APMA 0350: Homework 6

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Problem 1: Solve the system and draw a phase portrait

$$x' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} x$$

Eigenvalues:

$$\begin{vmatrix} -1 & -4 \\ 1 & -1 \end{vmatrix} = (-1-\lambda)^2 + 4 = \lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(5)}}{2} = -1 \pm 2i$$

Eigenvector $\lambda = -1 + 2i$:

$$\left.\begin{matrix} -2i & -4 \\ 1 & -2i \end{matrix}\,\right|\,\begin{matrix} 0 \\ \end{matrix} \sim \left.\begin{matrix} -2i & -4 \\ 0 & 0 \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix}$$

$$\implies -2ix_1 - 4x_2 = 0$$

$$\vec{v}_{\lambda=-1+2i} = \begin{bmatrix} 2 \\ -i \end{bmatrix}$$

Solution:

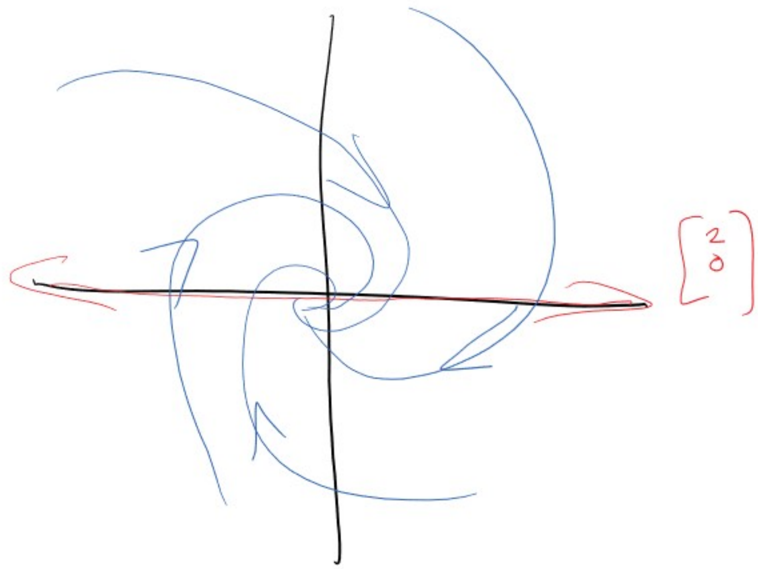
$$e^{(-1+2i)t} \begin{bmatrix} 2 + 0i \\ 0 - i \end{bmatrix} = e^{-t}\, e^{2it} \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

$$= e^{-t} \left(\cos 2t + i \sin 2t\right) \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

$$\boxed{\vec{x}(t) = C_1 e^{-t}\left(\cos 2t \begin{bmatrix}2\\0\end{bmatrix} - \sin 2t \begin{bmatrix}0\\-1\end{bmatrix}\right) + C_2 e^{-t}\left(\cos 2t \begin{bmatrix}0\\-1\end{bmatrix} + \sin 2t \begin{bmatrix}2\\0\end{bmatrix}\right)}$$

Phase portrait:

$$\vec{x}(t) = C_1 e^{-t}\left(\cos(2t)\begin{bmatrix}2\\0\end{bmatrix} - \sin(2t)\begin{bmatrix}0\\-2\end{bmatrix}\right) + C_2 e^{-t}\left(\cos(2t)\begin{bmatrix}0\\-2\end{bmatrix} + \sin(2t)\begin{bmatrix}2\\0\end{bmatrix}\right)$$



Problem 2: Solve the following system

1. $$x' = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Eigenvalues:

$$\begin{vmatrix} -3 & 2 \\ -1 & -1 \end{vmatrix} = (-2-\lambda)(-1-\lambda) + 2 = \lambda^2 + 4\lambda + 5 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16 - 4(5)}}{2} = -2 \pm i$$

Eigenvector for $\lambda = -2 - i$:

$$\left.\begin{matrix} -1+i & 2 \\ -1 & 1+i \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix} \sim \left.\begin{matrix} -1+i & 2 \\ 0 & 0 \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix}$$

$$\implies (-1+i)x_1 + 2x_2 = 0$$

$$\vec{v}_{\lambda=-2-i} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$$

Solution:

$$e^{(-2-i)t} \begin{bmatrix} 2+0i \\ 1-i \end{bmatrix} = e^{-2t} (\cos -2t + i \sin -2t) \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

$$\vec{x}(t) = C_1 e^{-2t} \left(\cos -2t \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \sin -2t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) + C_2 e^{-2t} \left(\cos -2t \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin -2t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$$

$$\vec{x}(0) = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$C_1 = \frac{1}{2}, \quad C_2 = \frac{5}{2}$$

$$\boxed{\vec{x}(t) = \frac{1}{2} e^{-2t} \left(\cos(-2t) \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \sin(-2t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) + \frac{5}{2} e^{-2t} \left(\cos(-2t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin(-2t) \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)}$$

