Partial Differential Equations: APMA 0360

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1 Lecture 1: Jan 25

Part I - Introduction

- Professor Peyam Tabrizian: drpeyam@brown.edu
- Office Hours: MWF 10:30-11:30
- Course site: sites.brown.edu/drpeyam
- Youtube: https://m.youtube.com/c/DrPeyam

Grading:

- Homework - 25% due Fridays 3pm
- Mini Project - 5% due Friday May 5
- Midterm 1 - 20% on Wednesday March 1
- Midterm 2 - 20% on Wednesday April 12
- Final - 30% on Tuesday May 16, 2-5pm

Part II - What is a PDE?

Partial Differential Equation: an equation relating a function u with one or more of its partial derivatives

Example: Laplace's Equation

$$\begin{cases} U = U(x,y) \\ \implies U_{xx} + U_{yy} = 0 \end{cases}$$

PhD advisor quote: "if you can solve all PDEs, you can solve the universe"

Part III - PDE Applications

1. Physical sciences
   e.g. Navier-Stokes
2. Geometry
   e.g. Poincare's Conjecture
3. Probability
4. Operations research
   e.g. Hamilton-Jacobian PDE for maximizing/minimizing
5. Image Processing
   e.g. Smartphones, MRIs
6. Money
   e.g. Black-Scholes Equation
7. Chemical Reactions
   e.g. Peyam's Dissertation

The main characters of the course for $U = U(x,t)$:

1. Transport equation
   $$U_t + 3U_x = 0$$
2. Heat/diffusion equation
   $$U_t = U_{xx}$$
3. Wave equation
   $$U_{tt} = U_{xx}$$
   ("much like an extra chromosome, an extra t is not necessarily such a good thing")

4. Laplace's equation ($U(x, y)$)

$$U_{xx} + U_{yy} = 0$$

Part III - Solution of PDE

Example 1: Is $U(x, t) = x^2 t^2$ a solution of $U_{tt} = U_{xx}$?

$$\begin{cases} U_{tt} = (x^2 t^2)_{tt} = 2x^2 \\ U_{xx} = (x^2 t^2)_{xx} = 2t^2 \end{cases}$$

So $\boxed{\text{No}}$

Example 2: Is $U(x, y) = e^x \cos(y)$ a solution of $U_{xx} + U_{yy} = 0$

$$\begin{aligned} U_{xx} + U_{yy} &= (e^x \cos y)_{xx} + (e^x \cos y)_{yy} \\ &= e^x \cos y + e^x(-\cos y) \\ &= 0 = RHS \end{aligned}$$

So $\boxed{\text{yes}}$

Part IV - Simple PDE

Note: $U = U(x, y)$

Example 3: $U_x = 0$ Because U can depend on y, this does NOT imply that $U = C$
Therefore:

$$\boxed{\text{U(x,y) = f(y)}}$$

Example 4: $U_{xx} = 0$

$$\implies U_x = f(y)$$

$$\implies U = \int f(y)\, dx = \boxed{x f(y) + g(y)}$$

(Where $g(y)$ is constant WRT x)

Example 5: $U_{xx} + U = 0$ Solving by Analogy: this is similar to ODE $y'' + y = 0 \implies y = A \cos x + B \sin x$

$$\boxed{U(x, y) = A(y) \cos x + B(y) \sin x}$$

2 Lecture 2: Jan 27

Part I - Simple PDE (Continued)

Example 1: $U_{xy} = 0$

$$\begin{aligned}(u_x)_y &= 0\\ u_x &= f(x)\\ u &= \int f(x)\,dx = \boxed{F(x) + G(y)}\end{aligned}$$

Part II - Classification of PDE

Order: the highest derivative that appears Examples:

1. $u_{xx} + 3u_y = 0$ (Second order)
2. $2u_x + 3u_y = 0$ (First order)
3. $u_{zzyzx} = 0$ (Fifth order)

Note: In general, third-order and higher are impossible to solve

Constant coefficient: if the coeffs are constant Example:

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y)$$

Note: this example is also the "general form"

Linear vs Nonlinear: if the coefficients depend on x and y but not u Examples:

1. $u_{xx} + u_{yy} = 0$ (Linear)
2. $(u_x)^2 + 3e^u + u_y = 0$ (Nonlinear)
3. $x^2u_{xx} + y^3u_y + 4u = 0$ (Linear)

All constant coefficient equations are also linear

Note: Nonlinear PDEs are VERY difficult and none of the normal PDE methods work to solve them

Interlude: the Linear Algebra View *Linear transformation:* a transformation L is linear if

1. $L(u + v) = L(u) + L(v)$

2. $L(cu) = cL(u)$

Linear PDE: a PDE of the form

$$L(u) = f$$

where L is linear and f doesn't depend on= u

Example 2: Check that the following PDE is linear

$$u_{xx} + x^2 u_{yy} = e^y$$

Solution: $L(u) = u_{xx} + x^2 u_{yy}$ so we just need to check that L is linear

$$\begin{aligned} L(u+v) &= (u+v)_{xx} + x^2(u+v)_{yy} \\ &= u_{xx} + v_{xx} + x^2 u_{yy} + x^2 v_{yy} \\ &= L(u) + L(v)\checkmark \end{aligned}$$

$$\begin{aligned} L(cu) &= (cu)_{xx} = x^2(cu)_{yy} \\ &= cu_{xx} + cx^2 u_{yy} \\ &= c(u_{xx} + x^2 u_{yy}) \\ &= cL(u)\checkmark \end{aligned}$$

Homogeneous/Inhomogeneous PDE: for linear PDE, Homogeneous if $f = 0$ and Inhomogeneous otherwise Examples:

1. $u_{xx} + u_{yy} = 0$ Homo
2. $u_{xx} + u_{yy} = 2x$ Not homo

Fun fact! For linear homgeneous PDE $L(u) = 0$, the sum of two solutions is still a solution **Why?** L is linear so solutions span a vector space

Part III - Types of Second-order PDE

Suppose you have a PDE of the form

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x,y)$$

Then, let $D = b^2 - 4ac$:

1. if $D < 0$ then the PDE is elliptic
2. if $D > 0$ then the PDE is hyperbolic
3. if $D = 0$ then the PDE is parabolic

