Ordinary Differential Equations Cheat Sheet

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1 The Fundamental Equation

For equations of the form:

$$y' = ky$$

Solution:

$$\boxed{y = Ce^{kt}}$$

2 Classification

1. *Order:* highest number of derivatives
2. *Homogeneous:* if the RHS is 0
3. *Inhomogeneous:* if the RHS is not 0
4. *Linear:* the coefficients depend on t but not y
5. *Nonlinear:* the coefficients depend on y

3 Solve by Separation

For equations of the form:

$$y' = \frac{f(t)}{g(t)}$$

Solution:

1. Rearrange so all y's on one side, all t's on other
2. Cross multiply by dx
3. Integrate each side (note: +C only needs to be on one side)
4. Isolate y

Example:

$$\int g(t)\ dy = \int f(t)\ dx$$

The Logistic Equation:

Equations of the form:

$$y' = \alpha y(1 - \frac{y}{\beta})$$

Solution:

$$\boxed{y = \frac{\beta C e^{At}}{1 + Ce^{\alpha t}}}$$

4 Integrating Factors:

For equations of the form:

$$y' + ky = f(t)$$

Solution:

1. Multiply each side by e^{kt}
2. Reduce the LHS to a product rule derivative
3. Integrate both sides
4. Isolate y

More general integrating factors

For equations of the form:

$$y' + P(t)y = f(t)$$

Use

$$e^{\int P(t)\, dt}$$

as the integrating factor with the method described above

5 Exact Equations:

For equations of the form:

$$P(x,y)\, dx + Q(x,y)\, dy = 0$$

Solution:

1. Check that $F = \langle P, Q \rangle$ is conservative ($P_y = Q_x$ – mnemonic "Peyam is quixotic")
2. Find the potential function f for which $F = \nabla f$ (integrate P and Q, creating a function of all terms)
3. The solution of the corresponding ODE will simply be

$$\boxed{f = C}$$

Non exact equations

For nonexact equations, multiply the entire ODE by the given integrating factor (found using PDEs)

6 Euler's Method

The value of y at a point can be approximated:

$$y_{n+1} \approx y_n + \left(\frac{t_n - t_0}{N} \right) y'(y_n, t_n)$$

7 Modelling Higher order equations as systems

Example: $y'' + 4y = 0$

Solution:

$$\begin{cases} x_1(t) = y \\ x_2(t) = y' \end{cases} \implies \begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = -4y = -4x_1 \end{cases}$$

$$\begin{cases} x_1' = 0x_1 + x_2 \\ x_2' = -4x_1 + 0x_2 \end{cases}$$

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \vec{x}(t)$$

8 Systems of ODE

Given an equation in the form

$$x' = Ax$$

The solution is of the form

$$\boxed{x(t) = C_1 e^{\lambda_1 t}\, \vec{v}_{\lambda_1} + C_2 e^{\lambda_2 t}\, \vec{v}_{\lambda_2}}$$

Complex Eigenvalues

Essential formula:

$$e^{it} = \cos t + i \sin t$$

To solve a system of ODE with complex eigenvalues $\pm\lambda$,

1. Find the eigenvector for one of the complex eigenvalues (say λ)
2. Trick: during the gaussian elimination, choose one of the two rows to 0 and solve the other to get the eigenvector
3. $$e^{(\lambda i)t} \begin{bmatrix} x_1 + x_3 i \\ x_2 + x_4 i \end{bmatrix} = (\cos(\lambda t) + i \sin(\lambda t)) \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + i \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \right)$$
4. Distribute the above equation to get a real term $\vec{x}_1$ and an imaginary term $\vec{x}_2$
5. The general solution will then be in the form

$$\boxed{\vec{x}(t) = C_1 \vec{x}_1 + C_2 \vec{x}_2}$$

Repeated Eigenvalues

To solve a system of ODE with a repeated eigenvalue λ:

1. Find the first eigenvector $\vec{v}$ as normal
2. Find the generalized eigenvector $\vec{w}$ by solving by row reduction $(A-\lambda I)\vec{w} = \vec{v}$
3. The general solution will be in the form

$$\boxed{x(t) = C_1 e^{\lambda t}\,\vec{v} + C_2(te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w})}$$

Note: do not rescale the generalized eigenvector

9 Matrix Exponentials

Note that equations in the form

$$x' = Ax$$

are analagous to equations of the form

$$y' = ky$$

Thus, to solve

$$\boxed{x = Ce^{At} = Pe^{Dt}P^{-1}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}}$$

