In Catilinam IV [⋆]

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Abstract

Cum M. Cicero consul Nonis Decembribus senatum in aede Iovis Statoris consuleret, quid de iis coniurationis Catilinae sociis fieri placeret, qui in custodiam traditi essent, factum est, ut duae potissimum sententiae proponerentur, una D. Silani consulis designati, qui morte multandos illos censebat, altera C. Caesaris, qui illos publicatis bonis per municipia Italiae distribuendos ac vinculis sempiternis tenendos existimabat.

Key words: Cicero; Catiline; orations.

1 Introduction

Gauge invariance is a central organizing principle of modern physics, expressing the idea that observable quantities should remain unchanged under certain local transformations of the mathematical description. In classical field theories and quantum electrodynamics, this concept ensures that different representations related by gauge transformations correspond to the same physical reality [3,5]. From an abstract viewpoint, gauge invariance provides a systematic way to separate physically meaningful dynamics from redundant coordinate or potential offsets. This symmetry-based perspective has proven essential for constructing robust and consistent models in physics, and it motivates the extension of invariance principles beyond their traditional domain toward engineered dynamical systems subject to measurement distortions and unknown offsets.

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2 Work

2.1 The System Model

Let $x \in \mathbb{R}^{n\times 1}$, $y \in \mathbb{R}^{p\times 1}$ and $w \in \mathbb{R}^{r\times 1}$ be the state, measurement, exogenous vectors, respectively, the SIMO/MIMO LTI system addressed in this paper is stated as,

$$\dot{x} = Ax + B_w w + B_u u,\ y = Cx + D_w w \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_w \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{p\times n}$ and $D_w \in \mathbb{R}^{p\times r}$. The following additional rank condition

$$\mathrm{rank}(C) = r < p \tag{2}$$

arises in sparse sensor applications[4], topologies used in Multi Agen Systems(MAS)[2] and distributed networks [7], and is also called Strictly Output Redundant(SOR) system [6]. Here, the system output is overdetermined, therefore,

$$\dim \mathcal{N}(C^T) = p - r \geq 1 \tag{3}$$

or using orthogonality $C^T C^\perp = 0$,

$$\dim \mathcal{N}(C^\perp) = p - r \geq 1 \tag{4}$$

which ensures nontrivial orthogonal component in the output space.

2.2 The Observer Models

The classical Luenberger Observer is defined as,

$$\dot{\hat{x}} = A\hat{x} + B_u u + L(y - \hat{y}),\ \hat{y} = C\hat{x} \tag{5}$$

where $L \in \mathbb{R}^{n\times p}$ is the observer gain. Assuming (A, C)-pair observable, the error dynamics are obtained for the error $e \triangleq x - \hat{x}$ as,

$$\dot{e} = (A - LC)e + (B_w - LD_w)w \tag{6}$$

The $\mathbb{H}_\infty$ optimal observer for objective function $z = C_z e$ is designed using the following LMI problem[1],

$$\begin{gathered}\min_{Y,P}(\gamma) \quad \text{s.t.} \\ \begin{bmatrix} (PA - YC) + \star^T & PB_w - YD_w & C_z^T \\ \star & -\gamma I & 0 \\ \star & 0 & -\gamma I \end{bmatrix} \prec 0 \\ P \succ 0 \end{gathered}$$

where the observer gain is recovered with $L = P^{-1}Y$.

2.3 The Projection

The projection operator defined using Gauge-Invariance principle is "proposed" as follows,

$$\Pi \triangleq I - C(C^TC)^{-1}C^T \tag{7}$$

which satisfies the following properties,

$$\begin{aligned} &\Pi C = (I - C(C^TC)^{-1}C^T)C = 0 \\ &\Pi C^\perp = (I - C(C^TC)^{-1}C^T)C^\perp = C^\perp,\ C^TC^\perp = 0 \\ &\Pi^2 = \Pi,\ \Pi \in \mathbb{R}^{p\times p} \end{aligned} \tag{8}$$