Example 3: What is the type of the PDE

$$5u_{xx} + 6u_{xy} - 4u_{yy} + 3u_x + 5u = x^2$$

Solution:

$$D = 6^2 - 4(5)(-4) = 36 + 80 = 116 > 0 \implies \boxed{\text{hyperbolic}}$$

Most famous PDE and their types:

1. Laplace's equation ($u_{xx} + u_{yy} = 0$) is elliptic
2. Wave equation ($u_{tt} - u_{xx} = 0$) is hyperbolic
3. Heat equation ($u_t = u_{xx} \implies u_{xx} + 0u_{tt-u_t} = 0$) is parabolic

Part IV - Review: Directional Derivatives

Gradient vector of $u = u(x, y)$: $\Delta u = (u_x, u_y)$

If $\vec{v}$ is a vector, then the *directional derivative* of u in the direction of $\vec{v}$ is

$$(\Delta u) \cdot \vec{v}$$

Intuitively, this measures the rate of change of u in the $\vec{v}$ direction. Normal convention is to have $\vec{v}$ as a unit vector but this is not actually necessary

Example 4: $u(x, y) = x^2 - y^2$ and $\vec{v} = (2, 3)$ Solution:

$$(\Delta u) \cdot \vec{v} = (2x, -2y) \cdot (2, 3) = \boxed{4x - 6y}$$

3 Lecture 3: Jan 30

Part I - The Constant Coefficient Case

Goal: solve a PDE of the form

$$au_x + bu_y = 0$$

Example 1: $2u_x + 3bu_y = 0$ Solution:

1. Observe the LHS is the same as

$$\langle u_x, u_y \rangle \cdot \langle 2, 3 \rangle = \nabla u \cdot \vec{v} = 0$$

   Note that this is the same as the directional derivative of u in the direction $\vec{v} = \langle 2, 3 \rangle$. This tells us that u is constant along lines parallel to $\langle 2, 3 \rangle$ (these are called *characteristic lines*)

2. Find the equation of each of the parallel lines

$$m = \frac{3}{2} \implies y = \frac{3}{2}x + C \implies 2y - 3x = C$$

3. Solution: $\boxed{u(x, y) = f(2y - 3x)}$ (where f is arbitrary)

Summary: the general solution of $au_x + bu_y = 0$ is

$$\boxed{\text{u(x, y) = f(ay- bx)}} \quad \text{where f is arbitrary}$$

Part II - The General Case

Example 2: $u_x + yu_y = 0$ Solution:

1. Directional Derivative

$$\nabla u \cdot (1, y) = 0$$

   So u is constant along curves with "slope" v

2. Characteristic lines On one hand, the slope of the directional derivative is y. On the other, assuming y is a function of x, the slope should be $y'(x)$

   Putting it together,

$$y' = y \implies y = Ce^x$$

   Why? Consider $g(x) = u(x, Ce^x)$ Then,

$$g'(x) = u_x(x, Ce^x) + Ce^x u_y(x, Ce^x) = u_x + yu_y = 0$$

3. Find the arbitrary function input that is constant on each curve $y = Ce^x$

$$y = Ce^x \implies ye^{-x} = C$$

4. Solution:

$$u(x, y) = f(ye^{-x})$$

Part III - More Practice

Example 3: $xu_x + yu_y = 0$ Directional derivative:

$$\nabla u \cdot \langle x, y \rangle = 0$$

ODE:

$$\begin{aligned} fracyx &= y'(x) \\ x\,dy &= y\,dx \\ \ln|y| &= \ln|x| + C \\ |y| &= |x|e^c \\ \frac{y}{x} &= C \end{aligned}$$

Solution: $\boxed{u(x,y) = f(\frac{y}{x})}$

4 Lecture 4: Feb 1

Part I - The Chain Rule

If $f = f(x,y)$ where $x = x(s,t)$ and $y = y(s,t)$ then,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Part II - Coordinate Method

Example: $2u_x + 3u_y = 0$

1. Define new variables x' and y'

$$\begin{cases} x' = 2x + 3y \\ y' = -3x + 2y \end{cases}$$

   Note: $(2,3)$ is the vector in the direction of the directional derivative and $(-3,2)$ is perpendicular

2. Rewrite in terms of x' and y' using chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = 2u_{x'} - 3u_{y'}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = 3u_{x'} + 2u_{y'}$$

3. Substitute definitions

$$\begin{aligned} 2u_x + 3u_y &= 0 \\ 2(2u_{x'} - 3u_{y'}) - 3(3u_{x'} + 2u_{y'}) &= 0 \\ 4u_{x'} - 6u_{y'} + 9u_{x'} + 6u_{y'} &= 0 \\ 13u_{x'} = 0 \implies u_{x'} &= 0 \end{aligned}$$

4. Solution

$$u_{x'} = 0 \implies u = f(y')$$
$$\boxed{u = f(2y - 3x)}$$

Part III - Transport equation

$$u_t + cu_x = 0$$

where $u = u(x,t)$ where x is position, t is time, and c is a speed constant. It models the density of a fluid that is transported at speed c

Derivation: The mass on an interval $[0,b]$ at time t is:

$$M = \int_0^b u(x,t)\,dx$$

At a later time, $t+h$, the fluid shifts from $[0,b]$ to $[ch, b+ch]$. Now, the mass is

$$M = \int_{ch}^{b+ch} u(x,t+h)\,dx$$

Since mass is conserved, get:

$$\int_0^b u(x,t)\,dx = \int_{ch}^{b+ch} u(x,t+h)\,dx$$

Differentiate with respect to b:

$$\frac{d}{db}\int_0^b u(x,t)\,dx = \frac{d}{db}\int_{ch}^{b+ch} u(x,t+h)\,dx$$

By the Fundamental Theorem of calculus:

$$u(b,t) = u(b+ch, t+h)$$

Differentiate with respect yo h:

$$0 = \frac{\partial u}{\partial x}\frac{\partial(b+ch)}{\partial h} + \frac{\partial u}{\partial t}\frac{\partial(t+h)}{\partial h}$$

$$0 = cu_x + u_t$$

Solving:

$$u_t + cu_x = 0 \implies cu_x - u_t = 0$$

Recall: the general solution to $au_x + bu_y = 0$

$$u(x,y) = f(ay - bx)$$

Note: this can also be written $f(bx - ay)$ but with different f

Therefore,

$$\boxed{\text{u(x, t) = f(x - ct)}}$$

5 Lecture 5: Feb 3

Part I - The Heat Equation

$$u_t = Du_{xx}$$

where $D > 0$ is a diffusion constant The equation gives the temperature of a metal rod at position x and time t.

