APMA 0350 Homework 3

Milan Capoor

TOTAL POINTS

20 / 20

QUESTION 1

Problem 1 4 pts

1.1 1a **2 / 2**

✓ **- 0 pts** Correct

1.2 1b **2 / 2**

✓ **- 0 pts** Correct

QUESTION 2

Problem 2 4 pts

2.1 2a **2 / 2**

✓ **- 0 pts** Correct

2.2 2b **2 / 2**

✓ **- 0 pts** Correct

QUESTION 3

Problem 3 7 pts

3.1 3a **1 / 1**

✓ **- 0 pts** Correct

3.2 3b **1 / 1**

✓ **- 0 pts** Correct

3.3 3c **2 / 2**

✓ **- 0 pts** Correct

3.4 3d **2 / 2**

✓ **- 0 pts** Correct

3.5 3e **1 / 1**

✓ **- 0 pts** Correct

QUESTION 4

4 Problem 4 **5 / 5**

✓ **- 0 pts** Correct

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30 September 2022

Problem 1: Solve the following ODE using integrating factors

- $y' - 2y = t^2 e^{2t}$
  Solution:

$$y' - 2y = t^2 e^{2t}$$
$$(y' - 2y)(e^{-2t}) = t^2$$
$$(ye^{-2t})' = t^2$$
$$ye^{-2t} = \frac{1}{3}t^3 + C$$
$$\boxed{y = \frac{1}{3}t^3 e^{2t} + Ce^{2t}}$$

- $ty' + 2y = \sin t$
  Solution:

$$y' + \frac{2}{t}y = \frac{\sin t}{t}$$
$$(y' + \frac{2}{t}y)e^{2\ln t} = \frac{e^{2\ln t}\sin t}{t}$$
$$(yt^2)' = t\sin t$$
$$yt^2 = \sin t - t\cos t + C$$
$$\boxed{y = \frac{\sin t - t\cos t + C}{t^2}}$$

1.1 1a 2 / 2

✓ **- 0 pts** Correct

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1.2 1b 2 / 2

✓ **- 0 pts** Correct

Problem 2: Solve the following exact ODE. Leave your answer in implicit form. Don't forget to check that your equations are exact

- $\frac{dy}{dx} = -\left(\frac{e^x \sin y - 2y \sin x}{e^x \cos y + 2\cos x}\right)$

  Solution:

$$e^x \cos y + 2\cos x \; dy = -e^x \sin y + 2y \sin x \; dx$$

$$e^x \sin y - 2y \sin x \; dx + e^x \cos y + 2\cos x \; dy = 0$$

$$P_y = e^x \cos y - 2\sin x = Q_x = e^x \cos y - 2\sin x \therefore \text{ exact}$$

$$f = \int e^x \sin y - 2y \sin x \; dx = e^x \sin y + 2y \cos x + g(y)$$

$$f = \int e^x \cos y + 2\cos x \; dy = e^x \sin y + 2y \cos x + g(x)$$

$$\boxed{e^x \sin y + 2y \cos x = C}$$

- $f(x)\; dx + g(y)\; dy = 0$

  Solution:

  Assuming $f_y(x) = g_x(y) = 0$,

$$f = \int f(x)\; dx := F(x)$$

$$f = \int g(y)\; dy := G(y)$$

$$\boxed{G(y) + F(x) = C}$$

2.1 2a 2 / 2

✓ - 0 pts Correct

Problem 2: Solve the following exact ODE. Leave your answer in implicit form. Don't forget to check that your equations are exact

- $\frac{dy}{dx} = -\left(\frac{e^x \sin y - 2y \sin x}{e^x \cos y + 2\cos x}\right)$

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$$\boxed{e^x \sin y + 2y \cos x = C}$$

- $f(x) \; dx + g(y) \; dy = 0$

  Solution:
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$$f = \int f(x) \; dx := F(x)$$

$$f = \int g(y) \; dy := G(y)$$

$$\boxed{G(y) + F(x) = C}$$

2.2 2b **2 / 2**

✓ **- 0 pts** Correct

Problem 3: Our goal in this problem is to develop a model for the bunny population at Brown by taking the following factors into account:

- The initial population on Jan 1 was 50 rabbits
- On average, every rabbit has 0.3 offspring per month
- On average, one tenth of the existing rabbit population dies per month
- 30 rabbits migrate into the Brown campus area per month.

