APMA 0350: Homework 9

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Problem 1: Find the general solutions

1. $y'' + 6y' + 25y = 0$

   Solution:

$$r^2 + 6r + 25 = 0$$
$$r = \frac{-6 \pm \sqrt{36 - 4(25)}}{2} = -3 \pm 4i$$
$$\boxed{y = Ae^{-3t}\cos(4t) + Be^{-3t}\sin(4t)}$$

2. $4y'' + 4y' + y = 0$

   Solution:

$$4r^2 + 4r + 1 = 0 \implies (2r+1)^2 = 0 \implies r = -\frac{1}{2}$$
$$\boxed{y = Ae^{-t/2} + Bte^{-t/2}}$$

3. An ODE with Aux equation

$$5r^2(r+4)^3(r+7)(r^2+9)^3(r^2+2r+10)^2 = 0$$

   Solution:

$$r = \{0,\ -4,\ -7\ \pm 3i,\ -1 \pm 3i\}$$
$$\boxed{\begin{aligned} &y = 1 + Ae^{-4t} + Bte^{-4t} + Ct^2e^{-4t} + De^{-7t} + E\cos(3t) \\ &+ F\cos(3t) + Gt\cos(3t) + Ht\cos(3t) + It^2\cos(3t) + Jt^2\cos(3t) \\ &+ Ke^{-t}\cos(3t) + Le^{-t}\sin(3t) + Mte^{-t}\cos(3t) + Nte^{-t}\sin(3t)\end{aligned}}$$

Problem 2: Solve the ODE

$$\begin{cases} y'' - 6y' + 13y = 0 \\ y(0) = 2 \\ y'(0) = 10 \end{cases}$$

Solution:

$$r^2 - 6r + 13 = 0$$

$$r = \frac{6 \pm \sqrt{36 - 52}}{2} = 3 \pm 2i$$

$$y = Ae^{3t}\cos(2t) + Be^{3t}\sin(2t)$$

$$y(0) = A = 2$$

$$y' = 3Ae^{3t}\cos(2t) - 2Ae^{3t}\sin(2t) + 3Be^{3t}\sin(2t) + 2Be^{3t}\cos(2t)$$

$$y'(0) = 3A + 2B = 10 = 6 + 2B \implies B = 2$$

$$\boxed{y = 2e^{3t}\cos(2t) + 2e^{3t}\sin(2t)}$$

Problem 3: Find the eigenvalues and eigenfunctions of

$$\begin{cases} y'' = \lambda y \\ y'(0) = 0 \\ y(3) = 0 \end{cases}$$

Aux equation:

$$r^2 = \lambda$$

Cases:

1. $\lambda > 0$ Then, $\lambda = \omega^2$ for some $\omega > 0$

$$r^2 = \lambda = \omega^2 \implies r = \pm\omega$$

$$y = Ae^{\omega t} + Be^{-\omega t}$$

$$y(3) = Ae^{3\omega} + Be^{-3\omega} = 0$$

$$y' = \omega Ae^{\omega t} - \omega Be^{-\omega t}$$

$$y'(0) = A\omega - B\omega = 0 \implies A = B$$

$$y = Ae^{3\omega} + Ae^{-3\omega} = 0$$

   Searching for a nonzero solution and dividing through by Ae^{ω}:

$$\implies e^3 + e^{-3} = 0$$

   But this is not true so there are no nonzero solutions for $\lambda > 0$

2. $\lambda = 0$

$$y = A + Bt$$

$$y'(0) = B = 0$$

$$y(3) = A = 0$$

$$y = 0$$

   So there are no nonzero solutions for $\lambda = 0$

3. $\lambda < 0$

$$r^2 = \lambda = -\omega^2 \implies r = \pm\omega i \quad (\omega > 0)$$

$$y = A\cos(\omega t) + B\sin(\omega t)$$

$$y' = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

$$y'(0) = B\omega = 0 \implies B = 0$$

$$y(3) = A\cos(3\omega) = 0$$

$$\cos(3\omega) = 0$$

$$3\omega = \frac{\pi}{2} + \pi m$$

$$\omega = \frac{\pi}{6} + \frac{\pi}{3}m \quad (m \in \mathbb{N}_0)$$

So, Eigenvalues:

$$\lambda = -\omega^2 = \boxed{-\left(\frac{\pi}{6} + \frac{\pi}{3}m\right)^2 \quad (m \in \mathbb{N}_0)}$$

Eigenfunctions:

$$\boxed{y = \cos(\left(\frac{\pi}{6} + \frac{\pi}{3}m\right)t) \quad (m \in \mathbb{N}_0)}$$

