# Partial Differential Equations: APMA 0360

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## Lecture 1, Jan 25: Introduction

### Part I - Introduction

- Professor Peyam Tabrizian: drpeyam@brown.edu
- Office Hours: MWF 10:30-11:30
- Course site: sites.brown.edu/drpeyam
- Youtube: https://m.youtube.com/c/DrPeyam

Grading:

- Homework - 25% due Fridays 3pm
- Mini Project - 5% due Friday May 5
- Midterm 1 - 20% on Wednesday March 1
- Midterm 2 - 20% on Wednesday April 12
- Final - 30% on Tuesday May 16, 2-5pm

### Part II - What is a PDE?

*Partial Differential Equation:* an equation relating a function $u$ with one or more of its partial derivatives

Example: Laplace's Equation

$$\begin{cases} U = U(x, y) \\ \implies U_{xx} + U_{yy} = 0 \end{cases}$$

PhD advisor quote: "if you can solve all PDEs, you can solve the universe"

## Part III - PDE Applications

1. Physical sciences
   e.g. Navier-Stokes
2. Geometry
   e.g. Poincare's Conjecture
3. Probability
4. Operations research
   e.g. Hamilton-Jacobian PDE for maximizing/minimizing
5. Image Processing
   e.g. Smartphones, MRIs
6. Money
   e.g. Black-Scholes Equation
7. Chemical Reactions
   e.g. Peyam's Dissertation

The main characters of the course for $U = U(x, t)$:

1. Transport equation
   $$U_t + 3U_x = 0$$
2. Heat/diffusion equation
   $$U_t = U_{xx}$$
3. Wave equation
   $$U_{tt} = U_{xx}$$
   ("much like an extra chromosome, an extra t is not necessarily such a good thing")

4. Laplace's equation ($U(x,y)$)

$$U_{xx} + U_{yy} = 0$$

## Part III - Solution of PDE

Example 1: Is $U(x,t) = x^2t^2$ a solution of $U_{tt} = U_{xx}$?

$$\begin{cases} U_{tt} = (x^2t^2)_{tt} = 2x^2 \\ U_{xx} = (x^2t^2)_{xx} = 2t^2 \end{cases}$$

So $\boxed{\text{No}}$

Example 2: Is $U(x,y) = e^x \cos(y)$ a solution of $U_{xx} + U_{yy} = 0$

$$\begin{aligned} U_{xx} + U_{yy} &= (e^x \cos y)_{xx} + (e^x \cos y)_{yy} \\ &= e^x \cos y + e^x(-\cos y) \\ &= 0 = RHS \end{aligned}$$

So $\boxed{\text{yes}}$

## Part IV - Simple PDE

Note: $U = U(x,y)$

Example 3: $U_x = 0$ Because U can depend on y, this does NOT imply that $U = C$ Therefore:

$$\boxed{\text{U(x,y) = f(y)}}$$

Example 4: $U_{xx} = 0$

$$\implies U_x = f(y)$$

$$\implies U = \int f(y)\, dx = \boxed{xf(y) + g(y)}$$

(Where $g(y)$ is constant WRT x)

Example 5: $U_{xx} + U = 0$ Solving by Analogy: this is similar to ODE $y'' + y = 0 \implies y = A\cos x + B\sin x$

$$\boxed{U(x,y) = A(y)\cos x + B(y)\sin x}$$

# Lecture 2, Jan 27: Classification of PDE

## Part I - Simple PDE (Continued)

Example 1: $U_{xy} = 0$

$$\begin{aligned}(u_x)_y &= 0\\ u_x &= f(x)\\ u &= \int f(x)\,dx = \boxed{F(x) + G(y)}\end{aligned}$$

## Part II - Classification of PDE

*Order:* the highest derivative that appears Examples:

1. $u_{xx} + 3u_y = 0$ (Second order)
2. $2u_x + 3u_y = 0$ (First order)
3. $u_{zzyzx} = 0$ (Fifth order)

Note: In general, third-order and higher are impossible to solve

*Constant coefficient:* if the coeffs are constant Example:

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y)$$

Note: this example is also the "general form"

*Linear vs Nonlinear:* if the coefficients depend on x and y but not u Examples:

1. $u_{xx} + u_{yy} = 0$ (Linear)
2. $(u_x)^2 + 3e^u + u_y = 0$ (Nonlinear)
3. $x^2u_{xx} + y^3u_y + 4u = 0$ (Linear)

All constant coefficient equations are also linear

Note: Nonlinear PDEs are VERY difficult and none of the normal PDE methods work to solve them

**Interlude: the Linear Algebra View** *Linear transformation:* a transformation L is linear if

1. $L(u + v) = L(u) + L(v)$

2. $L(cu) = cL(u)$

*Linear PDE:* a PDE of the form

$$L(u) = f$$

where L is linear and f doesn't depend on= u

Example 2: Check that the following PDE is linear

$$u_{xx} + x^2 u_{yy} = e^y$$

Solution: $L(u) = u_{xx} + x^2 u_{yy}$ so we just need to check that L is linear

$$\begin{aligned} L(u+v) &= (u+v)_{xx} + x^2(u+v)_{yy} \\ &= u_{xx} + v_{xx} + x^2 u_{yy} + x^2 v_{yy} \\ &= L(u) + L(v)\checkmark \end{aligned}$$

$$\begin{aligned} L(cu) &= (cu)_{xx} = x^2(cu)_{yy} \\ &= cu_{xx} + cx^2 u_{yy} \\ &= c(u_{xx} + x^2 u_{yy}) \\ &= cL(u)\checkmark \end{aligned}$$

*Homogeneous/Inhomogeneous PDE:* for linear PDE, Homogeneous if $f = 0$ and Inhomogeneous otherwise Examples:

1. $u_{xx} + u_{yy} = 0$ Homo
2. $u_{xx} + u_{yy} = 2x$ Not homo

**Fun fact!** For linear homgeneous PDE $L(u) = 0$, the sum of two solutions is still a solution **Why?** L is linear so solutions span a vector space

## Part III - Types of Second-order PDE

Suppose you have a PDE of the form

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y)$$

Then, let $D = b^2 - 4ac$:

1. if $D < 0$ then the PDE is elliptic
2. if $D > 0$ then the PDE is hyperbolic
3. if $D = 0$ then the PDE is parabolic

Example 3: What is the type of the PDE

$$5u_{xx} + 6u_{xy} - 4u_{yy} + 3u_x + 5u = x^2$$

Solution:

$$D = 6^2 - 4(5)(-4) = 36 + 80 = 116 > 0 \implies \boxed{\text{hyperbolic}}$$

**Most famous PDE and their types:**

1. Laplace's equation ($u_{xx} + u_{yy} = 0$) is elliptic
2. Wave equation ($u_{tt} - u_{xx} = 0$) is hyperbolic
3. Heat equation ($u_t = u_{xx} \implies u_{xx} + 0u_{tt-u_t} = 0$) is parabolic

## Part IV - Review: Directional Derivatives

*Gradient vector of* $u = u(x, y)$: $\Delta u = (u_x, u_y)$

If $\vec{v}$ is a vector, then the *directional derivative* of u in the direction of $\vec{v}$ is

$$(\Delta u) \cdot \vec{v}$$

Intuitively, this measures the rate of change of u in the $\vec{v}$ direction. Normal convention is to have $\vec{v}$ as a unit vector but this is not actually necessary

Example 4: $u(x, y) = x^2 - y^2$ and $\vec{v} = (2, 3)$ Solution:

$$(\Delta u) \cdot \vec{v} = (2x, -2y) \cdot (2, 3) = \boxed{4x - 6y}$$

# Lecture 3, Jan 30: First-order Linear PDE

## Part I - The Constant Coefficient Case

**Goal:** solve a PDE of the form

$$au_x + bu_y = 0$$

Example 1: $2u_x + 3bu_y = 0$ Solution:

1. Observe the LHS is the same as

$$\langle u_x, u_y \rangle \cdot \langle 2, 3 \rangle = \nabla u \cdot \vec{v} = 0$$

   Note that this is the same as the directional derivative of u in the direction $\vec{v} = \langle 2, 3 \rangle$. This tells us that u is constant along lines parallel to $\langle 2, 3 \rangle$ (these are called *characteristic lines*)

2. Find the equation of each of the parallel lines

$$m = \frac{3}{2} \implies y = \frac{3}{2}x + C \implies 2y - 3x = C$$

3. Solution: $\boxed{u(x, y) = f(2y - 3x)}$ (where f is arbitrary)

**Summary:** the general solution of $au_x + bu_y = 0$ is

$$\boxed{\text{u(x, y) = f(ay- bx)}} \quad \text{where f is arbitrary}$$

## Part II - The General Case

Example 2: $u_x + yu_y = 0$ Solution:

1. Directional Derivative

$$\nabla u \cdot (1, y) = 0$$

   So u is constant along curves with "slope" v

2. Characteristic lines On one hand, the slope of the directional derivative is y. On the other, assuming y is a function of x, the slope should be $y'(x)$

   Putting it together,

$$y' = y \implies y = Ce^x$$

   Why? Consider $g(x) = u(x, Ce^x)$ Then,

$$g'(x) = u_x(x, Ce^x) + Ce^x u_y(x, Ce^x) = u_x + yu_y = 0$$

3. Find the arbitrary function input that is constant on each curve $y = Ce^x$

$$y = Ce^x \implies ye^{-x} = C$$

4. Solution:

$$u(x, y) = f(ye^{-x})$$

## Part III - More Practice

Example 3: $xu_x + yu_y = 0$ Directional derivative:

$$\nabla u \cdot \langle x, y \rangle = 0$$

ODE:

$$\begin{aligned} fracyx &= y'(x) \\ x\,dy &= y\,dx \\ \ln|y| &= \ln|x| + C \\ |y| &= |x|e^c \\ \frac{y}{x} &= C \end{aligned}$$

Solution: $\boxed{u(x,y) = f(\frac{y}{x})}$

# 1 Lecture 4, Feb 1: Transport Equation

## Part I - The Chain Rule

If $f = f(x,y)$ where $x = x(s,t)$ and $y = y(s,t)$ then,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

## Part II - Coordinate Method

Example: $2u_x + 3u_y = 0$

1. Define new variables x' and y'

$$\begin{cases} x' = 2x + 3y \\ y' = -3x + 2y \end{cases}$$

   Note: $(2,3)$ is the vector in the direction of the directional derivative and $(-3,2)$ is perpendicular

2. Rewrite in terms of x' and y' using chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = 2u_{x'} - 3u_{y'}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = 3u_{x'} + 2u_{y'}$$

3. Substitute definitions

$$\begin{aligned} 2u_x + 3u_y &= 0 \\ 2(2u_{x'} - 3u_{y'}) - 3(3u_{x'} + 2u_{y'}) &= 0 \\ 4u_{x'} - 6u_{y'} + 9u_{x'} + 6u_{y'} &= 0 \\ 13u_{x'} = 0 \implies u_{x'} &= 0 \end{aligned}$$

4. Solution

$$u_{x'} = 0 \implies u = f(y')$$
$$\boxed{u = f(2y - 3x)}$$

## Part III - Transport equation

$$u_t + cu_x = 0$$

where $u = u(x,t)$ where x is position, t is time, and c is a speed constant. It models the density of a fluid that is transported at speed c

**Derivation:** The mass on an interval $[0, b]$ at time t is:

$$M = \int_0^b u(x,t)\, dx$$

At a later time, $t + h$, the fluid shifts from $[0, b]$ to $[ch, b + ch]$. Now, the mass is

$$M = \int_{ch}^{b+ch} u(x, t+h)\, dx$$

Since mass is conserved, get:

$$\int_0^b u(x,t)\, dx = \int_{ch}^{b+ch} u(x, t+h)\, dx$$

Differentiate with respect to b:

$$\frac{d}{db}\int_0^b u(x,t)\;dx = \frac{d}{db}\int_{ch}^{b+ch} u(x,t+h)\;dx$$

By the Fundamental Theorem of calculus:

$$u(b,t) = u(b+ch, t+h)$$

Differentiate with respect yo h:

$$0 = \frac{\partial u}{\partial x}\frac{\partial(b+ch)}{\partial h} + \frac{\partial u}{\partial t}\frac{\partial(t+h)}{\partial h}$$

$$0 = cu_x + u_t$$

**Solving:**

$$u_t + cu_x = 0 \implies cu_x - u_t = 0$$

Recall: the general solution to $au_x + bu_y = 0$

$$u(x,y) = f(ay - bx)$$

Note: this can also be written $f(bx - ay)$ but with different f

Therefore,

$$\boxed{\text{u(x, t) = f(x - ct)}}$$

# Lecture 5, Feb 3: Heat Equation Derivation

## Part I - The Heat Equation

$$u_t = Du_{xx}$$

where $D > 0$ is a diffusion constant The equation gives the temperature of a metal rod at position x and time t.

## Part II - Derivation

Note: can also use Fick's law from physics to derive it

1. Think about the rod as composed of particles that move in two dimensions (left or right)
2. Let $u = u(x,t)$ measure the concentration (#/length) of particles at x and t
3. Let $h = \Delta x$ and $\tau = \frac{h^2}{2D}$ (it will work!)
4. Focus on (x, t) (look at the small neighborhood of x: $[x - \frac{h}{2}, x + \frac{h}{2}]$)
5. Note that the length of the interval is h so the number of particles on the interval is roughly $hu(x,t)$
6. Divide the rod into more intervals of length h
7. Main assumption: as time increases from $t$ to $t + \tau$, each particle moves to the left or right with equal probability
8. $$hu(x, t+\tau) = hu(x,t) + \text{ change}$$
9. $$\text{change} = \text{in - out} = \begin{cases} \text{out} = \frac{1}{2}hu(x,t) + \frac{1}{2}hu(x,t) \\ \text{in} = \frac{1}{2}hu(x-h,t) + \frac{1}{2}hu(x+h,t) \end{cases}$$
$$\implies \frac{1}{2}hu(x-h,t) + \frac{1}{2}hu(x+h,t) - hu(x,t)$$
10. $$hu(x,t+\tau) = hu(x,t) + \frac{1}{2}hu(x-h,t) + \frac{1}{2}hu(x+h,t) - hu(x,t)$$
11. $$hu(x,t+\tau) - hu(x,t) = \frac{h}{2}\left(u(x-h,t) - 2u(x,t) + u(x+h,t)\right)$$
12. Make some more transformations to get into the right form:
$$\frac{u(x,t+\tau) - u(x,t)}{\tau} = \frac{h^2}{2\tau}\left(\frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2}\right)$$

13. Limits:
$$\lim_{\tau \to 0} \frac{u(x, t+\tau) - u(x,t)}{\tau} = u_t(x,t)$$
Then by double l'Hopital's:
$$\lim_{h \to 0} \left( \frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2} \right) = u_{xx}$$

14.
$$u_t = \left( \lim_{tau, h \to 0} \frac{h^2}{2\tau} \right) u_{xx}$$

15. Then using the definition of tau:
$$\boxed{u_t = Du_{xx}}$$

# 2 Lecture 6, Feb 6: Fourier Transform

## Part I - Behavior of Solutions

The Heat Equation:
$$u_t = Du_{xx}$$
Where $u(x,t)$ is the temperature of a metal rod at x and t and $D > 0$ is a diffusivity constant dependent on material

Notice that if $u_{xx} > 0$, then $u_t = Du_{xx} > 0$ whenever u is concave up in x, u will increase in time and vice versa. In other words, over time the graph will "flatten out"

