APMA 0360 Midterm 2 Review

Milan Capoor

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1 Trig Identities

1. $\sin^2 x + \cos^2 x = 1$
2. $1 + \tan^2 x = \sec^2 x$
3. $\cos(-x) = \cos(x), \quad \sin(-x) = -\sin(x)$
4. $\cos(2x) = \cos^2 x - \sin^2 x$
5. $\sin(2x) = 2\sin(x)\cos(x)$
6. $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$
7. $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$

2 Wave Equation

$$u_{tt} = c^2 u_{xx}$$

2.1 Factoring Method

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= \left[\left(\frac{\partial}{\partial t}\right)^2 - c^2\left(\frac{\partial}{\partial x}\right)^2\right] u \\ &= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) = 0 \end{aligned}$$

Let $v = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}$ si

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0 \implies v_t - cv_x = 0$$

Solving the transport PDE,

$$v(x,t) = f(x+ct)$$

Solving for u,

$$u_t + cu_x = f(x+ct)$$

Homogeneous solution:

$$u_0(x,t) = G(x-ct)$$

Undetermined coeffiecients:

$$u_p = h(x+ct)$$

$$\begin{aligned}(h(x+ct))_t + c(h(x+ct))_x &= f(x+ct)\\ ch' + ch' &= f\\ h' &= \frac{1}{2c}f\\ h(x+ct) &= \frac{1}{2c}F(x+ct)\end{aligned}$$

so

$$\boxed{u(x,t) = G(x-ct) + F(x+ct)}$$

2.2 Coordinate Method

Define orthogonal variables from the equation

$$\begin{cases}\xi = x - ct\\ \eta = x + ct\end{cases}$$

So

$$\begin{aligned}u_x &= u_\xi\,\xi_x + u_\eta\,\eta_x = u_\xi + u_\eta\\ u_{xx} &= (u_x)_x = (u_x)_\xi\,\xi_x + (u_x)_\eta\,\eta_x\\ &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\\ u_{tt} &= c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})\end{aligned}$$

Plug into the wave equation

$$
\begin{gathered}
u_{tt} = c^2 u_{xx} \\
c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) = c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = 4u_{\xi\eta} \\
u_{\xi\eta} = 0 \\
u_\xi = f(\xi) \\
u = F(\xi) + G(\eta)
\end{gathered}
$$

$$\boxed{u(x,t) = F(x-ct) + G(x+ct)}$$

2.3 D'Alembert's Formula Derivation

$$
\begin{cases}
u_{tt} = c^2 u_{xx} \\
u(x,0) = \phi(x) \\
u_t(x,0) = \psi(x)
\end{cases}
$$

From above, the wave equation has general solution

$$u(x,t) = F(x-ct) + G(x+ct)$$

so with initial conditions:

$$u(x,0) = \phi(x) = F(x) + G(x)$$

$$u_t(x,0) = \psi(x) = -cF'(x) + cG'(x) \implies -F'(x) + G'(x) = \frac{\psi(x)}{c}$$

Integrate over $[0,x]$:

$$
\begin{gathered}
\int_0^x -F'(s) + G'(s)\ ds = \int_0^x \frac{\psi(s)}{c}\ ds \\
-F(x) + G(x) - (-F(0) + G(0)) = \frac{1}{c}\int_0^x \psi(s)\ ds
\end{gathered}
$$

This gives the system of equations

$$
\begin{cases}
-F(x) + G(x) = A + \frac{1}{c}\int_0^x \psi(s)\ ds \\
F(x) + G(x) = \phi(x)
\end{cases}
$$

$$
\implies \begin{cases}
2G(x) = \phi(x) + A + \frac{1}{c}\int_0^x \psi(s)\ ds \\
2F(x) = \phi(x) - A - \frac{1}{c}\int_0^x x\psi(s)\ ds
\end{cases}
$$

