APMA 0350: Homework 11

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Problem 1: Use tabular integration to find $\mathcal{L}\{t^2\}$ Solution:

$$Lt^2 = \int_0^\infty t^2 e^{st}\, dt$$

$$\int_0^\infty t^2 e^{st}\, dt = \begin{array}{cccc} + & t^2 & & e^{-st} \\ - & 2t & \searrow & e^{-st}/(-s) \\ + & 2 & \searrow & e^{-st}/(-s)^2 \\ - & 0 & \searrow & e^{-st}/(-s)^3 \end{array}$$

$$\implies \left[t^2\left(\frac{e^{-st}}{-s}\right) - 2t\left(\frac{e^{st}}{s^2}\right) + 2\left(\frac{e^{-st}}{-s^3}\right)\right]_0^\infty = 0 - \left(\frac{2}{-s^3}\right)$$

$$\boxed{\mathcal{L}\{t^2\} = \frac{2}{s^3}}$$

Problem 2: Use complex exponentials to find $\mathcal{L}\{\cos(3t)\}$ and $\mathcal{L}\{\sin(3t)\}$

$$\mathcal{L}\{e^{-3it}\} = \mathcal{L}\{\cos(3t)\} + i\mathcal{L}\{\sin(3t)\}$$
$$\mathcal{L}\{e^{-3it}\} = \frac{1}{s+3i} = \frac{s-3i}{s^2+9} = \frac{s}{s^2+9} - i\frac{3}{s^2+9}$$

$$\boxed{\begin{aligned}\mathcal{L}\{\cos(3t)\} &= \frac{s}{s^2+9}\\ \mathcal{L}\{\sin(3t)\} &= \frac{3}{s^2+9}\end{aligned}}$$

Problem 3: Find examples of functions $f(t)$ and $g(t)$ with

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

$$\begin{cases} f(t) = 1 \\ g(t) = e^t \end{cases} \longrightarrow \begin{cases} \mathcal{L}\{f(t)\} = \frac{1}{s} \\ \mathcal{L}\{g(t)\} = \frac{1}{s-1} \end{cases}$$

$$\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{e^t\} = \frac{1}{s-1}$$

$$\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \left(\frac{1}{s}\right)\left(\frac{1}{s-1}\right) = \frac{1}{s^2-s}$$

$$\frac{1}{s-1} \neq \frac{1}{s^2-s}$$

$$\boxed{f(t) = 1, \quad g(t) = e^t}$$

Problem 4: Use Laplace Transforms to solve

$$\begin{cases} y'' + 9y = \cos(2t) \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Solution:

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{\cos(2t)\}$$
$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 9\mathcal{L}\{y\} = \frac{s}{s^2+4}$$
$$\mathcal{L}\{y\}(s^2+9) = s + \frac{s}{s^2+4}$$
$$\mathcal{L}\{y\} = \frac{s}{s^2+9} + \frac{s}{(s^2+9)(s^2+4)}$$

Partial Fraction:

$$\frac{s}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4}$$
$$s = A(s^3+4s) + B(s^2+4) + C(s^3+9s) + D(s^2+9)$$
$$s = s^3(A+C) + s^2(B+D) + s(4A+9C) + (4B+9D)$$

$$\begin{cases} A+C=0 \\ B+D=0 \\ 4A+9C=1 \\ 4B+9D=0 \end{cases} \implies \begin{cases} -4C+9C=5C=1 \\ -4D+9D=5D=0 \end{cases} \implies \begin{cases} A=-\frac{1}{5} \\ B=0 \\ C=\frac{1}{5} \\ D=0 \end{cases}$$

$$\begin{aligned} \mathcal{L}\{y\} &= \frac{s}{s^2+9} + \frac{s}{(s^2+9)(s^2+4)} \\ &= \mathcal{L}\{\cos(3t)\} + \frac{1}{10}\left(\frac{2}{s^2+4}\right) - \frac{1}{15}\left(\frac{3}{s^2+9}\right) \\ &= \mathcal{L}\{\cos(3t)\} + \frac{1}{10}\mathcal{L}\{\sin(2t)\} - \frac{1}{15}\mathcal{L}\{\sin(3t)\} \end{aligned}$$

$$\boxed{y = \cos(3t) + \frac{1}{10}\sin(2t) - \frac{1}{15}}$$

Problem 5: Find the laplace transform of

$$f(t) = \begin{cases} t & 0 \leq t < 2 \\ 2 & 2 \leq t < 5 \\ 7 - t & 5 \leq t < 7 \\ 0 & t \geq 7 \end{cases}$$

$$\begin{aligned} f(t) &= tU_0(t) + (2-t)U_2(t) + (5-t)U_5 + (t-7)U_7(t) \\ &= tU_0(t) - (t-2)U_2(t) - (t-5)U_5(t) + (t-7)U_7(t) \end{aligned}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{t}{s} - \frac{e^{-2s}}{s} + \frac{e^{-5s}}{s} + \frac{e^{-7s}}{s}}$$