## Part II - Interlude: The Gaussian Integral

Example:
$$\int_{-\infty}^{\infty} e^{-x^2}\, dx$$
Classically, $e^{-x^2}$ does not have an antiderivative and yet we can take the integral with the following method:

1. Trick: Consider
$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-y^2}\,dy > 0$$
(The variable does not matter)

2. Multiply:
$$\begin{aligned}
I^2 &= (I)(I) \\
&= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\,r\,dr\,d\theta \\
&= 2\pi\int_{0}^{\infty} re^{-r^2}\,dr \\
&= 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} \qquad (u=-r^2) \\
&= 2\pi\left(-\frac{1}{2}e^{-\infty+\frac{1}{2}e^0}\right) \\
&= \pi
\end{aligned}$$

3. Therefore $I^2 = \pi$ and since $I > 0$, we get $I = \sqrt{\pi}$ and so:
$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Note: this same method can be used to calculate $\int_{-\infty}^{\infty} \sin(x^2)\,dx$

# Part III - The Fourier Transform

The Fourier Transform functions in much the same way as the Laplace Transform of ODEs.
$$\widehat{f}(\kappa) = \int_{-\infty}^{\infty} f(x)e^{i\kappa x}\,dx$$

Notes:

- This is a function of $\kappa$ as x is integrated out
- Interpretation: changes functions from phase space to frequency space
- Application: essential for signal processing and imaging
- Often represented with $\xi$ instead of $\kappa$ and $e^{-i\kappa x}$ rather than $e^{i\kappa x}$

Example: Calculate $\widehat{f}$ where $f(x) = e^{-x^2}$ Solution:

$$\widehat{f}(\kappa) = \int_{-\infty}^{\infty} e^{-x^2} e^{i\kappa x}\, dx$$

1. Find a differential equation for $\widehat{f}$

$$\begin{aligned}\widehat{f}'(\kappa) &= \frac{d}{d\kappa}\int_{-\infty}^{\infty} e^{-x^2} e^{i\kappa x}\, dx \\ &= \int_{-\infty}^{\infty} e^{-x^2} e^{i\kappa x}(ix)\, dx \\ &= i\int_{-\infty}^{\infty} xe^{-x^2} e^{i\kappa x}\, dx\end{aligned}$$

2. Integrate by parts with respect to x:

$$\begin{cases} du = xe^{-x^2} \implies u = -\frac{1}{2}e^{-x^2} \\ v = e^{i\kappa x} \implies dv = e^{i\kappa x}(i\kappa) \end{cases}$$

Integrating:

$$\begin{aligned}&= i\left[-\frac{1}{2}e^{-x^2}e^{i\kappa x}\right]_{-\infty}^{\infty} - i\int_{-\infty}^{\infty} -\frac{1}{2}e^{-x^2}e^{i\kappa x}\, dx \\ &= 0 + \frac{i}{2}(i\kappa)\int_{-\infty}^{\infty} e^{-x^2}e^{i\kappa x}\, dx \\ &= -\frac{\kappa}{2}\widehat{f}(\kappa)\end{aligned}$$

Giving us a new ODE to solve in the next lecture of

$$\widehat{f}'(\kappa) = -\frac{\kappa}{2}\widehat{f}(\kappa)$$

## Part IV - The Schwartz Class

Notice that the infinite terms in the above example are 0 because $e^{-x^2}$ goes to 0 very quickly.

This is the easiest class of functions to apply the Fourier transform to

**Definition:** f is *Schwartz* if it is infinitely differentiable and for every n

$$\lim_{x \to \pm\infty} \left| \frac{f(x)}{x^n} \right| = 0$$

And same for all derivatives of f.

In other words, f and its derivatives go to 0 at $\pm\infty$ faster than any power function $x^n$. This allows us to ignore the infinite terms in the Fourier integration

# Lecture 7, Feb 8: Fourier Transform and Heat Equation

## Part I - Fourier Transform Example

$$\widehat{f}(\kappa) = \int_{-\infty}^{\infty} f(x) e^{i\kappa x} \, dx$$

Example: $\widehat{f}$ where $f(x) = e^{-x^2}$ Solution:

1. Find a Differential equation rather than try to solve directly

$$\widehat{f}'(\kappa) = -\frac{\kappa}{2} f(\kappa)$$

2. Solve the ODE

$$\widehat{f}' + \frac{\kappa}{2} f = 0$$

$$\left( \widehat{f} e^{\frac{\kappa^2}{4}} \right)' = 0$$

$$\widehat{f}(\kappa) = C e^{-\frac{\kappa^2}{4}}$$

3. Find C

$$\kappa = 0 \implies \widehat{f}(\kappa) = Ce^0 = C$$

$$C = \widehat{f}(\kappa) = \int_{-\infty}^{\infty} e^{-x^2} e^{i0x}\, dx = \sqrt{\pi}$$

4. Answer

$$\boxed{\widehat{f}(\kappa) = \sqrt{\pi} e^{-\frac{\kappa^2}{4}}}$$

Note that if you apply the fourier to a gaussian, you get another gaussian!

More generally, The Fourier transform of $f(x) = e^{-ax^2}$ is

$$\boxed{\widehat{f}(\kappa) = \sqrt{\frac{\pi}{a}} e^{-\frac{\kappa^2}{4a}}}$$

## Part II - Fourier Transform and Derivatives

**Recall:** The Laplace transform turns derivatives into products

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} = y(0)$$

**Fact:**

$$\boxed{\widehat{f'}(\kappa) = (-i\kappa)\widehat{f}(\kappa)}$$

**Proof:**

$$\begin{aligned}
\widehat{f'}(\kappa) &= \int_{-\infty}^{\infty} f'(x) e^{i\kappa x}\, dx \\
&\overset{\text{IBP}}{=} \left[ f(x) e^{i\kappa x} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \frac{d}{dx} e^{i\kappa x}\, dx \\
&= 0 - \int_{-\infty}^{\infty} f(x) \frac{d}{dx} e^{i\kappa x}\, dx = -i\kappa \int_{-\infty}^{\infty} f(x) \frac{d}{dx} e^{i\kappa x}\, dx \\
&= -i\kappa \widehat{f}(\kappa)
\end{aligned}$$

## Part III - Fourier transform and the Heat Equation

Example: Solve

$$\begin{cases} u_t = Du_{xx} \\ u(x,0) = f(x) \quad \text{(given)} \end{cases}$$

Solution:

1. Apply the x fourier Transform

$$\widehat{u_t} = D\widehat{u_{xx}}$$

$$\widehat{u}(\kappa, t) = \int_{-\infty}^{\infty} u(x,t)e^{i\kappa x}\, dx$$

$$\begin{aligned} \widehat{u_{xx}}(\kappa, t) &\overset{\text{fact}}{=} (-i\kappa)\widehat{u_x}(\kappa, t) \\ &\overset{\text{fact}}{=} (-i\kappa)(-i\kappa)\widehat{u}(\kappa, t) \\ &= -\kappa^2\widehat{u}(\kappa, t) \end{aligned}$$

   For $u_t$, do directly:

$$\begin{aligned} \widehat{u_t} &= \int_{-\infty}^{\infty} u_t(x,t)e^{i\kappa x}\, dx \\ &= \int_{-\infty}^{\infty} \frac{d}{dt}(u(x,t)e^{i\kappa x})\, dx \\ &= \frac{d}{dt} int_{-\infty}^{\infty} u(x,t)e^{i\kappa x}\, dx \\ &= \frac{d}{dt}\widehat{u}(\kappa, t) \end{aligned}$$

2. Solve the new ODE

$$\widehat{u_t} = D\widehat{u_{xx}} \implies \frac{d}{dt}\widehat{u}(\kappa, t) = -D\kappa^2\widehat{u}(\kappa, t)$$

   Recall:

$$y' = ay \implies y = Ce^{at} = y(0)e^{at}$$

   Similarly,

$$\widehat{u}(\kappa, t) = \widehat{u}(\kappa, 0)e^{-D\kappa^2 t}$$

Note:

$$u(x,0) = f(x) \overset{\text{fourier}}{\Longrightarrow} \widehat{u}(\kappa, 0) = \widehat{f}(\kappa)$$

Therefore,

$$\boxed{\widehat{u}(\kappa, t) = \widehat{f}(\kappa) e^{-D\kappa^2 t}}$$

**Problem:** But how do we go from $\widehat{u}$ to $u$?

# Lecture 8, Feb 10: Convolution

## Part I - Convolution

**Definition:**

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$$

**Example:** $(f \star g)(x)$ where $f(x) = e^x$ and

$$g(x) = \begin{cases} 1 & [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$\begin{aligned} (f \star g)(x) &= \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy \\ &= \int_0^1 e^{x-y}\, dy \\ &= e^x \int_0^1 e^{-y}\, dy \\ &= e^x [-e^{-y}]_0^1 = \boxed{(1-e^{-1})e^x} \end{aligned}$$

**Fact:**

$$\widehat{f \star g}(\kappa) = \widehat{f}(\kappa) \cdot \widehat{g}(\kappa)$$

## Part II - Solving the Heat Equation

**Example:** Use the fourier transform to solve

$$\begin{cases} u_t = Du_{xx} \\ u(x,0) = f(x) \end{cases}$$

Solution: (Via ODEs)

$$\widehat{u}(\kappa, t) = \widehat{f}(\kappa) e^{-D\kappa^2 t}$$

Next, we wish to write $e^{-D\kappa^2 t}$ as a fourier transform. Note that for most equations this is impossible or VERY difficult but not for the Gaussian!

**Recall:**

$$\widehat{e^{-ax^2}} = \sqrt{\frac{\pi}{a}} e^{-\frac{\kappa^2}{4a}} \implies e^{-\frac{-\kappa^2}{4a}} = \widehat{\sqrt{\frac{a}{\pi}} e^{-ax^2}}$$

Therefore find a such that

$$e^{-\frac{\kappa^2}{4a}} = e^{-D\kappa^2 t}$$
$$\longrightarrow a = \frac{1}{4Dt}$$

So,

$$\sqrt{\frac{a}{\pi}} = \frac{1}{\sqrt{4\pi Dt}}$$
$$\longrightarrow e^{-\kappa^2 Dt} = \mathcal{F}\left(\frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}\right)$$

Or,

$$e^{-\kappa^2 Dt} = \widehat{g}(\kappa, t) \quad g(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

**Grand Finale!**

$$\widehat{u}(\kappa, t) = \widehat{f}(\kappa) e^{-\kappa^2 t} = \widehat{f}(\kappa)\widehat{g}(\kappa, t)$$
$$\widehat{u}(\kappa, t) = \mathcal{F}((f \star g)(\kappa, t))$$
$$u(x,t) = (f \star g) = \int_{-\infty}^{\infty} f(y)\, g(x-y, t)\, dy$$

where

$$g(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

Solving:

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} f(y)\, e^{-\frac{(x-y)^2}{4Dt}}\, dy} \quad (t > 0)$$

## Part III - The Heat Kernel

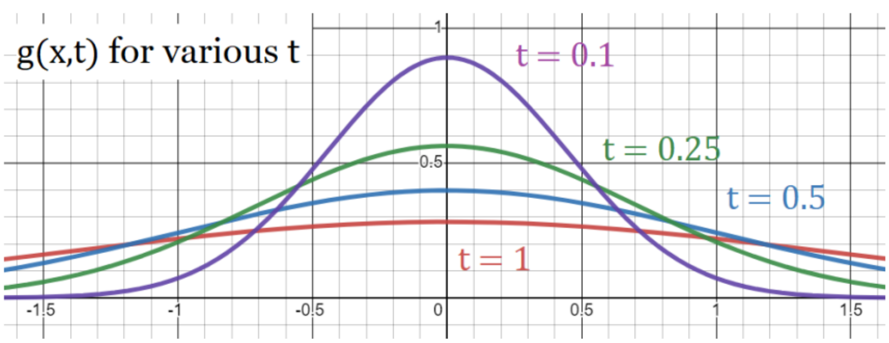
**Definition:** Heat kernel (AKA Fundamental sol of the heat equation)

$$g(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

Properties:

1. g itself solves $g_t = D g_{xx}$
2. $\int_{-\infty}^{\infty} g(x,t)\, dx = 1$ for all t

**Picture:** For every $t$, $g(x,t)$ looks like a bell-curve $e^{-x^2}$ but that gets more and more spread out as you increase $t$:




Note that as $t \to 0^+$, $g(x,t)$ is the Dirac delta at $x = 0$

## Part V - Convolution Intuition

**Example:** What is the coefficient of $x^2$ in

$$(x^2 + 2x + 3)(2x^2 + 4x + 1)$$

Generally, the coeff of $x^2$ in $(a_2x^2 + a_1x + a_0)(b_2x^2 + b_1x + b_2)$ is

$$C_2 = a_0b_2 + a_1b_1 + a_2b_0$$

and more generally, the coefficient of $x^k$ in $(a_nx^n + ...a_0)(b_nx^n + ... + b_0)$ is

$$C_k = \sum_{i=0}^{k} a_i b_{k-1}$$

Note the parallel to

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

# Lecture 9, Feb 13: Heat Equation Properties

## Part I - Heat Equation Example

**Example 1:** Solve

$$\begin{cases} u_t = Du_{xx} \\ u(x,0) = e^{-x} \end{cases}$$

**Solution:**

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} e^{-y}\, dy$$

Looking at the exponent:

$$\frac{-(x-y)^2}{4Dt} - y = -\frac{(x-y)^2 + 4Dty}{4Dt}$$

Expand the numerator:

$$= -\frac{x^2 - 2xy - y^2 + 4Dty}{4Dt}$$

Note the numerator is a quadratic in y:

$$\begin{aligned} y^2 + (4Dt - 2x)y + x^2 &= (y + 2Dt - x)^2 - (2Dt - x)^2 + x^2 \\ &= (y + 2Dt - x)^2 - 4D^2t^2 + 4Dtx - x^2 + x^2 \\ &= (y + 2Dt - x)^2 + 4Dt(x - Dt) \end{aligned}$$

So the full numerator is

$$\frac{-(x-y)^2}{4Dt} = -\left(\frac{(y+2Dt-x)^2 + 4Dt(x-Dt)}{4Dt}\right) = -\left(\frac{(y+2Dt-x)^2}{4Dt} + (x-Dt)\right)$$

Substituting back in,

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\left(\frac{(y+2Dt-x)^2}{4Dt} + (x-Dt)\right)}\, dy \\ &= \frac{e^{Dt-x}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-frac(y+2Dt-x)^2 4Dt}\, dy \\ &= \frac{e^{Dt-x}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\left(\frac{y+2Dt-x}{\sqrt{4Dt}}\right)^2}\, dy \end{aligned}$$

Now use u-sub with

$$p = \frac{y+2Dt-x}{\sqrt{4Dt}}$$

so

$$dp = \frac{dy}{\sqrt{4Dt}} \implies dy = \sqrt{4Dt}\, dp$$

$$u(x,t) = \frac{e^{Dt-x}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-p^2}\sqrt{4Dt}\, dp = \frac{e^{Dt-x}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\, dp$$

$$\boxed{u(x,t) = e^{Dt-x}}$$

## Part II - Infinite speed of propagation

Remember the heat equation solution is:

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\, dy$$

with an initial condition $u(x,0) = f(x)$

**Property 1:** If $f \geq 0$ is positive somewhere and continuous, then $u(x,t)$ is positive everywhere.

This means that heat propagates at infinite speed because heat at one place affects heat everywhere else instantly. Note that the transport equation implies a finite speed of propagation.

