APMA 0350: Homework 10

Milan Capoor

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Problem 1: Use undetermined coefficients to find the general solution of $x' = Ax + f$ where

$$A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \quad f = \begin{bmatrix} e^{-2t} \\ -2e^t \end{bmatrix}$$

Eigenvalues:

$$\det(A - \lambda I) = 0 \implies (1-\lambda)(-2-\lambda) - 4 = 0$$
$$\lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2) = 0$$
$$\lambda = \{2,\ -3\}$$

Eigenvector $\lambda = 2$:

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 4 & -4 & 0 \end{array}\right] \implies -x_1 + x_2 = 0$$
$$\vec{v}_{\lambda=2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvector $\lambda = -3$:

$$\left[\begin{array}{cc|c} 4 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right] \implies 4x_1 + x_2 = 0$$
$$\vec{v}_{\lambda=-3} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Homogeneous solution:

$$x_0 = Ae^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + Be^{-3t}\begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Roots of inhomogeneous term: $r = \{-2,\ 1\} \longrightarrow$ (no resonance)

Particular solution:

$$f = e^{-2t}\begin{bmatrix}1\\0\end{bmatrix} + e^{t}\begin{bmatrix}0\\-2\end{bmatrix}$$

$$x_p = e^{-2t}\begin{bmatrix}A\\B\end{bmatrix} + e^{t}\begin{bmatrix}C\\D\end{bmatrix} = \begin{bmatrix}Ae^{-2t}+Ce^{t}\\Be^{-2t}+De^{t}\end{bmatrix}$$

$$x_p' = Ax_p + f$$

$$\begin{aligned}
\begin{bmatrix}-2Ae^{2t}+Ce^{t}\\-2Be^{-2t}+De^{t}\end{bmatrix} &= \begin{bmatrix}1 & 1\\4 & -2\end{bmatrix}\begin{bmatrix}Ae^{-2t}+Ce^{t}\\Be^{-2t}+De^{t}\end{bmatrix} + \begin{bmatrix}e^{-2t}\\-2e^{t}\end{bmatrix}\\
&= \begin{bmatrix}Ae^{-2t}+Be^{-2t}+Ce^{t}+De^{t}\\4Ae^{-2t}+4Ce^{t}-2Be^{-2t}-2De^{t}\end{bmatrix} + \begin{bmatrix}e^{-2t}\\-2e^{t}\end{bmatrix}\\
&= \begin{bmatrix}Ae^{-2t}+Be^{-2t}+Ce^{t}+De^{t}+e^{-2t}\\4Ae^{-2t}+4Ce^{t}-2Be^{-2t}-2De^{t}-2e^{t}\end{bmatrix}\\
&= \begin{bmatrix}(A+B+1)e^{-2t}+(C+D)e^{t}\\(4A-2B)e^{-2t}+(4C-2D-2)e^{t}\end{bmatrix}
\end{aligned}$$

$$\begin{cases}A+B+1=-2A\\C+D=C\\4A-2B=-2B\\4C-2D-2=D\end{cases} \implies \begin{cases}A=0\\B=-1\\C=1/2\\D=0\end{cases}$$

$$x_p = \begin{bmatrix}Ae^{-2t}+Ce^{t}\\Be^{-2t}+De^{t}\end{bmatrix}$$

$$x_p = \begin{bmatrix}\frac{1}{2}e^{t}\\-e^{-2t}\end{bmatrix}$$

General solution:

$$x = x_0 + x_p$$

$$x = Ae^{2t}\begin{bmatrix}1\\1\end{bmatrix} + Be^{-3t}\begin{bmatrix}1\\-4\end{bmatrix} + \begin{bmatrix}\frac{1}{2}e^{t}\\-e^{-2t}\end{bmatrix}$$

$$\boxed{x = \begin{bmatrix}Ae^{2t}+Be^{-3t}+\frac{1}{2}e^{t}\\Ae^{2t}-4Be^{3t}-e^{-2t}\end{bmatrix}}$$

