Partial Differential Equations: APMA 0360

Milan Capoor

Spring 2022

1 Jan 25: Introduction

Part I - Introduction

- Professor Peyam Tabrizian: drpeyam@brown.edu
- Office Hours: MWF 10:30-11:30
- Course site: sites.brown.edu/drpeyam
- Youtube: https://m.youtube.com/c/DrPeyam

Grading:

- Homework - 25% due Fridays 3pm
- Mini Project - 5% due Friday May 5
- Midterm 1 - 20% on Wednesday March 1
- Midterm 2 - 20% on Wednesday April 12
- Final - 30% on Tuesday May 16, 2-5pm

Part II - What is a PDE?

Partial Differential Equation: an equation relating a function u with one or more of its partial derivatives

Example: Laplace's Equation

$$\begin{cases} U = U(x,y) \\ \implies U_{xx} + U_{yy} = 0 \end{cases}$$

PhD advisor quote: "if you can solve all PDEs, you can solve the universe"

Part III - PDE Applications

1. Physical sciences
   e.g. Navier-Stokes
2. Geometry
   e.g. Poincare's Conjecture
3. Probability
4. Operations research
   e.g. Hamilton-Jacobian PDE for maximizing/minimizing
5. Image Processing
   e.g. Smartphones, MRIs
6. Money
   e.g. Black-Scholes Equation
7. Chemical Reactions
   e.g. Peyam's Dissertation

The main characters of the course for $U = U(x,t)$:

1. Transport equation
   $$U_t + 3U_x = 0$$
2. Heat/diffusion equation
   $$U_t = U_{xx}$$
3. Wave equation
   $$U_{tt} = U_{xx}$$
   ("much like an extra chromosome, an extra t is not necessarily such a good thing")

4. Laplace's equation ($U(x,y)$)

$$U_{xx} + U_{yy} = 0$$

Part III - Solution of PDE

Example 1: Is $U(x,t) = x^2t^2$ a solution of $U_{tt} = U_{xx}$?

$$\begin{cases} U_{tt} = (x^2t^2)_{tt} = 2x^2 \\ U_{xx} = (x^2t^2)_{xx} = 2t^2 \end{cases}$$

So $\boxed{\text{No}}$

Example 2: Is $U(x,y) = e^x \cos(y)$ a solution of $U_{xx} + U_{yy} = 0$

$$\begin{aligned} U_{xx} + U_{yy} &= (e^x \cos y)_{xx} + (e^x \cos y)_{yy} \\ &= e^x \cos y + e^x(-\cos y) \\ &= 0 = RHS \end{aligned}$$

So $\boxed{\text{yes}}$

Part IV - Simple PDE

Note: $U = U(x,y)$

Example 3: $U_x = 0$ Because U can depend on y, this does NOT imply that $U = C$ Therefore:

$$\boxed{\text{U(x,y) = f(y)}}$$

Example 4: $U_{xx} = 0$

$$\implies U_x = f(y)$$

$$\implies U = \int f(y)\, dx = \boxed{xf(y) + g(y)}$$

(Where $g(y)$ is constant WRT x)

Example 5: $U_{xx} + U = 0$ Solving by Analogy: this is similar to ODE $y'' + y = 0 \implies y = A\cos x + B\sin x$

$$\boxed{U(x,y) = A(y)\cos x + B(y)\sin x}$$

2 Jan 27: Classification of PDE

Part I - Simple PDE (Continued)

Example 1: $U_{xy} = 0$

$$\begin{aligned} (u_x)_y &= 0 \\ u_x &= f(x) \\ u &= \int f(x)\,dx = \boxed{F(x) + G(y)} \end{aligned}$$

Part II - Classification of PDE

Order: the highest derivative that appears Examples:

1. $u_{xx} + 3u_y = 0$ (Second order)
2. $2u_x + 3u_y = 0$ (First order)
3. $u_{zzyzx} = 0$ (Fifth order)

Note: In general, third-order and higher are impossible to solve

Constant coefficient: if the coeffs are constant Example:

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y)$$

Note: this example is also the "general form"

Linear vs Nonlinear: if the coefficients depend on x and y but not u Examples:

1. $u_{xx} + u_{yy} = 0$ (Linear)
2. $(u_x)^2 + 3e^u + u_y = 0$ (Nonlinear)
3. $x^2u_{xx} + y^3u_y + 4u = 0$ (Linear)

All constant coefficient equations are also linear

Note: Nonlinear PDEs are VERY difficult and none of the normal PDE methods work to solve them