Part II - Derivation

Note: can also use Fick's law from physics to derive it

1. Think about the rod as composed of particles that move in two dimensions (left or right)
2. Let $u = u(x,t)$ measure the concentration (#/length) of particles at x and t
3. Let $h = \Delta x$ and $\tau = \frac{h^2}{2D}$ (it will work!)
4. Focus on (x, t) (look at the small neighborhood of x: $[x - \frac{h}{2}, x + \frac{h}{2}]$)
5. Note that the length of the interval is h so the number of particles on the interval is roughly $hu(x,t)$
6. Divide the rod into more intervals of length h
7. Main assumption: as time increases from t to $t+\tau$, each particle moves to the left or right with equal probability
8. $$hu(x,t+\tau) = hu(x,t) + \text{ change}$$
9. $$\text{change} = \text{in - out} = \begin{cases} \text{out} = \frac{1}{2}hu(x,t) + \frac{1}{2}hu(x,t) \\ \text{in} = \frac{1}{2}hu(x-h,t) + \frac{1}{2}hu(x+h,t) \end{cases}$$
$$\implies \frac{1}{2}hu(x-h,t) + \frac{1}{2}hu(x+h,t) - hu(x,t)$$
10. $$hu(x,t+\tau) = hu(x,t) + \frac{1}{2}hu(x-h,t) + \frac{1}{2}hu(x+h,t) - hu(x,t)$$
11. $$hu(x,t+\tau) - hu(x,t) = \frac{h}{2}\left(u(x-h,t) - 2u(x,t) + u(x+h,t)\right)$$
12. Make some more transformations to get into the right form:
$$\frac{u(x,t+\tau) - u(x,t)}{\tau} = \frac{h^2}{2\tau}\left(\frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2}\right)$$

13. Limits:
$$\lim_{\tau \to 0} \frac{u(x,t+\tau) - u(x,t)}{\tau} = u_t(x,t)$$
Then by double l'Hopital's:
$$\lim_{h \to 0} \left(\frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2} \right) = u_{xx}$$

14.
$$u_t = \left(\lim_{tau,h \to 0} \frac{h^2}{2\tau} \right) u_{xx}$$

15. Then using the definition of tau:
$$\boxed{u_t = Du_{xx}}$$

6 Lecture 6: Feb 6

Part I - Behavior of Solutions

The Heat Equation:
$$u_t = Du_{xx}$$
Where $u(x,t)$ is the temperature of a metal rod at x and t and $D > 0$ is a diffusivity constant dependent on material

Notice that if $u_{xx} > 0$, then $u_t = Du_{xx} > 0$ whenever u is concave up in x, u will increase in time and vice versa. In other words, over time the graph will "flatten out"

Part II - Interlude: The Gaussian Integral

Example:
$$\int_{-\infty}^{\infty} e^{-x^2} \, dx$$
Classically, e^{-x^2} does not have an antiderivative and yet we can take the integral with the following method:

1. Trick: Consider
$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-y^2}\,dy > 0$$
(The variable does not matter)
2. Multiply:
$$\begin{aligned}
I^2 &= (I)(I) \\
&= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\,r\,dr\,d\theta \\
&= 2\pi\int_{0}^{\infty} re^{-r^2}\,dr \\
&= 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} \qquad (u=-r^2) \\
&= 2\pi\left(-\frac{1}{2}e^{-\infty+\frac{1}{2}e^0}\right) \\
&= \pi
\end{aligned}$$
3. Therefore $I^2 = \pi$ and since $I > 0$, we get $I = \sqrt{\pi}$ and so:
$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Note: this same method can be used to calculate $\int_{-\infty}^{\infty} \sin(x^2)\,dx$

Part III - The Fourier Transform

The Fourier Transform functions in much the same way as the Laplace Transform of ODEs.
$$\widehat{f}(\kappa) = \int_{-\infty}^{\infty} f(x)e^{i\kappa x}\,dx$$

Notes:

- This is a function of κ as x is integrated out
- Interpretation: changes functions from phase space to frequency space
- Application: essential for signal processing and imaging
- Often represented with ξ instead of κ and $e^{-i\kappa x}$ rather than $e^{i\kappa x}$

Example: Calculate $\widehat{f}$ where $f(x) = e^{-x^2}$ Solution:

$$\widehat{f}(\kappa) = \int_{-\infty}^{\infty} e^{-x^2} e^{i\kappa x}\, dx$$

1. Find a differential equation for $\widehat{f}$

$$\begin{aligned}\widehat{f}'(\kappa) &= \frac{d}{d\kappa}\int_{-\infty}^{\infty} e^{-x^2} e^{i\kappa x}\, dx\\ &= \int_{-\infty}^{\infty} e^{-x^2} e^{i\kappa x}(ix)\, dx\\ &= i\int_{-\infty}^{\infty} xe^{-x^2} e^{i\kappa x}\, dx\end{aligned}$$

2. Integrate by parts with respect to x:

$$\begin{cases} du = xe^{-x^2} \implies u = -\frac{1}{2}e^{-x^2} \\ v = e^{i\kappa x} \implies dv = e^{i\kappa x}(i\kappa) \end{cases}$$

Integrating:

$$\begin{aligned}&= i\left[-\frac{1}{2}e^{-x^2} e^{i\kappa x}\right]_{-\infty}^{\infty} - i\int_{-\infty}^{\infty} -\frac{1}{2}e^{-x^2} e^{i\kappa x}\, dx\\ &= 0 + \frac{i}{2}(i\kappa)\int_{-\infty}^{\infty} e^{-x^2} e^{i\kappa x}\, dx\\ &= -\frac{\kappa}{2}\widehat{f}(\kappa)\end{aligned}$$

Giving us a new ODE to solve in the next lecture of

$$\widehat{f}'(\kappa) = -\frac{\kappa}{2}\widehat{f}(\kappa)$$

Part IV - The Schwartz Class

Notice that the infinite terms in the above example are 0 because e^{-x^2} goes to 0 very quickly.