Matrix Exponentials wih repeated eigenvalues

For 2×2 matrices with one eigenvalue,

$$(A - \lambda I)^2 = 0$$

Thus, in the Taylor expansion of the matrix exponentiation, all higher terms for to zero.

$$e^{At} = e^{\lambda t}(I + (A - \lambda I)t)$$

So,

$$\boxed{x = (I + (A - \lambda I)t)\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^{\lambda t}}$$

10 Equilibrium points of nonlinear systems

For systems of the form

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

Solution:

1. Find the nullclines ($x' = 0$ and $y' = 0$)
2. Solve for the equilibrium points
3. Find the Jacobian/linearization

$$\nabla F = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

4. Classify the equilibrium points

CLassifying equilibrium points

For the equilibrium point (x, y),

- If both eigenvalues of $F(x, y)$ are positive, it is unstable
- If both eigenvalues are negative, it is stable
- If one positive, one negative, it is a semistable saddle point

11 2nd Order ODEs

For an equation of the form,

$$Ay'' + By' + Cy = 0$$

The auxiliary equation is

$$Ar^2 + Br + C = 0$$

with roots r_1 and r_2

The solution will then be in the form

$$\boxed{y = C_1 e^{r_1 t} + C_2 e^{r_2 t}}$$

Complex roots

For equations with auxiliary roots of the form $a+bi$, the solution will be in the form:

$$\boxed{y = e^{at}\left(C_1\cos(bt) + C_2\sin(bt)\right)}$$

Repeated roots

For equations with with repeated auxiliary roots, the solution will be of the form

$$\boxed{y = C_1e^{rt} + C_2te^{rt}}$$

12 Boundary value problems

Find the eigenvalues and eigenfunctions of equations of the form

$$y'' = \lambda y$$

Solution:

1. Identify the auxiliary equation $r^2 = \lambda$
2. In the case $\lambda > 1$, search for nonzero solutions where $r = \pm\omega$, for a positive number ω
3. Case $\lambda = 0$: search for nonzero solutions
4. Case $\lambda < 0$: $r = \pm\omega i \quad (\omega > 0)$
5. Parameterize the infinite solutions with aa positive integer m

13 Inhomogeneous equations

Solution:

1. Find the homogeneous solution y_0 using the auxiliary equation
2. Find any particular solution y_p
3. The general solution will then be of the form

$$\boxed{y = y_0 + y_p}$$

Undetermined coefficients

To find the particular solution, make an initial guess based on the form of the inhomogeneous term:

- if RHS is e^{rt}, guess Ae^{rt}
- if RHS is a polynomial, guess the general degree polynomial
- if RHS is sin or cos, guess $A\cos(rt) + B\sin(rt)$

The substitute the guess for y in the original ODE and solve for the constants

Resonance

If the guess of the particular solution from the undetermined coefficients method has the same root(s) as the homoegenous solution, add a resonance term

Example: if the naive guess is Ae^{rt} but the root r coincides, guess

$$Ate^{rt}$$

Variation of parameters

This is an alternative method to Undetermined coefficients where the particular solution to an equation of the form

$$Ay'' + By' + Cy = f(t)$$

will be of the form

$$y_p = u(t)e^{r_2 t} + v(t)e^{r_2 t}$$

To solve for u and v, solve the Wronskian matrix

$$\begin{bmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

using Cramer's rule where f is the inhomogeneous term/

Then integrate u' and v' to find the final coefficients.

Variation of parameters for matrix systems

For equations of the form

$$x' = Ax + f$$

The method of variation of parameters will be the same but the var par equation is instead

$$A \begin{bmatrix} u' \\ v' \end{bmatrix} = \vec{f}$$

14 Laplace transform

Fundamental equation:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

Laplace transform table:

$f(t)$	$\mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$u_c f(t-c)$	$e^{cs}\mathcal{L}\{f(t)\}$
$e^{at}f(t)$	$f(s-a)$
$\delta(t-c)$	e^{-cs}
y'	$s\mathcal{L}\{y\} - y(0)$
y''	$s^2\mathcal{L}\{y\} - sy(0) - y'(0)$

$$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

To solve an ODE with laplace transforms:

1. Take the laplace of each term
2. rearrange to the form
$$\mathcal{L}\{y\} = f(s)$$
3. Manipulate the RHS into a sum of known laplace compositions
4. Take the inverse laplace of those transforms