**Why?** Suppose $f(x_0) > 0$ for some $x_0$. Then because f is continuous it is actually positive for all x in an interval around $x_0$ Also

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy$$

and we know the integrand is non-negative so we have

$$\frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy \geq \frac{1}{\sqrt{4\pi Dt}} \int_{a}^{b} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy$$

But the integrand of the second is also positive so

$$u(x,t) > 0$$

## Part III - Smoothness

**Property 2:** $u(x,t)$ is infinitely differentiable (for $t > 0$) even if $f(x)$ might not be

**Why?** All the derivatives fall of $\exp(-\frac{(x-y)^2}{4Dt})$ and not on f:

$$\frac{d}{dx}u(x,t) = \frac{d}{dx}\frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \frac{d}{dx}e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy$$

But the term

$$e^{-\frac{(x-y)^2}{4Dt}}$$

is infinitely differentiable and

$$\frac{d}{dt}u(x,t) = Du_{xx}$$

but $u_{xx}$ is also smooth

## Part IV - Irreversibility

**Property 3:** The heat equation is irreversible ($u(x,0)$ cannot be determined from $u(x,1)$)

**Why?** "something something entropy"

Suppose $u(x,1) = |x|$ but by smoothness, $u(x,t)$ must be smooth for all t so $|x|$ must be smooth but this is a contradiction

# Lecture 10, Feb 15: Inverse Fourier Transform

## Part I - Long-time behavior of the heat kernel

$$u(x,t) = \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\, dy$$

**Property 4:**

$$\lim_{t\to\infty} u(x,t) = 0$$

"heat dissipates over time"

**Why?**

$$\begin{aligned}
|u(x,t)| &= \left| \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\, dy \right| \\
&\le \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} |f(y)| \underbrace{e^{-\frac{(x-y)^2}{4Dt}}}_{\le 1}\, dy \\
&\le \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} |f(y)|\, dy \\
&= C \xrightarrow{t\to\infty} 0
\end{aligned}$$

## Part II - Boundedness

"u(x, t) does not blow up"

**Property 5:** If $|f(x) \le M$ for some M (and all x) then for all x and t we have

$$|u(x,t) \le M$$

**Why?**

$$\begin{aligned}
|u(x,t)| &= \left| \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\, dy \right| \\
&\leq \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} \underbrace{|f(y)|}_{\leq M} e^{-\frac{(x-y)^2}{4Dt}}\, dy \\
&\leq \frac{M}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} e^{-\left(\frac{y-x}{\sqrt{4\pi Dt}}\right)^2} dy \quad u = \frac{y-x}{\sqrt{4Dt}} \implies du = \frac{dy}{\sqrt{4Dt}} \\
&= \frac{M}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} e^{-u^2} \sqrt{4Dt}\, dy \qquad = \frac{M}{\sqrt{4}\pi Dt} \sqrt{4Dt}\sqrt{\pi} \\
&= M
\end{aligned}$$

# Part III - Conservation of Mass

"The area under the curve of u – no matter its shape – is always the same"

$$\int_{-\infty}^{\infty} u(x,t)\, dx = \int_{-\infty}^{\infty} f(x)\, dx$$

**Why?**

*Lemma:*

$$\lim_{x \to \pm\infty} u_x(x,t) = 0$$

Then,

$$u_t = Du_{xx}$$

$$\int_{-\infty}^{\infty} u_t(x,t)\, dx = \int_{-\infty}^{\infty} Du_{xx}(x,t)\, dx$$

and by FTC

$$\frac{d}{dt} \int_{-\infty}^{\infty} u(x,t)\, dx = D\left[u_x(xmt)\right]_{-\infty}^{\infty}$$

Thus by the lemma,

$$\frac{d}{dt} \int_{-\infty}^{\infty} u(x,t)\, dx = D(0-0) = 0$$

So the integral is constant with respect to time:

$$\int_{-\infty}^{\infty} u(x,t)\, dx = \int_{-\infty}^{\infty} u(x,0)\, dx = \int_{-\infty}^{\infty} f(x)\, dx$$

# Part IV - Inverse Fourier Transform

Note that for the heat equation, we were very lucky to be able to write the Gaussian as a fourier transform

$$e^{-D\kappa^2 t} = \mathcal{F}\left(\frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}\right)$$

**But what do we do in general?**

Example: Solve

$$\begin{cases} u_t = -u_{xxxx} \\ u(x,0) = f(x) \end{cases}$$

Solution:

1. Fourier transform it

$$\mathcal{F}\left(u_t\right) = \mathcal{F}\left(-u_{xxxx}\right)$$

$$\frac{d}{dt}\widehat{u} = -(-i\kappa)^4\widehat{u} = -\kappa^4\widehat{u}$$

2. Solve the ODE

$$\widehat{u} = u(x,0)e^{-\kappa^4 t} = \widehat{f}(\kappa)e^{-\kappa^4 t}$$

3. Write the exponential term as a fourier transform

   **Definition:** *Inverse Fourier Transform*

$$\boxed{\check{f}(x) = \mathcal{F}^{-1}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\kappa)e^{-i\kappa x}\, d\kappa}$$

   So in this example,

$$e^{-\kappa^4 t} = \widehat{g}(\kappa) \quad g(x,t) = \mathcal{F}^{-1}\left(e^{-\kappa^4 t}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\kappa^4 t}e^{-i\kappa x}\, d\kappa$$

4. Convolution

5. So now we have

$$\begin{aligned} \widehat{u}(\kappa,t) &= \widehat{f}(\kappa)e^{-\kappa^4 t} \\ &= \widehat{f}(\kappa)\widehat{g}(\kappa,t) \\ &= \mathcal{F}\left(f \star g\right)(\kappa,t) \end{aligned}$$

Therefore,

$$\boxed{\begin{cases} u(x,t) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy \\ \text{where} \quad g(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\kappa^4 t}e^{-i\kappa x}\,d\kappa \end{cases}}$$

# Lecture 11, Feb 17: Wave Equation Derivation

## Part I - The Wave Equation

$$u_{tt} = c^2 u_{xx}$$

where $u = u(x,t)$ gives the displacement of a vibrating string at position x and time t and c is a constant giving the speed of the wave

Note: despite the only difference between this and the heat equation is an extra time derivative, the derivation and solution will be *completely* different

## Part II - Derivation

1. Setting: start with a thin string of infinite length and consider a minute subpiece from $x$ to $x + \Delta x$

   **Assumption:** points on the string only move vertically

2. By Newton's second law of motion,

   $$F = ma$$

   By the assumption above and the definition of u, the displacement vector is

   $$s(x,t) = \langle 0, u(x,t)\rangle$$

   Therefore, acceleration is

   $$a(x,t) = s_{tt}(x,t) = \langle 0, u_{tt}\rangle$$

   **Assumption:** the string has constant density $\rho$

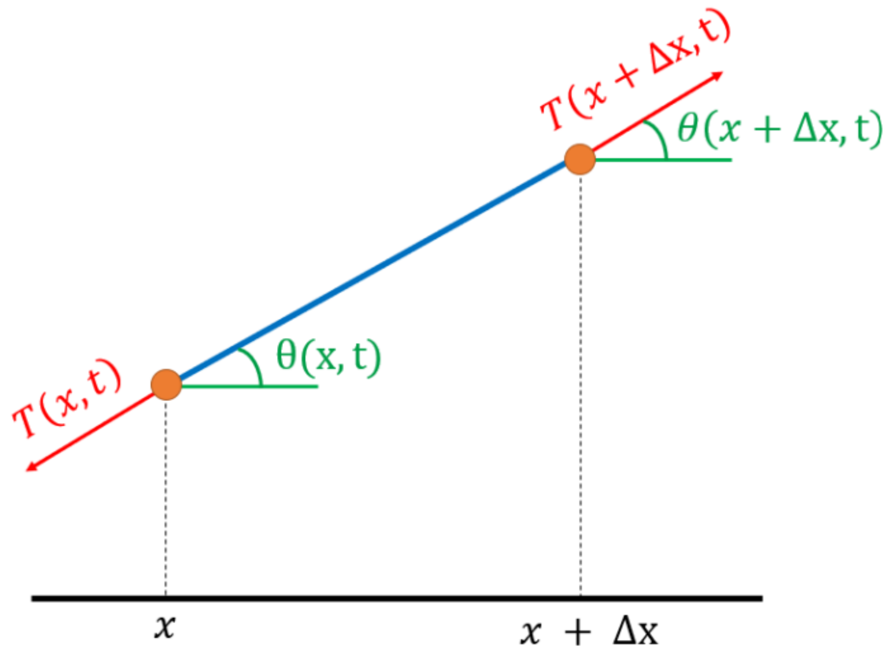
   Then, the mass of the string is density times length (which can be taken by assuming the length is the hypotenuse of a right triangle with legs $\Delta x$ and $\Delta u$). Thus,

   $$m = \rho\sqrt{(\Delta x)^2 + (\Delta u)^2}$$

   So,

   $$F = ma = \langle 0, \rho\sqrt{(\Delta x)^2 + (\Delta u)^2}\rangle u_{tt}$$

3. Study of the Force: **Assumption: The only force acting on the string is the tension** So if $T(x,t)$ is the magnitude of the tension vector and $\theta(x,t)$ is the angle of the tension vector:




Then from trig, we can calculate the tension force via components of the resultant:

$$\begin{cases} x = T(x,t)\cos(\theta(x,t)) \\ y = T(x,t)\sin(\theta(x,t)) \end{cases} \implies -\langle T\cos(\theta), T\sin(\theta)\rangle(x,t)$$

Note: the minus comes from T pointing the opposite direction of the string

Then in the same way, the force at $(x+\Delta x)$ is

$$\langle T\cos(\theta), T\sin(\theta)\rangle(x+\Delta x, t)$$

so the net force is

$$F(x,t) = \langle T\cos(\theta), T\sin(\theta)\rangle(x+\Delta x, t) - \langle T\cos(\theta), T\sin(\theta)\rangle(x,t)$$

4. Then using $F = ma$ and comparing the components,

$$\begin{cases} T\cos(\theta)(x+\Delta x,t) - T\cos(\theta)(x,t) = 0 \\ T\sin(\theta)(x+\Delta x,t) - T\sin(\theta)(x,t) = \rho\sqrt{(\Delta x)^2 + (\Delta u)^2}u_{tt}(x,t) \end{cases}$$

Note, however, that both these LHS look like derivatives. Starting with the cos terms,

$$(T\cos(\theta))_x = 0$$

so $T(x,t)\cos(\theta(x,t))$ is constant in x. But $|\theta(x,t)| << 1$ so $\cos(\theta(x,t)) \approx 1$ and

$$T(x,t)\cos(\theta(x,t)) = T(x,t)$$

which is constant in x so $T(x,t) = T(t)$

**Assumption:** Tension is also constant in time $T(t) = T$

Then the sin terms,

$$\begin{aligned} (T\sin(\theta))_x &= \rho u_{tt}\left(\frac{\sqrt{(\Delta x)^2 + (\Delta u)^2}}{\Delta x}\right) \\ &= \rho u_{tt}\sqrt{\frac{(\Delta x)^2 + (\Delta u)^2}{\Delta x}} \\ &= \rho u_{tt}\sqrt{1 + \left(\frac{\Delta u}{\Delta x}\right)^2} \\ &= \rho u_{tt}\sqrt{1 + (u_x)^2} \end{aligned}$$

**Assumption:** if the displacements $\Delta u/\Delta x$ are small, then

$$\theta(x,t) = \tan^{-1}\frac{\Delta u}{\Delta x}$$

is small, proving the inequality above.

Then, as $\Delta x \to 0$, $|u_x| << 1$ so

$$\sqrt{1 + (u_x)^2} \approx 1$$

and

$$(T\sin(\theta))_x = \rho u_{tt}$$

but

$$\sin\theta = \tan\theta\cos\theta = \frac{\Delta u}{\Delta x}\cos\theta \to u_x$$

so

$$(Tu_x)_x = Tu_{xx} \quad \text{(assuming T is constant)}$$

and at last,

$$Tu_{xx} = \rho u_{tt} \longrightarrow u_{tt} = \frac{T}{\rho}u_{xx}$$

Set, $c = \sqrt{T/\rho} > 0$ and

$$\boxed{u_{tt} = c^2 u_{xx}}$$

# Lecture 12, Feb 22: Wave Equation Solution

**Goal:** Solve $u_{tt} = c^2 u_{xx}$

## Part I - Factoring Method

But this kind of looks like

$$t^2 - c^2x^2 = (t - cx)(t + cx)$$

**Definition:** Differential operator

$$\frac{\partial}{\partial t}u = u_t$$

$$\left(\frac{\partial}{\partial t}\right)^2 u = u_{tt}$$

Using this operator we can more rigorously "factor" the PDE.

1. Apply the differential operator

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= \left[\left(\frac{\partial}{\partial t}\right)^2 - c^2\left(\frac{\partial}{\partial x}\right)^2\right]u \\ &= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) \end{aligned}$$

2. Solve the equation
$$u_{tt} - c^2 u_{xx} = 0$$
$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) = 0$$
Let $v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u$ so
$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0 \Longrightarrow v_t - cV_x = 0$$

3. Solve the transport PDE
$$v(x,t) = f(x+ct)$$

4. Solve for u
$$v := \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = u_t + cu_x$$
$$u_t + cu_x = f(x+ct)$$
But this is just an inhomogeneous transport equation! The homogeneous solution is just
$$u_0(x,t) = G(x-ct)$$
And a particular solution can be found using undetermined coefficients. Notice that the RHS is a function of $x+ct$ so we can guess
$$u_p = h(x+ct)$$
so
$$(h(x+ct))_t + c(h(x+ct))_x = f(x+ct)$$
$$ch'(x+ct) + ch'(x+ct) = f(x+ct)$$
$$2ch'(x+ct) = f(x+ct) \Longrightarrow h' = \frac{1}{2c}f'$$
$$h(x+ct) = \frac{1}{2c}F(x+ct)$$
where $F$ is an antiderivative of $f$ Thus giving the general solution
$$u(x,t) = G(x-ct) + \frac{1}{2c}F(x+ct)$$
$$\boxed{u(x,t) = G(x-ct) + F(x+ct)}$$

**Interpretation:** A wave is a sum of two functions, one moving to the left at speed c and the other to the right at speed c

## Part II - Coordinate Method

1. Define variables
$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$

2. Chain rule
$$u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi + u_\eta$$
and
$$\begin{aligned} u_{xx} = (u_x)_x &= \frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= u_{\xi_\xi} + u_{\eta_\eta} = u_{\xi\xi} + u_{\eta\xi} + u_{\xi\eta} + u_{\eta\eta} \\ &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \end{aligned}$$
Similarly,
$$u_t t = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$

3. Plug into wave equation:
$$u_{tt} = c^2 u_{xx}$$
$$c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) = c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = 4u_{\xi\eta}$$
$$\boxed{u_{\xi\eta} = 0}$$

# Lecture 13, Feb 24: D'Alembert's Formula

## Part I - Solving the wave equation (continued)

$$u_{tt} = c^2 u_{xx}$$
Using the coordinate method with the choices
$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$
we get the equation
$$u_{\xi\eta} = 0$$
so
$$u_\xi = f(\xi) \Longrightarrow u = F(\xi) + G(\eta)$$
thus
$$\boxed{u(x,t) = F(x - ct) + G(x + ct)}$$