Using this information:

- Clearly identify your dependent and independent variables including their units
- Write down a differential equation for the rabbit population that takes all factors listed above into account
- Perform a qualitative analysis: Find the equilibrium solutions, draw the bifurcation diagram, plot the equilibrium solution and at least two other solutions, and determine if the equilibrium solution is stable/unstable/bistable
- Solve the ODE, including the initial condition
- Use your solution to figure out what happens to the population of bunnies as $t \to \infty$

Solution:

1. Independent: time (months), Dependent: Population size (number of rabbits)
2. ODE:
$$\begin{cases} y(0) = 50 \\ y' = 0.3y - 0.1y + 30 \implies \boxed{y' = 0.2y + 30} \end{cases}$$
3. Equilibrium solutions: $0 = 0.2y + 30 \implies \boxed{y = -150}$

3.1 3a 1 / 1

✓ **- 0 pts** Correct

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3.2 3b **1 / 1**

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$$\begin{cases} y(0) = 50 \\ y' = 0.3y - 0.1y + 30 \implies \boxed{y' = 0.2y + 30} \end{cases}$$
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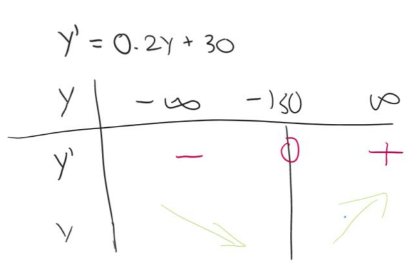
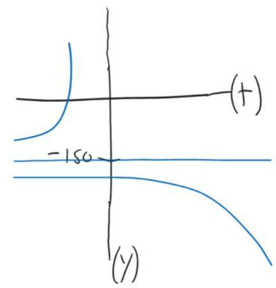




Thus, the equilibrium solution is unstable.

4. Solving the ODE $y' = 0.2y + 30$:

$$y' - 0.2y = 30$$

$$(y' - 0.2y)e^{-0.2t} = 30e^{-0.2t}$$

$$(ye^{-0.2t})' = 30e^{-0.2t}$$

$$ye^{-0.2t} = -150e^{-0.2t} + C$$

$$y = -150 + Ce^{0.2t}$$

$$50 = -150 + C \implies C = 200$$

$$\boxed{y = -150 + 200e^{0.2t}}$$

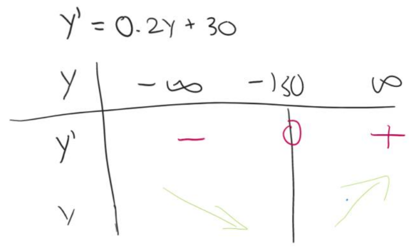
5. End conditions

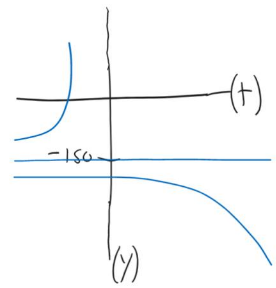
$$\lim_{t\to\infty} -150 + 200e^{0.2t} = \boxed{\infty}$$

So the population will explode.

3.3 3c 2 / 2

✓ - 0 pts Correct





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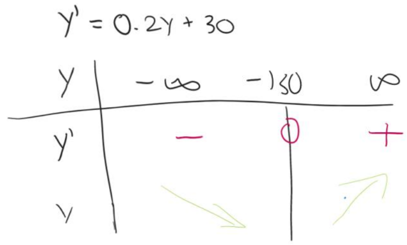
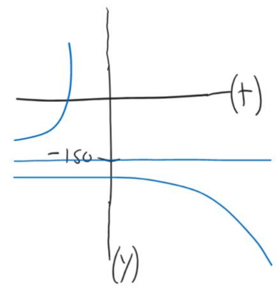
So the population will explode.

3.4 3d **2 / 2**

✓ **- 0 pts** Correct





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$$y' - 0.2y = 30$$
$$(y' - 0.2y)e^{-0.2t} = 30e^{-0.2t}$$
$$(ye^{-0.2t})' = 30e^{-0.2t}$$
$$ye^{-0.2t} = -150e^{-0.2t} + C$$
$$y = -150 + Ce^{0.2t}$$
$$50 = -150 + C \implies C = 200$$
$$\boxed{y = -150 + 200e^{0.2t}}$$

5. End conditions

$$\lim_{t\to\infty} -150 + 200e^{0.2t} = \boxed{\infty}$$

So the population will explode.

3.5 3e **1 / 1**

✓ **- 0 pts** Correct

Problem 4: Peyam baked a Yam Pie, and he only wants to eat it once the temperature reaches 25 degrees Celsius (so it will not burn him). Initially he measured the temperature and the pie was 40 degrees Celsius. One minute later, the pie was 35 degrees. The (ambient) temperature of the room is 22 degrees. When should Peyam start eating the pie? Hint: Newton's law of cooling states that the rate at which the temperature of the pie is changing is proportional to the difference between the ambient temperature and the temperature of the pie.

Solution:

$$\begin{cases} T(0) = 40 \\ T(1) = 35 \\ T' = k(22 - T) \end{cases}$$

$$T' + kT = 22k$$

$$(Te^{kt})' = 22ke^{kt}$$

$$Te^{kt} = 22e^{kt} + C$$

$$T = 22 + Ce^{-kt}$$

$$40 = 22 + C \implies C = 18$$

$$35 = 22 + 18e^{-k} \implies k \approx 0.3254$$

$$T = 22 + 18e^{-0.3254t}$$

$$25 = 22 + 18e^{-0.3254t} \implies \boxed{t \approx 5.506}$$

Therefore, Peyam should eat the Yam Pie sometime shortly after 5.5 minutes.

4 Problem 4 **5 / 5**

✓ **- 0 pts** Correct