Solution:

$$\begin{aligned}
u(x,t) &= F(x-ct) + G(x+ct) \\
&= \frac{1}{2}\phi(x-ct) - \frac{A}{2} - \frac{1}{2c}\int_0^{x-ct} \psi(s)\ ds + \frac{1}{2}\phi(x+ct) + \frac{A}{2} + \frac{1}{2c}\int_0^{x+ct} \psi(s)\ ds \\
&= \frac{1}{2}(\phi(x-ct) + \psi(x+ct)) + \frac{1}{2c}\left(\int_{x-ct}^{0} \psi(s)\ ds + \int_0^{x+ct} \psi(s)\ ds\right)
\end{aligned}$$

Which at last gives us d'Alembert's equation:

$$\boxed{u(x,t) = \frac{1}{2}(\phi(x-ct) + \psi(x+ct)) + \frac{1}{2c}\left(\int_{x-ct}^{0} \psi(s)\ ds + \int_0^{x+ct} \psi(s)\ ds\right)}$$

3 Energy Methods

3.1 Wave Equation

Steps:

1. Multiply by a clever function (usually u or u_t)
2. Integrate with respect to x

Example:

$$\begin{aligned}
u_{tt} &= c^2 u_{xx} \\
u_{tt}u_t &= c^2 u_{xx}u_t \\
\int_{-\infty}^{\infty} u_{tt}u_t\ dx &=
\end{aligned}$$

LHS by chain rule:

$$\int_{-\infty}^{\infty} u_{tt}u_t\ dx = \frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} (u_t)^2\ dx\right)$$

RHS by parts:

$$
\begin{aligned}
c^2 \int_{-\infty}^{\infty} u_{xx} u_t \, dx &= [u_x u_t]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_x u_{xt} \, dx \\
&= - \int_{-\infty}^{\infty} u_x u_{xt} \, dx \\
&= - \int_{-\infty}^{\infty} \frac{d}{dt} \left(\frac{1}{2} (u_x)^2 \right) \, dx \\
&= \frac{d}{dt} \left(-\frac{1}{2} \int_{-\infty}^{\infty} (u_x)^2 \, dx \right)
\end{aligned}
$$

Then from $u_{tt} u_t = c^2 u_{xx} u_t$,

$$
\frac{d}{dt} \left(\frac{1}{2} \int_{-\infty}^{\infty} (u_t)^2 \, dx \right) = c^2 \frac{d}{dt} \left(-\frac{1}{2} \int_{-\infty}^{\infty} (u_x)^2 \, dx \right)
$$

$$
\frac{d}{dt} \left(\frac{1}{2} \int_{-\infty}^{\infty} (u_t)^2 + c^2 (u_x)^2 \, dx \right) = 0
$$

$$
\frac{d}{dt} E(t) = 0
$$

Which proves that the energy function is constant. ■

3.2 Heat Equation

Suppose u solves the PDE

$$
\begin{cases}
u_t = D u_{xx} \\
u(x,0) = 0 \\
u(0,t) = 0 \\
u(l,t) = 0
\end{cases}
$$

Then $u(x,t) = 0 \quad \forall x, \, t$.

Proof: Start with $u_t = D u_{xx}$ and multiply by u:

$$
u_t u = D u_{xx} u
$$

Integrate WRT x on $[0, l]$

$$
\int_0^l u_t u \, dx = D \int_0^l u_{xx} u \, dx
$$

$$\frac{d}{dt}\left(\int_0^l u^2\ dx\right) = D\left[u_x(l,t)u(l,t) - u_x(0,t) - \int_0^l u_x u_x\ dx\right]$$

$$\frac{d}{dt}\left(\int_0^l u^2\ dx\right) = -D\int_0^l (u_x)^2\ dx$$

Define

$$E(t) = \frac{1}{2}\int_0^l u^2\ dx$$

and notice that $-D\int_0^l (u_x)^2\ dx \leq 0$ so

$$\frac{d}{dt}E(t) \leq 0$$

which means that $E(t) \leq E(0)$ and

$$E(t) = \frac{1}{2}\int_0^l (u(x,t))^2\ dx \leq E(0) = \frac{1}{2}\int_0^l (u(x,0))^2\ dx = 0$$

which means that $0 \leq E(t) \leq E(0) = 0 \implies E(t) = 0$ so $u(x,t) = 0$ for all x and t. ■