## Part II - D'Alembert's Formula

**Example:**

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

**Solution:**

1. General Solution
$$u(x,t) = F(x-ct) + G(x+ct)$$

2. Plug in the initial condition
$$u(x,0) = \phi(x) = F(x) + G(x)$$

3. Differentiate with t
$$u_t(x,t) = -cF'(x-ct) + cG(x+ct)$$
$$u_t(x,0) = \psi(x) = -cF'(x) + cG'(x)$$
$$-F'(x) + G'(x) = \frac{\psi(x)}{c}$$

4. Integrate over $[0,x]$
$$\int_0^x -F'(s) + G'(s)\ ds = \int_0^x \frac{\psi(s)}{c}\ ds$$
$$-F(x) + G(x) - (-F(0) + G(0)) = \frac{1}{c}\int_0^x \psi(s)\ ds$$

   This gives us the system of equations
$$\begin{cases} -F(x) + G(x) = A + \frac{1}{c}\int_0^x \psi(s)\ ds \\ F(x) + G(x) = \phi(x) \end{cases}$$

$$\implies \begin{cases} 2G(x) = \phi(x) + A + \frac{1}{c}\int_0^x \psi(s)\ ds \\ 2F(x) = \phi(x) - A - \frac{1}{c}\int_0^x \psi(s)\ ds \end{cases} \implies \begin{cases} F(X) = \frac{1}{2}\phi(x) - \frac{A}{2} - \frac{1}{2c}\int_0^x \psi(s)\ ds \\ G(X) = \frac{1}{2}\phi(x) + \frac{A}{2} + \frac{1}{2c}\int_0^x \psi(s)\ ds \end{cases}$$

5. Solution

$$
\begin{aligned}
u(x,t) &= F(x-ct)+G(x+ct) \\
&= (\frac{1}{2}\phi(x-ct) - \frac{A}{2} - \frac{1}{2c}\int_0^{x-ct}\psi(s)\,ds) \\
&+ (\frac{1}{2}\phi(x+ct)\frac{A}{2} + \frac{1}{2c}\int_0^{x+ct}\psi(s)\,ds) \\
&= \frac{1}{2}(\phi(x-ct)+\phi(x+ct)) + \frac{1}{2c}\left(\int_{x-ct}^{0}\psi(s)\,ds + \int_0^{x+ct}\psi(s)\,ds\right)
\end{aligned}
$$

Which at last gives us d'Alembert's equation to solve the wave equation with initial conditions:

$$\boxed{u(x,t) = \frac{1}{2}(\phi(x-ct)+\phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds}$$

## Part III - Example

$$
\begin{cases} u_{tt} = u_{xx} \\ u(x,0) = 0 \\ u_t(x,0) = \cos(x) \end{cases} \implies \begin{cases} c = 1 \\ \phi(x) = 0 \\ \psi(x) = \cos(x) \end{cases}
$$

Then using D'Alembert's:

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}(\phi(x-ct)+\phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds \\
&= \frac{1}{2}(0+0) + \frac{1}{2}\int_{x-t}^{x+t}\cos(s)\,ds \\
&= \frac{1}{2}(\sin(x+t)-\sin(x-t)) \\
&= \frac{1}{2}(\sin x\cos t + \cos x \sin t - \sin x \cos -t - \cos x \sin -t) \\
&= \frac{1}{2}(2\cos x \sin t)
\end{aligned}
$$

$$\boxed{u(x,t) = \sin(t)\cos(x)}$$

(Or, the wave takes the shape of cos with amplitude sin)

# 3 Lecture 14, Feb 27: Midterm Review

## Part I - First Order PDE

$$\begin{cases} (1+x^2)u_x + e^y u_y = 0 \\ u(0,y) = y^2 \end{cases}$$

$$\nabla u \cdot (1+x^2, e^y) = 0$$
$$y' = \frac{e^y}{1+x^2}$$
$$\frac{1}{e^y}\, dy = \frac{1}{1+x^2}\, dx$$
$$\tan^{-1} x = -e^{-y} + C \implies \tan^{-1} x + e^{-y} = C$$
$$u(x,y) = f(\tan^{-1}(x) + e^{-y})$$
$$u(0,y) = f(\tan^{-1}(0) + e^{-y}) = y^2 = f(e^{-y})$$
$$z := e^{-y} \implies -y = \ln z \implies y = -\ln z$$
$$f(z) = f(e^{-y}) = y^2 = (-\ln(z))^2 = (\ln z)^2$$
$$\boxed{u(x,y) = \ln(\tan^{-1} x + e^{-y})^2}$$

## Part II - Coordinate Method

$$au_x + bu_y + cu = 0$$

Solution:

$$\begin{cases} \xi = ax + by \\ \eta = ay - bx \end{cases}$$

$$\begin{cases} u_x = u_\xi \xi_x + u_\eta \eta_x = au_\xi - bu_\eta \\ u_y = u_\xi \xi_y + u_\eta \eta_y = bu_\xi + au_\eta \end{cases}$$

$$\begin{aligned} au_x + bu_y + cu &= a(au_\xi - bu_\eta) + b(bu_\xi + au_\eta) + cu \\ &= a^2 u_\xi - abu_\eta + b^2 u_\xi + abu_\eta + cu \\ &= (a^2+b^2)u_\xi + cu \\ &= u_\xi + \frac{c}{a^2+b^2} u = 0 \end{aligned}$$

$$u = f(\eta)e^{-\frac{c}{a^2+b^2}\xi}$$

$$\boxed{u(x,y) = f(ay-bx)e^{-\left(\frac{c}{a^2+b^2}\right)(ax+by)}}$$

## Part III - Fourier Transform

$$\begin{cases} au_x + bu_t + cu = 0 \\ u(x,0) = f(x) \end{cases}$$

Solution:

$$\mathcal{F}(au_x) + \mathcal{F}(bu_t) + \mathcal{F}(cu) \implies \mathcal{F}(bu_t) = -\mathcal{F}(au_x) - \mathcal{F}(cu)$$

$$\begin{aligned} b\frac{d}{dt}\widehat{u} &= -a(-i\kappa)\widehat{u} - c\widehat{u} \\ \frac{d}{dt}\widehat{u} &= \left(\frac{ai\kappa - c}{b}\right)\widehat{u} \\ \widehat{u} &= \widehat{f}(\kappa)e^{\frac{ai\kappa-c}{b}t} \\ &= \widehat{f}(\kappa)e^{i\kappa\left(\frac{a}{b}\right)t}e^{-\frac{ct}{b}} \\ &= e^{-\frac{ct}{b}}\mathcal{F}\left(f(x-\frac{a}{b}t)\right) \\ &= \mathcal{F}\left(e^{-\frac{ct}{b}}f(x-\frac{a}{b}t)\right) \end{aligned}$$

$$\boxed{u(x,t) = e^{-\frac{ct}{b}}f(x-\frac{a}{b}t)}$$

## Part IV - Wave Equation Factoring Method

$$3u_{tt} + 10u_{xt} + 3u_{xx} = 0$$

Solution:

$$3t^2 + 10xt + 3x^2 \implies (x+3t)(3x+t)$$

$$\begin{aligned} 3u_{tt} + 10u_{xt} + 3u_{xx} &= 3\frac{\partial^2}{\partial t^2} + 10\frac{\partial^2}{\partial x\partial t} + 3\frac{\partial^2}{\partial x^2} \\ &= \left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial t}\right)\left(3\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u \end{aligned}$$

Let

$$v = \left(3\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) u$$

so

$$\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial t}\right) v = 0$$

$$v_x + 3v_t = 0 \implies v = f(x - \frac{t}{3})$$

Then

$$\left(3\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) u = 3u_x + u_t = f(x - \frac{t}{3})$$

Homogeneous solution:

$$3u_x + u_t = 0 \implies u_0(x,t) = G(x - 3t)$$

Particular solution:

$$f(x - 3t) \implies u_p(x,t) = h(x - \frac{t}{3})$$

$$3(h(x - \frac{t}{3}))_x + (h(x - \frac{t}{3}))_t = f(x - \frac{t}{3})$$

$$\implies h(s) = \frac{3}{8}F(S)$$

$$u_p = \frac{3}{8}F(x - \frac{t}{3})$$

$$u(x,t) = G(x - 3t) + \frac{3}{8}F(x - \frac{t}{3})$$

$$\boxed{u(x,t) = G(x - 3t) + F(x - \frac{t}{3})}$$

# 4 Lecture 15, March 3: Energy Methods

## Energy Method for Waves

**Example 1: Conservation of Energy** Suppose u solves the wave equation $u_{tt} = c^2 u_{xx}$. Then the following is constant

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u_t)^2 + c^2(u_x)^2 \, dx$$

where $(u_t)^2 = \frac{1}{2}mv^2$ is the kinetic energy and the second term $c^2(u_x)^2$ is the potential energy. Thus, for the wave equation, the total energy is conserved.

*Method 1:* Calculate $E'(t)$ and show it is 0 This is easier but requires you know E a priori.

*Method 2:* Energy Method

1. Start with $u_t = c^2 u_{xx}$ **Trick:** Multiply the PDE by a clever function, here by $u_t$
$$u_{tt}u_t = c^2 u_{xx} u_t$$
Integrate with respect to x:
$$\int_{-\infty}^{\infty} u_{tt}u_t \, dx = c^2 \int_{-\infty}^{\infty} u_{xx}u_t \, dx$$

2. Study A From calculus,
$$y''y' = \left[\frac{1}{2}(y')^2\right]'$$
Therefore,
$$u_{tt}u_t = \frac{d}{dt}\left[\frac{1}{2}(u_t)^2\right]$$
so
$$A = \int_{-\infty}^{\infty} u_{tt}u_t \, dx = \frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} (u_t)^2 \, dx\right)$$

3. Study B
$$B = \int_{-\infty}^{\infty} u_{xx}u_t \, dx$$

Integrate by parts WRT x:

$$
\begin{aligned}
A &= \int_{-\infty}^{\infty} u_{xx} u_t \, dx \\
&\overset{\text{IBP}}{=} [u_x u_t]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_x u_{xt} \, dx \\
&= -\int_{-\infty}^{\infty} u_x u_{xt} \, dx \\
&= -\int_{-\infty}^{\infty} \frac{d}{dt}\left(\frac{1}{2}(u_x)^2\right) \, dx \\
&= \frac{d}{dt}\left(-\frac{1}{2}\int_{-\infty}^{\infty} (u_x)^2 \, dx\right)
\end{aligned}
$$

4. Then $A = c^2 B$ implies

$$
\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} (u_t)^2 \, dx\right) = c^2 \frac{d}{dt}\left(-\frac{1}{2}\int_{-\infty}^{\infty} (u_x)^2 \, dx\right)
$$

$$
\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} (u_t)^2 + c^2 (u_x)^2 \, dx\right) = 0
$$

$$
\frac{d}{dt} E(t) = 0
$$

so E is constant

**Note:** How do we know which function to multiply the PDE by? It's an art! Here we multiplied by $u_t$ to make a time-derivative appear

## Application: Uniqueness

**Lemma:** Suppose $w$ solves the following PDE

$$
\begin{cases}
w_{tt} = c^2 w_{xx} \\
w(x,0) = 0 \\
w_t(x,0) = 0
\end{cases}
$$

Then $w(x,t) = 0$ for all x and t.

**Why?**

1. the energy $E(t)$ is constant so

$$E(t) = E(0)$$

$$\frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2 + c^2(w_x)^2 \, dx = \frac{1}{2}\int_{-\infty}^{\infty} (w_t(x,0))^2 + c^2(w_x(x,0))^2 \, dx$$

By assumption, we have $w_t(x,0) = 0$ and moreover

$$w(x,0) = 0 \implies (w(x,0))_x = 0 = 0_x \implies w_x(x,0) = 0$$

Therefore the above becomes

$$\frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2 + c^2(w_x)^2 = 0$$

2. **Fact:** if $f \geq 0$ and $\int_{-\infty}^{\infty} f(x)\, dx = 0$ then f must be the zero-function

Therefore $(w_t)^2 + c^2(w_x)^2 = 0$ So $w_t = 0$ and $w_x = 0$ Hence $w(x,t) = C$ But plugging in $t = 0$ we get $C = w(x,0) = 0$ and $w(x,t) = 0$ $\square$

**Application:** There is at most one solution of

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

**Proof:** Standard trick: suppose there are two solutions u and v and let $w = u - v$. Then we check that w solves

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = u(x,0) - v(x,0) = \phi(x) - \phi(x) = 0 \\ u_t(x,0) = u_t(x,0) - v_t(x,0) = \psi(x) - \psi(x) = 0 \end{cases}$$

By the lemma above, we get $w = 0$ so $u - v = 0$ so $u = v$ and thus there is exactly one solution of the above wave equation

## Energy Method for Heat

Suppose we have a finite rod of length l with intitial temperature 0 and insulated.

**Example 2:** Suppose u solves the PDE

$$\begin{cases} u_t = Du_{xx} \\ u(x,0) = 0 \quad \text{Initial} \\ u(0,t) = 0 \quad \text{Endpoint} \\ u(l,t) = 0 \quad \text{Endpoint} \end{cases}$$

Then $u(x,t) = 0$ for all x and t

**Proof:**

1. Start with $u_t = Du_{xx}$. Multiply by u

$$u_t u = Du_{xx}u$$

   Integrate with respect to x on $[0, l]$

$$\int_0^l u_t u \, dx = F \int_0^l u_{xx} u \, dx$$

2. Study A

$$A = \int_0^l \frac{d}{dt}\left(\frac{1}{2}u^2\right) \, dx = \frac{d}{dt}\left(\int_0^l u^2 \, dx\right)$$

3. Study B

$$\int_0^l u_{xx}u \, dx = u_x(l,t)u(l,t) - u_x(0,t) - \int_0^l u_x u_x \, dx = -\int_0^l (u_x)^2 \, dx$$

4. So $A = DB$ and

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^l u^2 \, dx\right) = -D\int_0^l (u_x)^2 \, dx \leq 0$$

   Then if you define

$$E(t) = \frac{1}{2}\int_0^l u^2 \, dx$$

you have $E'(t) \leq 0$ so the energy is decreasing

**Interpretation:** heat is dissipative. An insulated metal rod generally gets cooler with time

This also means that

$$E(t) \leq E(0)$$

so

$$E(t) = \frac{1}{2} \int_0^l (u(x,t)^2)\, dx \leq E(0) = \frac{1}{2} \int_0^l (u(x,0))^2\, dx = 0$$

5. Finale But $E(t) \geq 0$ by definition so $0 \leq E(t) \leq E(0) = 0 \implies E(t) = 0$ and

$$\frac{1}{2} \int_0^l (u(x,t))^2\, dx = 0$$

Therefore $(u(x,t))^2 = 0$ so $u(x,t) = 0$ for all x and t

# 5 Lecture 16, March 6: Heat vs Wave Equations

## Part I - Energy Method Application

**Example:**

$$\begin{cases} u_t = Du_{xx} + f(x,t) \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(l,t) = h(t) \end{cases}$$

**Trick:** $w := u - v$ and via the initial conditions, $w = 0$ so $u = v$ and there is only one solution

## Part II - The Infinite Rod

**Example:**

$$\begin{cases} u_t = Du_{xx} \\ u(x,0) = f(x) \end{cases}$$

has many solutions including $u = f \star g$

**Note:** we have uniqueness if

$$|u(x,t)| \leq Ce^{ax^2}$$

for some $C > 0$ and $a > 0$ (meaning the PDE is in the Schwartz class)

### Part III - Comparison of Waves and Diffusions

| Heat Equation $\begin{cases} u_t = Du_{xx} \\ u(x,0) = f(x) \end{cases}$ | Wave Equation $\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$ |
|---|---|
| Existence via Fourier Transform $u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\, dy$ | Existence via Factoring $u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\, ds$ |
| Uniqueness (given u is Schwartz) | Uniqueness by energy method |
| Smooth (infinite differentiability of heat kernel) | Not smooth (initial condition splits but will not smooth) |
| Infinite speed of propagation | Finite speed of propagation |
| Not reversible | Reversible (if wave travels backwards) |
| Goes to zero over time (energy is decaying) | Does not go to zero (energy is conserved) |

# 6 Lecture 17, March 8: Separation of Variables I

This is the most important PDE technique in the course.