2.

$$x' = \begin{bmatrix} 3 * -2 & \\ 8 & -5 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Eigenvalues:

$$\begin{vmatrix} 3 & -2 \\ 8 & -5 \end{vmatrix} = (-3-\lambda)(-5-\lambda) + 16 = \lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda+1)^2 = 0 \implies \lambda = -1$$

Eigenvector for $\lambda = -1$:

$$\begin{array}{cc|c} 4 & -2 & 0 \\ 8 & -4 & 0 \end{array} \sim \begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}$$

$$\implies 2x_1 - x_2 = 0$$

$$\vec{v}_{\lambda=-1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{array}{cc|c} 4 & -2 & 1 \\ 8 & -4 & 2 \end{array} \sim \begin{bmatrix} 4 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\implies 4x_1 - 2x_2 = 1$$

$$\vec{w} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$$

Solution:

$$\vec{x}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \left(te^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-t} \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \right)$$

$$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} C_2 \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$C_1 = 2, \quad C_2 = 4$$

$$\boxed{\vec{x}(t) = e^{-t} \begin{bmatrix} 2+4t \\ 2+8t \end{bmatrix}}$$

Problem 3: Use matrix exponentials to solve the following systems.

1.
$$x' = \begin{bmatrix} 7 & -2 \\ 10 & -2 \end{bmatrix} x$$

Eigenvalues:

$$\begin{vmatrix} 7 & -2 \\ 10 & -2 \end{vmatrix} = (7-\lambda)(-2-\lambda) + 20 = \lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 3)(\lambda - 2) = 0 \implies \lambda = \{3, 2\}$$

Eigenvector $\lambda = 3$:

$$\left.\begin{matrix} 4 & -2 \\ 10 & -5 \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix} \implies 2x_1 - x_2 = 0$$

$$\vec{v}_{\lambda=3} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eigenvector $\lambda = 2$:

$$\left.\begin{matrix} 5 & -2 \\ 10 & -4 \end{matrix}\,\right|\,\begin{matrix} 0 \\ 0 \end{matrix} \implies 5x_1 - 2x_2 = 0$$

$$\vec{v}_{\lambda=2} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

$$\vec{x}(t) = e^{At} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Pe^{Dt}P^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} e^{3t} & -2e^{3t} \\ -2e^{2t} & 5e^{2t} \end{bmatrix} = \begin{bmatrix} e^{3t} - 4e^{2t} & -2e^{3t} + 10e^{2t} \\ 2e^{3t} - 10e^{2t} & -4e^{3t} + 25e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\boxed{\vec{x}(t) = C_1 \begin{bmatrix} e^{3t} - 4e^{2t} \\ 2e^{3t} - 10e^{2t} \end{bmatrix} + C_2 \begin{bmatrix} -2e^{3t} + 10e^{2t} \\ -4e^{3t} + 25e^{2t} \end{bmatrix}}$$

2.

$$x' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} x$$

Eigenvalues:

$$\begin{vmatrix} 1 & -4 \\ 4 & -7 \end{vmatrix} = (1-\lambda)(-7-\lambda) + 16 = \lambda^2 + 6\lambda + 9$$

$$\lambda = -3$$

$$(A + 3I)^2 = 0$$

$$x = e^{At} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = e^{-3t} e^{(A+3I)t} = e^{-3t}(I + (A+3I)t)$$

$$= e^{-3t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} t = e^{-3t} \begin{bmatrix} 1+4t & -4t \\ 4t & 1-4t \end{bmatrix}$$