Using the projection operator on the measured output yields,

$$\Pi y = \Pi(Cx + D_w w) = \Pi D_w w \tag{9}$$

The projection eliminates the state dependent component of the output, isolating the exogenous component establishing state invariance. This is an algebraic measurement of the disturbance reflected to the output.The exogenous input is recovered under the following condition,

$$\text{rank}(\Pi D_w) = r \tag{10}$$

or equivalently,

$$\ker(\Pi D_w) = 0. \tag{11}$$

with the least square estimator,

$$\hat{\omega} = (\Pi D_w)^\dagger(\Pi y) \tag{12}$$

where $(.)^\dagger$ is the Moore-Penrose pseudoinverse. The exogenous signal recovery fails if

$$D_w w \in \text{Im}(C), \tag{13}$$

otherwise,

$$\hat{\omega} \to \omega \tag{14}$$

2.4 The GI-Luenberger Observer

The classical Luenberger observer is fed with the sensed exogenous input forming the GI-Luenberger observer, the observer expression becomes,

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_w\hat{\omega} + B_u u + L(y - \hat{y}) \\ \hat{y} &= C\hat{x} + D_w\hat{\omega} \\ \hat{\omega} &= (\Pi D_w)^\dagger(\Pi y) \end{aligned} \tag{15}$$

the error dynamics are obtained as,

$$\dot{e} = (A - LC)e \tag{16}$$

The exogenous input is recovered since,

$$\Pi y = \Pi Cx + \Pi D_w w = \Pi D_w w \tag{17}$$

and,

$$(\Pi D_w)^\dagger \Pi y = (\Pi D_w)^\dagger(\Pi D_w)w = w \tag{18}$$

under the following condition, where $D_w w$ is partitioned as,

$$D_w = C\alpha + C^\perp\beta,\ \beta \neq 0 \tag{19}$$

since,

$$\Pi D_w = \Pi C\alpha + \Pi C^\perp\beta = \Pi C^\perp\beta,\ \beta \neq 0 \tag{20}$$

and finally,

$$\Pi D_w \neq 0 \tag{21}$$

2.5 Robust Observer

Let the system be given as,

$$\dot{x} = Ax + B_w w,\ y = (C_0 + \Delta C_0)x + D_w w \tag{22}$$

where $||\Delta||_\infty \leq 1$, the projection,

$$\Pi \triangleq I - C_0(C_0^T C_0)^{-1}C_0^T \tag{23}$$

on the output gives,

$$\Pi(C_0x + \Delta C_0 x + D_w w) = \Pi\Delta C_0 x + \Pi D_w w \tag{24}$$

The uncertainty model,

$$\Delta C_0 = H\Delta E \tag{25}$$

where $\Delta^T\Delta \leq I$ or $||\Delta||_\infty \leq 1$. Hence,

$$\Pi y = \Pi H\Delta E x + \Pi D_w w \tag{26}$$

For

$$C_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{27}$$

Case 1: $H = C_0$, $\Delta = \delta$, and $E = 1$ hence,

$$C = C_0 + C_0\delta = C_0(1+\delta) \tag{28}$$

Case 2:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tag{29}$$

and

$$\Delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \tag{30}$$

and $E = 1$ hence,

$$C = C_0 + \begin{bmatrix} 2 \\ 0 \end{bmatrix}\delta_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\delta_2 \tag{31}$$

where $\delta_1 \neq \delta_2$.

2.6 Recovery of w and Δ with projection

The system output with uncertainty is given as,

$$y = (C_0 + H\Delta E)x + D_w w \tag{32}$$

where the projection

$$\Pi \triangleq I - C_0(C_0^T C_0)^{-1}C_0^T \tag{33}$$

is applied to the output, hence,

$$\begin{aligned} \Pi y &= \Pi C_0 + \Pi H\Delta E x + \Pi D_w w \\ \Pi y &= \Pi H\Delta E x + \Pi D_w w \end{aligned} \tag{34}$$

is obtained. The following is obtained to factor out $\Delta E x$,

$$\begin{aligned} (\Pi H)^\dagger \Pi y &= (\Pi H)^\dagger \Pi H\Delta E x + (\Pi H)^\dagger \Pi D_w w \\ H^\dagger \Pi^\dagger \Pi y &= \Delta E x + (\Pi H)^\dagger \Pi D_w w \\ H^\dagger \Pi y &= \Delta E x + H^\dagger \Pi^\dagger \Pi D_w w \\ H^\dagger \Pi y &= \Delta E x + H^\dagger \Pi D_w w \end{aligned} \tag{35}$$

