Homework 2

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Problem 1: Solve

$$\begin{cases} (1+x^2)u_x + u_y = 0 \\ u(0,y) = y^3 \end{cases}$$

Using the directional derivative:

$$(1+x^2)u_x + u_y = \nabla u \cdot \langle 1+x^2, 1\rangle$$

So u is constant along curves with "slope" $\frac{1}{1+x^2}$ Taking the characteristic lines:

$$y'(x) = \frac{1}{1+x^2}$$

$$y = \tan^{-1} x + C \implies y - \tan^{-1} x = C$$

$$u(x,y) = f(y - \tan^{-1} x)$$

Initial condition:

$$u(0,y) = y^3 = f(y - \tan^{-1} 0) = f(y) \implies f = y^3$$

$$\boxed{u(x,y) = (y - \tan^{-1} x)^3}$$

Problem 2: Solve $u_x + u_y = 1$ [1]

Homogeneous solution:

$$u_x + u_y = 0 \implies \nabla u \cdot \langle 1, 1 \rangle$$

$$\frac{1}{1} = y' \implies m = 1 \implies y = x + C$$

$$y - x = C$$

$$u(x, y) = f(y - x)$$

Particular solution:

$$g(x, y) = x$$

$$g_x + g_y = 1 + 0 = 1 \checkmark$$

General solution:

$$\boxed{u(x, y) = f(y - x) + x}$$

[1] Note: Just like for linear ODE, it is enough to find the general solution of $u_x + u_y = 0$ and one particular solution to $u_x + u_y = 1$ (which you can guess) and add the two together

Problem 3: Solve $au_x + bu_y + cu = 0$ where a, b, c are constants and $a \neq 0$ [2]

Note that

$$u_x + \frac{c}{a}u = -\frac{b}{a}u_y$$

Which by ODE integrating factors is

$$\left(ue^{\frac{cx}{a}}\right)_x = -\frac{b}{a}u_y e^{\frac{cx}{a}}$$

Let $v(x,y) = u(x,y)e^{\frac{cx}{a}}$ giving

$$v_x = -\frac{b}{a}u_y e^{\frac{cx}{a}}$$
$$v_x = -\frac{b}{a}v_y$$
$$av_x + bv_y = 0$$
$$v(x,y) = f(ay - bx)$$

But because we already know v:

$$v(x,y) = f(ay - bx) = u(x,y)e^{\frac{cx}{a}}$$

Thus, rearranging

$$\boxed{u(x,y) = e^{-\frac{cx}{a}}f(ay - bx)}$$

Checking:

$$u_x = -\frac{c}{a}e^{-\frac{cx}{a}}f(ay - bx) - be^{-\frac{cx}{a}}f'(ay - bx)$$
$$u_y = ae^{-\frac{cx}{a}}f'(ay - bx)$$

$$au_x + bu_y + cu = 0$$
$$-ce^{-\frac{cx}{a}}f(ay - bx) - abe^{-\frac{cx}{a}}f'(ay - bx) + abe^{-\frac{cx}{a}}f'(ay - bx) + ce^{-\frac{cx}{a}}f(ay - bx) = 0$$
$$-abe^{-\frac{cx}{a}}f'(ay - bx) + abe^{-\frac{cx}{a}}f'(ay - bx) = 0 = 0 \quad \checkmark$$

[2] Hint: Let $v(x,y) = u(x,y)e^{\frac{cx}{a}}$ and find a PDE for v. For this, solve for u in terms of v and calculate u_x and u_y

Note: if you're completely stuck, check out: Transform method

Problem 4: Use the coordinate method to solve [3]

$$u_x + 2u_y + (2x-y)u = 0$$

$$\begin{cases} x' = x + 2y \\ y' = -2x + y \end{cases}$$

Chain rule:

$$u_x = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = u_{x'} - 2u_{y'}$$

$$u_y = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = 2u_{x'} + u_{y'}$$

Substituting,

$$u_x + 2u_y + (2x-y)u = 0$$

$$(u_{x'} - 2u_{y'}) + 2(2u_{x'} + u_{y'}) + (2x-y)u = 0$$

$$5u_{x'} + (2x-y)u = 0$$

Solving via ODE:

$$u_{x'} + \frac{1}{5}(2x-y)u = 0$$

$$(ue^{\int \frac{2x-y}{5}\, dx'})_{x'} = 0$$

$$(ue^{-\frac{1}{5}\int y'\, dx'})_{x'} = 0$$

$$ue^{-\frac{1}{5}y'x'} = f(y')$$

$$u = e^{\frac{1}{5}y'x'} f(y')$$

$$\boxed{u(x,y) = e^{\frac{1}{5}(-2x+y)(x+2y)} \cdot f(-2x+y)}$$

Check:

$$u_x = -2e^{-0.4x^2-0.6xy+0.4y^2} f'(-2x+y) + (-0.8x - 0.6y)u$$

$$2u_y = 2e^{-0.4x^2-0.6xy+0.4y^2} f'(-2x+y) + 2(-0.6x + 0.8y)u$$

$$u_x + 2u_y = (-0.8x - 0.6y)u + 2(-0.6x+0.8y)u = u(y-2x)$$

$$u_x + 2u_y + (2x-y)u = (y-2x)u + (2x-y)u = yu - 2xu + 2xu - yu = 0 = RHS\checkmark$$

[3] Hint: One of the coordinates is in the equation, for the other one, think perpendicular

Problem 5: Solve the transport equation

$$\begin{cases} u_t + 3u_x = 0 \\ u(x,0) = x^2 \end{cases}$$

Sketch u(x, 0), u(x, 1), u(x, 2) on the same graph and convince yourself that the solutions are moving to the right with speed 3

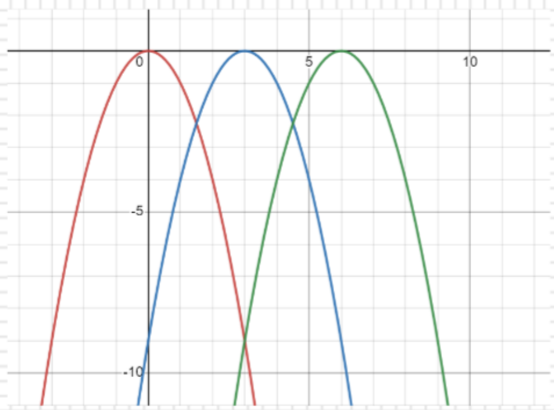
$$u_t + 3u_x = 0 \implies u(x,t) = f(3t - x)$$

$$u(x,0) = f(-x) = x^2 \implies f(x) = -x^2$$

$$\boxed{u(x,t) = -(3t - x)^2}$$

$$\boxed{u(x,t) = -(3t-x)^2}$$

x	t	u(x,t)
x	0	$-x^2$
x	1	$-(3-x)^2$
x	2	$-(6-x)^2$



Problem 6: Derive the transport equation $u_t + cu_x = 0$ using an approach similar to what we did for the heat equation in class. The notation is the same as in class, but this time assume that during the time interval from t to $t + \tau$, all particles in an interval move to the nearest interval to their right, like in the figure below.

1. Let $u = u(x, t)$ measure the concentration of particles at x and t
2. Let $h = \Delta x$
3. Note that the number of particles on an interval of the rod of length h is roughly $hu(x, t)$
4. Divide the rod into intervals of length h
5. Note the main assumption that with each time interval $[t, t + \tau]$, each particle moves from their position x_i to $x_i + h$
6. Thus, the number of particles at a given position x at time $t + \tau$ will be
$$hu(x, t + \tau) = hu(x, t) + \delta$$
where δ is a quantity representing the change in the number of particles at that point during that time step
7. Finding delta:
$$\delta = \text{in - out} = hu(x - h, t) - hu(x, t)$$
8. So,
$$hu(x, t + \tau) = hu(x, t) + hu(x - h, t) - hu(x, t)$$
9. Rearranging for the limit form:
$$hu(x, t + \tau) - hu(x, t) = hu(x - h, t) - hu(x, t)$$
$$u(x, t + \tau) - u(x, t) = h\left(\frac{u(x - h, t) - hu(x, t)}{h}\right)$$
$$\frac{u(x, t + \tau) - u(x, t)}{\tau} = \frac{h}{\tau}\left(\frac{u(x - h, t) - hu(x, t)}{h}\right)$$
10. Taking the limits:
$$\lim_{\tau \to 0} \frac{u(x, t + \tau) - u(x, t)}{\tau} = u_t$$
$$\lim_{h \to 0} \frac{u(x - h, t) - hu(x, t)}{h} = -u_x$$

11. Combining:

$$u_t = -\frac{h}{\tau} u_x$$

12. Then, setting the constant $c := \frac{h}{\tau}$,

$$\boxed{u_t + c u_x = 0}$$