### Part I - Boundary Value Problem

**Example 1:** Find all the values of $\lambda$ for which the following ODE has a nonzero solution X

$$\begin{cases} X''(X) = \lambda X(x) \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

Case 1: $\lambda > 0$ Then $\lambda = \omega^2$ for positive $\omega$ and the aux equation is

$$r^2 = \omega^2 \implies r = \pm\omega$$

so

$$X(x) = Ae^{\omega x} + Be^{-\omega x}$$
$$X(0) = A + B = 0 \implies B = -A$$
$$X(x) = Ae^{\omega x} - Ae^{-\omega x}$$
$$X(\pi) = 0 \implies Ae^{\omega\pi} - Ae^{-\omega pi} = 0 \implies 2\pi\omega = 0$$

But then $\lambda = 0$ which contradicts $\lambda > 0$ so no nonzero solutions

Case 2: $\lambda = 0$ Aux: $r^2 = 0 \implies r = 0$

$$X(x) = A + Bx \implies X(0) = A = 0 \implies X(x) = Bx$$

$$X(\pi) = 0 \implies B\pi = 0 \implies B = 0$$

But then

$$X(x) = 0$$

So there are no nonzero solutions here either

Case 3: $\lambda < 0$

$$r^2 = \lambda = -\omega^2 \implies r = \pm\omega i$$
$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$
$$X(0) = A = 0 \implies X(x) = B\sin(\omega x)$$
$$X(\pi) = 0 \implies \sin(\omega\pi) = \implies \omega\pi = \pi m \implies \omega = m$$

Where m is any positive integer. So our eigenvalues are

$$\lambda = -m^2 \quad (m = 1, 2, ...)$$

and our eigenfunctions are

$$X(x) = \sin(mx) \quad (m = 1, 2, ...)$$

## Part II - Separation of Variables

**Example 2:**

$$\begin{cases} u_t = Du_{xx} \\ u(0,t) = 0 \\ u(\pi,t) = 0 \\ u(x,0) = x^2 \end{cases}$$

Note that this is the same finite rod problem for which we proved uniqueness via energy methods. Then we just need to find one nonzero solution.

Solution:

1. Separation of variables Assume u is the form $u(x,t) = X(x)T(t)$ Plug into the PDE:

$$u_t = Du_{xx}$$
$$(X(x)T(t))_t = D(X(x)T(t))_{xx}$$
$$X(x)T'(t) = DX''(x)T(t)$$
$$\frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)}$$

   Note: always put extra terms on the T side to keep the X equation simple

2. Constant Note that

$$\left(\frac{X''(x)}{X(x)}\right)_t = 0$$

   and

$$\left(\frac{X''(x)}{X(x)}\right)_x = \left(\frac{T'(t)}{DT(t)}\right)_x = 0$$

   so

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{DT(t)} = \text{Constant} = \lambda$$

   Then

$$X''(x) = \lambda X(x)$$
$$T'(t) = D\lambda T(t)$$

   and instead of a PDE we have two ODEs!

Focusing on the X equation,

$$X''(x) = \lambda X(x)$$

we can use the boundary conditions

$$u(0,t) = 0 \overset{u=XT}{\Longleftarrow} X(0)T(t) = 0 \implies X(0) = 0$$

(you can cancel the T because fo $T = 0$ the problem is not interesting)

$$u(\pi, t) = 0 \implies X(pi) = 0$$

So

$$\begin{cases} X''(x) & = \lambda X(x) \\ X(0) & = 0 \\ X(\pi) & = 0 \end{cases}$$

which is the same ODE from part I!

$$\begin{cases} \lambda = -m^2 \quad (m = 1, 2, ...) \\ X(x) = \sin(mx) \quad (m = 1, 2, ...) \end{cases}$$

3. T equation

$$\begin{cases} \frac{T'}{DT} = \lambda = -m^2 \\ T' = -m^2 DT \\ T(t) = e^{-m^2 Dt} \end{cases}$$

Conclusion: For every $m = 1, 2, ...$

$$u(x,t) = X(x)T(t) = \sin(mx)e^{-m^2 Dt}$$

is a solution to the PDE

4. Linearity Since the PDE is linear, any linear combo of the above solution is also a solution!

$$u(x,t) = \sum_{m=1}^{\infty} A_m \sin(mx) e^{-m^2 Dt}$$

5. Initial Condition

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(mx) = x^2$$

**Question:** How do you find $A_m$ (can you write $x^2$ as a linear combo of sines?)

### Part III - Easier Problem

$$\begin{cases} u_t = Du_{xx} \\ u(0,t) = 0 \\ u(\pi,t) = 0 \\ u(x,0) = 3\sin(2x) + 4\sin(3x) \end{cases}$$

The steps above give

$$u(x,t) = \sum_{m=1}^{\infty} A_m \sin(mx) e^{-m^2 Dt}$$

so

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(mx) = 3\sin(2x) + 4\sin(3x)$$

and

$$\boxed{u(x,t) = 3\sin(2x)e^{-4Dt} + 4\sin(3x)e^{-9Dt}}$$

**Interpretation:** Initially, the solution starts out as the sum of sines but eventually dies down as the exponential terms go to 0

## Lecture 18, March 10: Separation of Variables II

### Part I - Setting

**Example 1:**

$$\begin{cases} u_{tt} = c^2 u_{xx} \quad 0 < x < 1 \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = e^x \end{cases}$$

Note that you cannot use d'Alembert's formula because the bounds are not $-\infty < x < \infty$

## Part II - Separation of variables

Assume that

$$u(x,t) = X(x)T(t)$$

Then plug this in

$$(X(x)T(t))_{tt} = c^2((X(x)T(t))_{xx})$$
$$XT'' = c^2 X'' T$$
$$\frac{X''}{X} = \frac{T''}{c^2 T}$$

Note that

$$\left(\frac{X''}{X}\right)_t = 0$$
$$\left(\frac{X''}{X}\right)_x = \left(\frac{T''}{c^2 T}\right)_x = 0$$

so

$$\frac{X''}{X} = \frac{T''}{c^2 T} = \lambda$$

with $\lambda$ constant. Then

$$X'' = \lambda X$$
$$T'' = c^2 \lambda T$$

Using the boundary conditions

$$u(0,t) = 0 \implies X(0)T(t) = 0 \implies X(0) = 0$$

$$u(1,t) = 0 \implies X(1)T(t) = 0 \implies X(1) = 0$$

Then looking at the X ODE:

$$\begin{cases} X'' = \lambda X \\ X(0) = 0 \\ X(1) = 0 \end{cases}$$

$$r^2 = \lambda \implies r = \pm\omega$$

Case $\lambda > 0$:

$$X = Ae^{\omega x} + Be^{-\omega x}$$

Initial conditions show no nonzero solutions

Case $\lambda = 0$:

$$X(x) = A + Bx \longrightarrow X(0) = A = 0 \longrightarrow X(x) = 0$$

Case $\lambda < 0$:

$$r^2 = \lambda = -\omega^2 \implies r = \pm\omega i$$
$$X(x) = A\cos(\omega X) + B\sin(\omega X)$$
$$X(0) = A = 0 \implies X(x) = B\sin(\omega x)$$
$$X(1) = B\sin(\omega) = 0$$
$$\sin(\omega) = 0 \implies \omega = \pi m \quad (m = 1, 2, ...)$$

$$\lambda = -\omega^2 = -(\pi m)^2 \quad (eigenvalues)$$
$$X(x) = \sin(\pi m x) \quad (eigenfunction)$$

Now going all the way back to the T equation:

$$T'' = c^2\lambda T$$
$$\frac{T''}{c^2 T} = \lambda = -(\pi m)^2$$
$$T'' = -(\pi m)^2 c^2 T$$
$$T'' = -(\pi mc)^2 T$$
$$T'' - (\pi mc)^2 T = 0$$

Auxiliary:

$$r^2 - (\pi mc)^2 = 0$$
$$r = \pm\pi mci$$
$$T = A\cos(\pi mct) + B\sin(\pi mct)$$

Conclusion:

$$u(x,t) = X(x)T(t) = (A\cos(\pi mct) + B\sin(\pi mct))\sin(\pi mx)$$

General solution via linearity:

$$\boxed{u(x,t) = \sum_{m=1}^{\infty}(A_m\cos(\pi mct) + B_m\sin(\pi mct))\sin(\pi mx)}$$

Initial conditions:

$$u(x,0) = \sum_{m=1}^{\infty} (A_m \cos(0) + B_m \sin(0)) \sin(\pi m x) = \sum_{m=1}^{\infty} A_m \sin(\pi m x) = x^2$$

The question of representing an arbitrary function as a sum of sin waves deals with Fourier series and will come up later.

$$u_t = \sum_{m=1}^{\infty} (-A_m \pi m c \sin(\pi m c t) + B_m \pi m c \cos(\pi m c t)) \sin(\pi m x)$$

$$u_t(x,0) = \sum_{m=1}^{\infty} B_m \pi m c \sin(\pi m x) = e^x$$

## Part III - Easier initial conditions

**Example 2:**

$$\begin{cases} u_{tt} = c^2 u_{xx} \quad 0 < x < 1 \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = 2\sin(2\pi x) + 3\sin(3\pi x) \\ u_t(x,0) = 4\sin(2\pi x) \end{cases}$$

General solution:

$$u(x,t) = \sum_{m=1}^{\infty} (A_m \cos(\pi m c t) + B_m \sin(\pi m c t)) \sin(\pi m x)$$

Initials:

$$\begin{aligned} u(x,0) &= \sum_{m=1}^{\infty} A_m \sin(\pi m x) \\ &= A_1 \sin(\pi x) + A_2 \sin(2\pi x) + A_3 \sin(3\pi x) + ... \\ &= 0 \sin(\pi x) + 1 \sin(2\pi x) + 3 \sin(3\pi x) + 0 \sin(4\pi x) + ... \end{aligned}$$

therefore,

$$A_1 = 0 \quad A_2 = 1 \quad A_3 = 3 \quad A_m = 0 \ (m \geq 4)$$

And from the other initial condition

$$u_t(x,0) = \sum_{m=1}^{\infty} (\pi mc) B_m \sin(\pi mx) = \pi c B_1 \sin(\pi x) + 2\pi c B_2 \sin(2\pi x) + 3\pi c B_3 \sin(3\pi x) = 4\sin(2\pi x)$$

so

$$B_1 = 0 \quad B_2 = \frac{2}{\pi c} \quad B_m = 0 \ (m \geq 3)$$

and

$$\begin{aligned} u(x,t) &= \sum_{m=1}^{\infty} = (A_m \cos(\pi mct) + B_m \sin(\pi mct)) \sin(\pi mx) \\ &= (A_2 \cos(2\pi ct) + B_2 \sin(2\pi ct)) \sin(2\pi x) + (A_3 \cos(3\pi ct) + B_3 \sin(3\pi ct)) \sin(3\pi x) \end{aligned}$$

At long last,

$$\boxed{u(x,t) = \left(\cos(2\pi ct) + \frac{2}{\pi c}\right) \sin(2\pi x) + (2\cos(3\pi ct)) \sin(3\pi x)}$$

# Lecture 20, March 13: Separation of Variables III

## Part I - Setting

Heat equation but assuming velocity at the endpoints is 0 ("rate in = rate out")

$$\begin{cases} u_t = Du_{xx} \\ u_x(0,t) = 0 \\ u_x(\pi,t) = 0 \\ u(x,0) = x^2 \end{cases}$$

## Part II - Separation of variables

Suppose $u(x,t) = X(x)T(t)$ Plug in

$$(X(x)T(t))_t = D(X(x)T(t))_{xx}$$

$$\begin{aligned} XT' &= DX''T \\ \frac{T'}{DT} &= \frac{X''}{X} = \lambda \end{aligned}$$

$$\begin{cases} X'' = \lambda X \\ T' = \lambda D T \end{cases}$$

X boundary conditions:

$$\begin{aligned} u &= XT \\ u_x &= X'T \\ u_x(0,t) = 0 \implies X'(0) &= 0 \\ u_x(\pi,t) = 0 \implies X'(\pi) &= 0 \end{aligned}$$

So

$$\begin{cases} X'' = \lambda X \\ X'(0) = 0 \\ X'(\pi) = 0 \end{cases}$$

Boundary value problem: Case 1 $\lambda > 0$:

$$r^2 = \lambda = \omega^2 \implies r = \pm\omega$$

$$X = Ae^{\omega x} + Be^{-\omega x}$$

$$X'(0) = A - B = 0 \implies A = B$$

$$X = Ae^{\omega x} + Ae^{-\omega x}$$

$$X'(pi) = 0 \implies \omega\pi = -\omega\pi \implies \omega = 0$$

No nonzero solutions

Case 2 $\lambda = 0$

$$r^2 = 0 \implies X = A + Bx$$

$$X'(0) = 0 \implies B = 0 \implies X = A$$

So $\lambda = 0$ is an eigenvalue with eigenfunction $X(x) = A$

Case $\lambda < 0$:

$$r^2 = \lambda = -\omega^2 \implies r = \pm\omega i$$

$$X = A\cos(\omega x) + B\sin(\omega x)$$

$$X'(0) = B\omega = 0 \implies X = A\cos(\omega x)$$

$$X'(\pi) = 0 \implies \sin(\omega\pi) = 0 \implies \omega = m$$

$$\lambda = -\omega^2 = -m^2$$

So with the above case, the eigenvalues are

$$\lambda = -m^2 \quad (m = 0, 1, 2, ...)$$

and the eigenfunctions are

$$X(x) = \cos(mx) \quad (m = 0, 1, 2, ...)$$

Back to the T equation:

$$\begin{aligned} \frac{T'}{DT} &= \lambda = -m^2 \\ T' &= -m^2 DT \\ T(t) &= e^{-m^2 Dt} \end{aligned}$$

Conclusion: for every $m = 0, 1, 2, ...$ the following solves the PDE:

$$u(x,t) = X(x)T(t) = e^{-m^2 Dt}\cos(mx)$$

Taking linear combinations we get

$$u(x,t) = \sum_{m=0\infty} A_m e^{-m^2 Dt}\cos(mx)$$

So with the initial condition

$$x^2 = \sum_{m=0}^{\infty} A_m \cos(mx)$$

## Part III - Inhomogeneous Problems

**Example 2:**

$$\begin{cases} u_t = Du_{xx} \\ u(0,t) = 7 \\ u(\pi,t) = 7 \\ u(x,0) = x^2 \end{cases}$$

Trick: Let $v(x,t) = u(x,t) - 7$ Then

$$v_t = (u-7)_t = u_{tt} = Du_{xx} = D(v+7)_{xx} = Dv_{xx}$$

$$\begin{aligned} v(0,t) &= u(0,t) - 7 = 0 \\ v(\pi,t) &= u(\pi,t) - 7 = 0 \\ v(x,0) &= u(x,0) - 7 = x^2 - 7 \end{aligned}$$