Interlude: the Linear Algebra View *Linear transformation:* a transformation L is linear if

1. $L(u + v) = L(u) + L(v)$

2. $L(cu) = cL(u)$

Linear PDE: a PDE of the form

$$L(u) = f$$

where L is linear and f doesn't depend on= u

Example 2: Check that the following PDE is linear

$$u_{xx} + x^2 u_{yy} = e^y$$

Solution: $L(u) = u_{xx} + x^2 u_{yy}$ so we just need to check that L is linear

$$\begin{aligned} L(u+v) &= (u+v)_{xx} + x^2(u+v)_{yy} \\ &= u_{xx} + v_{xx} + x^2 u_{yy} + x^2 v_{yy} \\ &= L(u) + L(v)\checkmark \end{aligned}$$

$$\begin{aligned} L(cu) &= (cu)_{xx} = x^2(cu)_{yy} \\ &= cu_{xx} + cx^2 u_{yy} \\ &= c(u_{xx} + x^2 u_{yy}) \\ &= cL(u)\checkmark \end{aligned}$$

Homogeneous/Inhomogeneous PDE: for linear PDE, Homogeneous if $f = 0$ and Inhomogeneous otherwise Examples:

1. $u_{xx} + u_{yy} = 0$ Homo
2. $u_{xx} + u_{yy} = 2x$ Not homo

Fun fact! For linear homogeneous PDE $L(u) = 0$, the sum of two solutions is still a solution **Why?** L is linear so solutions span a vector space

Part III - Types of Second-order PDE

Suppose you have a PDE of the form

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y)$$

Then, let $D = b^2 - 4ac$:

1. if $D < 0$ then the PDE is elliptic
2. if $D > 0$ then the PDE is hyperbolic
3. if $D = 0$ then the PDE is parabolic

Example 3: What is the type of the PDE

$$5u_{xx} + 6u_{xy} - 4u_{yy} + 3u_x + 5u = x^2$$

Solution:

$$D = 6^2 - 4(5)(-4) = 36 + 80 = 116 > 0 \implies \boxed{\text{hyperbolic}}$$

Most famous PDE and their types:

1. Laplace's equation ($u_{xx} + u_{yy} = 0$) is elliptic
2. Wave equation ($u_{tt} - u_{xx} = 0$) is hyperbolic
3. Heat equation ($u_t = u_{xx} \implies u_{xx} + 0u_{tt-u_t} = 0$) is parabolic

Part IV - Review: Directional Derivatives

Gradient vector of $u = u(x, y)$: $\Delta u = (u_x, u_y)$

If $\vec{v}$ is a vector, then the *directional derivative* of u in the direction of $\vec{v}$ is

$$(\Delta u) \cdot \vec{v}$$

Intuitively, this measures the rate of change of u in the $\vec{v}$ direction. Normal convention is to have $\vec{v}$ as a unit vector but this is not actually necessary

Example 4: $u(x, y) = x^2 - y^2$ and $\vec{v} = (2, 3)$ Solution:

$$(\Delta u) \cdot \vec{v} = (2x, -2y) \cdot (2, 3) = \boxed{4x - 6y}$$

3 Jan 30: First-order Linear PDE

Part I - The Constant Coefficient Case

Goal: solve a PDE of the form

$$au_x + bu_y = 0$$

Example 1: $2u_x + 3bu_y = 0$ Solution:

1. Observe the LHS is the same as

$$\langle u_x, u_y \rangle \cdot \langle 2, 3 \rangle = \nabla u \cdot \vec{v} = 0$$

Note that this is the same as the directional derivative of u in the direction $\vec{v} = \langle 2, 3 \rangle$. This tells us that u is constant along lines parallel to $\langle 2, 3 \rangle$ (these are called *characteristic lines*)

2. Find the equation of each of the parallel lines

$$m = \frac{3}{2} \implies y = \frac{3}{2}x + C \implies 2y - 3x = C$$

3. Solution: $\boxed{u(x, y) = f(2y - 3x)}$ (where f is arbitrary)

Summary: the general solution of $au_x + bu_y = 0$ is

$\boxed{\text{u(x, y) = f(ay- bx)}}$ where f is arbitrary

Part II - The General Case

Example 2: $u_x + yu_y = 0$ Solution:

1. Directional Derivative

$$\nabla u \cdot (1, y) = 0$$

So u is constant along curves with "slope" v

2. Characteristic lines On one hand, the slope of the directional derivative is y. On the other, assuming y is a function of x, the slope should be $y'(x)$