Problem 6: Find a function whose Laplace transform is

1. $\frac{8}{s^2-4s+4}$

$$\frac{8}{s^2-4s+4} = \frac{8}{(s-2)^2} = 8\left(\frac{1}{(s-2)^2}\right)$$
$$= 8\mathcal{L}\{te^{2t}\}$$
$$\boxed{f(t) = 8te^{2t}}$$

2. $\frac{(s-2)e^{-s}}{s^2-4s+5}$

$$\frac{(s-2)e^{-s}}{s^2-4s+5} = \left(\frac{s-2}{s^2-4s+5}\right)e^{-s}$$
$$\frac{s-2}{s^2-4s+5} = \frac{s-2}{(s-2)^2+1} = \mathcal{L}\{e^{2t}\cos(t)\}$$
$$\mathcal{L}\{y\} = \mathcal{L}\{e^{2t}\sin(t)\}e^{-s} = \mathcal{L}\{\}$$
$$\boxed{y = e^{2t-1}\sin(t-1)U_1(t)}$$

Problem 7: Use Python to

1. Find the laplace transform of $e^{2t} + 4t^3$
2. Find a function whose laplace transform is $\frac{2(s-1)e^{-3s}}{s^2-2s+2}$
3. Solve the following ODE and plot it for $0 \leq t \leq 10$

Solution:

```python
from sympy import *

s, t = symbols('s t',positive="True")
x = Function('x')
y = Function('y')

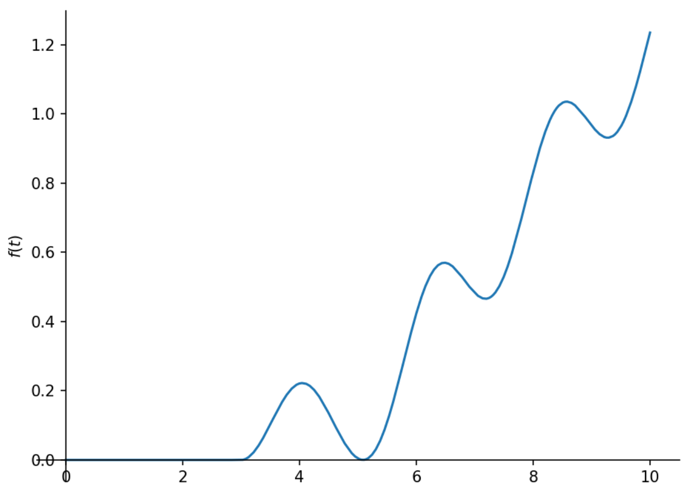
a_eq = exp(2*t) + 4*(t**3)
P7_a = laplace_transform(a_eq, t, s)[0]

b_eq = (2*(s-1)*exp(-3*s))/((s**2)-(2*s)+2)
P7_b = inverse_laplace_transform(b_eq, s, t)

c_eq = diff(y(t), t, 2) + 9*y(t) - Heaviside(t - 3) - 2*(t - 5) * Heaviside(t - 5)
P7_c = dsolve(c_eq, y(t), ics={y(0):0, diff(y(t), t).subs(t, 0):0})

print(f"Part A: {P7_a}")
print(f"Part B: {P7_b}")
print(f"Part C: {P7_c.rhs}")
plot(P7_c.rhs, (t, 0, 10))

```




```
Part A: (s**4 + 24*s - 48)/(s**4*(s - 2))
Part B: 2*exp(t - 3)*cos(t - 3)*Heaviside(t - 3)
Part C: 2*t*Heaviside(t - 5)/9 - 2*sin(3*t - 15)*Heaviside(t - 5)/27 - cos(3*t - 9)*Heaviside(t - 3)/9 - 10*Heaviside(t - 5)/9 + Heaviside(t - 3)/9
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