Student Information

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Answer 1

a)

- We know that the variable X takes one and only one value x. This makes events $\{X = x\}$ disjoint and exhaustive, and therefore we get the following,

$$\sum_{x \in S} P(x) = \sum_{x \in S} \mathbf{P}\{X = x\} = 1 \text{ where } S = \{1, 2, 3, 4, 5\}$$

- Applying this to our variable x,

$$\sum_{x \in S} P(x) = \sum_{x \in S} \mathbf{P}\{X = x\} = N + \frac{N}{2} + \frac{N}{3} + \frac{N}{4} + \frac{N}{5} = 1$$

$$137N = 60 \rightarrow N = \frac{60}{137} \approx 0.438$$

b)

$$\begin{aligned}
\mathbf{E}(X) = \mu_x = \sum_{x \in S} xP(x) &= 1 \cdot P(1) + 2 \cdot P(2) + 3 \cdot P(3) + 4 \cdot P(4) + 5 \cdot P(5) \\
&= 1 \cdot \frac{60}{137} + 2 \cdot \frac{30}{137} + 3 \cdot \frac{20}{137} + 4 \cdot \frac{15}{137} + 5 \cdot \frac{12}{137} = \frac{300}{137} \approx 2.190
\end{aligned}$$

c)

$$\begin{aligned}
\mathbf{E}(X^2) = \sum_{x \in S} x^2 P(x) &= 1 \cdot P(1) + 4 \cdot P(2) + 9 \cdot P(3) + 16 \cdot P(4) + 25 \cdot P(5) \\
&= 1 \cdot \frac{60}{137} + 4 \cdot \frac{30}{137} + 9 \cdot \frac{20}{137} + 16 \cdot \frac{15}{137} + 25 \cdot \frac{12}{137} = \frac{900}{137} \approx 6.569
\end{aligned}$$

$$Var(X) = \mathbf{E}(X^2) - \mu_x^2 = 6.569 - (2.19)^2 \approx 1.774$$

d)

$$\begin{aligned}
\mathbf{E}(Y) = \sum_{y \in S} yP(y) &= 1 \cdot P(1) + 2 \cdot P(2) + 3 \cdot P(3) + 4 \cdot P(4) + 5 \cdot P(5) \\
&= 1 \cdot \frac{1}{15} + 2 \cdot \frac{2}{15} + 3 \cdot \frac{3}{15} + 4 \cdot \frac{4}{15} + 5 \cdot \frac{5}{15} = \frac{55}{15} \approx 3.667
\end{aligned}$$

$$\mathbf{E}(XY) = \sum_{y\in S}\sum_{x\in S} xyP(x,y) = \sum_{y\in S}\sum_{x\in S} xyP(x)P(y) \approx 8.029$$

$$Cov(X,Y) = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y) = 8.029 - 3.667 \cdot 2.190 \approx 0.000$$

- Since we've found the covariance of X and Y as zero, we can say that these two events are relatively independent.

Answer 2

a)

- We know the probability of the event "at least one attempt is successful in 1000 trials" is 95%, which is equal to:

$$\mathbf{P}\{X \geq 1\} = 1 - \mathbf{P}\{X = 0\} = 1 - \binom{1000}{0} p^0 q^{1000} = 0.95$$

- Here, p denotes the success and q denotes the failure.
- Solving for q we get:

$$\begin{aligned} q^{1000} &= 0.05 \rightarrow q \approx 0.997 \\ p = 1 - q &= 1 - 0.997 \approx 0.003 \end{aligned}$$

- Therefore, we get the success rate as approximately 0.003.

b)

- For part i) we can interpret the problem as follows:

$$\begin{aligned} \mathbf{P}\{X > 500\} = \mathbf{P}\{\text{more than 500 games needed to get 2 wins}\} \\ \mathbf{P}\{\text{there are fewer than 2 wins in 500 games}\} \\ \mathbf{P}\{Y < 2\} = \mathbf{P}\{Y \leq 1\} \approx 0.558 \text{ using binocdf on octave.} \end{aligned}$$

