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Answer 1

- Let us denote the number of edges in a cube graph as a_n.
- While constructing a cube graph, we are using the previous edges with a repeating manner. For example, while constructing Q_2, we take two copies of Q_1 and add 2 edges (since it has 2^{2-1} vertices) to it in order to connect the copies we use.
- Recursively, we can always take two copies of the previous cube graphs, and connect the vertices they have. This leads us having $2a_{n-1}$ edges coming from the previous cube graphs and 2^{n-1} edges to connect the vertices of the copies.
- Therefore, we get the following recurrence relation:

$$a_n = 2a_{n-1} + 2^{n-1},\ n \geq 1$$

Answer 2

- The generating function $< 1, 4, 7, 10, 13, ... >$ can be written as the summation of $< 1, 1, 1, 1, 1, ... >$ and $< 0, 3, 6, 9, 12, ... >$. Now let's find the closed form of these generating functions:

$$\frac{1}{1-x} \leftrightarrow < 1, 1, 1, 1, ... > \text{ from Table 1 on Section 8.4}$$
$$\frac{d}{dx}\left(\frac{1}{1-x}\right) \leftrightarrow < 1, 2, 3, 4, ... > \text{ taking the derivative of the function}$$
$$\frac{1}{(1-x)^2} \leftrightarrow < 1, 2, 3, 4, ... > \text{ from Table 1 on Section 8.4}$$
$$\frac{x}{(1-x)^2} \leftrightarrow < 0, 1, 2, 3, ... > \text{ shifting once to the right by multiplying with } x$$
$$\frac{3x}{(1-x)^2} \leftrightarrow < 0, 3, 6, 9, ... > \text{ multiplying with 3}$$

- Now we can sum these functions:

$$F(x) = \frac{1}{1-x} + \frac{3x}{(1-x)^2}$$
$$F(x) = \frac{1+2x}{(1-x)^2}$$

Answer 3

- We can denote $F(x) = \sum_{n=0}^{\infty} a_n x^n$, plugging into the equation we get:

$$\begin{aligned} \sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} (a_{n-1} + 2^n) x^n \\ &= x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=1}^{\infty} 2^n x^n \end{aligned}$$

- We can use the equations on Table 1 of Section 8.4:

$$F(x) - a_0 = xF(x) + \frac{1}{1-2x} - 1$$

- Since $a_0 = 1$ we get $F(x)$ as follows:

$$\begin{aligned} F(x) - 1 = xF(x) + \frac{1}{1-2x} - 1 \\ F(x) = xF(x) + \frac{1}{1-2x} \\ (1-x)F(x) = \frac{1}{1-2x} \\ F(x) = \frac{1}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x} \\ A - 2Ax + B - Bx = 1 \\ A + B = 1 \text{ and} - 2A - B = 0 \\ \text{we get } A = -1 \text{ and } B = 2 \\ F(x) = \frac{2}{1-2x} - \frac{1}{1-x} \end{aligned}$$

- Now we can use the corresponding generating functions:

$$\begin{aligned} \frac{1}{1-2x} \leftrightarrow < 1, 2, 4, 8, ..., 2^n, ... > \text{from Table 1 on Section 8.4} \\ \frac{1}{1-x} \leftrightarrow < 1, 1, 1, 1, ..., 1, ... > \text{from Table 1 on Section 8.4} \end{aligned}$$

- Multiply the first one by 2 and subtract the second one from it:

$$F(x) = \frac{2}{1-2x} - \frac{1}{1-x} \leftrightarrow < 1, 3, 7, 15, ..., 2^{n+1} - 1, ... >$$

- Since the n-th term will be equal to a_n, we get $a_n = 2^{n+1} - 1$.