Then solve the above system for v using separation of variables and use

$$u(x,t) = v(x,t) + 7$$

to solve for u

**Example 3:**

$$\begin{cases} u_t = Du_{xx} \\ u_x(0,t) = 7 \\ u_x(\pi,t) = 7 \\ u(x,0) = x^2 \end{cases}$$

Trick: Use $v(x,t) = u(x,t) - 7x$ Then

$$v_t = (u - 7x)_t = u_t = Du_{xx} = D(v + 7x)_{xx} = Dv_{xx}$$

$$\begin{aligned} v_t &= Dv_{xx} \\ v_x(0,t) &= u_x(0,t) - 7 = 0 \\ v_x(\pi,t) &= u_x(\pi,t) - 7 = 0 \\ v(x,0) &= u(x,0) - 7x = x^2 - 7x \end{aligned}$$

Solve for v using separation of variables and use

$$u(x,t) = v(x,t) + 7x$$

**Example 4:**

$$\begin{cases} u_t = Du_{xx} \\ u(0,t) = 1 \\ u(\pi,t) = 3 \\ u(x,0) = x^2 \end{cases}$$

Let $v(x,t) = u(x,t) - f(x)$ Where f is a linear function such that $f(0) = 1$ and $f(\pi) = 3$ Then

$$f(x) = \left(\frac{3-1}{\pi - 0}\right)(x - 0) + 1 = \frac{2}{\pi}x + 1$$

$$v(x,t) = u(x,t) - frac2\pi x - 1$$
$$v_t = (u - f(x))_t = u_t = D(v + f(x))_{xx} = Dv_{xx}$$
Here we used $f''(x)$ since f is linear
$$v(0,t) = u(0,t) - f(0) = 1 - 1 = 0$$
$$v(\pi,t) = u(\pi,t) - f(\pi) = 3 - 3 = 0$$
$$v(x,0) = u(x,0) - f(x) = x^2 - \frac{2}{\pi}x - 1$$

Then solve
$$\begin{cases} v_t = Dv_{xx} \\ v(0,t) = 0 \\ v(\pi,t) = 0 \\ v(x,0) = x^2 - \frac{2}{\pi}x - 1 \end{cases}$$
Then
$$u(x,t) = v(x,t) + f(x) = v(x,t) + \frac{2}{\pi}x + 1$$
Note: we could in theory also solve the case where $u_x(0,t) = 1$ and $u_x(\pi,t) = 3$ by subtracting a function whose *derivative* is $\frac{2}{\pi}x + 1$ except you would get an inhomogeneous wave equation for v

# Lecture 21, March 15: Fourier Series (I)

## Part I - Prelude

**Goal:** given a function $f(x)$ on $(0,\pi)$ find $A_m$ such that
$$f(x) = \sum_{m=1}^{\infty} A_m \sin(mx)$$
This is known as a Fourier sine series and is similar to a Taylor series

## Part II - Orthogonality

**Definition:** Let $u,\ v,\ w$ be vectors. Then $u$ and $v$ are orthogonal if
$$u \cdot v = 0$$

and $\{u, v, w\}$ is orthogonal if any two different vectors in that set are orthogonal

**Fact:** If a set is orthogonal and some vector x is in the span i.e.

$$x = au + bv + cw$$

for some a, b, c, then:

$$a = \frac{x \cdot u}{u \cdot u}, \quad b = \frac{x \cdot v}{v \cdot v}, \quad x = \frac{x \cdot w}{w \cdot w}$$

**Proof:**

$$x \cdot u = (au + bv + cu) \cdot u = a(u \cdot u) + b\underbrace{(v \cdot u)}_{0} + c\underbrace{(w \cdot u)}_{0} = x \cdot u = a(u \cdot u)$$

and the same for b and c.

## Part III - Fourier Series

**Definition:**

$$\boxed{f \cdot g = \int_0^\pi f(x)g(x)\, dx}$$

**Example:** $(x^2) \cdot (x^3) = \int_0^\pi x^5\, dx = \frac{\pi^2}{6}$

**Fact:**

$$\{\sin(mx) | m = 1, 2, ...\} = \{\sin(x), \sin(2x), \sin(3x), ...\}$$

is orthogonal

**Proof:** Check that for $m \neq n$ then

$$\sin(mx) \cdot \sin(nx) = 0 \implies \int_0^\pi \sin(mx)\sin(nx)\, dx = 0$$

**Consequence:** In particular, if

$$f(x) = \sum_{m=1}^{\infty} A_m \sin(mx)$$

then

$$A_m = \frac{f \cdot \sin(mx)}{\sin(mx) \cdot \sin(mx)} = \frac{\int_0^\pi f(x)\sin(mx)\, dx}{\int_0^\pi \sin^2(mx)\, dx} = \frac{\int_0^\pi f(x)\ \sin(mx)\, dx}{\frac{\pi}{2}}$$

**Fact:**

1. If $f(x) = \sum_{m=1}^{\infty} A_m \sin(mx)$ on $(0, \pi)$ then
$$A_m = \frac{2}{\pi} \int_0^{\pi} f(x)\ \sin(mx\ dx)$$

2. If $f(x) = \sum_{m=1}^{\infty} A_m \sin(\frac{\pi m x}{L})$ on $(0, L)$ then
$$\boxed{A_m = \frac{2}{L} \int_0^{L} f(x) \sin(\frac{\pi m x}{L})\ dx}$$

# Part IV - Examples

**Example 2:** Find $A_m$ such that on $(0, \pi)$
$$x = \sum_{m=1}^{\infty} A_m \sin(mx)$$

$$\begin{aligned} A_m &= \frac{2}{\pi} \int_0^{\pi} x \sin(mx)\ dx \\ &= \frac{2}{\pi} \left( \left[ x(\frac{-\cos(mx)}{m}) \right]_0^{\pi} - \int_0^{\pi} (\frac{-\cos(mx)}{m})\ dx \right) \\ &= \frac{2}{\pi} (-\frac{\pi}{m} \cos(\pi m) + 0 + [\frac{1}{m^2} \sin(mx)]_0^{\pi}) \\ &= \frac{2}{\pi} (-\frac{\pi}{m} (-1)^m) \\ &= \frac{2}{m} (-1)^{m+1} \end{aligned}$$

so
$$x = \sum_{m=1}^{\infty} (-1)^{m+1} \left( \frac{2}{m} \right) \sin(mx) = 2 \sin x - \sin(2x) + \frac{2}{3} \sin(3x) - \frac{1}{2} \sin(4x) + ...$$

**Consequence 1:** If you let $x = \frac{\pi}{2}$ in the above
$$\frac{\pi}{2} = 2 \sin(\frac{\pi}{2}) - \sin(\pi) + \frac{2}{3} \sin(\frac{3\pi}{2}) + ...$$
$$\frac{\pi}{2} = 2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + ...$$

$$\implies 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

**Consequence 2: Example 3:**

$$\begin{cases} u_t = Du_{xx} \\ u(0,t) = 0 \\ u(\pi,t) = 0 \\ u(x,0) = x \end{cases}$$

$$u(x,t) = \sum_{m=1}^{\infty} A_m e^{-m^2 Dt} \sin(mx)$$

$$u(x,0) = x = \sum_{m=1}^{\infty} A_m \sin(mx)$$

Then from the above:

$$\boxed{u(x,t) = \sum_{m=1}^{\infty} \frac{2}{m}(-1)^{m+1} e^{-m^2 Dt}}$$

# Lecture 22, March 17: Fourier Series (II)

## Part I - Fourier Cosine Series

**Goal:** Find $A_m$ such that on the interval $(0,\pi)$

$$f(x) = \sum_{m=0}^{\infty} A_m \cos(mx)$$

**Application:** This was needed to solve the Neumann problem

Luckily, this is practically the same problem as the cosine series so

$$A_m = \frac{f \cdot \cos(mx)}{\cos(mx) \cdot \cos(mx)} = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(mx)\, dx \quad (m = 1, 2, ...)$$

Note however, that $A_0$ corresponds to $\cos(0x)$ so

$$A_0 = \frac{f \cdot \cos(0)}{\cos(0) \cdot \cos(0)} = \frac{\int_0^{\pi} f(x) \cos(0)\, dx}{\int_0^{\pi} 1\, dx} = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx$$

So, the Fourier Cosine Series is:

$$\boxed{\begin{gathered} \text{If } f(x) = \sum_{m=0}^{\infty} A_m \cos(\frac{\pi m x}{L}) \quad 0 < x < L: \\ A_m = \frac{2}{L}\int_0^L f(x)\cos(\frac{\pi m x}{L})\,dx \\ A_0 = \frac{1}{L}\int_0^L f(x)\,dx \end{gathered}}$$

## Part II - Tabular Integration

**Example 1:** Find $A_m$ on $(0,1)$ such that

$$x^3 = \sum_{m=0}^{\infty} A_m \cos(\pi m x)$$

First isolate $m = 0$:

$$A_0 = \int_0^1 x^3\,dx = \frac{1}{4}$$

$$A_m = 2\int_0^1 x^3 \cos(\pi m x)\,dx$$

Method:

1. Put $x^3$ on the left and $\cos(\pi m x)$ on the right
2. Differentiate $x^3$ to 0 and integrate $\cos(\pi m x)$
3. Cross multiply alternating signs

$$\begin{array}{ccc} x^3 & & \cos(\pi m x) \\ & \searrow & \\ -3x^2 & & \sin(\pi m x)/\pi m \\ & \searrow & \\ 6x & & -\cos(\pi m x)/(\pi m)^2 \\ & \searrow & \\ 6 & & -\sin(\pi m x)/(\pi m)^3 \\ & \searrow & \\ 0 & & \cos(\pi m x)/(\pi m)^4 \end{array}$$

so

$$
\begin{aligned}
A_m &= 2\left[x^3\left(\frac{\sin(\pi m x)}{\pi m}\right) - 3x^2\left(-\frac{\cos(\pi m x)}{(\pi m)^2}\right) + 6x\left(-\frac{\sin(\pi m x)}{(\pi m)^3}\right) - 6\left(\frac{\cos(\pi m x)}{(\pi m)^4}\right)\right]_0^1 \\
&= 6\left(\frac{(-1)^m}{(\pi m)^2}\right) - 12\left(\frac{(-1)^m}{(\pi m)^4}\right) + \left(\frac{12}{(\pi m)^2}\right) \\
&= 6\left(\frac{(-1)^m}{(\pi m)^2}\right) + \left(\frac{12}{(\pi m)^4}(\underbrace{(-1)^{m+1} + 1}_{0 \text{ or } 2})\right)
\end{aligned}
$$

thus

$$
A_0 = \frac{1}{4} \qquad A_m = \begin{cases} 6/(\pi m)^2 & m \text{ even} \\ -6/(\pi m)^2 + 24/(\pi m)^4 & m \text{ odd} \end{cases}
$$

## Part III - Full Fourier Series

**Goal:** Find $A_m$ and $B_m$ on $(-\pi, \pi)$ such that

$$
f(x) = \sum_{m=0}^{\infty} A_m \cos(mx) + B_m \sin(mx)
$$

Now the dot product is defined

$$
f \cdot g = \int_{-\pi}^{\pi} f(x) g(x)\, dx
$$

and

$$
\begin{aligned}
A_m &= \frac{f \cdot \cos(mx)}{\cos(mx) \cdot \cos(mx)} = \frac{\int_{-\pi}^{\pi} f(x)\cos(x)\, dx}{\int_{-\pi}^{\pi} \cos^2(mx)\, dx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\, dx \\
B_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\, dx
\end{aligned}
$$

**Exception:** $m = 0$

$$
\begin{aligned}
A_0 &= \frac{f \cdot \cos(0)}{\cos(0) \cdot \cos(0)} = \frac{\int_{-\pi}^{\pi} f(x)\, dx}{\int_{-\pi}^{\pi} 1\, dx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx \\
B_0 &= 0 \quad \text{(by convention because } \sin(0) = 0)
\end{aligned}
$$

Thus the full fourier series on $(-L, L)$ is:

$$\boxed{\begin{aligned} f(x) &= \sum_{m=0}^{\infty} A_m \cos(\frac{\pi m x}{L}) + B_m \sin(\frac{\pi m x}{L}) \\ A_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{\pi m x}{L})\ dx \\ B_m &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{\pi m x}{L})\ dx \\ A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\ dx \\ B_0 &= 0 \end{aligned}}$$

**Example 2:** Find $A_m$ and $B_m$ such that on $(-\pi, \pi)$ we have

$$x = \sum_{m=0}^{\infty} A_m \cos(mx) + B_m \sin(mx)$$

Solution:

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\ dx = 0$$

$$B_0 = 0$$

$$A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x\cos(mx)}_{\text{Odd}}\ dx = 0$$

$$B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x\sin(mx)}_{\text{Even}}\ dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin(mx)\ dx = \left(\frac{2}{\pi}\right)(-1)^{m+1}$$

$$x = \sum_{m=1}^{\infty} \left(\frac{2}{m}\right)(-1)^{m+1}\sin(mx)$$

Which is the sine series!

Note: In fact, this generalizes: If f is odd on $(-\pi, \pi)$ then the Full Fourier series is a sine series, and similar if f is even. But in practice you have to calculate both $A_m$ and $B_m$

# Lecture 23, March 20: Fourier Series (III)

## Part I - Complex Fourier Series

**Goal:** find $C_m$ such that

$$f(x) = \sum_{m=-\infty}^{\infty} C_M e^{imx} \qquad -\pi < x < \pi$$

We define a new dot product

$$f \cdot g = \int_{-\pi}^{\pi} f(x)\overline{g(x)}$$

where

$$\overline{a+bi} = a - bi$$

then

$$C_m = \frac{f \cdot e^{imx}}{e^{imx} \cdot e^{imx}} = \frac{\int_{-\pi}^{\pi} f(x)\overline{e^{imx}}\, dx}{\int_{-\pi}^{\pi} e^{imx}\overline{e^{imx}}\, dx} = \frac{\int_{-\pi}^{\pi} f(x)e^{-imx}\, dx}{\int_{-\pi}^{\pi} e^{imx}e^{-imx}\, dx}$$

or

$$C_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}\, dx$$

Generally,

$$\boxed{\begin{array}{c} \text{If} \quad f(x) = \sum_{-\infty}^{\infty} C_m e^{i\left(\frac{\pi m x}{L}\right)} \quad \text{on } (-L, L): \\ C_m = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\left(\frac{\pi m x}{L}\right)\, dx} \end{array}}$$

**Example:** Find $C_m$ such that on $(-\pi, \pi)$

$$e^x = \sum_{m=-\infty}^{\infty} C_m e^{imx}$$

$$
\begin{aligned}
C_m &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-imx}\, dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-im)x}\, dx \\
&= \frac{1}{2\pi}\left[\frac{e^{(1-im)x}}{1-im}\right]_{-\pi}^{\pi} \\
&= \frac{1}{(2\pi)(2-im)}\left(e^{(1-im)\pi} - e^{(1-im)(-\pi)}\right) \\
&= \frac{1}{(2\pi)(2-im)}\left(e^{\pi}e^{-im\pi} - e^{-\pi}e^{-im\pi}\right)
\end{aligned}
$$

Then notice that

$$
\begin{aligned}
e^{\pi mi} &= \cos(\pi m) + i\sin(\pi m) = (-1)^m \\
e^{-\pi mi} &= \cos(-\pi m) + i\sin(-\pi m) = (-1)^m
\end{aligned}
$$