3.3 Uniqueness

Fact: there is at most one solution of

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Proof: Suppose there are two functions u and v that solve the above PDE. Then, checking if w solves the PDE too,

$$w_{tt} = c^2 w_{xx}$$
$$w(x,0) = u(x,0) - v(x,0) = \phi(x) - \phi(x) = 0$$
$$w_t(x,0) = u_t(x,0) - v_t(x,0) = \psi(x) - \psi(x) = 0$$

Using the energy method as above, we show that

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2 + c^2(w_x)^2\ dx$$

is constant so $E(t) = E(0)$ and

$$\frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2 + c^2(w_x)^2 \, dx = \frac{1}{2}\int_{-\infty}^{\infty} (w_t(x,0))^2 + c^2(w_x(x,0))^2 \, dx$$

by the initial conditions $w_t(x,0) = 0$ and

$$w(x,0) = 0 \implies (w(x,0))_x = 0_x \implies w_x(x,0) = 0$$

so the above becomes

$$\frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2 + c^2(w_x)^2 \, dx = 0$$

Then as $(w_t)^2 + c^2(w_x)^2 \geq 0$, and the integral is positive,

$$(w_t)^2 + c^2(w_x)^2 = 0$$

so

$$\begin{cases} w_t = 0 \\ w_x = 0 \end{cases} \implies w(x,t) = C$$

but as $w(x,0) = 0$, we know that

$$w(x,t) = 0 \implies u - v = 0 \implies u = v$$

so there is at most one solution. $\blacksquare$

4 Separation of Variables

4.1 Wave

Example:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = e^x \end{cases}$$

Solution: Assume that $u(x,t) = X(x)T(t)$ so

$$XT'' = c^2 X'' T$$
$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$$

Initial conditions:

$$\begin{cases} X'' = \lambda X \\ u(0,t) = 0 \implies X(0) = 0 \\ u(1,t) = 0 \implies X(1) = 0 \end{cases}$$

Boundary Value Problem: $\lambda > 0$:

$$X = Ae^{\omega x} + Be^{-\omega x}$$

$$X(0) = A + B = 0 \implies X = Ae^{\omega x} - Ae^{\omega x}$$

$$X(1) = Ae^{\omega} - Ae^{-\omega} = 0 \implies \omega = -\omega \implies \omega = 0$$

$\lambda = 0$:

$$X(x) = A + Bx$$

$$X(0) = A = 0 \implies X(x) = Bx$$

$$X(1) = B = 0 \implies X(x) = 0$$

$\lambda < 0$:

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

$$X(0) = A = 0 \implies X(x) = B\sin(\omega x)$$

$$X(1) = 0 \implies \sin(\omega) = 0 \implies \omega = \pi m \quad (m = 1, 2, ...)$$

So $X(x) = \sin(\pi m x)$ corresponding to $\lambda = -(\pi m)^2 \quad (m = 1, 2, ...)$

Conclusion

$$\frac{T''}{c^2 T} = \lambda = -(\pi m)^2$$

$$T'' + (\pi m c)^2 T = 0$$

$$T(t) = A\cos(\pi m c t) + b\sin(\pi m c t)$$

So via linearity and the definition of X and T,

$$u(x,t) = \sum_{m=1}^{\infty} (A_m \cos(\pi m c t) + B_m \sin(\pi m c t)) \sin(\pi m x)$$

Initial conditions:

$$u(x,0) = \boxed{\sum_{m=1}^{\infty} A_m \sin(\pi m x) = x^2}$$

$$u_t(x,t) = \sum_{m=1}^{\infty} (-A_m \pi mc \sin(\pi mct) + B_m \pi mc \cos(\pi mct)) \cos(\pi mx)$$

$$u_t(x,0) = \boxed{\sum_{m=1}^{\infty} B_m \pi mc \sin(\pi mx) = e^x}$$

4.2 Heat

Example:

$$\begin{cases} u_t = Du_{xx} \\ u_x(0,t) = 0 \\ u_x(\pi,t) = 0 \\ u(x,0) = x^2 \end{cases}$$