This is the easiest class of functions to apply the Fourier transform to

Definition: f is *Schwartz* if it is infinitely differentiable and for every n

$$\lim_{x \to \pm\infty} \left| \frac{f(x)}{x^n} \right| = 0$$

And same for all derivatives of f.

In other words, f and its derivatives go to 0 at $\pm\infty$ faster than any power function x^n. This allows us to ignore the infinite terms in the Fourier integration

7 Lecture 7: Feb 8

Part I - Fourier Transform Example

$$\widehat{f}(\kappa) = \int_{-\infty}^{\infty} f(x) e^{i\kappa x}\, dx$$

Example: $\widehat{f}$ where $f(x) = e^{-x^2}$ Solution:

1. Find a Differential equation rather than try to solve directly

$$\widehat{f}'(\kappa) = -\frac{\kappa}{2} f(\kappa)$$

2. Solve the ODE

$$\widehat{f}' + \frac{\kappa}{2} f = 0$$

$$\left(\widehat{f} e^{\frac{\kappa^2}{4}}\right)' = 0$$

$$\widehat{f}(\kappa) = C e^{-\frac{\kappa^2}{4}}$$

3. Find C

$$\kappa = 0 \implies \widehat{f}(\kappa) = Ce^0 = C$$

$$C = \widehat{f}(\kappa) = \int_{-\infty}^{\infty} e^{-x^2} e^{i0x}\, dx = \sqrt{\pi}$$

4. Answer

$$\boxed{\widehat{f}(\kappa) = \sqrt{\pi} e^{-\frac{\kappa^2}{4}}}$$

Note that if you apply the fourier to a gaussian, you get another gaussian!

More generally, The Fourier transform of $f(x) = e^{-ax^2}$ is

$$\boxed{\widehat{f}(\kappa) = \sqrt{\frac{\pi}{a}} e^{-\frac{\kappa^2}{4a}}}$$

Part II - Fourier Transform and Derivatives

Recall: The Laplace transform turns derivatives into products

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} = y(0)$$

Fact:

$$\boxed{\widehat{f'}(\kappa) = (-i\kappa)\widehat{f}(\kappa)}$$

Proof:

$$\begin{aligned}
\widehat{f'}(\kappa) &= \int_{-\infty}^{\infty} f'(x) e^{i\kappa x}\, dx \\
&\overset{\text{IBP}}{=} \left[f(x)e^{i\kappa x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \frac{d}{dx} e^{i\kappa x}\, dx \\
&= 0 - \int_{-\infty}^{\infty} f(x) \frac{d}{dx} e^{i\kappa x}\, dx = -i\kappa \int_{-\infty}^{\infty} f(x) \frac{d}{dx} e^{i\kappa x}\, dx \\
&= -i\kappa \widehat{f}(\kappa)
\end{aligned}$$

Part III - Fourier transform and the Heat Equation

Example: Solve

$$\begin{cases} u_t = D u_{xx} \\ u(x,0) = f(x) \quad \text{(given)} \end{cases}$$

Solution:

1. Apply the x fourier Transform

$$\widehat{u_t} = D\widehat{u_{xx}}$$

$$\widehat{u}(\kappa, t) = \int_{-\infty}^{\infty} u(x,t)e^{i\kappa x}\, dx$$

$$\begin{aligned}
\widehat{u_{xx}}(\kappa, t) &\overset{\text{fact}}{=} (-i\kappa)\widehat{u_x}(\kappa, t) \\
&\overset{\text{fact}}{=} (-i\kappa)(-i\kappa)\widehat{u}(\kappa, t) \\
&= -\kappa^2\widehat{u}(\kappa, t)
\end{aligned}$$

For u_t, do directly:

$$\begin{aligned}
\widehat{u_t} &= \int_{-\infty}^{\infty} u_t(x,t)e^{i\kappa x}\, dx \\
&= \int_{-\infty}^{\infty} \frac{d}{dt}(u(x,t)e^{i\kappa x})\, dx \\
&= \frac{d}{dt} int_{-\infty}^{\infty} u(x,t)e^{i\kappa x}\, dx \\
&= \frac{d}{dt}\widehat{u}(\kappa, t)
\end{aligned}$$

2. Solve the new ODE

$$\widehat{u_t} = D\widehat{u_{xx}} \implies \frac{d}{dt}\widehat{u}(\kappa, t) = -D\kappa^2\widehat{u}(\kappa, t)$$

Recall:

$$y' = ay \implies y = Ce^{at} = y(0)e^{at}$$

Similarly,

$$\widehat{u}(\kappa, t) = \widehat{u}(\kappa, 0)e^{-D\kappa^2 t}$$

Note:

$$u(x, 0) = f(x) \overset{\text{fourier}}{\implies} \widehat{u}(\kappa, 0) = \widehat{f}(\kappa)$$

Therefore,

$$\boxed{\widehat{u}(\kappa, t) = \widehat{f}(\kappa)e^{-D\kappa^2 t}}$$

Problem: But how do we go from $\widehat{u}$ to u?

8 Lecture 8: Feb 10

Part I - Convolution

Definition:

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$$

Example: $(f \star g)(x)$ where $f(x) = e^x$ and

$$g(x) = \begin{cases} 1 & [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$\begin{aligned} (f \star g)(x) &= \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy \\ &= \int_0^1 e^{x-y}\, dy \\ &= e^x \int_0^1 e^{-y}\, dy \\ &= e^x[-e^{-y}]_0^1 = \boxed{(1-e^{-1})e^x} \end{aligned}$$

Fact:

$$\widehat{f \star g}(\kappa) = \widehat{f}(\kappa) \cdot \widehat{g}(\kappa)$$

Part II - Solving the Heat Equation

Example: Use the fourier transform to solve

$$\begin{cases} u_t = D u_{xx} \\ u(x,0) = f(x) \end{cases}$$

Solution: (Via ODEs)

$$\widehat{u}(\kappa, t) = \widehat{f}(\kappa) e^{-D\kappa^2 t}$$

Next, we wish to write $e^{-D\kappa^2 t}$ as a fourier transform. Note that for most equations this is impossible or VERY difficult but not for the Gaussian!