So

$$
C_m = \frac{1}{2\pi(1-im)}\left(e^{\pi}(-1)^m - e^{-\pi}(-1)^m\right) = \frac{(-1)^m}{\pi(1-\pi m)}\left(\frac{e^{\pi}-e^{-\pi}}{2}\right)
$$

so

$$
C_m = \frac{(-1)^m}{\pi(1-im)}\sinh(\pi)
$$

and then

$$
\boxed{e^x = \sum_{m=-\infty}^{\infty}\left(\frac{(-1)^m}{\pi(1-im)}\sinh(\pi)\right)e^{imx}}
$$

## Part II - Norms

**Definition:** $||u|| = \sqrt{u \cdot u}$ which is the length of u and $||cu|| = |c|||u||$ where c is any constant.

**Pythagorean theorem:** If u and v are orthogonal then

$$
||u+v||^2 = ||u||^2 + ||v||^2
$$

**Consequence:** If $\{u, v, w\}$ is orthogonal then

$$
||u+v+w||^2 = ||u||^2 + ||v||^2 + ||w||^2
$$

## Part III - Parseval's Identity

Apply the Pythagorean theorem to Fourier series: Suppose that on $(0, \pi)$,

$$f(x) = \sum_{m=1}^{\infty} A_m \sin(mx)$$

then since $\{\sin(mx)\}$ is orthogonal

$$||f||^2 = \left|\left|\sum_{m=1}^{\infty} A_m \sin(mx)\right|\right|^2 = \sum_{m=1}^{\infty} ||A_m \sin(mx)||^2 = \sum_{m=1}^{\infty} |A_m|^2 \; ||\sin(mx)||^2$$

but remember that

$$||f||^2 = f \cdot f = \int_0^{|} pi(f(x))^2 \, dx$$

$$||\sin(mx)||^2 = \int_0^{\pi} \sin^2(mx) \, dx = \frac{\pi}{2}$$

So

$$\int_0^{\pi} (f(x))^2 = \frac{\pi}{2} \sum_{m=1}^{\infty} |A_m|^2$$

which gives us Parseval's identity:

$$\boxed{\begin{array}{l} \text{If} \quad f(x) = \displaystyle\sum_{m=1}^{\infty} A_m \sin(mx) \qquad \text{on } (0, \pi) \\[2em] \text{Then} \quad \displaystyle\sum_{m=1}^{\infty} |A_m|^2 = \frac{2}{\pi} \int_0^{\pi} (f(x))^2 \, dx \end{array}}$$

## Part IV- Examples

**Example 1:** $f(x) = x$ on $(0, \pi)$ From an earlier lecture,

$$x = \sum_{m=1}^{\infty} A_m \sin(mx) \qquad A_m = \frac{2(-1)^m}{m}$$

$$\int_0^\pi (f(x))^2 \, dx = \int_0^\pi x^2 \, dx = \frac{\pi^3}{3}$$

$$\sum_{m=1}^{|} infty|A_m|^2 = \sum_{m=1}^{\infty} \left| \frac{2}{m}(-1)^m \right|^2 = \sum_{m=1}^{|} infty \frac{4}{m^2}$$

By Parseval's,

$$4 \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{2}{\pi} \left( \frac{\pi^3}{3} \right) = \frac{2}{3}\pi^2$$

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{1}{4}(\frac{2}{3}\pi^2) = \frac{\pi^2}{6}$$

**Example 2:** Cosine Series This works exactly the same as sin except for the exception $m = 0$ where

$$||\cos(0)||^2 = \int_0^\pi \cos^2(0) \, dx = \pi$$

so for

$$f(x) = \sum_{m=0}^{\infty} A_m \cos(mx)$$

on $(0, \pi)$, then Parseval's says

$$2||A_0||^2 + \sum_{m=1}^{\infty} |A_m|^2 = \frac{2}{\pi} \int_0^\pi (f(x))^2 \, dx$$

# Lecture 24, March 22: Laplace Equation (I)

## Part I - Complex Parseval

Note that

$$||e^{imx}||^2 = e^{imx} \cdot e^{imx} = \int_{-\pi}^{\pi} e^{imx} e^{-imx} \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi$$

$$f \cdot f = \int_{-\pi}^{\pi} f(x)\overline{f(x)} \, dx = \int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

Therefore Parseval's identity becomes:

$$\boxed{\begin{aligned} &\text{If } f(x) = \sum_{-\infty}^{\infty} C_m e^{imx} in(-\pi,\pi) \text{ then} \\ &\sum_{-\infty}^{\infty} |C_m|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx \end{aligned}}$$

**Example:** $f(x) = e^x$ on $(-\pi, \pi)$ Before, we found

$$C_m = \frac{1}{\pi}\left(\frac{1}{1-im}\right)(-1)^m \sinh(\pi)$$

So

$$|C_m|^2 = \left(\frac{1}{\pi^2}\right)\left(\frac{1}{|1-im|^2}\right)|(-1)^m||\sinh(\pi)|^2 = \frac{\sinh^2(\pi)}{\pi^2(1+m^2)}$$

Then by $|a+bi|^2 = a^2+b^2$:

$$\sum_{m=-\infty}^{\infty} |C_m|^2 = \sum_{-\infty}^{\infty} \frac{\sinh^2(\pi)}{\pi^2(m^2+1)} = \frac{\sinh^2(\pi)}{\pi^2}\sum_{-\infty}^{\infty}\frac{1}{m^2+1}$$

And

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{1}{2\pi}\int_{-\pi}^{|} pi|e^x|^2 \, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2x} \, dx = \frac{1}{2\pi}\sinh(2\pi)$$

Therefore Parseval says,

$$\frac{\sinh^2(\pi)}{\pi}\int_{-\pi}^{\pi}\frac{1}{m^2+1} = \frac{1}{2\pi}\sinh(2\pi)$$

$$\int_{-\pi}^{\pi}\frac{1}{m^2+1} = \frac{\pi}{2}\left(\frac{\sinh(2\pi)}{\sinh^2(\pi)}\right)$$

Then notice that

$$\sum_{m=1}^{\infty}\frac{1}{m^2+1} = \frac{\pi}{4}\left(\frac{\sinh(2\pi)}{\sinh^2(\pi)}\right) - \frac{1}{2}$$

## Part II - Convergence of Fourier Series

**Question:** When does a function $f$ equal its Fourier series $\mathcal{F}$

**Fact:** There is a (finite) function f for which $\mathcal{F}(x) = \pm\infty$ *everywhere*

But

$$\boxed{\begin{array}{l} \text{1) If f is continuous at x then } \mathcal{F}(x) = f(x) \\ \text{2) If f has a jump discontinuity at x then} \\ \qquad \mathcal{F}(x) = \dfrac{f(x^-) + f(x^+)}{2} \quad \text{(Average of jumps)} \end{array}}$$

Where $f(x^-)$ and $f(x^+)$ are the left and right limits of f at x.

**Example:** Let $f(x) = x$ on $(-\pi, \pi)$, draw the graph of F(x) on all of $\mathbb{R}$

Notice: $\mathcal{F}(x) = \sum A_m \cos(mx) + B_m \sin(mx)$ is periodic of period $2\pi$ so we first need to "repeat" f and then apply the rules above

**Aside:** There's a whole math subject called Harmonic Analysis just dedicated to the question of convergence of Fourier series, just to show how delicate this question is!

## Part III - Laplace Equation

Laplace's equation is

$$\boxed{u_{xx} + u_{yy} = 0}$$

Note: This is sometimes written as $\Delta u = 0$ where $\Delta u = u_{xx} + u_{yy}$ Interpretation: $u(x, y)$ gives you the temperature of a metal plate $\Omega$ at $(x, y)$ after a long time

Notice: There is no t in Laplace's equation, so no initial conditions. On the other hand, the boundary conditions are more complicated: you need to specify u on the whole boundary $\partial\Omega$ of the metal plate.

## Part IV - Derivation

**Setting:** The temperature of a metal plate at (x, y) and any time t is given by the 2D heat equation, which is

$$u_t = D(u_{xx} + u_{yy})$$

Where$u = u(x,y,t)$

**Derivation:** See homework

After a long time, we assume u is becomes constant in t and so $u_t = 0$, Hence

$$0 = D(u_{xx} + u_{yy}) \implies u_{xx} + u_{yy} = 0$$

## Part V - Applications

1. Physics - $u(x,y)$ is the temperature of a metal plate after a long time
2. Image processing - can convert a pixelated image into a smooth one
3. Music - Solutions are called harmonics

   Suppose you have a region $\Omega$, think the surface of a drum and consider the eigenvalue problem: "For which $\lambda$ does the following have a nonzero solution?"

   $$\begin{cases} -\Delta u = \lambda u \in \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

   **Fact:** There is an infinite sequence of eigenvalues $\lambda_n$ with

   $$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ...$$

   For which the above has a nonzero solution. Then $\lambda_1$ is the *principal harmonic* and the others are *overtones*

# Lecture 25, March 24: Laplace Equation (II)

## Part I - Application 3: Music

$$\begin{cases} -\Delta u = \lambda u & \in \Omega \\ \quad u = 0 & \partial\Omega \end{cases}$$

Where $\Omega$ is some domain (for example a drum head) and $\partial\Omega$ is the boundary.

Note: The negative sign is to get positive eigenvalues

**Fact:** There is an infinite sequence of eigenvalues which give a nonzero solution

**Famous question:** "Can you hear the shape of a drum?" i.e. Given the eigenvalues can you determine $\Omega$? *Answer:* Yes in 2D is $\Omega$ is smooth

## Part II - "OMG Application:" Brownian Motion

Suppose that you take a Brownian motion random walk on $(x, y)$ until you hit a boundary point $(x^*, y^*)$ and pay a gain/loss $g(x^*, y^*)$ Let $u(x, y)$ be the expected value of the gain/loss starting at $(x, y)$

Then, *that expected value will solve*

$$\begin{cases} \Delta u = 0 & \in \Omega \\ u = g & \in \partial\Omega \end{cases}$$

**Insane Consequence** Suppose $g = 0$ everywhere except $g(x^*, y^*) > 0$ at some point $(x^*, y^*)$ on the boundary (say some far away treasure).

Then by infinite speed of propagation, $u(x, y) > 0$ everywhere.

Note that because u is an expected value, this means that no matter how far away we are, there is always a positive chance of hitting $(x^*, y^*)$

## Part III - Application 5: the Heat Equation

Consider the 1-dimension domain of the boundary. Start at point $x$ at time t, do Brownian motion, stop at point $x^*$ at time $T$, gain $g(x^*)$

Consider the average value $u(x, t)$ of the gain/loss function starting at $(x, t)$. **Fact:** u solves

$$\begin{cases} u_t = u_x \\ u(x, T) = g(x) \end{cases}$$

Note: this is interesting because there is no guarantee that a terminal value problem has a solution.

**Consequence:** If $g(x^*)$ is positive somewhere than $u(x, t)$ is positive everywhere. This implies that it's always possible to reach any $x^*$ (no matter how far) at any time T (no matter how small) no matter where you start

## Part IV - Rotation Invariance

**Goal:** find the "fundamental solution" of Laplace's equation

**Setting:** Suppose $u(x, y)$ solves

$$u_{xx} + u_{yy} = 0$$

in $\mathbb{R}^2$ Notice that u is invariant under rotations

More precisely, let $\theta$ be constant and let

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So

$$\begin{cases} x' = \cos(\theta)x - \sin(\theta)y \\ y' = \sin(\theta)x + \cos(\theta)y \end{cases}$$

**Fact:**

$$u_{x'x'} + u_{y'y'} = u_{xx} + u_{yy} = 0$$

**Proof:**

$$\begin{aligned} u_x &= u_{x'} \cdot x'_x + u_{y'} \cdot y'_x \\ &= (u_{x'})\cos(\theta) + (u_{y'})\sin(\theta) \\ u_{xx} &= (u_x)_{x'} \cdot x'_x + (u_x)_{y'} \cdot y'_x \\ &= u_{x'x'}\cos^2(\theta) + 2u_{x'y'}\sin(\theta)\cos(\theta) + u_{y'y'}\sin^2(\theta) \end{aligned}$$

# Lecture 26, April 3: Laplace's Equation (III)

## Part I - Review

**The Laplace Equation in $\mathbb{R}^2$:**

$$u_{xx} + u_{yy} = 0$$

**Important Fact:** u is invariant under rotations. This means that we can use the coordinate method in polar coordinates to get nice results.

## Part II - Polar Coordinates

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \implies \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(\frac{y}{x}) \end{cases}$$

**Goal:** write $u_{xx} + u_{yy} = 0$ in terms of r and $\theta$ *Step 1:* Prep work

$$\frac{\partial r}{\partial x} = (\sqrt{x^2 + y^2})_x = \frac{x}{\sqrt{x^2 + y^2}} = \cos(\theta)$$

Similarly,

$$\frac{\partial r}{\partial y} = \sin\theta$$
$$\frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{r}$$
$$\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

*Step 2:* A very long coordinate method problem

$$\begin{aligned}
u_x &= u_r \cdot r_x + u_\theta \cdot \theta_x = u_r \cos\theta + u_\theta(-\frac{\sin\theta}{r}) \\
u_{xx} &= (u_x)_r \cdot r_x + (u_x)_\theta \cdot \theta_x \\
&= \left(u_r \cos\theta - u_\theta(\frac{\sin\theta}{r})\right)_r \cos(\theta) + \left(u_r\cos\theta - u_\theta\left(\frac{\sin\theta}{r}\right)\right)_\theta \left(-\frac{\sin\theta}{r}\right) \\
&= \left(u_{rr}\cos(\theta) - u_{\theta r}\left(\frac{\sin\theta}{r}\right) + u_\theta\left(\frac{\sin\theta}{r^2}\right)\right)\cos(\theta) \\
&+ \left(u_{r\theta}\cos\theta - u_r \sin\theta - u_{\theta\theta}\frac{\sin\theta}{r} - u_\theta \frac{\cos\theta}{r}\right)\left(\frac{-\sin\theta}{r}\right) \\
&= u_{rr}\cos^2\theta - u_{r\theta}\frac{\sin\theta\cos\theta}{r} + u_\theta\frac{\sin\theta\cos\theta}{r^2} - u_{r\theta}\frac{\cos\theta\sin\theta}{r} + u_r\frac{\sin^2\theta}{r} + u_{\theta\theta}\frac{\sin^2\theta}{r^2} + u_\theta\frac{\cos\theta\sin\theta}{r^2} \\
&= u_{rr}\cos^2\theta - 2u_{r\theta}\frac{\sin\theta\cos\theta}{r} + 2u_\theta\frac{\sin\theta\cos\theta}{r^2} + u_r\frac{\sin^2\theta}{r} + u_{\theta\theta}\frac{\sin^2\theta}{r^2}
\end{aligned}$$

Similarly,

$$u_{yy} = u_{rr}\sin^2\theta + 2u_{r\theta}\frac{\cos\theta\sin\theta}{r} - 2u_\theta\frac{\cos\theta\sin\theta}{r^2} + u_r\frac{\cos^2\theta}{r} + u_{\theta\theta}\frac{\cos^2\theta}{r^2}$$

*Step 3:* Combine

$$u_{xx} + u_{yy} = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}$$

So at long last, the **Polar Laplace** is:

$$\boxed{u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0}$$

## Part II - Fundamental Solution

Because of rotational invariance, we can look for **radial solutions** (solutions that only depend on r) Note that there are many non-radial solutions but this gives a place to start.

