APMA 0350: Homework 4

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Problem 1: Write the ODE in the form $\vec{x}' = A\vec{x} + \vec{f}$:

$$y''' + 2y'' - 4y' + y = 5t$$

Solution:

$$y''' + 2y'' - 4y' + y = 5t$$

$$\begin{cases} x_1 = y \\ x_2 = y' \\ x_3 = y'' \end{cases} \implies \begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = x_3 \\ x_3' = y''' = -2x_3 + 4x_2 - x_1 + 5t \end{cases}$$

$$\boxed{\vec{x'} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 4 & -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5t \end{bmatrix}}$$

Problem 2: Solve the following systems

1. $\vec{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \vec{x}$

   Solution:

   (a) Eigenvalues

   $$\begin{aligned} (1-\lambda)(-4-\lambda)+6 &= 0 \\ \lambda^2+3\lambda+2 &= 0 \\ (\lambda+2)(\lambda+1) &= 0 \\ \lambda &= \{-2,\ -1\} \end{aligned}$$

   (b) Eigenvector for $\lambda = -2$

   $$\begin{bmatrix} 3 & -2 & 0 \\ 3 & -2 & 0 \end{bmatrix} \implies 3x_1 - 2x_2 = 0$$

   So $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = -2$

   (c) Eigenvector for $\lambda = -1$

   $$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \implies 2x_1 - 2x_2 = 0$$

   so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = -1$

   Therefore,

   $$\boxed{x(t) = C_1 e^{-2t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

2. Solve $\vec{x}' = A\vec{x}$ where

   $$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

   Solution:

(a) Eigenvalues

$$(5-\lambda)(1-\lambda)+3=0$$
$$\lambda^2-6\lambda+8=0$$
$$(\lambda-4)(\lambda-2)=0$$
$$\lambda=\{2,\ 4\}$$

(b) Eigenvectors for $\lambda=2$

$$\begin{bmatrix}3 & -1\\ 3 & -1\end{bmatrix} \implies 3x_1-x_2=0$$

So $\begin{bmatrix}1\\3\end{bmatrix}$ is an eigenvector

(c) Eigenvectors for $\lambda=4$

$$\begin{bmatrix}1 & -1\\ 3 & -3\end{bmatrix} \implies x_1-x_2=0$$

so $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector

Therefore,

$$x(t)=C_1e^{2t}\begin{bmatrix}1\\3\end{bmatrix}+C_2e^{4t}\begin{bmatrix}1\\1\end{bmatrix}$$
$$x(0)= \implies C_1=-\frac{3}{2},\quad C_2=\frac{7}{2}$$

Therefore,

$$\boxed{x(t)=-\frac{3}{2}e^{2t}\begin{bmatrix}1\\3\end{bmatrix}+\frac{7}{2}e^{4t}\begin{bmatrix}1\\1\end{bmatrix}}$$

Problem 3: Solve the system and draw a phase portrait by hand

$$\vec{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \vec{x}$$

Solution:

1. Eigenvalues

$$(3-\lambda)(-2-\lambda)+4=0$$
$$-2-\lambda+\lambda^2=0$$
$$(\lambda-2)(\lambda+1)=0$$
$$\lambda=\{-1,\ 2\}$$

2. Eigenvector for $\lambda = -1$

$$\begin{bmatrix} 4 & -2 & 0 \\ 2 & -1 & 0 \end{bmatrix} \implies 2x_1 - x_2 = 0$$

So $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector

3. Eigenvector for $\lambda = 2$

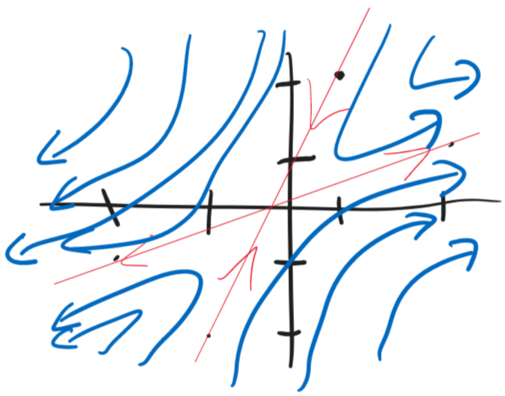
$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \implies x_1 - 2x_2 = 0$$

So $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector

Therefore,

$$\boxed{x(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

$$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



Problem 4: Solve $y'' - 5y' + 6y = 0$ as a system.

$$\begin{cases} x_1 = y \\ x_2 = y' \end{cases} \implies \begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = 5x_2 - 6x_1 \end{cases}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Eigenvalues:

$$(0 - \lambda)(5 - \lambda) + 6 = 0$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$\lambda = \{3, 2\}$$

Eigenvector for $\lambda = 3$

$$\begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix} \implies -3x_1 + x_2 = 0 \implies \vec{v_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Eigenvector for $\lambda = 2$:

$$\begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix} \implies -2x_1 + x_2 = 0 \implies v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

So,

$$\boxed{x(t) = C_1 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

Problem 5: Consider the following system of interconnected tanks that have an inflow and outflow of salt-water mixture.

- Tank 1 has input 4L/min water with 0.5 kg/L salt
- Tank 2 has input 3L/min water with 1 kg/L salt
- Tank 1 has $Q_1(t)$ salt and 20 L water
- Tank 2 has $Q_2(t)$ salt with 10 L water
- 4 L/min flows from Tank 1 to Tank 2
- 1 L/min flows from Tank 2 to Tank 1
- Tank 1 has output 1 L/min
- Tank 2 has output 6 L/min

Set up but do not solve a system of the form $\vec{Q}'(t) = A\vec{Q}(t) + \vec{b}$

Solution:

$$Q_1'(t) = 2 - \frac{5}{20}Q_1(t) + \frac{1}{10}Q_2(t)$$

$$Q_2'(t) = 3 - \frac{7}{10}Q_2(t) + \frac{4}{20}Q_1(t)$$

$$\boxed{\vec{Q}(t) = \begin{bmatrix} -0.25 & 0.1 \\ 0.2 & -0.7 \end{bmatrix} \begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$