Problem 4: Use undetermined coefficients to solve

$$\begin{cases} y'' - 5y' + 4y = 20\cos(2t) + 30\sin(2t) \\ y(0) = 1 \\ y'(0) = 3 \end{cases}$$

Homogeneous solution:

$$r^2 - 5r + 4 = 0 \implies (r-4)(r-1) = \implies r = \{1, 4\}$$

$$y_0 = Ae^t + Be^{4t}$$

Particular solution:

$$20\cos(2t) + 30\sin(2t) = 0 \implies r = 2i$$

There is no resonance so:

$$y_p = A\cos(2t) + B\sin(2t) + C\cos(2t) + D\sin(2t) = A\cos(2t) + B\sin(2t)$$

$$(A\cos(2t) + B\sin(2t))'' - 5(A\cos(2t) + B\sin(2t))' + 4(A\cos(2t) + B\sin(2t))$$
$$= 20\cos(2t) + 30\sin(2t)$$

$$-4A\cos(2t) - 4B\sin(2t) + 10A\sin(2t) - 10B\cos(2t) + 4A\cos(2t) + 4B\sin(2t)$$
$$= 20\cos(2t) + 30\sin(2t)$$

$$(-4A - 10B + 4A)\cos(2t) + (-4B + 10A + 4B)\sin(2t) = 20\cos(2t) + 30\sin(2t)$$

$$\begin{cases} -10B = 20 \\ 10A = 30 \end{cases} \implies A = 3, \quad B = -2$$

$$y_p = 3\cos(2t) - 2\sin(2t)$$

General solution:

$$y = Ae^t + Be^{4t} + 3\cos(2t) - 2\sin(2t)$$

$$y(0) = A + B + 3 = 1$$

$$y' = Ae^t + 4Be^t - 6\sin(2t) - 4\cos(t)$$

$$y'(0) = A + 4B - 4 = 3$$

$$\begin{cases} A + B = -2 \\ A + 4B = 7 \end{cases}$$

$$(7 - 4B) + B = -2 \implies B = 3 \implies A = -5$$

$$\boxed{y = -5e^t + 3e^{4t} + 3\cos(2t) - 2\sin(2t)}$$

Problem 5: Guess the form of the particular solution

1. $y'' - 3y' + 2y$ Aux equation:

$$r^2 - 3r + 2 = 0 \implies r = \{1, 2\}$$

   There is resonance so the form of the particular solution is

$$\boxed{Ate^t}$$

2. $y'' - 3y' + 2y = t^2e^{2t}$ There is also resonance here so the particular solution is

$$\boxed{t(At^2 + Bt + C)e^{2t}}$$

3. $y'' - 2y' + 5y = \sin(2t)$ Aux equation:

$$r^2 - 2r + 5 \implies r = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$$

   No resonance so the form of y_p is

$$\boxed{A\cos(2t) + B\sin(2t)}$$

4. $y'' - 2y' + 5y = e^t\cos(2t)$ There is no resonance so the form of y_p is

$$\boxed{Ae^t\cos(2t) + Be^t\sin(2t)}$$

Problem 6: Use the dsolve command in Python to solve the following ODE. Please include a screenshot of your code and your solution/plot.

1. Solve but do not plot

$$2y'' + 4y' + y = 0$$

Solution:

```
dsolve.py > ...
from sympy import *

t = symbols('t')
y = Function('y')
deq = 2*diff(y(t), t, 2) + 4*diff(y(t), t) + y(t)
ysoln = dsolve(deq, y(t))
print(ysoln.rhs)
```

```
PROBLEMS 5   OUTPUT   DEBUG CONSOLE   TERMINAL   ...   Code
[Running] python -u "c:\Users\capoo\Documents\Github\APMA 0350\dsolve.py"
C1*exp(t*(-1 + sqrt(2)/2)) + C2*exp(-t*(sqrt(2)/2 + 1))
```

2. Solve and plot for $-20 \le t \le 1$

$$\begin{cases} y'' + 4y = 2e^{3t} + 2t + 6\cos t \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

Solution:

```
dsolve.py > ...
from sympy import *

t = symbols('t')
y = Function('y')
deq = diff(y(t), t, 2) + 4*y(t) - 2*exp(3*t) - 2*t - 6*cos(t)
ysoln = dsolve(deq, y(t),
    ics={y(0):1, diff(y(t), t).subs(t, 0):1})
print(ysoln.rhs)
plot(ysoln.rhs, (t, -20, 1))
```

```
PROBLEMS 4   OUTPUT   DEBUG CONSOLE   TERMINAL   ...   Code
[Running] python -u "c:\Users\capoo\Documents\Github\APMA 0350\dsolve.py"
t/2 + 2*exp(3*t)/13 + sin(2*t)/52 + 2*cos(t) - 15*cos(2*t)/13
```