1 Gauge-Invariance

Let

$$\dot{x} = Ax,\ y = Cx + b,\ \dot{b} = 0$$

be given. The bias term is modeled as,

$$b \triangleq \alpha C + \beta C^{\perp}$$

where $C^T C^{\perp} = (C^{\perp})^T C = 0$ or alternatively $C \perp C^{\perp}$, and the projection matrix is chosen as,

$$\Pi \triangleq I - \frac{CC^T}{C^T C}$$

The following can be stated,

$$\begin{aligned} \Pi C &= \left(I - \frac{CC^T}{C^T C} \right) C \\ &= C - \frac{CC^T C}{C^T C} \\ &= C - C \\ \Pi C &= 0 \end{aligned}$$

and

$$\begin{aligned} \Pi C^{\perp} &= \left(I - \frac{CC^T}{C^T C} \right) C^{\perp} \\ &= C^{\perp} - \frac{CC^T C^{\perp}}{C^T C} \\ \Pi C^{\perp} &= C^{\perp} \end{aligned}$$

The projection matrix applied on the bias Πb gives,

$$\begin{aligned} \Pi b &= \Pi(\alpha C + \beta C^{\perp}) \\ &= \alpha \Pi C + \beta \Pi C^{\perp} \\ &= \beta C^{\perp} \end{aligned}$$

Π applied to the output gives,

$$\begin{aligned} \Pi y &= \Pi(Cx + \alpha C + \beta C^{\perp}) \\ &= \Pi Cx + \alpha \Pi C + \beta \Pi C^{\perp} \\ &= \beta \Pi C^{\perp} \\ &= \beta C^{\perp} \end{aligned}$$

and β can be recovered using,

$$\begin{aligned}
(C^{\perp})^T(\Pi y &= \beta C^{\perp}) \\
(C^{\perp})^T \Pi y &= (C^{\perp})^T \beta C^{\perp} \\
(C^{\perp})^T \Pi y &= \beta (C^{\perp})^T C^{\perp} \\
\beta &= \frac{(C^{\perp})^T \Pi y}{(C^{\perp})^T C^{\perp}}
\end{aligned}$$

Here, the bias perpendicular to the output vector C can be recovered but, the bias aligned with the output can not be recovered via the projection. This can also be observed by the following,

$$\begin{aligned}
\dot{x} = Ax,\ y = Cx + b,\ \dot{b} &= 0 \\
\dot{x} = Ax,\ y = Cx + \alpha C + \beta C^{\perp},\ \dot{\alpha C + \beta C^{\perp}} &= 0 \\
\dot{x} = Ax,\ y = C(x + \alpha) + \beta C^{\perp},\ \dot{\alpha} C + \dot{\beta} C^{\perp} &= 0 \\
\dot{x} = Ax,\ y = C(x + \alpha) + \beta C^{\perp},\ \dot{\alpha} = 0,\ \dot{\beta} &= 0
\end{aligned}$$

Defining $z \triangleq x + \alpha$,

$$\dot{z} = Az - \alpha A,\ y = Cz + \beta C^{\perp},\ \dot{\alpha} = 0,\ \dot{\beta} = 0$$

gives

$$\begin{bmatrix} \dot{z} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} A & -A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ \alpha \end{bmatrix},\ y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} z \\ \alpha \end{bmatrix} + \beta C^{\perp},\ x = \begin{bmatrix} I & -I \end{bmatrix} \begin{bmatrix} z \\ \alpha \end{bmatrix}$$

An observer as follows,

$$\begin{bmatrix} \dot{\hat{z}} \\ \dot{\hat{\alpha}} \end{bmatrix} = \begin{bmatrix} A & -A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{z} \\ \hat{\alpha} \end{bmatrix} + \begin{bmatrix} L \\ L_{\alpha} \end{bmatrix} (y - \hat{y}),\ \hat{y} = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} \hat{z} \\ \hat{\alpha} \end{bmatrix} + \Pi y,\ \hat{x} = \begin{bmatrix} I & -I \end{bmatrix} \begin{bmatrix} \hat{z} \\ \hat{\alpha} \end{bmatrix}$$

may be used to determine x and α where β is obtained using the projection.