Recall:

$$\widehat{e^{-ax^2}} = \sqrt{\frac{\pi}{a}} e^{-\frac{\kappa^2}{4a}} \implies e^{-\frac{-\kappa^2}{4a}} = \widehat{\sqrt{\frac{a}{\pi}} e^{-ax^2}}$$

Therefore find a such that

$$e^{-\frac{\kappa^2}{4a}} = e^{-D\kappa^2 t}$$
$$\longrightarrow a = \frac{1}{4Dt}$$

So,

$$\sqrt{\frac{a}{\pi}} = \frac{1}{\sqrt{4\pi Dt}}$$
$$\longrightarrow e^{-\kappa^2 Dt} = \mathcal{F}(\frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}})$$

Or,

$$e^{-\kappa^2 Dt} = \widehat{g}(\kappa, t) \quad g(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

Grand Finale!

$$\widehat{u}(\kappa, t) = \widehat{f}(\kappa) e^{-\kappa^2 t} = \widehat{f}(\kappa)\widehat{g}(\kappa, t)$$
$$\widehat{u}(\kappa, t) = \mathcal{F}((f \star g)(\kappa, t))$$
$$u(x,t) = (f \star g) = \int_{-\infty}^{\infty} f(y)\ g(x-y,t)\ dy$$

where

$$g(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

Solving:

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} f(y)\, e^{-\frac{(x-y)^2}{4Dt}}\ dy} \quad (t > 0)$$

Part III - The Heat Kernel

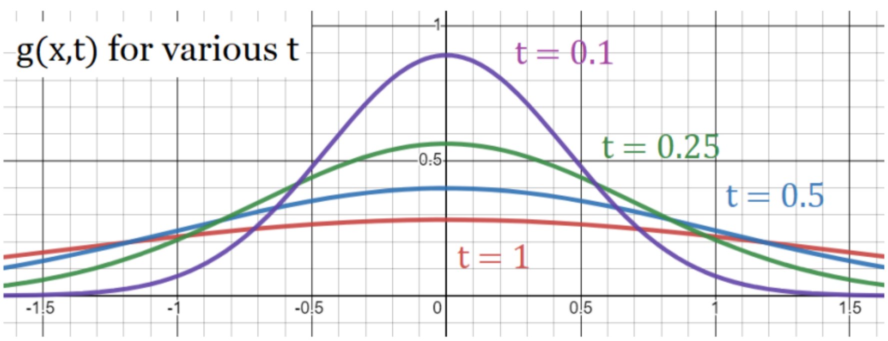
Definition: Heat kernel (AKA Fundamental sol of the heat equation)

$$g(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

Properties:

1. g itself solves $g_t = Dg_{xx}$
2. $\int_{-\infty}^{\infty} g(x,t)\, dx = 1$ for all t

Picture: For every t, $g(x,t)$ looks like a bell-curve e^{-x^2} but that gets more and more spread out as you increase t:



Note that as $t \to 0^+$, $g(x,t)$ is the Dirac delta at $x = 0$

Part V - Convolution Intuition

Example: What is the coefficient of x^2 in

$$(x^2 + 2x + 3)(2x^2 + 4x + 1)$$

Generally, the coeff of x^2 in $(a_2x^2 + a_1x + a_0)(b_2x^2 + b_1x + b_2)$ is

$$C_2 = a_0b_2 + a_1b_1 + a_2b_0$$

and more generally, the coefficient of x^k in $(a_nx^n + ...a_0)(b_nx^n + ... + b_0)$ is

$$C_k = \sum_{i=0}^{k} a_ib_{k-1}$$

Note the parallel to

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

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Part I - Heat Equation Example

Example 1: Solve

$$\begin{cases} u_t = Du_{xx} \\ u(x,0) = e^{-x} \end{cases}$$

Solution:

$$u(x,t) = \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} e^{-y}\, dy$$

Looking at the exponent:

$$\frac{-(x-y)^2}{4Dt} - y = -\frac{(x-y)^2 + 4Dty}{4Dt}$$

Expand the numerator:

$$= -\frac{x^2 - 2xy - y^2 + 4Dty}{4Dt}$$

Note the numerator is a quadratic in y:

$$\begin{aligned} y^2 + (4Dt - 2x)y + x^2 &= (y + 2Dt - x)^2 - (2Dt - x)^2 + x^2 \\ &= (y + 2Dt - x)^2 - 4D^2t^2 + 4Dtx - x^2 + x^2 \\ &= (y + 2Dt - x)^2 + 4Dt(x - Dt) \end{aligned}$$

So the full numerator is

$$\frac{-(x-y)^2}{4Dt} = -\left(\frac{(y+2Dt-x)^2 + 4Dt(x-Dt)}{4Dt}\right) = -\left(\frac{(y+2Dt-x)^2}{4Dt} + (x - Dt)\right)$$

Substituting back in,

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\left(\frac{(y+2Dt-x)^2}{4Dt} + (x-Dt)\right)}\, dy \\
&= \frac{e^{Dt-x}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-frac(y+2Dt-x)^2 4Dt}\, dy \\
&= \frac{e^{Dt-x}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\left(\frac{y+2Dt-x}{\sqrt{4Dt}}\right)^2}\, dy
\end{aligned}$$

Now use u-sub with

$$p = \frac{y+2Dt-x}{\sqrt{4Dt}}$$

so

$$dp = \frac{dy}{\sqrt{4Dt}} \implies dy = \sqrt{4Dt}\, dp$$

$$u(x,t) = \frac{e^{Dt-x}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-p^2} \sqrt{4Dt}\, dp = \frac{e^{Dt-x}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\, dp$$

$$\boxed{u(x,t) = e^{Dt-x}}$$

Part II - Infinite speed of propagation

Remember the heat equation solution is:

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\, dy$$

with an initial condition $u(x,0) = f(x)$

Property 1: If $f \geq 0$ is positive somewhere and continuous, then $u(x,t)$ is positive everywhere.