Solution: Suppose $u(x,t) = X(x)T(t)$. Then

$$XT' = DX''T$$
$$\frac{X''}{X} = \frac{T'}{DT} = \lambda$$

$$\begin{cases} X'' = \lambda X \\ u_x(0,t) = 0 \implies X'(x) = 0 \\ u_x(\pi,t) = 0 \implies X'(\pi) = 0 \end{cases}$$

Boundary Value Problem: $\lambda > 0$:

$$X = Ae^{\omega x} + Be^{-\omega x}$$

$$X'(0) = A\omega - B\omega = 0 \implies A = B$$
$$X'(\pi) = A\omega e^{\pi\omega} - A\omega e^{\pi\omega} = 0 \implies \omega = 0$$

$\lambda = 0$:

$$X = A + Bx$$
$$X'(0) = B = 0 \implies X = A$$

So $\lambda = 0$ is an eigenvalue with eigenfunction $X(x) = A$

$\lambda < 0$:

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

$$X'(0) = -A\omega\sin(0) + B\omega\cos(0) = 0 \implies B = 0$$

$$X(x) = A\cos(\omega x)$$

$$X'(\pi) = 0 \implies \sin(\omega\pi) = 0$$

So $\lambda = -m^2$ are the eigenvalues corresponding to eigenfunction $X(x) = \cos(mx)$

T equation

$$\frac{T'}{DT} = \lambda = -m^2 \implies T' = -m^2DT$$

$$T(t) = e^{-m^2Dt}$$

so

$$u(x,t) = X(x)T(t) = e^{-m^2Dt}\cos(mx) \quad (m = 0, 1, 2, ...)$$

Initial Conditions By linearity,

$$u(x,t) = \sum_{m=0}^{\infty} A_m e^{-m^2Dt}\cos(mx)$$

$$u(x,0) = \boxed{x^2 = \sum_{m=0}^{\infty} A_m\cos(mx)}$$

4.3 Laplace

Note: For Laplace's equation, you don't always start with the X equation, sometimes you start with Y . Always choose the variable that gives you a 0 boundary condition

Example:

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = 0 \\ u(\pi,y) = 0 \\ u(x,0) = x \\ u(x,1) = 3 \end{cases}$$

Solution: Assume $u(x, y) = X(x)Y(y)$.

$$X''Y + XY'' = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

Initial conditions:

$$u(0, y) = 0 \implies X(0) = 0$$
$$u(\pi, y) = 0 \implies X(\pi) = 0$$

Boundary Value:

$$X'' = \lambda X$$

$\lambda > 0$:

$$X = Ae^{\omega x} + Be^{\omega x}$$
$$X(0) = A + B = 0 \implies A = -B$$
$$X(\pi) = Ae^{\omega\pi} - Ae^{-\omega\pi} = 0 \implies \omega = 0$$

$\lambda = 0$:

$$X = A + Bx$$
$$X(0) = A = 0$$
$$X(\pi) = B\pi = 0 \implies B = 0$$

$\lambda < 0$:

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$
$$X(0) = A = 0 \implies X(x) = B\sin(\omega x)$$
$$X(\pi) = B\sin(\pi\omega) = 0 \implies \sin(\pi m) = 0 \quad (m = 1, 2, ...)$$