If u is radial, then

$$u_{rr} + \frac{1}{r}u_r = 0$$

By integrating factors,

$$\exp(\int \frac{1}{r}\,dr) = e^{\ln r} = r$$

so

$$ru_{rr} + u_r = 0 \implies (ru_r)_r = 0$$

$$ru_r = A \implies u_r = \frac{A}{r}$$

$$u = \int \frac{A}{r}\,dr = A\ln r + B$$

Therefore, in 2 dimensions,

$$\boxed{u(x,y) = A\ln(\sqrt{x^2+y^2}) + B}$$

is a solution to laplace's equation.

**Definition:** the *Fundamental Solution* of $u_{xx} + u_{yy} = 0$ is

$$\boxed{\Phi(x,y) = -\frac{1}{2\pi}\ln(\sqrt{x^2+y^2})}$$

# Lecture 27, April 5: Laplace's Equation (IV)

## Part I - Setting

**Example 1:**

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = 0 \\ u(\pi,y) = 0 \\ u(x,0) = x \\ u(x,1) = 3 \end{cases}$$

This gives us a rectangle of width $\pi$ and height 1 with left and right edges at temperature 0, bottom edge at temperature x, and top edge at temperature 3.

## Part II - Separation of Variables

Assume that u is of the form $u(x,y) = X(x)Y(y)$. So

$$u_{xx} + u_{yy} = X''Y + XY'' = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y}$$

Then note that as neither side of the equation depends on the other variable, both sides are constant and equal to some value $\lambda$. Cross multiplying,

$$\begin{cases} X'' = \lambda X \\ Y'' = -\lambda Y \end{cases}$$

Because X yields the zero boundary condition, we start there.

Looking at the boundary conditions,

$$u(0,y) = 0 \implies X(0) = 0$$

$$u(\pi,y) = 0 \implies X(\pi) = 0$$

This gives the ODE

$$\begin{cases} X'' = \lambda X \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

which has eigenfunction

$$X(xi) = \sin(mx) \quad (m = 1, 2, ...)$$

corresponding to eigenvalues $\lambda = -m^2 \quad (m = 1, 2, ...)$

Then for the Y equation,

$$\begin{aligned} Y''(y) &= -\lambda Y(y) \\ &= -(-m^2)Y \\ &= m^2 Y \end{aligned}$$

$$\implies Y = Ae^{my} + Be^{-my}$$

**Analogy:** For the wave equation,

$$T = A\cos(mct) + B\sin(mct)$$

so at $T(0) = A$

**Recall:**

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\implies \begin{cases} e^x = \cosh(x) + \sinh(x) \\ e^{-x} = \cosh(x) - \sinh(x) \end{cases}$$

So

$$\begin{aligned} Y = Ae^{my} + Be^{-my} & \\ &= A(\cosh(my) + \sinh(my)) + B(\cosh(-my) + \sinh(-my)) \\ &= A\cosh(my) + A\sinh(my) + B\cosh(-my) + B\sinh(-my) \\ &= (A+B)\cosh(my) + (A-B)\sinh(my) \\ &= A\cosh(my) + b\sinh(my) \end{aligned}$$

and just like in the analogy, $Y(0) = A$!

**Conclusion:** For every $m = 1, 2, ...$

$$u(x, y) = X(x)Y(y) = (A\cosh(my) + B\sinh(my))\sin(mx)$$

## 6.1 Part III - Initial conditions

$$u(x, y) = \sum_{m=1}^{\infty} [A_m \cosh(my) + B_m \sinh(my)] \sin(mx)$$

For $u(x, 0) = x$,

$$u(x, 0) = \sum_{m=1}^{\infty} [A_m \underbrace{\cosh(0)}_{1} + B_m \underbrace{\sinh(0)}_{0}] \sin(mx)$$

$$x = \sum_{m=1}^{\infty} A_m \sin(mx) \quad (0 < x < \pi)$$

Using fourier series,

$$A_m = \frac{2}{\pi} \int_0^{\pi} x \sin(mx)\, dx = \frac{2}{m}(-1)^m$$

And for $u(x, 1) = 3$

$$u(x,1) = \sum_{m=1}^{\infty} \left[ \frac{2}{m}(-1)^m \cosh(m) + B_m \sinh(m) \right] \sin(mx) = 3$$

Make the substitution

$$\tilde{B}_m = A_m \cosh(m) + B_m \sinh(m)$$

so

$$3 = \sum_{m=1}^{\infty} \tilde{B}_m \sin(mx)$$

and

$$\tilde{B}_m = \frac{2}{\pi} \int_0^{\pi} 3 \sin(mx)\, dx = \frac{2}{\pi} \left[ \frac{-3\cos(mx)}{m} \right]_0^{\pi} = \frac{6}{\pi m}[(-1)^{m+1} + 1]$$

so from $\tilde{B}_m$,

$$A_m \cosh(m) + B_m \sinh(m) = \frac{6}{\pi m}[(-1)^{m+1} + 1]$$

$$B_m = \frac{\frac{6}{\pi m}[(-1)^{m+1} + 1] - (\frac{2}{m})(-1)^m \cosh(m)}{\sinh(m)}$$

## Part IV - Conclusion

$$\boxed{u(x,y) = \sum_{m=1}^{\infty} \left[ (\frac{2}{m})(-1)^m \cosh(my) + \frac{\frac{6}{\pi m}[(-1)^{m+1} + 1] - (\frac{2}{m})(-1)^m \cosh(m)}{\sinh(m)} \sinh(my) \right] \sin(mx)}$$

## Part V - Variation

If one of the boundary conditions were not zero, you would not be able to solve the equation.

**Trick:** $u = v + w$ where $v$ solves the situation where the left and right are 0 and where $w$ solves the situation where the top and bottom are 0.

# 7 Lecture 28, April 7: Laplace's Equation (V)

## Fundamental Solution

**Recall:** Polar Laplace

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Searching for the radial solution gives the general solution

$$\boxed{u(x,y) = A\ln(\sqrt{x^2+y^2}) + B}$$

This also gives the *fundamental solution*

$$\Phi(x,y) = -\frac{1}{2\pi}\ln(\sqrt{x^2+y^2})$$

**Note:** $-1/2\pi$ is chosen so that $\Delta\Phi = -\delta(0,0)$ (the Dirac at 0)

**Fact:** A solution to Poisson's equation

$$u_{xx} + u_{yy} = -f(x,y)$$

is

$$\boxed{u(x,y) = \Phi * f = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(x-s,y-t)\,f(s,t)\,ds\,dt}$$

## Part II - Two Dimensional Laplace

Suppose $u(x,y,z)$ solves $u_{xx}+u_{yy}+u_{zz}=0$ Using spherical coordinates, the radial component is

$$u_{\rho\rho} + \frac{2}{\rho}u_\rho + \ \dots\ = 0$$

where $\rho = \sqrt{x^2+y^2+z^2}$ and the ellipses obscure terms that don't depend on $\rho$.

Solving for radial solutions gives

$$u = \frac{A}{\rho} + B$$

**Fact:** Laplace's Equation in 3 dimensions is solved by

$$\boxed{u(x,y,z) = \frac{A}{\sqrt{x^2+y^2+z^2}} + B}$$

**Note:** The fundamental n-dimensional solution is of the form

$$\Phi(x_1,...,x_n) = \frac{C_n}{r^{n-2}}$$

## Part III - Properties of Laplace's Equation

### Property 1) Mean-Value Formula

**Recall:** The average value of $f$ from $a$ to $b$ is

$$⨍ f(x)\ dx = \frac{1}{b-a}\int_a^b f(x)\ dx$$

**Notice:** $b - a = |[a,b]|$ = the size/length of $[a,b]$

**Definition:** The average value of $f$ on $\Omega$ is

$$⨍_\Omega f(x)\ dx$$

where $|\Omega|$ is the area/volume of $\Omega$.

**Notation:** $B(x,r)$ is a ball centered at x with radius r.

**Mean Value Formula:** If $\Delta u = 0$ then for every x and every $r > 0$ we have

$$\boxed{⨍_{B(x,r)} u(y)\ dy = \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\ dy = u(x)}$$

**Interpretation:** The average value of u over the bal l is just the value at the center!

**Note:** Solutions to $\Delta u = 0$ are called *isotropic* (same from every direction)

**Consequences:**

1. Solutions to $\Delta u = 0$ are infinitely differentiable

   **Proof:** $u = ⨍_{B(x,r)} u(y)\ dy \longrightarrow$ one level smoother (because of integral)

2. *Liouville's Theorem:* If $\Delta u = 0$ and u is bounded ($|u| \leq c$) then $u$ must be constant.

   **Corollary:** If u is not constant then it must blow up somewhere

### 7.0.1 Property 2) Strong Max Principle

*Question:* where is a metal plate hottest/coldest?

**Strong Max Principle:** If $\Delta u = 0$ in $\Omega$, then $\max u$ and $\min u$ are attained on $\partial\Omega$ and ONLY on $\partial\Omega$ (except if u is constant)

**Proof:** Suppose u has a max M at some point x in $\Omega$. But by mean value formula we get

$$\fint_{B(x,r)} u(y)\, dy = u(x) = M$$

(the highest value is equal to the average value so u is constant)

# Lecture 29, April 10: Midterm 2 Review

## Part I - Separation of Variables

**Example 1:**

$$\begin{cases} tu_t = u_{xx} - u \\ u(0,t) = 0 \\ u(\pi,t) = 0 \\ u(x,1) = 1 \end{cases}$$

Step 1: Separation of Variables

$$\begin{gathered} u(x,t) = X(x)T(t) \\ tXT' = X''T - XT \\ \frac{tT'}{T} = \frac{X''}{X} - 1 \\ \frac{tT'}{T} + 1 = \frac{X''}{X} = \lambda \end{gathered}$$

Step 2: Boundary Value

$$u(0,t) = X(0)T(t) = 0 \implies X(0) = 0$$
$$u(\pi,t) = 0 \implies X(\pi) = 0X'' = \lambda X$$

(skipping the three cases)

$$\{X(x) = \sin(mx),\ \lambda = -m^2 \quad |m = 1, 2, ...\}$$

Step 3: T Equation

$$t\frac{T'}{T} = (-m^2 - 1)$$
$$\frac{T'}{T} = -\frac{m^2+1}{t}$$
$$(\ln|T|)' = -\frac{m^2+1}{t}$$
$$\ln|T| = -(m^2+1)\ln|t| + C$$
$$|T| = t^{-(m^2+1)}e^C$$
$$T(t) = Ct^{-(m^2+1)}$$

Step 4: Final u

$$u = XT$$
$$u(x,t) = Ct^{-(m^2+1)}\sin(mx)$$
$$u(x,t) = \sum_{m=1}^{\infty} A_m t^{-(m^2+1)}\sin(mx)$$
$$u(x,1) = \sum_{m=1}^{\infty} A_m \sin(mx) = 1$$

$$\begin{aligned} A_m &= \frac{2}{\pi}\int_0^{\pi} \sin(mx)\ dx \\ &= \frac{2}{\pi}\left[-\frac{\cos(mx)}{m}\right]_0^{\pi} \\ &= \frac{2}{\pi m}\left[-\cos(\pi m) + 1\right] \\ &= \frac{2}{\pi m}[(-1)^{m+1} + 1] \end{aligned}$$

$$\boxed{u(x,t) = \sum_{m=1}^{\infty} \frac{2}{\pi m} \left[(-1)^{m+1} + 1\right] t^{-(m^2+1)} \sin(mx)}$$

## Part II - Parseval

**Ex 2:** Derive Parseval's identity om $(0, \pi)$ for

$$x = \sum_{m=0}^{\infty} A_m \sin\left(\left(\frac{2m+1}{2}\right) x\right)$$

using only the Orthogonality of the sine functions. Use this to calculate

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^4} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \ ...$$

Taking the norms:

$$\begin{aligned}
||x||^2 &= \left|\left|\sum_{m=0}^{\infty} A_m \sin\left(\left(\frac{2m+1}{2}\right) x\right)\right|\right|^2 \\
&= \sum_{m=0}^{\infty} \left|\left|A_m \sin\left(\left(\frac{2m+1}{2}\right) x\right)\right|\right|^2 \\
&= \sum_{m=0}^{\infty} |A_m|^2 \left|\left|\sin\left(\left(\frac{2m+1}{2}\right) x\right)\right|\right|^2
\end{aligned}$$

LHS:

$$||x||^2 = \int_0^{\pi} x^2 \ dx = \frac{\pi^3}{3}$$

RHS:

$$\begin{aligned}
\left\|\sin\left(\left(\frac{2m+1}{2}\right)x\right)\right\|^2 &= \int_0^\pi \sin^2\left(\left(\frac{2m+1}{2}\right)x\right)\,dx \\
&= \int_0^\pi \frac{1}{2} - \frac{1}{2}\cos((2m+1)x)\,dx \\
&= \frac{\pi}{2} - \frac{1}{2}\int_0^\pi \cos((2m+1)x)\,dx \\
&= \frac{\pi}{2} - \frac{1}{2}\left[\frac{\sin(2m+1)x}{2m+1}x\right]_0^\pi \\
&= \frac{\pi}{2}
\end{aligned}$$

Putting together,

$$\frac{\pi^3}{3} = \sum_{m=0}^{\infty} |A_m|^2 \frac{\pi}{2}$$

$$\sum_{m=0}^{\infty} |A_m|^2 = \frac{2\pi^2}{3}$$

Calculating the coefficients,

$$\begin{aligned}
A_m &= \frac{x\cdot\sin\left(\left(\frac{2m+1}{2}\right)x\right)}{\sin\left(\left(\frac{2m+1}{2}\right)x\right)\cdot\sin\left(\left(\frac{2m+1}{2}\right)x\right)} \\
&= \frac{2}{\pi}\left[\frac{\pi\cos(2m+1)}{2m+1} - \frac{\sin(2m+1)}{(2m+1)^2}\right] \\
&\overset{IBP}{=} \frac{8}{\pi}\frac{1}{(2m+1)^2}(-1)^m
\end{aligned}$$

Then,

$$\sum_{m=0}^{\infty} |A_m|^2 = \frac{2\pi^2}{3}$$

$$\sum_{m=0}^{\infty} \frac{64}{\pi^2}\frac{1}{(2m+1)^4}|(-1)^m|^2 = \frac{2\pi^2}{3}$$

$$\frac{64}{\pi^2}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^4} = \frac{2\pi^2}{3}$$

$$\boxed{\sum_{m=0}^{\infty} \frac{1}{(2m+1)^4} = \frac{\pi^4}{96}}$$

## Part III - Energy Method + Laplace

**Ex 3:** Use energy methods to show that if

$$\begin{cases} \Delta u = 0 & \in \Omega \\ u = 0 & \in \partial\Omega \end{cases}$$

Then $u = 0$ for all $x$ and $t$.

**Hint:** use that for all $v$ with $v = 0$ on $\partial\Omega$,

$$\int_\Omega (\Delta u) v \, dx = -\int_\Omega (\nabla u) \cdot (\nabla v) \, dx$$

Starting with energy methods,

$$\Delta u = 0$$

$$(\Delta u) u = 0$$

$$\int_\Omega (\Delta u) u \, dx = 0$$

$$-\int_\Omega (\nabla u) \cdot (\nabla u) \, dx = 0$$

$$-\int_\Omega \underbrace{||\nabla u||^2}_{\geq 0} \, dx = 0$$

So

$$||\nabla u||^2 = 0 \implies \nabla u = 0 \implies u = C \implies u = 0 \quad \blacksquare$$