1 Luenberger Observer

Let the following system be given,

$$\dot{x} = Ax + B_w(C\alpha + C^{\perp}\beta),\; y = Cx + (C\alpha + C^{\perp}\beta)$$

The classical Luenberger observer is given by,

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}),\; \hat{y} = C\hat{x}$$

Using the projection

$$\Pi = I - \frac{CC^T}{C^TC}$$

the GI-Luenberger observer can be defined as,

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}) + B_w\Pi y,\; \hat{y} = C\hat{x} + \Pi y$$

which is in explicit form,

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}) + B_wC^{\perp}\beta,\; \hat{y} = C\hat{x} + C^{\perp}\beta$$

where $\Pi C = 0$, $\Pi C^{\perp} = C^{\perp}$ and $C^TC^{\perp} = 0$. The error dynamics of the classical Luenberger observer is obtained as,

$$\begin{aligned}
\dot{e} &= Ax + B_wC\alpha + B_wC^{\perp}\beta - A\hat{x} - L(Cx + C\alpha + C^{\perp}\beta - C\hat{x})\\
\dot{e} &= (A - LC)e + (B_wC - LC)\alpha + (B_wC^{\perp} - LC^{\perp})\beta\\
\dot{e} &= \boxed{(A - LC)e + (B_w - L)C\alpha + (B_w - L)C^{\perp}\beta}
\end{aligned}$$

where the GI-Luenberger observer is obtained as,

$$\begin{aligned}
\dot{e} &= Ax + B_wC\alpha + B_wC^{\perp}\beta - A\hat{x} - L(Cx + C\alpha + C^{\perp}\beta - C\hat{x} - \Pi y) - B_w\Pi y\\
\dot{e} &= Ax + B_wC\alpha + B_wC^{\perp}\beta - A\hat{x} - L(Cx + C\alpha + C^{\perp}\beta - C\hat{x} - C^{\perp}\beta) - B_wC^{\perp}\beta\\
\dot{e} &= \boxed{(A - LC)e + (B_w - L)C\alpha}
\end{aligned}$$

From the error dynamics of GI-Luenberger observer stating,

$$||(B_w - L)C|| = 0$$

the ideal case is obtained as

$$B_w = L$$

for complete invariance respect to α, but practically the following $\mathcal{H}_\infty$ problem is more applicable,

$$\begin{aligned} \min_L \gamma\, s.t. \\ P \succ 0 \\ \begin{bmatrix} (A-LC)^T P + P(A-LC) & P(B_w - L)C \\ PC^T(B_w - L)^T & -\gamma I \end{bmatrix} \prec 0 \end{aligned}$$

which is results in the following LMI problem,

$$\begin{aligned} \min_L \rho\, s.t. \\ P \succ 0 \\ \begin{bmatrix} A^T P + PA - YC - C^T Y^T & (PB_w - Y)C \\ C^T(PB_w)^T - Y^T & -\rho I \end{bmatrix} \prec 0 \\ L = P^{-1}Y,\, \gamma = \sqrt{\rho} \end{aligned}$$

2 PI Observer

Let the following system be given,

$$\dot{x} = Ax + B_w(C\alpha + C^\perp \beta),\ y = Cx + (C\alpha + C^\perp\beta)$$

The classical PI observer is given by,

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}) + L_i q,\ \dot{q} = y - \hat{y},\ \hat{y} = C\hat{x}$$

Using the projection

$$\Pi = I - \frac{CC^T}{C^T C}$$

the GI-PI observer can be defined as,

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}) + L_i q + B_w \Pi y,\ \hat{y} = C\hat{x} + \Pi y,\ ,\dot{q} = y - \hat{y}$$

which is in explicit form,

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}) + L_i q + B_w C^\perp \beta,\ \hat{y} = C\hat{x} + C^\perp\beta,\ ,\dot{q} = y - \hat{y}$$

where $\Pi C = 0$, $\Pi C^\perp = C^\perp$ and $C^T C^\perp = 0$.

The error dynamics of the classical PI observer is obtained as,

$$
\begin{aligned}
\dot{e} &= Ax + B_w C\alpha + B_w C^{\perp}\beta - A\hat{x} - L(Cx + C\alpha + C^{\perp}\beta - C\hat{x}) - L_i q \\
\dot{e} &= (A - LC)e + (B_w C - LC)\alpha + (B_w C^{\perp} - LC^{\perp})\beta - L_i q \\
\dot{e} &= \boxed{(A - LC)e + (B_w - L)C\alpha + (B_w - L)C^{\perp}\beta - L_i q}
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{q} &= Cx + C\alpha + C^{\perp}\beta - C\hat{x} - L_i q \\
\dot{q} &= \boxed{Ce - L_i q + C\alpha + C^{\perp}\beta}
\end{aligned}
$$

where the GI-PI observer is obtained as,

$$
\begin{aligned}
\dot{e} &= Ax + B_w C\alpha + B_w C^{\perp}\beta - A\hat{x} - L(Cx + C\alpha + C^{\perp}\beta - C\hat{x} - \Pi y) - B_w \Pi y - L_i q \\
\dot{e} &= Ax + B_w C\alpha + B_w C^{\perp}\beta - A\hat{x} - L(Cx + C\alpha + C^{\perp}\beta - C\hat{x} - C^{\perp}\beta) - B_w C^{\perp}\beta - L_i q \\
\dot{e} &= \boxed{(A - LC)e + (B_w - L)C\alpha - L_i q}
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{q} &= Cx + C\alpha + C^{\perp}\beta - C\hat{x} - C^{\perp}\beta - L_i q \\
\dot{q} &= \boxed{Ce - L_i q + C\alpha}
\end{aligned}
$$