So $X(x) = \sin(mx)$ corresponding to $\lambda = -m^2$

Back to Laplace:

$$Y''(y) = -\lambda Y(y) = m^2 Y(y)$$

$$\begin{aligned} Y(y) &= Ae^{my} + Be^{-my} \\ &= A(\cosh(my) + \sinh(my)) + B(\cosh(my) - \sinh(my)) \\ &= (A + B)\cosh(my) + (A - B)\sinh(my) \\ &= A\cosh(my) + B\sinh(my) \end{aligned}$$

$$u(x,y) = X(x)Y(y) = (A\cosh(my) + B\sinh(my))\sin(mx)$$

$$u(x,y) = \sum_{m=1}^{\infty}(A_m\cosh(my) + B_m\sinh(my))\sin(mx)$$

$$u(x,0) = \boxed{\sum_{m=1}^{\infty} A_m \sin(mx) = x}$$

$$u(x,1) = \boxed{\sum_{m=1}^{\infty}(A_m\cosh(m) + B_m\sinh(m))\sin(mx) = 3}$$

5 Fourier Series

5.1 Sine series

Because $\{\sin(mx) \mid m = 1, 2, ...\}$ is orthogonal, for

$$f(x) = \sum_{m=1}^{\infty} A_m \sin(mx)$$

on $(0, \pi)$ we have

$$A_m = \frac{f \cdot \sin(mx)}{\sin(mx)\cdot\sin(mx)} = \frac{\int_0^\pi f(x)\sin(mx)\ dx}{\int_0^\pi \sin^2(mx)\ dx} = \frac{2}{\pi}\int_0^\pi f(x)\ \sin(mx)\ dx.$$

More generally for

$$f(x) = \sum_{m=1}^{\infty} A_m \sin(\frac{\pi m x}{L})$$

on $(0, L)$ we have

$$\boxed{A_m = \frac{2}{L}\int_0^L f(x)\ \sin(\frac{\pi m x}{L})\ dx}$$

5.2 Cosine series

Similarly, for

$$f(x) = \sum_{m=0}^{\infty} A_m \cos(mx)$$

on $(0, \pi)$,

$$A_m = \frac{f \cdot \cos(mx)}{\cos(mx) \cdot \cos(mx)} = \frac{2}{\pi} \int_0^\pi f(x) \cos(mx)\, dx$$

but

$$A_0 = \frac{1}{\pi} \int_0^\pi f(x)\, dx.$$

Generally, for

$$f(x) = \sum_{m=0}^{\infty} A_m \cos(\frac{\pi m x}{L}) \quad 0 < x < L$$

we have

$$\boxed{\begin{aligned} A_m &= \frac{2}{L} \int_0^L f(x) \cos(\frac{\pi m x}{L})\, dx \\ A_0 &= \frac{1}{L} \int_0^L f(x)\, dx \end{aligned}}$$

5.3 Full series

When the interval is extended to $(-\pi, \pi)$ we redefine the dot product as

$$f \cdot g = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

so for

$$f(x) = \sum_{m=0}^{\infty} A_m \cos(\frac{\pi m x}{L}) + B_m \sin(\frac{\pi m x}{L})$$

on $(-\pi, \pi)$ we have

$$\boxed{\begin{aligned} A_m &= \frac{1}{L} \int_{-L}^L f(x) \cos(\frac{\pi m x}{L})\, dx \\ B_m &= \frac{1}{L} \int_{-L}^L f(x) \sin(\frac{\pi m x}{L})\, dx \\ A_0 &= \frac{1}{2L} \int_{-L}^L f(x)\, dx \\ B_0 &= 0 \end{aligned}}$$

5.4 Complex series

For complex numbers, we again redefine the dot product such that

$$f \cdot g = \int_{-\pi}^{\pi} f(x)\overline{g(x)}$$

where

$$\overline{a + bi} = a - bi$$

so on $-\pi < x < \pi$,

$$f(x) = \sum_{m=-\infty}^{\infty} C_m e^{imx}$$

we have

$$C_m = \frac{f \cdot e^{imx}}{e^{imx} \cdot e^{imx}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-imx}\, dx.$$

Generally, with

$$f(x) = \sum_{m=-\infty}^{\infty} C_m e^{i\left(\frac{\pi m x}{L}\right)}$$

on $(-L, L)$:

$$\boxed{C_m = \frac{1}{2L} \int_{-L}^{L} f(x)e^{-i\left(\frac{\pi m x}{L}\right)}\, dx}$$