Problem 2: Guess the form of a particular solution to $x' = Ax + f$ where

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$$

$$\lambda = \{1,\ 3\}$$

1.
$$\mathbf{f} = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$$

Solution:

$$f = e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (\lambda = 2)$$

No resonance so

$$\boxed{x_p = \begin{bmatrix} Ae^{2t} \\ Be^{2t} \end{bmatrix}}$$

2.
$$\mathbf{f} = \begin{bmatrix} 3e^{2t} \\ -e^{4t} \end{bmatrix}$$

Solution:

$$f = e^{2t} \begin{bmatrix} 3 \\ 0 \end{bmatrix} + e^{4t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad (\lambda = \{2,\ 4\})$$

No resonance so

$$\boxed{x_p = e^{2t} \begin{bmatrix} A \\ B \end{bmatrix} + e^{4t} \begin{bmatrix} C \\ D \end{bmatrix}}$$

3.
$$\mathbf{f} = \begin{bmatrix} \cos(2t) \\ \sin(3t) \end{bmatrix}$$

Solution:

$$\boxed{x_p = \cos(2t) \begin{bmatrix} A \\ B \end{bmatrix} + \sin(2t) \begin{bmatrix} C \\ D \end{bmatrix} + \cos(3t) \begin{bmatrix} E \\ F \end{bmatrix} + \sin(3t) \begin{bmatrix} G \\ H \end{bmatrix}}$$

4.

$$\mathbf{f} = \begin{bmatrix} t^2 \\ te^{5t} \end{bmatrix}$$

Solution:

$$\begin{bmatrix} At^2 + Ct \\ bt^2 + Dt \end{bmatrix} e^{5t}$$

Problem 3: Use variation of parameters to find the general solution of the following ODE. Simplify your answers.

1. $$\mathbf{y'' + 4y' + 4y = \frac{e^{-2t}}{t^2}}$$

   Homogeneous equation:

$$r^2 + 4r + 4 = (r+2)^2 = 0 \implies r = -2 \quad \text{(repeated)}$$

$$y_0 = Ae^{-2t} + Bte^{-2t}$$

   Var par equations:

$$\begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^{-2t}}{t^2} \end{bmatrix}$$

$$u'(t) = \frac{\begin{vmatrix} 0 & te^{-2t} \\ \frac{e^{-2t}}{t^2} & e^{-2t} - 2te^{-2t} \end{vmatrix}}{\begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix}} = \frac{-\frac{e^{-4t}}{t}}{e^{-4t} - 2te^{-4t} + 2te^{-4t}}$$

$$= \frac{-e^{-4t}}{te^{-4t} - 2t^2e^{-4t} + 2t^2e^{-4t}} = \frac{1}{t}$$

$$v'(t) = \frac{\begin{vmatrix} e^{-2t} & 0 \\ -2e^{-2t} & \frac{e^{-2t}}{t^2} \end{vmatrix}}{\begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix}} = \frac{\frac{e^{-4t}}{t^2}}{e^{-4t} - 2te^{-4t} + 2te^{-4t}} = \frac{1}{t^2}$$

$$\begin{cases} u'(t) = -\frac{1}{t} \\ v'(t) = \frac{1}{t^2} \end{cases} \implies \begin{cases} u(t) = -\ln|t| \\ v(t) = -\frac{1}{t} \end{cases}$$

   Particular solution:

$$y_p(t) = u(t)e^{-2t} + v(t)te^{-2t}$$

$$y_p(t) = -\ln|t|e^{-2t} - e^{-2t}$$

   General solution:

$$\boxed{y = Ae^{-2t} + Bte^{-2t} - \ln|t|e^{-2t} - e^{-2t}}$$

2.

$$\mathbf{t^2 y'' - t(t+2)y' + (t+2)y = 2t^3}$$

Problem 4: Use variation of parameters to find the general solution of $x' = Ax + f$ where

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \quad \mathbf{f} = \mathbf{e^t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Homogeneous solution:

$$\det(A - \lambda I) = (2 - \lambda)(-2 - \lambda) + 3 = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1) = 0$$

$$\lambda = \{-1,\ 1\}$$

Eigenvector $\lambda = -1$:

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right] \implies 3x_1 - x_2 = 0$$

$$\vec{v}_{\lambda=-1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Eigenvector $\lambda = 1$:

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right] \implies x_1 - x_2 = 0$$

$$\vec{v}_{\lambda=1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_0 = Ae^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + Be^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} = A \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix} + B \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

