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Answer 1

a)

- **Base Case ($n = 1$):**

$$2^3 - 3 = 5$$

  - Since 5 is divisible by itself, we can see that the base case holds.

- **Inductive Step ($n \geq 1$):**
  - Assume that the property holds for some integer $k \geq 1$. We have the following:

$$5 \mid 2^{3k} - 3^k$$

  - Then we can see that for some integer c:

$$2^{3k} - 3^k = 5c$$

  - Let's also consider for the integer $k + 1$:

$$2^{3(k+1)} - 3^{(k+1)} = 2^3 \cdot 2^{3k} - 3 \cdot 3^k$$

  - We can replace 2^{3k} with $3^k + 5c$ (inductive hypothesis):

$$8 \cdot (3^k + 5c) - 3 \cdot 3^k = 5 \cdot 3^k + 40c = 5(3^k + 8c)$$

  - We can see that $5(3^k + 8)$ is divisible by 5.

- Therefore, we have shown that $2^{3n} - 3^n$ is divisible by 5 for all integers $n \geq 1$ by using mathematical induction.

b)

- **Base Case ($n = 2$):**

$$4^2 - 7 \cdot 2 - 1 = 1 > 0$$

  - We can see that the base case holds.

- **Inductive Step ($n \geq 2$):**
  - Assume that the property holds for some integer $k \geq 2$. We have the following:

$$4^k - 7k - 1 > 0$$

  - Let's also consider for the integer $k+1$.

$$4^{k+1} - 7(k+1) - 1 = 4 \cdot 4^k - 7k - 8$$

  - We can manipulate the inductive hypothesis as follows:

$$\begin{aligned} 4^k &> 7k + 1 \\ 4 \cdot 4^k &> 28k + 4 \end{aligned}$$

  - Going back to our equation using the inductive hypothesis:

$$4 \cdot 4^k - 7k - 8 > 28k + 4 - 7k - 8 = 21k - 4 > 0$$

  - Since $k \geq 2$ for all k values, we can see that $21k - 4 > 0$ for all k values and therefore, $4^{k+1} - 7(k+1) - 1 > 0$.
- Therefore, we have shown that $4^n - 7n - 1 > 0$ for all integers $n \geq 2$ by using mathematical induction.

Answer 2

a)

- In order to find the number of bit strings of length 10 which have at least seven 1s, we can consider the cases where:
  - Bit string has 7 1's.
  - Bit string has 8 1's.
  - Bit string has 9 1's.
  - Bit string has 10 1's.
- Let's calculate each one of the cases:
  - We can choose 7 positions for the ones, and the rest of them will be zeros.

$$C(10,7) = \binom{10}{7} = \frac{10!}{(10-7)! \cdot 7!} = 120$$

- We can choose 8 positions for the ones, and the rest of them will be zeros.

$$C(10,8) = \binom{10}{8} = \frac{10!}{(10-8)! \cdot 8!} = 45$$

- We can choose 9 positions for the ones, and one position to be zero.

$$C(10,9) = \binom{10}{9} = \frac{10!}{(10-9)! \cdot 9!} = 10$$

- We can set all of the positions for the ones.

$$C(10,10) = \binom{10}{10} = 1$$

- Summing all of the results up, we get $120 + 45 + 10 + 1 = 176$ bit strings.

b)

- Since 2 positions are taken by 1 Discrete Mathematics textbook and 1 Statistical Methods textbook, we have 2 positions left.
- We are going to pick 2 textbooks for our collections which can be done by either:
  - 1 Discrete Mathematics textbook and 1 Statistical Methods textbook
  - 2 Discrete Mathematics textbooks
  - 2 Statistical Methods textbooks
- So, there are 3 ways to form our collection of Discrete Mathematics and Statistical Methods textbooks.

c)

- We can calculate the number of onto functions from set from $A \to B$ where $|A| = m$ and $|B| = n$ by using inclusion-exclusion principle with the following formula:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m$$

- Plugging in $m = 5$, $n = 3$ we get:

$$\binom{3}{0} 3^5 - \binom{3}{1} (3-1)^5 + \binom{3}{2} (3-2)^5 - \binom{3}{3} (3-3)^5 = 243 - 96 + 3 - 0 = 150$$

- So, there are 150 onto functions that can be defined from a set with 5 elements to a set with 3 elements.

Answer 3

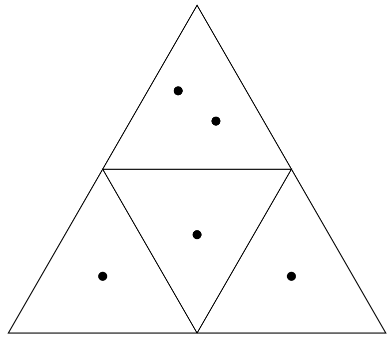

Figure 1. An equilateral triangle with side length of 500 meters.

- From Figure 1. it can be observed that there are 4 holes (equilateral triangle spots with side length of 250 meters) and 5 pigeons (children). By the Pigeonhole Principle, there must be at least 2 pigeons in the same hole.
- When 2 children are at the same hole, their distance can not exceed 250 meters, their max distance will be 250 meters and this occurs when they're at the corners of the hole.

Answer 4

- Let's consider the recurrence:

$$a_n = 3a_{n-1} + 5^{n-1}$$
$$a_n - 3a_{n-1} = 5^{n-1}$$

a)

- Homogeneous equation: $a_n - 3a_{n-1} = 0$
- Characteristic polynomial can be given as $\lambda - 3 = 0$ so the only characteristic root is $\lambda = 3$.
- Therefore, the solution is of the form $a_n^{(h)} = c_1 \cdot 3^n$.

b)

- Since 5 is not a characteristic root, the particular solution will be of the form $a_n^{(p)} = c_2 \cdot 5^n$.
- Plugging into the recurrence relation:

$$\begin{aligned} c_2 \cdot 5^n - 3(c_2 \cdot 5^{n-1}) &= 5^{n-1} \\ c_2 \cdot 5^n - \frac{3c_2 \cdot 5^n}{5} &= \frac{5^n}{5} \\ \frac{2c_2 \cdot 5^n}{5} &= \frac{5^n}{5} \\ c_2 &= \frac{1}{2} \end{aligned}$$

- Since we found both homogeneous and particular solutions, we can sum them up and find the total solution.

$$a_n = a_n^{(h)} + a_n^{(p)} = c_1 \cdot 3^n + \frac{5^n}{2} \text{ with the initial condition } a_1 = 4$$

$$a_1 = 3c_1 + \frac{5}{2} = 4 \text{ we get } c_1 = \frac{1}{2} \to a_n = \frac{3^n}{2} + \frac{5^n}{2}$$

c)

- **Base Case ($n = 1$):**

$$\frac{3^1}{2} + \frac{5^1}{2} = 4$$

  - We can see that the base case satisfies the initial condition $a_1 = 4$.

- **Inductive Step ($n \geq 1$):**
  - Assume that the property holds for some integer $k \geq 1$.
  - Then we have the following:

$$a_k = \frac{3^k + 5^k}{2}$$

  - Now using inductive hypothesis, let's show that it holds for the integer $k + 1$:

$$a_{k+1} - 3a_k = 5^k$$

$$a_{k+1} = \frac{3(3^k + 5^k)}{2} + 5^k = \frac{3^{k+1} + 3 \cdot 5^k + 2 \cdot 5^k}{2} = \frac{3^{k+1} + 5^{k+1}}{2}$$

- Therefore, we have shown that $a_n = \dfrac{3^n}{2} + \dfrac{5^n}{2}$ is a solution for all $n \geq 1$.