Answer 4

a)

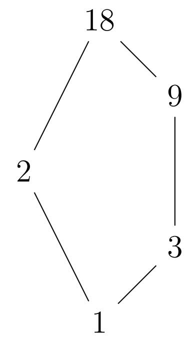

Figure 1: Hasse Diagram for relation R

b)

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 9 & 18 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 9 \\ 18 \end{array} &
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

Figure 2: Matrix representation for relation R

c)

- Let's see for every pair if we have a unique least upper bound (LUB) and unique greatest lower bound (GLB):
  - For (1,2) LUB is 2, GLB is 1.
  - For (1,3) LUB is 3, GLB is 1.
  - For (1,9) LUB is 9, GLB is 1.
  - For (1,18) LUB is 18, GLB is 1.
  - For (2,3) LUB is 18, GLB is 1.
  - For (2,9) LUB is 18, GLB is 1.
  - For (2,18) LUB is 18, GLB is 2.
  - For (3,9) LUB is 9, GLB is 3.

- For (3,18) LUB is 18, GLB is 3.
- For (9,18) LUB is 18, GLB is 9.

- We don't have to do these steps for the reflexive ones since their LUB and GLB will be the same. And also we don't have to do these steps for the pairs that are ordered the other way around since their LUB and GLB will be the same ones we've found before.
- Therefore we've shown that (A, R) is a lattice.

d)

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 9 & 18 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 9 \\ 18 \end{array} &
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\end{array}
$$

Figure 3: Matrix representation for the symmetric closure of R, R_s

e)

- By the definition of comparability, in order to identify 2 elements a and b as comparable with respect to the binary relation R, we need to have aRb or bRa as true. In other words we need to have (a, b) or (b, a) in our relation. Let's check this for 2 and 9:
  - $2R9 \rightarrow$ false since we don't have $(2, 9)$ in our relation.
  - $9R2 \rightarrow$ false since we don't have $(9, 2)$ in our relation.
- Now let's check it for 3 and 18:
  - $3R18 \rightarrow$ true since we have $(3, 18)$ in our relation.
  - $18R3 \rightarrow$ false since we don't have $(18, 3)$ in our relation.
- Therefore, we can say that 2 and 9 are not comparable, but 3 and 18 are comparable.

Answer 5

a)

- A reflexive relation on a set A should include all pairs (a, a) for each element in A. There are $\binom{n}{1} = n$ pairs. However, these pairs will not contribute to the number of relations that are reflexive and symmetric since they will already be included.

- For a relation to be symmetric, if it includes the pair (a, b), then it should also include the pair (b, a) where $a \neq b$ since we already included the ones where $a = b$. The number of distinct unordered pairs from a set with n elements is $\binom{n}{2} = \frac{n(n-1)}{2}$.
- For each pair we selected, we can either include both (a, b) and (b, a) or we don't include any of them, leaving us with 2 choices for each pair. There are $2^{\frac{n(n-1)}{2}}$ choices in total.
- Therefore, we can say that there are $2^{\frac{n(n-1)}{2}}$ relations that are both reflexive and symmetric.

b)

- The same thing for reflexivity will be applied here. The relations in the question should include all pairs (a, a) for each element in the set A.
- For a relation to be antisymmetric, if it contains the pair (a, b), then it shouldn't contain the pair (b, a) unless $a = b$. Again we have $\binom{n}{2} = \frac{n(n-1)}{2}$ distinct unordered pairs.
- For each pair we selected, we have 3 options:
  - We can include (a, b).
  - We can include (b, a).
  - We don't include any of them.
- There are $3^{\frac{n(n-1)}{2}}$ choices in total.
- Therefore, we can say that there are $3^{\frac{n(n-1)}{2}}$ ways to form a relation that is both reflexive and antisymmetric.

Answer 6

- No, the transitive closure of an antisymmetric relation is not always antisymmetric. Let's disprove this claim by giving a counterexample.
- Let's consider an antisymmetric relation R which is $R = \{(1, 2), (3, 4), (4, 1), (2, 3)\}$ on a set $A = \{1, 2, 3, 4\}$.
- We can denote the transitive closure of R as R_T and R_T is equal to:

$$R_T = \{(1, 2), (3, 4), (4, 1), (2, 3), (1, 3), (3, 1), (2, 4), (4, 2)\}$$

- We can see that R_T is not antisymmetric since it has $(1, 3)$ and $(3, 1)$ at the same time but $1 \neq 3$. It also has $(2, 4)$ and $(4, 2)$ but $4 \neq 2$.
- Therefore, we can say that the transitive closure of an antisymmetric relation is not always antisymmetric.