &= Ax + B_w(C\alpha + C^{\perp}\beta) \\ y &= Cx + (C\alpha + C^{\perp}\beta) \\ \dot{\alpha} &= \dot{\beta} = 0 \end{aligned}$$

The classical Luenberger observer is given by,

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + B_w(C\hat{\alpha} + C^{\perp}\hat{\beta}) + L(y - \hat{y})\\ \dot{\hat{\alpha}} &= L_w(y - \hat{y})\\ \dot{\hat{\beta}} &= L_w(y - \hat{y})\\ \hat{y} &= C\hat{x} + C\hat{\alpha} + C^{\perp}\hat{\beta}\end{aligned}$$

The error dynamics are,

$$\begin{bmatrix}\dot{e}\\ \dot{e}_\alpha\\ \dot{e}_\beta\end{bmatrix} = \begin{bmatrix}A - LC & (B_w - L)C & (B_w - L)C^{\perp}\\ -L_wC & -L_wC & -L_wC^{\perp}\\ -L_wC & -L_wC & -L_wC^{\perp}\end{bmatrix}\begin{bmatrix}e\\ e_\alpha\\ e_\beta\end{bmatrix}$$

Since,

$$\Pi y = \Pi C x + \Pi C\alpha + \Pi C^{\perp}\beta = C^{\perp}\beta$$

the GI Luenberger observer is given by,

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + B_wC\hat{\alpha} + L(y - \hat{y}) + B_w\Pi y\\ \dot{\hat{\alpha}} &= L_w(y - \hat{y})\\ \hat{y} &= C\hat{x} + C\hat{\alpha} + \Pi y\end{aligned}$$

and the corresponding error dynamics are given by,

$$\begin{aligned}\dot{e} &= Ax + B_w(C\alpha + C^{\perp}\beta) - A\hat{x} - B_wC\hat{\alpha} - L(y - \hat{y}) - B_w\Pi y\\ \dot{e} &= Ax + B_w(C\alpha + C^{\perp}\beta) - A\hat{x} - B_wC\hat{\alpha}\\ &\quad - L(Cx + C\alpha + C^{\perp}\beta - C\hat{x} - C\hat{\alpha} - C^{\perp}\beta) - B_wC^{\perp}\beta\\ \dot{e} &= (A - LC)e + (B_w - L)Ce_\alpha\end{aligned}$$

and

$$\begin{aligned}\dot{e}_\alpha &= -L_w(Cx + C\alpha + C^{\perp}\beta - C\hat{x} - C\hat{\alpha} - C^{\perp}\beta)\\ \dot{e}_\alpha &= -L_wCe + -L_wCe_\alpha\end{aligned}$$

hence,

$$\begin{bmatrix}\dot{e}\\ \dot{e}_\alpha\end{bmatrix} = \begin{bmatrix}A - LC & (B_w - L)C\\ -L_wC & -L_wC\end{bmatrix}\begin{bmatrix}e\\ e_\alpha\end{bmatrix}$$