Particular solution:

$$x_p = u(t) \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix} + v(t) \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

$$\begin{bmatrix} e^{-t} & e^t \\ 3e^{-t} & e^t \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$$

$$u' = \frac{\begin{vmatrix} e^t & e^t \\ e^{-t} & e^t \end{vmatrix}}{\begin{vmatrix} e^{-t} & e^t \\ 3e^{-t} & e^t \end{vmatrix}} = \frac{e^{2t} - 1}{-2} = \frac{-e^{2t}}{2} + \frac{1}{2}$$

$$v' = \frac{\begin{vmatrix} e^{-t} & e^{t} \\ 3e^{-t} & e^{-t} \end{vmatrix}}{\begin{vmatrix} e^{-t} & e^{t} \\ 3e^{-t} & e^{t} \end{vmatrix}} = \frac{e^{-2t} - 3}{-2} = \frac{-e^{2t}}{2} + \frac{3}{2}$$

$$\begin{cases} u'(t) = \frac{-e^{2t}}{2} + \frac{1}{2} \\ v'(t) = \frac{-e^{2t}}{2} + \frac{3}{2} \end{cases} \implies \begin{cases} u = \int \frac{-e^{2t}}{2} + \frac{1}{2}\, dt = \frac{-e^{2t}}{4} + \frac{t}{2} \\ v = \int \frac{-e^{2t}}{2} + \frac{4}{2}\, dt = \frac{-e^{2t}}{4} + \frac{3t}{2} \end{cases}$$

$$x_p = \left(\frac{-e^{2t}}{4} + \frac{t}{2}\right) \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix} + \left(\frac{-e^{2t}}{4} + \frac{3t}{2}\right) \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}$$

$$x_p = \begin{bmatrix} -\frac{1}{4}e^{t} + \frac{1}{2}te^{-t} \\ -\frac{3}{4}e^{t} + \frac{3}{2}te^{-t} \end{bmatrix} + \begin{bmatrix} -\frac{1}{4}e^{t} + \frac{3}{2}te^{-t} \\ -\frac{3}{4}e^{t} + \frac{9}{2}te^{-t} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}e^{t} + 2te^{-t} \\ -\frac{3}{2}e^{t} + 6te^{-t} \end{bmatrix}$$

General solution:

$$\boxed{x = A \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix} + B \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}e^{t} + 2te^{-t} \\ -\frac{3}{2}e^{t} + 6te^{-t} \end{bmatrix}}$$

Problem 5: Find the general solution of $y'' + y = cos(t)$

1. Using undetermined coefficients

$$r^2 + r = 0 \implies r(r+1) = 0 \implies r = \{0,\ 1\}$$
$$\cos t \implies r = 1$$

So we have resonance!

$$y_0 = A + Bte^t$$

$$x_p = A\cos(t) + B\sin(t)$$
$$(A\cos(t) + B\sin(t))'' + (A\cos(t) + B\sin(t)) = \cos t$$
$$-A\cos(t) - B\sin t + A\cos t + B\sin t$$
$$\cos t = 0$$

2. Using variation of parameters

$$r^2 + r = 0 \implies r(r+1) = 0 \implies r = \{0,\ 1\}$$
$$y_0 = A + Be^t$$
$$y_p = u(t) + v(t)e^t$$
$$\begin{bmatrix} 1 & e^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ cos(t) \end{bmatrix}$$

$$u' = \frac{\begin{vmatrix} 0 & e^t \\ \cos t & e^t \end{vmatrix}}{\begin{vmatrix} 1 & e^t \\ 0 & e^t \end{vmatrix}} = \frac{-e^t \cos t}{e^t} = -\cos t \implies u = -\sin t$$

$$v' = \frac{\begin{vmatrix} 1 & 0 \\ 0 & \cos t \end{vmatrix}}{\begin{vmatrix} 1 & e^t \\ 0 & e^t \end{vmatrix}} = \frac{-\cos t}{e^t} \implies v = \frac{1}{2}e^{-t}(\cos t - \sin t)$$

$$y_p = -\sin t + \frac{1}{2}\cos t - \frac{1}{2}\sin t = -\frac{3}{2}\sin t + \frac{1}{2}\cos t$$

General solution:

$$\boxed{y = A + Be^t - \frac{3}{2}\sin t + \frac{1}{2}\cos t}$$