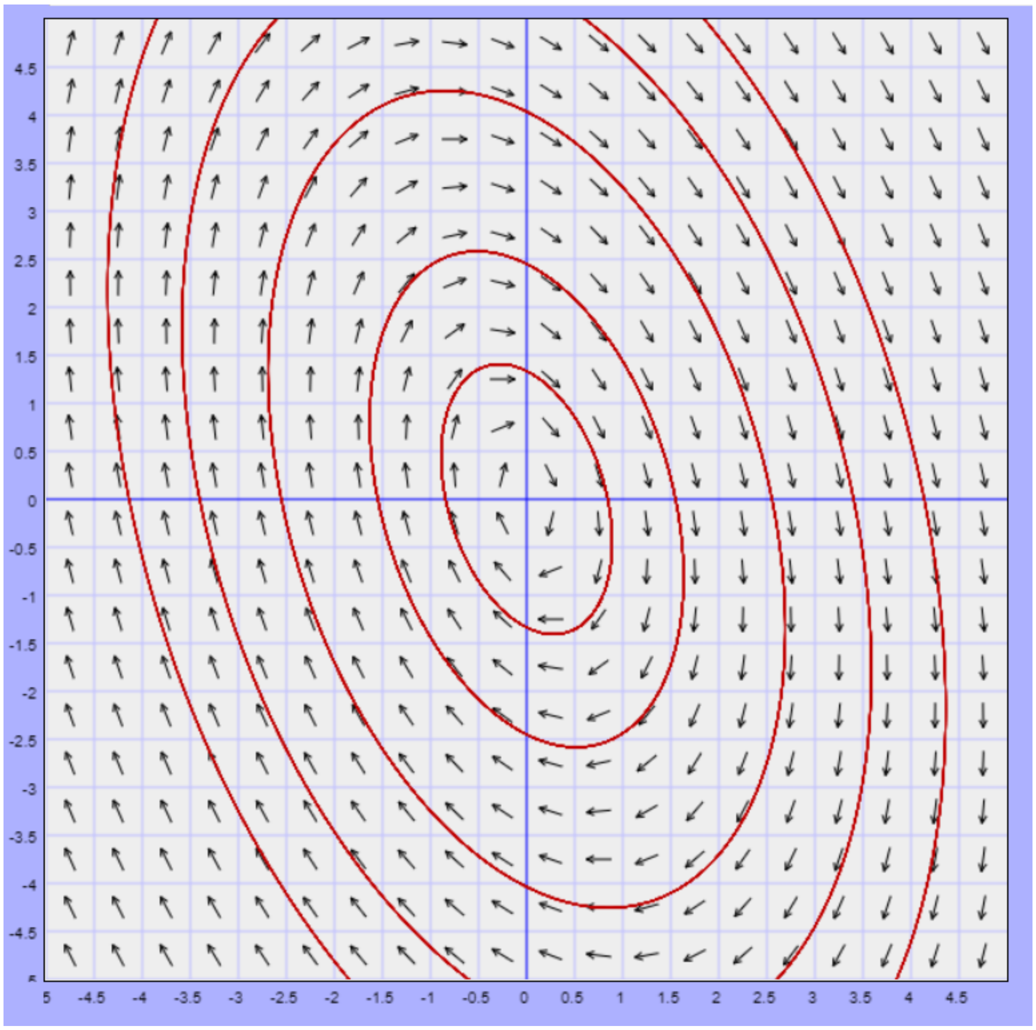
$$\boxed{\vec{x}(t) = C_1 e^{-3t} \begin{bmatrix} 1+4t \\ 4t \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} -4t \\ 1-4t \end{bmatrix}}$$

Problem 4: Use the pplane app to plot the phase portraits.

1. (Click on at least 5 solutions)