This means that heat propagates at infinite speed because heat at one place affects heat everywhere else instantly. Note that the transport equation implies a finite speed of propagation.

Why? Suppose $f(x_0) > 0$ for some x_0. Then because f is continuous it is actually positive for all x in an interval around x_0 Also

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\, dy$$

and we know the integrand is non-negative so we have

$$\frac{1}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy \geq \frac{1}{\sqrt{4}\pi Dt}\int_{a}^{b} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy$$

But the integrand of the second is also positive so

$$u(x,t) > 0$$

Part III - Smoothness

Property 2: $u(x,t)$ is infinitely differentiable (for $t > 0$) even if $f(x)$ might not be

Why? All the derivatives fall of $\exp(-\frac{(x-y)^2}{4Dt})$ and not on f:

$$\frac{d}{dx}u(x,t) = \frac{d}{dx}\frac{1}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy = \frac{1}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} \frac{d}{dx}e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy$$

But the term

$$e^{-\frac{(x-y)^2}{4Dt}}$$

is infinitely differentiable and

$$\frac{d}{dt}u(x,t) = Du_{xx}$$

but u_{xx} is also smooth

Part IV - Irreversibility

Property 3: The heat equation is irreversible ($u(x,0)$ cannot be determined from $u(x,1)$)

Why? "something something entropy"

Suppose $u(x,1) = |x|$ but by smoothness, $u(x,t)$ must be smooth for all t so $|x|$ must be smooth but this is a contradiction

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Part I - Long-time behavior of the heat kernel

$$u(x,t) = \frac{1}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy$$

Property 4:

$$\lim_{t\to\infty} u(x,t) = 0$$

"heat dissipates over time"

Why?

$$\begin{aligned}
|u(x,t)| &= \left| \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\, dy \right| \\
&\leq \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} |f(y)| \underbrace{e^{-\frac{(x-y)^2}{4Dt}}}_{\leq 1}\, dy \\
&\leq \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} |f(y)|\, dy \\
&= C \xrightarrow{t\to\infty} 0
\end{aligned}$$

Part II - Boundedness

"u(x, t) does not blow up"

Property 5: If $|f(x) \leq M$ for some M (and all x) then for all x and t we have

$$|u(x,t) \leq M$$

Why?

$$\begin{aligned}
|u(x,t)| &= \left| \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\, dy \right| \\
&\leq \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} \underbrace{|f(y)|}_{\leq M} e^{-\frac{(x-y)^2}{4Dt}}\, dy \\
&\leq \frac{M}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} e^{-\left(\frac{y-x}{\sqrt{4\pi Dt}}\right)^2}\, dy \quad u = \frac{y-x}{\sqrt{4Dt}} \implies du = \frac{dy}{\sqrt{4Dt}} \\
&= \frac{M}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} e^{-u^2} \sqrt{4Dt}\, dy \qquad = \frac{M}{\sqrt{4}\pi Dt} \sqrt{4Dt}\sqrt{\pi} \\
&= M
\end{aligned}$$

Part III - Conservation of Mass

"The area under the curve of u – no matter its shape – is always the same"

$$\int_{-\infty}^{\infty} u(x,t)\; dx = \int_{-\infty}^{\infty} f(x)\; dx$$

Why?

Lemma:

$$\lim_{x\to\pm\infty} u_x(x,t) = 0$$

Then,

$$u_t = Du_{xx}$$

$$\int_{-\infty}^{\infty} u_t(x,t)\; dx = \int_{-\infty}^{\infty} Du_{xx}(x,t)\; dx$$

and by FTC

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\; dx = D\left[u_x(xmt)\right]_{-\infty}^{\infty}$$

Thus by the lemma,

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\; dx = D(0-0) = 0$$

So the integral is constant with respect to time:

$$\int_{-\infty}^{\infty} u(x,t)\; dx = \int_{-\infty}^{\infty} u(x,0)\; dx = \int_{-\infty}^{\infty} f(x)\; dx$$

Part IV - Inverse Fourier Transform

Note that for the heat equation, we were very lucky to be able to write the Gaussian as a fourier transform

$$e^{-D\kappa^2 t} = \mathcal{F}\left(\frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}\right)$$

But what do we do in general?

Example: Solve

$$\begin{cases} u_t = -u_{xxxx} \\ u(x,0) = f(x) \end{cases}$$

Solution:

1. Fourier transform it

$$\mathcal{F}\left(u_t\right) = \mathcal{F}\left(-u_{xxxx}\right)$$
$$\frac{d}{dt}\widehat{u} = -(-i\kappa)^4\widehat{u} = -\kappa^4\widehat{u}$$

2. Solve the ODE

$$\widehat{u} = u(x,0)e^{-\kappa^4 t} = \widehat{f}(\kappa)e^{-\kappa^4 t}$$

3. Write the exponential term as a fourier transform

   Definition: *Inverse Fourier Transform*

$$\boxed{\check{f}(x) = \mathcal{F}^{-1}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\kappa)e^{-i\kappa x}\, d\kappa}$$

   So in this example,

$$e^{-\kappa^4 t} = \widehat{g}(\kappa) \quad g(x,t) = \mathcal{F}^{-1}\left(e^{-\kappa^4 t}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\kappa^4 t}e^{-i\kappa x}\, d\kappa$$

4. Convolution
5. So now we have

$$\begin{aligned}\widehat{u}(\kappa,t) &= \widehat{f}(\kappa)e^{-\kappa^4 t}\\ &= \widehat{f}(\kappa)\widehat{g}(\kappa,t)\\ &= \mathcal{F}\left(f \star g\right)(\kappa,t)\end{aligned}$$

   Therefore,

$$\boxed{\begin{cases}u(x,t) = \int_{-\infty}^{\infty} f(y)g(x-y)\, dy\\ \text{where} \quad g(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\kappa^4 t}e^{-i\kappa x}\, d\kappa\end{cases}}$$

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Part I - The Wave Equation

$$u_{tt} = c^2 u_{xx}$$

where $u = u(x, t)$ gives the displacement of a vibrating string at position x and time t and c is a constant giving the speed of the wave

Note: despite the only difference between this and the heat equation is an extra time derivative, the derivation and solution will be *completely* different

