Student Information

Name : Solution

ID :

Answer 1

a)

To find k, we must use the fact that PDF sums up to 1;

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy &= 1 \\
\int_{0}^{1}\int_{0}^{1} x + ky^3 dxdy &= 1 \\
\int_{0}^{1} \frac{x^2}{2} + ky^3 x \Big]_{x=0}^{x=1} dy &= 1 \\
\int_{0}^{1} \frac{1}{2} + ky^3 dy &= 1 \\
\frac{y}{2} + \frac{ky^4}{4}\Big]_{y=0}^{y=1} &= 1 \\
\frac{1}{2} + \frac{k}{4} &= 1 \\
\mathbf{k} &\mathbf{= 2}
\end{aligned}$$

b)

Since there are infinitely many outcomes associated with a continuous random variable, the probability of a specific outcome is 0. Hence, $P(X = \frac{1}{2}) = 0$

c)

$$
\begin{aligned}
P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2}) &= \int_0^{1/2} \int_0^{1/2} x + 2y^3 dx dy \\
&= \int_0^{1/2} \frac{x^2}{2} + 2y^3 x \Big]_{x=0}^{x=1/2} dy \\
&= \int_0^{1/2} \frac{1}{8} + y^3 dy \\
&= \frac{y}{8} + \frac{y^4}{4} \Big]_{y=0}^{y=1/2} \\
&= \frac{1}{16} + \frac{1}{64} = \frac{5}{64}
\end{aligned}
$$

Answer 2

a)

For $f(y)$ we need to integrate over x;

$$
\begin{aligned}
f(y) &= \int_0^{\infty} f_{X,Y}(x, y) dx \\
&= \int_0^{\infty} \frac{e^{(-y - \frac{x}{y})}}{y} dx
\end{aligned}
$$

Using the substitution $u = -y - \frac{x}{y}$; the integration becomes $\int_0^{\infty} e^u du = e^u$.

Reversing the substitution; $f(y) = e^{(-y - \frac{x}{y})} \Big]_{x=0}^{x=\infty}$.

Putting the values of x; $f(y) = 0 - (-e^{-y}) = e^{-y}$.
Here, Y follows an exponential distribution with $\lambda = 1$.

b)

Although the expected value of Y can be obtained by calculating $\int_0^{\infty} y e^{-y} dy$, its also simple to observe that Y follows an exponential distribution with $\lambda = 1$. Thus the expected value, $E(Y) = \frac{1}{\lambda} = 1$

Answer 3

a)

We can apply central limit theorem here. Lets first calculate the expected value and the variance;

$E(x) = 1000 * 0.1 = 100$ and $\sigma = \sqrt{1000 * 0.1 * 0.9} = 9.487$

Lets calculate the Z value;

$$Z = \frac{\hat{X} - E(x)}{\sigma} = \frac{89.5 - 100}{9.487} = -1.107$$

$P(Z \geq -1.11) = P(Z \leq 1.11) = 0.866$
Note that the values are approximate and different answers due to rounding are also accepted.

b)

The main difference will occur in the calculation of Z. The absolute value of the nominator will be multiplied by two, whereas the value of the denominator will be multiplied by the square root of two. This will result in a greater absolute Z value, and an increase in the probability. This makes sense, since the increase in the sample would make it closer to the population and the major's expected threshold was lower than the original percentage in the population. You can observe that if the whole population is included, the probability of achieving the threshold is 1.
Calculating the Z value;

$$Z = \frac{179.5 - 200}{\sqrt{180} = 1.53}$$

$P(Z \leq 1.53) = 0.937$

You can see the increase in probability.

Answer 4

a)

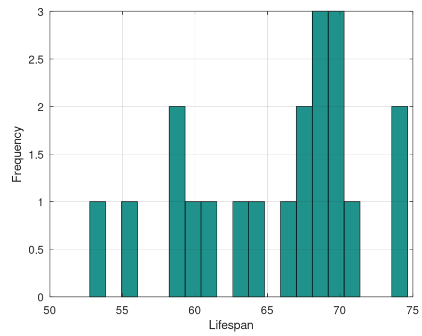
For $X_1 = 60,\ Z_1 = \dfrac{60 - 65}{6} = -0.833$
$P_1 = P(Z_1 \geq -0.833) = 0.7967$

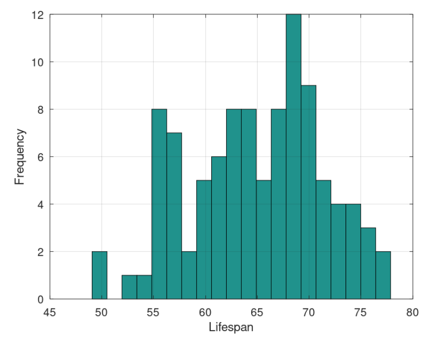
For $X_2 = 70,\ Z_1 = \dfrac{75 - 65}{6} = 1.666$
$P_2 = P(Z_2 \geq 1.666) = 1 - 0.9525 = 0.0475$

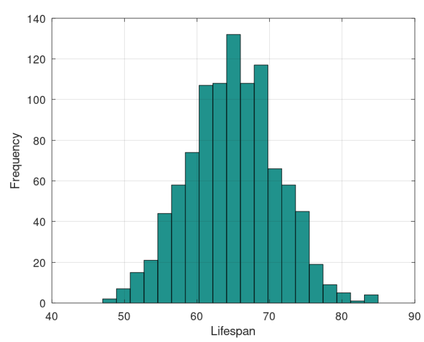
The probability that a randomly selected elephant will live more than 60 years, but less than 75 years is $P_1 - P_2 = 0.7967 - 0.0475 = 0.7492$

b)

```
n = 1000;
mean = 65;
var = 6;
lifespan = normrnd(mean, var, 1, n);
hist(lifespan, 25);
xlabel(”Lifespan”);
ylabel(”Frequency”);
grid on;
```







You can see smoothing of the histograms as sample size becomes larger.

c)

```
n = 100;
iter = 1000;
mean = 65;
var = 6;
lower = 60;
upper = 75;
count70 = 0;
count85 = 0;

for i = 1:iter
lifespan = normrnd(mean, var, 1, n);
x = sum(lifespan >= lower & lifespan <= upper) / n;
if(x >= 0.7)
count70++;
endif
if(x >= 0.85)
count85++;
endif
endfor

count70
count85
```

The output of the code is;
count70 = 909
count85 = 12

Let r denote the probability that a given elephant's lifespan would be in the range (60-75). It's calculated to be approximately 75% in part **a**. On a population of 100 elephants distribution of r would also follow the normal distribution with a mean around 75%. Observe that the limit of 70 is below this mean and would include the majority of the simulations; however, 85 corresponds to the right tail, making it a harder threshold.