$$x' = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} x$$

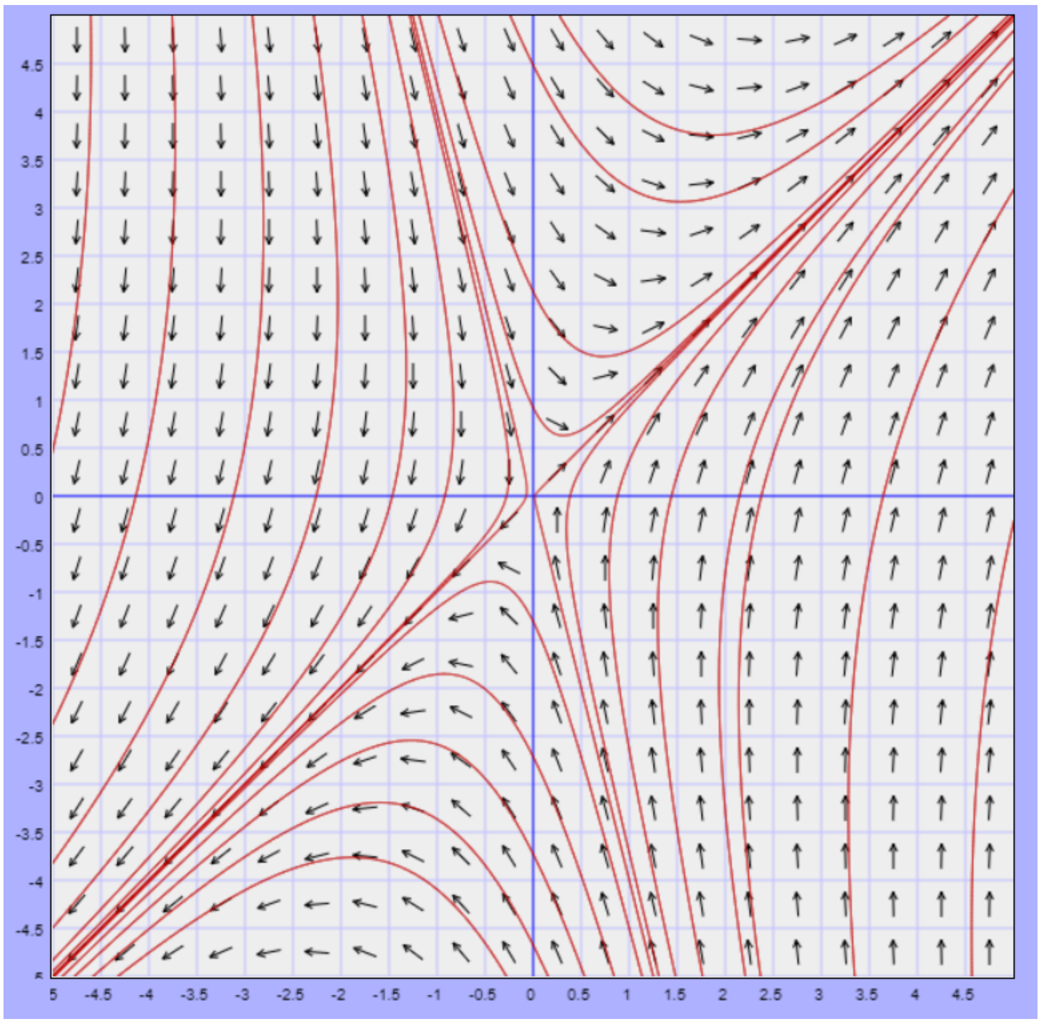
Solution:



2. (Click on three solutions in each region)

$$x' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} x$$

Solution:



Use the dsolve command in Python to solve the following systems

1.

$$x' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} x$$

Solution:

APMA 0350 > HW 6 > dsolve.py > ...

```python
import sympy as sp

t = sp.symbols('t')
x1 = sp.Function('x1')
x2 = sp.Function('x2')
deq1 = sp.diff(x1(t),t) - 5*x1(t)+1*x2(t)
deq2 = sp.diff(x2(t),t) - 3*x1(t)-1*x2(t)
print(sp.dsolve([deq1,deq2]))
```

PROBLEMS OUTPUT ··· Code

```
[Running] python -u "c:\Users\capoo\Documents\Github\APMA 0350\HW 6\dsolve.py"
[Eq(x1(t), C1*exp(2*t)/3 + C2*exp(4*t)), Eq(x2(t), C1*exp(2*t) + C2*exp(4*t))]
```

2.

$$x' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Solution:

APMA 0350 > HW 6 > dsolve.py > ...

```python
import sympy as sp

t = sp.symbols('t')
x1 = sp.Function('x1')
x2 = sp.Function('x2')
deq1 = sp.diff(x1(t),t) - x1(t) + 4*x2(t)
deq2 = sp.diff(x2(t),t) - 4*x1(t) + 7*x2(t)
print(sp.dsolve([deq1,deq2],ics=({x1(0):3,x2(0):2})))
```

PROBLEMS OUTPUT ··· Code

```
[Running] python -u "c:\Users\capoo\Documents\Github\APMA 0350\HW 6\dsolve.py"
[Eq(x1(t), 4*t*exp(-3*t) + 3*exp(-3*t)), Eq(x2(t), 4*t*exp(-3*t) + 2*exp(-3*t))]
```