5.5 Parseval's Identity

Definition: $||u|| = \sqrt{u \cdot u}$ and $||cu|| = |c|\ ||u||$

Pythagorean Theorem: If $\{u, v, w\}$ is orthogonal,

$$||u + v + w||^2 = ||u||^2 + ||v||^2 + ||w||^2$$

Then on $(0, \pi)$, because $\{\sin(mx)\}$ is orthogonal,

$$\begin{aligned}
f(x) &= \sum_{m=1}^{\infty} A_m \sin(mx) \\
||f||^2 &= \left|\left|\sum_{m=1}^{\infty} A_m \sin(mx)\right|\right|^2 \\
&= \sum_{m=1}^{\infty} ||A_m \sin(mx)||^2 \\
&= \sum_{m=1}^{\infty} |A_m|^2 || \sin(mx)||^2 \\
\int_0^{\pi} (f(x))^2 \, dx &= \sum_{m=1}^{\infty} |A_m|^2 \int_0^{\pi} \sin^2(mx) \, dx \\
&= \frac{\pi}{2} \sum_{m=1}^{\infty} |A_m|^2
\end{aligned}$$

so

$$\boxed{\sum_{m=1}^{\infty} |A_m|^2 = \frac{2}{\pi} \int_0^{\pi} (f(x))^2 \, dx}$$

6 Laplace Equation

6.1 Derivation

From the 2D heat equation where $u = u(x, y, t)$

$$u_t = D(u_{xx} + u_{yy})$$

We assume that $\lim_{t \to \infty} u_t = 0$ so

$$0 = D(u_{xx} + u_{yy}) \implies u_{xx} + u_{yy} = 0$$

6.2 Rotational Invariance

Theorem:

Let $\Delta u(x, y) = 0$. Then for some constant θ where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$u_{x'x'} + u_{y'y'} = u_{xx} + u_{yy} = 0$$

Proof:

$$\begin{cases} x' = \cos(\theta)x - \sin(\theta)y \\ y' = \sin(\theta)x + \cos(\theta)y \end{cases}$$

$$\begin{aligned} u_x &= u_{x'} \cdot x'_x + u_{y'} \cdot y'_x = (u_{x'})\cos(\theta) + (u_{y'})\sin(\theta) \\ u_{xx} &= (u_x)_{x'} \cdot x'_x + (u_x)_{y'} \cdot y'_x = u_{x'x'}\cos^2(\theta) + 2u_{x'y'}\sin(\theta)\cos(\theta) + u_{y'y'}\sin^2(\theta) \\ u_y &= u_{x'} \cdot x'_y + u_{y'} \cdot y'_y = -u_{x'}\sin(\theta) + u_{y'}\cos(\theta) \\ u_{yy} &= (u_y)_{x'} \cdot x'_y + (u_y)_{y'} \cdot y'_y = u_{x'x'}\sin^2(\theta) - 2u_{y'x'}\cos(\theta)\sin(\theta) + u_{y'y'}\cos^2(\theta) \end{aligned}$$

$$u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'} \quad \blacksquare$$

6.3 Fundamental Solution

Example: Use the Polar Laplace

$$u_{rr} + \frac{1}{r}u_r = 0$$

with constants $A = -\frac{1}{2\pi}$ and $B = 0$ to derive the fundamental solution to the Laplace equation.

Solution: *Integrating factors:*

$$e^{\int \frac{1}{r}\,dr} = e^{\ln r} = r$$

$$ru_{rr} + u_r = 0 \implies (ru_r)_r = 0$$

$$ru_r = A \implies u_r = \frac{A}{r}$$

$$u = A\ln r + B$$

$$u(x, y) = A\ln(\sqrt{x^2 + y^2}) + B$$

$$\boxed{\Phi(x, y) = -\frac{1}{2\pi}\ln(\sqrt{x^2 + y^2})}$$