Homework 1

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Problem 1: Find the order of each equation and state if it is linear or non-linear. If it's linear, state if it is homogeneous or inhomogeneous. No justification needed

1. $u_t - u_{xxt} + uu_x = 0$ - $\boxed{\text{3rd order, Nonlinear}}$
2. $u_x + e^y u_y = 2$ - $\boxed{\text{1st order, Linear inhomogeneous}}$
3. $u_{xx} - 2u_{yyy} = 0$ - $\boxed{\text{3rd order, Linear homogeneous}}$

Problem 2: Use the definition of linear transformations to show that the following PDE is linear

$$u_x + x^2 u_{yy} = e^{x+y}$$

Solution:

$$L(u) = u_x + x^2 u_{yy}$$

$$\begin{aligned} L(u+v) &= (u+v)_x + x^2(u+v)_{yy} \\ &= u_x + v_x + x^2 u_{yy} + x^2 v_{yy} \\ &= u_x + x^2 u_{yy} + v_x + x^2 v_{yy} \\ &= L(u) + L(v) \checkmark \end{aligned}$$

$$\begin{aligned} L(cu) &= (cu)_x + x^2(cu)_{yy} \\ &= cu_x + cx^2 u_{yy} \\ &= c(u_x + x^2 u_{yy}) = cL(u) \checkmark \end{aligned}$$

Problem 3: Show that $u(x, y) = f(x)\ g(y)$ solves the PDE

$$uu_{xy} = u_x u_y$$

Solution:

$$\begin{aligned}
(f(x)g(y))(f(x)g(y))_{xy} &= (f(x)g(y))_x(f(x)g(y))_y \\
(f(x)g(y))(f'(x)g(y))_y &= (f'(x)g(y)(f(x)g(y)))_y \\
(f(x)g(y))(f'(x)g'(y)) &= (f'(x)g(y))(f(x)g'(y)) \\
(f(x)g(y))(f'(x)g'(y)) &= (f'(x)g'(y))(f(x)g(y)) \checkmark
\end{aligned}$$

Problem 4: Solve and check

$$3u_y + u_{xy} = 0$$

Solution:

$$v = u_y \implies 3v + v_x = 0$$

By ODE analogy:

$$v = f(y)e^{-3x}$$

$$u_y = f(y)e^{-3x}$$

$$u = \int f(y)e^{-3x}\, dy = \boxed{F(y)e^{-3x} + G(x)}$$

Check:

$$\begin{aligned} u &= F(y)e^{-3x} + G(x) \\ u_y &= f(y)e^{-3x} \\ u_{xy} &= -3f(y)e^{-3x} 3u_y + u_{xy} = 3f(y)e^{-3x} - 3f(y)e^{-3x} = 0\checkmark \end{aligned}$$

Problem 5: Solve

$$u_{yy} - 5u_y + 6u = 0$$

By ODE analogy:

$$r^2 - 5r + 6 = 0 \implies (r-3)(r-2) = 0 \implies r = \{2, 3\}$$

$$\boxed{u(x,y) = A(x)e^{2y} + B(x)e^{3y}}$$

Problem 6: What is the type of the following 2nd order PDE:

1. $u_{xx} - 4u_{xy} + u_{yy} + 2u_y + 4u = 0$

$$D = (-4)^2 - 4(1)(1) = 12 > 0 \implies \boxed{\text{Hyperbolic}}$$

2. $9u_{xx} + 6u_{xt} + u_{tt} + u_x = x^2$

$$D = (-6)^2 + 4(9)(1) = 0 \implies \boxed{\text{parabolic}}$$

Problem 7: Find all real numbers α such that

$$u(x, y, z) = (x^2 + y^2 + z^2)^\alpha$$

is a solution of $u_{xx} + u_{yy} + u_{zz} = 0$

Solution:

$$\begin{aligned}
u_x &= 2x\alpha(x^2 + y^2 + z^2)^{\alpha-1} \\
u_y &= 2y\alpha(x^2 + y^2 + z^2)^{\alpha-1} \\
u_z &= 2z\alpha(x^2 + y^2 + z^2)^{\alpha-1}
\end{aligned}$$

$$\begin{aligned}
u_{xx} &= 2\alpha(x^2 + y^2 + z^2)^{\alpha-1} + 4x^2\alpha(\alpha - 1)(x^2 + y^2 + z^2)^{\alpha-2} \\
u_{yy} &= 2\alpha(x^2 + y^2 + z^2)^{\alpha-1} + 4y^2\alpha(\alpha - 1)(x^2 + y^2 + z^2)^{\alpha-2} \\
u_{zz} &= 2\alpha(x^2 + y^2 + z^2)^{\alpha-1} + 4z^2\alpha(\alpha - 1)(x^2 + y^2 + z^2)^{\alpha-2}
\end{aligned}$$

$$\begin{aligned}
0 &= u_{xx} + u_{yy} + u_{zz} \\
&= 6\alpha(x^2 + y^2 + z^2)^{\alpha-1} + 4\alpha(\alpha - 1)(x^2 + y^2 + z^2)^{\alpha-2}(x^2 + y^2 + z^2) \\
&= 6\alpha(x^2 + y^2 + z^2)^{\alpha-1} + 4\alpha(\alpha - 1)(x^2 + y^2 + z^2)^{\alpha-1} \\
&= (6\alpha + 4\alpha^2 - 4\alpha)(x^2 + y^2 + z^2)^{\alpha-1} \\
&= (2\alpha)(1 + 2\alpha)(x^2 + y^2 + z^2)^{\alpha-1}
\end{aligned}$$

The above equation is undefined for $\alpha <= 0 \cap (x, y, z) = (0, 0, 0)$ and positive for all other values. Thus, the only real numbers for which $u(x, y, z) = (x^2 + y^2 + z^2)^\alpha$ satisfies $u_{xx} + u_{yy} + u_{zz} = 0$ are

$$\boxed{\alpha = \{0, -\frac{1}{2}\} \text{ given that } (x, y, z) \neq (0, 0, 0) \text{ and } \alpha = -\frac{1}{2} \text{ otherwise}}$$