and to factor out w,

$$\begin{aligned} (\Pi D_w)^\dagger \Pi y &= (\Pi H)^\dagger \Pi H\Delta E x + (\Pi H)^\dagger \Pi D_w w \\ D_w^\dagger \Pi^\dagger \Pi y &= (\Pi D_w)^\dagger \Pi H\Delta E x + (\Pi D_w)^\dagger \Pi D_w w \\ D_w^\dagger \Pi y &= (\Pi D_w)^\dagger \Pi H\Delta E x + w \\ &= D_w^\dagger \Pi^\dagger \Pi H\Delta E x + w \\ D_w^\dagger \Pi y &= D_w^\dagger \Pi H\Delta E x + w \end{aligned} \tag{36}$$

Both Eq 35 and Eq 36 are combined into,

$$\begin{bmatrix} I & H^\dagger \Pi D_w \\ D_w^\dagger \Pi H & I \end{bmatrix} \begin{bmatrix} \Delta E x \\ w \end{bmatrix} = \begin{bmatrix} H^\dagger \\ D_w^\dagger \end{bmatrix} \Pi y \tag{37}$$

The singularity,

$$\begin{vmatrix} I & H^\dagger \Pi D_w \\ D_w^\dagger \Pi H & I \end{vmatrix} = I - H^\dagger \Pi D_w D_w^\dagger \Pi H \tag{38}$$

is obtained. w is recovered with,

$$\begin{aligned} w &= \frac{\begin{vmatrix} I & H^\dagger \\ D_w^\dagger \Pi H & D_w^\dagger \end{vmatrix}}{\begin{vmatrix} I & H^\dagger \Pi D_w \\ D_w^\dagger \Pi H & I \end{vmatrix}} \\ &= \frac{D_w^\dagger - H^\dagger D_w^\dagger \Pi H}{I - H^\dagger \Pi D_w D_w^\dagger \Pi H} \end{aligned} \tag{39}$$

3 Numerical Example

3.1 Example 1

The system is given as, $A = -1$, $B_w = 1$,

$$C_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, D_w = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, E = 1 \tag{40}$$

The uncertainty is modeled as,

$$\begin{aligned} C_0 + H\Delta E &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \\ &= \begin{bmatrix} 1+\delta_1 \\ 2+2\delta_2 \end{bmatrix} \end{aligned} \tag{41}$$

The projection is calculated as,

$$\begin{aligned} \Pi_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 0.2 \end{bmatrix} \end{aligned} \tag{42}$$

The output is explicitly written as,

$$y = \begin{bmatrix} 1+\delta_1 \\ 2+2\delta_2 \end{bmatrix} x + \begin{bmatrix} 3 \\ 3 \end{bmatrix} w \tag{43}$$

The projection is applied and the following is obtained,

$$\begin{aligned} \Pi_0 y &= \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 0.2 \end{bmatrix} \begin{bmatrix} 1+\delta_1 \\ 2+2\delta_2 \end{bmatrix} x + \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 0.2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} w \\ &= \begin{bmatrix} 0.8\delta_1 - 0.8\delta_2 \\ 0.4\delta_2 - 0.4\delta_1 \end{bmatrix} + \begin{bmatrix} 1.2 \\ -0.6 \end{bmatrix} w \end{aligned} \tag{44}$$

Premultiplying the equation with

$$(\Pi_0 H)^\dagger = \begin{bmatrix} 0.5 & -0.25 \\ -0.5 & 0.25 \end{bmatrix} \tag{45}$$

gives,

$$(\Pi_0 H)^\dagger \Pi_0 y = \begin{bmatrix} 0.5\delta_1 - 0.5\delta_2 \\ 0.5\delta_2 - 0.5\delta_1 \end{bmatrix} x + \begin{bmatrix} 0.7500 \\ -0.7500 \end{bmatrix} w \tag{46}$$

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A A summary of Latin grammar

B Some Latin vocabulary