Part II - Derivation

1. Setting: start with a thin string of infinite length and consider a minute sub-piece from x to $x + \Delta x$

   Assumption: points on the string only move vertically

2. By Newton's second law of motion,

   $$F = ma$$

   By the assumption above and the definition of u, the displacement vector is

   $$s(x, t) = \langle 0, u(x, t) \rangle$$

   Therefore, acceleration is

   $$a(x, t) = s_{tt}(x, t) = \langle 0, u_{tt} \rangle$$

   Assumption: the string has constant density ρ

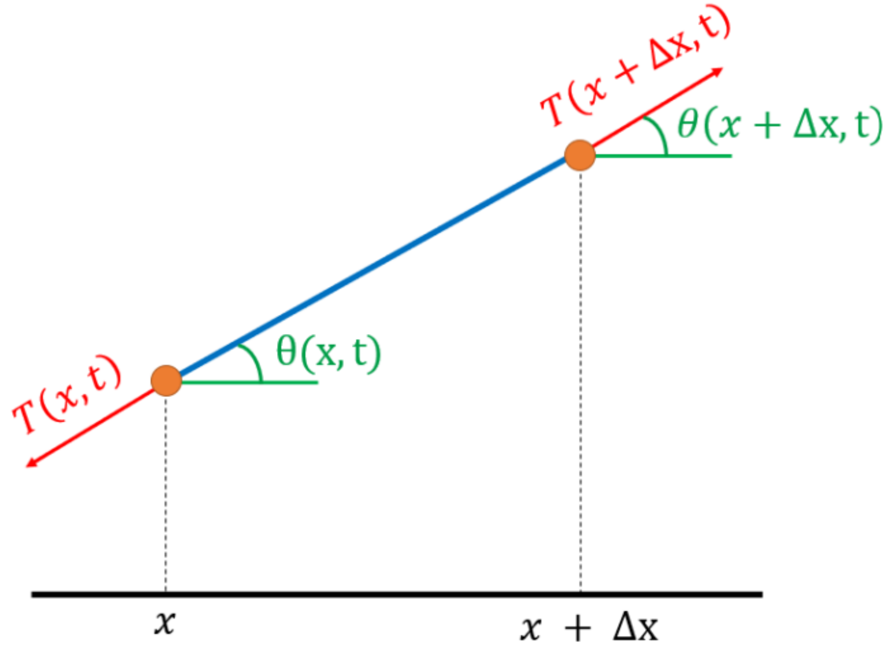
   Then, the mass of the string is density times length (which can be taken by assuming the length is the hypotenuse of a right triangle with legs Δx and Δu). Thus,

   $$m = \rho \sqrt{(\Delta x)^2 + (\Delta u)^2}$$

   So,

   $$F = ma = \langle 0, \rho \sqrt{(\Delta x)^2 + (\Delta u)^2} \rangle u_{tt}$$

3. Study of the Force: **Assumption: The only force acting on the string is the tension** So if $T(x, t)$ is the magnitude of the tension vector and $\theta(x, t)$ is the angle of the tension vector:



Then from trig, we can calculate the tension force via components of the resultant:

$$\begin{cases} x = T(x,t)\cos(\theta(x,t)) \\ y = T(x,t)\sin(\theta(x,t)) \end{cases} \implies -\langle T\cos(\theta), T\sin(\theta)\rangle(x,t)$$

Note: the minus comes from T pointing the opposite direction of the string

Then in the same way, the force at $(x + \Delta x)$ is

$$\langle T\cos(\theta), T\sin(\theta)\rangle(x + \Delta x, t)$$

so the net force is

$$F(x,t) = \langle T\cos(\theta), T\sin(\theta)\rangle(x + \Delta x, t) - \langle T\cos(\theta), T\sin(\theta)\rangle(x,t)$$

4. Then using $F = ma$ and comparing the components,

$$\begin{cases} T\cos(\theta)(x + \Delta x, t) - T\cos(\theta)(x,t) = 0 \\ T\sin(\theta)(x + \Delta x, t) - T\sin(\theta)(x,t) = \rho\sqrt{(\Delta x)^2 + (\Delta u)^2}u_{tt}(x,t) \end{cases}$$

Note, however, that both these LHS look like derivatives. Starting with the cos terms,

$$(T\cos(\theta))_x = 0$$

so $T(x,t)\cos(\theta(x,t))$ is constant in x. But $|\theta(x,t)| << 1$ so $\cos(\theta(x,t)) \approx 1$ and

$$T(x,t)\cos(\theta(x,t)) = T(x,t)$$

which is constant in x so $T(x,t) = T(t)$

Assumption: Tension is also constant in time $T(t) = T$

Then the sin terms,

$$\begin{aligned}
(T\sin(\theta))_x &= \rho u_{tt}\left(\frac{\sqrt{(\Delta x)^2 + (\Delta u)^2}}{\Delta x}\right) \\
&= \rho u_{tt}\sqrt{\frac{(\Delta x)^2 + (\Delta u)^2}{\Delta x}} \\
&= \rho u_{tt}\sqrt{1 + \left(\frac{\Delta u}{\Delta x}\right)^2} \\
&= \rho u_{tt}\sqrt{1 + (u_x)^2}
\end{aligned}$$

Assumption: if the displacements $\Delta u/\Delta x$ are small, then

$$\theta(x,t) = \tan^{-1}\frac{\Delta u}{\Delta x}$$

is small, proving the inequality above.

Then, as $\Delta x \to 0$, $|u_x| << 1$ so

$$\sqrt{1 + (u_x)^2} \approx 1$$

and

$$(T\sin(\theta))_x = \rho u_{tt}$$

but

$$\sin\theta = \tan\theta\cos\theta = \frac{\Delta u}{\Delta x}\cos\theta \to u_x$$

so

$$(Tu_x)_x = Tu_{xx} \quad \text{(assuming T is constant)}$$

and at last,

$$Tu_{xx} = \rho u_{tt} \longrightarrow u_{tt} = \frac{T}{\rho} u_{xx}$$

Set, $c = \sqrt{T/\rho} > 0$ and

$$\boxed{u_{tt} = c^2 u_{xx}}$$

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Goal: Solve $u_{tt} = c^2 u_{xx}$

Part I - Factoring Method

But this kind of looks like

$$t^2 - c^2x^2 = (t - cx)(t + cx)$$

Definition: Differential operator

$$\frac{\partial}{\partial t} u = u_t$$

$$\left(\frac{\partial}{\partial t}\right)^2 u = u_{tt}$$

Using this operator we can more rigorously "factor" the PDE.