- Or we can calculate it in the following way:

$$\begin{aligned} \mathbf{P}\{Y \leq 1\} = \mathbf{P}\{Y = 0\} + \mathbf{P}\{Y = 1\} \\ \mathbf{P}\{Y = 0\} = \binom{500}{0} p^0 q^{500} = (0.997)^{500} \approx 0.223 \\ \mathbf{P}\{Y = 1\} = \binom{500}{1} p^1 q^{499} = 500 \cdot (0.003) \cdot (0.997)^{499} \approx 0.335 \\ \mathbf{P}\{Y \leq 1\} = \mathbf{P}\{Y = 0\} + \mathbf{P}\{Y = 1\} = 0.223 + 0.335 \approx 0.558 \end{aligned}$$

- For part ii) we can apply the same procedure:

$$\begin{aligned}\mathbf{P}\{X > 10,000\} = \mathbf{P}\{&\text{more than 10,000 games needed to get 2 wins}\}\\ \mathbf{P}\{&\text{there are fewer than 2 wins in 10,000 games}\}\\ \mathbf{P}\{Y < 2\} = \mathbf{P}\{Y \leq 1\} &\approx 0.736 \text{ using binocdf on octave.}\end{aligned}$$

- Or we can calculate it in the following way:

$$\mathbf{P}\{Y \leq 1\} = \mathbf{P}\{Y = 0\} + \mathbf{P}\{Y = 1\}$$
$$\mathbf{P}\{Y = 0\} = \binom{10,000}{0} p^0 q^{10,000} = (0.9999)^{10,000} \approx 0.368$$
$$\mathbf{P}\{Y = 1\} = \binom{10,000}{1} p^1 q^{9,999} = 10,000 \cdot (0.0001) \cdot (0.9999)^{9,999} \approx 0.368$$
$$\mathbf{P}\{Y \leq 1\} = \mathbf{P}\{Y = 0\} + \mathbf{P}\{Y = 1\} = 0.368 + 0.368 \approx 0.736$$

c)

- Let X be the number of days that we're not feeling sick. We can say that X is binomial with the values $n = 366$ and $p = 0.98$. We can't apply Poisson approximation on p since it's too large. However, we can apply Poisson approximation on $q = 0.02$ since it's value is small enough. Therefore we get the following equations:

$$\lambda = nq = (366) \cdot (0.02) = 7.32 \approx 7.5$$
$$\mathbf{P}\{X \geq 360\} = \mathbf{P}\{Y \leq 6\} = F_Y(6) = 0.378 \text{ from Table A3.}$$

Answer 3

a)

- We found our answer in **Q2c** by approximating the λ value to match the table's values, it's different from what we get in Octave. We used the λ value as 7.5 and found the result of approximately 0.378 in **Q2c**.
- However, when we use the λ value as 7.32 on Octave using the command **poisscdf(6, 7.32)**, we get the result of 0.403 and it's higher than what we've found in **Q2c**.
- Increasing the value of λ widens the Poisson distribution probability mass function (pmf), resulting in longer tails. This leads to lower cumulative distribution function (cdf) values at certain points. This is why we got a result which is higher that what we've found in **Q2c**.

b)

```
octave:1> p = 0.98;
ns = 50:400;
binomial_probabilities = binocdf(6, ns, 1-p);
poisson_probabilities = poisscdf(6, ns*(1-p));
close all;
plot(ns, binomial_probabilities, 'linewidth', 2);
hold on;
plot(ns, poisson_probabilities, '-.', 'linewidth', 2);
saveas(1, "p=0.98.png");
```

Figure 1: Code of the plot when $p = 0.98$.