3 Luenberger Observer

Let the following system be given,

$$\dot{x} = Ax + B_w w, \; y = Cx + w, \; \dot{w} = 0$$

The classical Luenberger observer is given by,

$$\dot{\hat{x}} = A\hat{x} + B_w \hat{w} + L(y - \hat{y}), \; \dot{\hat{w}} = L_w(y - \hat{y}), \; \hat{y} = C\hat{x} + \hat{w}$$

The error dynamics are obtained as,

$$
\begin{aligned}
\dot{e} &= Ax + B_w w - A\hat{x} - L(Cx + w - C\hat{x} - \hat{w}) \\
\dot{e} &= (A - LC)e + (B_w - L)e_w
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{e}_w &= -L_w(Cx + w - C\hat{x}) \\
\dot{e}_w &= -L_w Ce
\end{aligned}
$$

or put together the following is obtained,

$$\begin{bmatrix} \dot{e} \\ \dot{e}_w \end{bmatrix} = \begin{bmatrix} A - LC & B_w - L \\ -L_w C & 0 \end{bmatrix} \begin{bmatrix} e \\ e_w \end{bmatrix}$$

Using the projection

$$\Pi = I - \frac{CC^T}{C^T C}$$

the signal w is decomposed as

$$w = C\alpha + C^{\perp}\beta$$

and the system is re-stated as

$$\begin{aligned} \dot{x} &= Ax + B_w(C\alpha + C^{\perp}\beta) \\ y &= Cx + (C\alpha + C^{\perp}\beta) \\ \dot{\alpha} &= \dot{\beta} = 0 \end{aligned}$$

The classical Luenberger observer is given by,

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_w(C\hat{\alpha} + C^{\perp}\hat{\beta}) + L(y - \hat{y}) \\ \dot{\hat{\alpha}} &= L_w(y - \hat{y}) \\ \dot{\hat{\beta}} &= L_w(y - \hat{y}) \\ \hat{y} &= C\hat{x} + C\hat{\alpha} + C^{\perp}\hat{\beta} \end{aligned}$$

The error dynamics are,

$$\begin{bmatrix} \dot{e} \\ \dot{e}_\alpha \\ \dot{e}_\beta \end{bmatrix} = \begin{bmatrix} A - LC & (B_w - L)C & (B_w - L)C^{\perp} \\ -L_w C & -L_w C & -L_w C^{\perp} \\ -L_w C & -L_w C & -L_w C^{\perp} \end{bmatrix} \begin{bmatrix} e \\ e_\alpha \\ e_\beta \end{bmatrix}$$

Since,

$$\Pi y = \Pi C x + \Pi C\alpha + \Pi C^{\perp}\beta = C^{\perp}\beta$$

the GI Luenberger observer is given by,

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_w C\hat{\alpha} + L(y - \hat{y}) + B_w \Pi y \\ \dot{\hat{\alpha}} &= L_w(y - \hat{y}) \\ \hat{y} &= C\hat{x} + C\hat{\alpha} + \Pi y \end{aligned}$$

and the corresponding error dynamics are given by,

$$
\begin{aligned}
\dot{e} &= Ax + B_w(C\alpha + C^{\perp}\beta) - A\hat{x} - B_w C\hat{\alpha} - L(y - \hat{y}) - B_w \Pi y \\
\dot{e} &= Ax + B_w(C\alpha + C^{\perp}\beta) - A\hat{x} - B_w C\hat{\alpha} \\
&\quad - L(Cx + C\alpha + C^{\perp}\beta - C\hat{x} - C\hat{\alpha} - C^{\perp}\beta) - B_w C^{\perp}\beta \\
\dot{e} &= (A - LC)e + (B_w - L)Ce_{\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{e}_{\alpha} &= -L_w(Cx + C\alpha + C^{\perp}\beta - C\hat{x} - C\hat{\alpha} - C^{\perp}\beta) \\
\dot{e}_{\alpha} &= -L_w Ce + -L_w Ce_{\alpha}
\end{aligned}
$$

hence,

$$
\begin{bmatrix} \dot{e} \\ \dot{e}_{\alpha} \end{bmatrix} = \begin{bmatrix} A - LC & (B_w - L)C \\ -L_w C & -L_w C \end{bmatrix} \begin{bmatrix} e \\ e_{\alpha} \end{bmatrix}
$$