1. Apply the differential operator

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= \left[\left(\frac{\partial}{\partial t}\right)^2 - c^2 \left(\frac{\partial}{\partial x}\right)^2\right] u \\ &= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) \end{aligned}$$

2. Solve the equation

$$u_{tt} - c^2 u_{xx} = 0$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) = 0$$

Let $v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u$ so

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0 \Longrightarrow v_t - cV_x = 0$$

3. Solve the transport PDE

$$v(x,t) = f(x+ct)$$

4. Solve for u

$$v := \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = u_t + cu_x$$
$$u_t + cu_x = f(x+ct)$$

But this is just an inhomogeneous transport equation! The homogeneous solution is just

$$u_0(x,t) = G(x-ct)$$

And a particular solution can be found using undetermined coefficients. Notice that the RHS is a function of $x+ct$ so we can guess

$$u_p = h(x+ct)$$

so

$$(h(x+ct))_t + c(h(x+ct))_x = f(x+ct)$$
$$ch'(x+ct) + ch'(x+ct) = f(x+ct)$$
$$2ch'(x+ct) = f(x+ct) \Longrightarrow h' = \frac{1}{2c}f'$$
$$h(x+ct) = \frac{1}{2c}F(x+ct)$$

where F is an antiderivative of f Thus giving the general solution

$$u(x,t) = G(x-ct) + \frac{1}{2c}F(x+ct)$$

$$\boxed{u(x,t) = G(x-ct) + F(x+ct)}$$

Interpretation: A wave is a sum of two functions, one moving to the left at speed c and the other to the right at speed c

Part II - Coordinate Method

1. Define variables
$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$

2. Chain rule
$$u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi + u_\eta$$
and
$$\begin{aligned} u_{xx} = (u_x)_x &= \frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= u_{\xi_\xi} + u_{\eta_\eta} = u_{\xi\xi} + u_{\eta\xi} + u_{\xi\eta} + u_{\eta\eta} \\ &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \end{aligned}$$
Similarly,
$$u_t t = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$

3. Plug into wave equation:
$$u_t t = c^2 u_{xx}$$
$$c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) = c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = 4u_{\xi\eta}$$
$$\boxed{u_{\xi\eta} = 0}$$

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Part I - Solving the wave equation (continued)

$$u_{tt} = c^2 u_{xx}$$
Using the coordinate method with the choices
$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$
we get the equation
$$u_{\xi\eta} = 0$$
so
$$u_\xi = f(\xi) \implies u = F(\xi) + G(\eta)$$
thus
$$\boxed{u(x,t) = F(x - ct) + G(x + ct)}$$

Part II - D'Alembert's Formula

Example:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Solution:

1. General Solution
$$u(x,t) = F(x-ct) + G(x+ct)$$

2. Plug in the initial condition
$$u(x,0) = \phi(x) = F(x) + G(x)$$

3. Differentiate with t
$$u_t(x,t) = -cF'(x-ct) + cG(x+ct)$$
$$u_t(x,0) = \psi(x) = -cF'(x) + cG'(x)$$
$$-F'(x) + G'(x) = \frac{\psi(x)}{c}$$

4. Integrate over $[0,x]$
$$\int_0^x -F'(s) + G'(s)\ ds = \int_0^x \frac{\psi(s)}{c}\ ds$$
$$-F(x) + G(x) - (-F(0) + G(0)) = \frac{1}{c}\int_0^x \psi(s)\ ds$$

This gives us the system of equations

$$\begin{cases} -F(x) + G(x) = A + \frac{1}{c}\int_0^x \psi(s)\ ds \\ F(x) + G(x) = \phi(x) \end{cases}$$

$$\implies \begin{cases} 2G(x) = \phi(x) + A + \frac{1}{c}\int_0^x \psi(s)\ ds \\ 2F(x) = \phi(x) - A - \frac{1}{c}\int_0^x \psi(s)\ ds \end{cases} \implies \begin{cases} F(X) = \frac{1}{2}\phi(x) - \frac{A}{2} - \frac{1}{2c}\int_0^x \psi(s)\ ds \\ G(X) = \frac{1}{2}\phi(x) + \frac{A}{2} + \frac{1}{2c}\int_0^x \psi(s)\ ds \end{cases}$$

5. Solution

$$\begin{aligned}
u(x,t) &= F(x-ct) + G(x+ct) \\
&= (\frac{1}{2}\phi(x-ct) - \frac{A}{2} - \frac{1}{2c}\int_0^{x-ct} \psi(s)\ ds) \\
&\quad + (\frac{1}{2}\phi(x+ct)\frac{A}{2} + \frac{1}{2c}\int_0^{x+ct} \psi(s)\ ds) \\
&= \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\left(\int_{x-ct}^{0} \psi(s)\ ds + \int_0^{x+ct} \psi(s)\ ds\right)
\end{aligned}$$

Which at last gives us d'Alembert's equation to solve the wave equation with initial conditions:

$$\boxed{u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds}$$

Part III - Example

$$\begin{cases} u_{tt} = u_{xx} \\ u(x,0) = 0 \\ u_t(x,0) = \cos(x) \end{cases} \implies \begin{cases} c = 1 \\ \phi(x) = 0 \\ \psi(x) = \cos(x) \end{cases}$$

Then using D'Alembert's:

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds \\
&= \frac{1}{2}(0+0) + \frac{1}{2}\int_{x-t}^{x+t} \cos(s)\ ds \\
&= \frac{1}{2}(\sin(x+t) - \sin(x-t)) \\
&= \frac{1}{2}(\sin x \cos t + \cos x \sin t - \sin x \cos -t - \cos x \sin -t) \\
&= \frac{1}{2}(2\cos x \sin t)
\end{aligned}$$

$$\boxed{u(x,t) = \sin(t)\cos(x)}$$

(Or, the wave takes the shape of cos with amplitude sin)