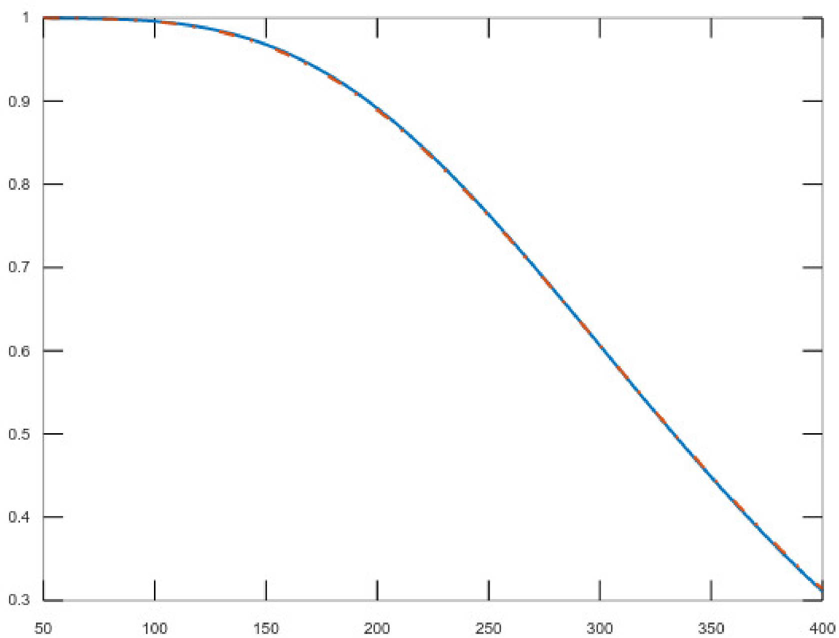

Figure 2: Graph of the plot when $p = 0.98$.

c)

```
octave:10> p = 0.78;
ns = 50:400;
binomial_probabilities = binocdf(6, ns, 1-p);
poisson_probabilities = poisscdf(6, ns*(1-p));
close all;
plot(ns, binomial_probabilities, 'linewidth', 2);
hold on;
plot(ns, poisson_probabilities, '-.', 'linewidth', 2);
saveas(1, "p=0.78.png");
```

Figure 3: Code of the plot when $p = 0.78$.

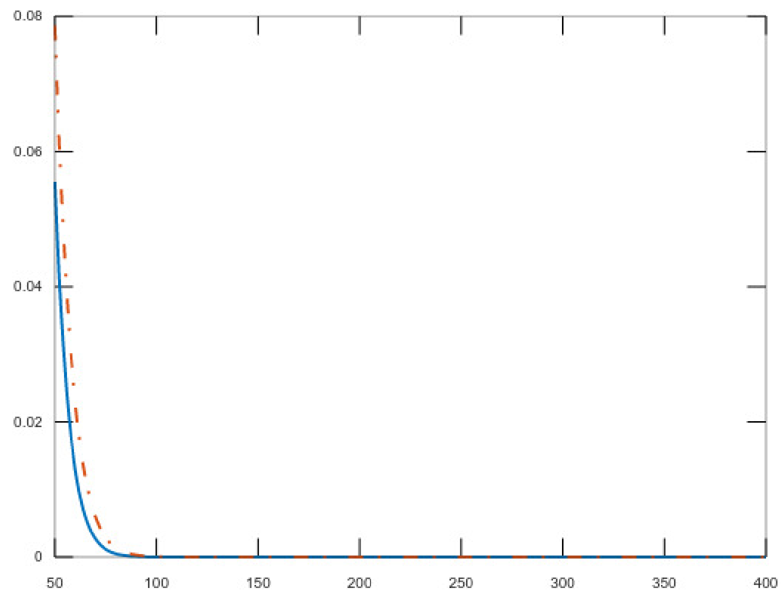

Figure 4: Graph of the plot when $p = 0.78$.

- In order to use Poisson distribution's approximation to the Binomial distribution we need to have a large n value and small p value. Since our question gave us the p value as 0.98 and it's too large, we used the q value as our success rate and interpreted the problem in a different way.

- Since our q is small, we can see that in the graph of part b), Poisson distribution's approximation is really close to the actual Binomial distribution. However, when our q value gets larger from 0.02 to 0.22, we can see that it's not as accurate as before. This outcome validates our understanding that as the sample size n increases and the success rate decreases, Poisson distribution tends to approximate the Binomial distribution.