Putting it together,

$$y' = y \implies y = Ce^x$$

Why? Consider $g(x) = u(x, Ce^x)$ Then,

$$g'(x) = u_x(x, Ce^x) + Ce^x u_y(x, Ce^x) = u_x + yu_y = 0$$

3. Find the arbitrary function input that is constant on each curve $y = Ce^x$

$$y = Ce^x \implies ye^{-x} = C$$

4. Solution:

$$u(x, y) = f(ye^{-x})$$

Part III - More Practice

Example 3: $xu_x + yu_y = 0$ Directional derivative:

$$\nabla u \cdot \langle x, y \rangle = 0$$

ODE:

$$\begin{aligned} fracyx &= y'(x) \\ x\,dy &= y\,dx \\ \ln|y| &= \ln|x| + C \\ |y| &= |x|e^c \\ \frac{y}{x} &= C \end{aligned}$$

Solution: $\boxed{u(x,y) = f(\frac{y}{x})}$

4 Feb 1: Transport Equation

Part I - The Chain Rule

If $f = f(x,y)$ where $x = x(s,t)$ and $y = y(s,t)$ then,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Part II - Coordinate Method

Example: $2u_x + 3u_y = 0$

1. Define new variables x' and y'

$$\begin{cases} x' = 2x + 3y \\ y' = -3x + 2y \end{cases}$$

   Note: $(2,3)$ is the vector in the direction of the directional derivative and $(-3,2)$ is perpendicular

2. Rewrite in terms of x' and y' using chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = 2u_{x'} - 3u_{y'}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = 3u_{x'} + 2u_{y'}$$

3. Substitute definitions

$$\begin{aligned} 2u_x + 3u_y &= 0 \\ 2(2u_{x'} - 3u_{y'}) - 3(3u_{x'} + 2u_{y'}) &= 0 \\ 4u_{x'} - 6u_{y'} + 9u_{x'} + 6u_{y'} &= 0 \\ 13u_{x'} = 0 \implies u_{x'} &= 0 \end{aligned}$$

4. Solution

$$u_{x'} = 0 \implies u = f(y')$$
$$\boxed{u = f(2y - 3x)}$$

Part III - Transport equation

$$u_t + cu_x = 0$$

where $u = u(x,t)$ where x is position, t is time, and c is a speed constant. It models the density of a fluid that is transported at speed c

Derivation: The mass on an interval $[0, b]$ at time t is:

$$M = \int_0^b u(x,t)\, dx$$

At a later time, $t + h$, the fluid shifts from $[0, b]$ to $[ch, b + ch]$. Now, the mass is

$$M = \int_{ch}^{b+ch} u(x, t + h)\, dx$$

Since mass is conserved, get:

$$\int_0^b u(x,t)\, dx = \int_{ch}^{b+ch} u(x, t + h)\, dx$$

Differentiate with respect to b:

$$\frac{d}{db}\int_0^b u(x,t)\;dx = \frac{d}{db}\int_{ch}^{b+ch} u(x,t+h)\;dx$$

By the Fundamental Theorem of calculus:

$$u(b,t) = u(b+ch, t+h)$$

Differentiate with respect yo h:

$$0 = \frac{\partial u}{\partial x}\frac{\partial(b+ch)}{\partial h} + \frac{\partial u}{\partial t}\frac{\partial(t+h)}{\partial h}$$

$$0 = cu_x + u_t$$

Solving:

$$u_t + cu_x = 0 \implies cu_x - u_t = 0$$

Recall: the general solution to $au_x + bu_y = 0$

$$u(x,y) = f(ay - bx)$$

Note: this can also be written $f(bx - ay)$ but with different f

Therefore,

$$\boxed{\text{u(x, t) = f(x - ct)}}$$

5 Feb 3: Heat Equation Derivation

Part I - The Heat Equation

$$u_t = Du_{xx}$$

where $D > 0$ is a diffusion constant The equation gives the temperature of a metal rod at position x and time t.

Part II - Derivation

Note: can also use Fick's law from physics to derive it

1. Think about the rod as composed of particles that move in two dimensions (left or right)
2. Let $u = u(x, t)$ measure the concentration (#/length) of particles at x and t
3. Let $h = \Delta x$ and $\tau = \frac{h^2}{2D}$ (it will work!)
4. Focus on (x, t) (look at the small neighborhood of x: $[x - \frac{h}{2}, x + \frac{h}{2}]$)
5. Note that the length of the interval is h so the number of particles on the interval is roughly $hu(x, t)$
6. Divide the rod into more intervals of length h
7. Main assumption: as time increases from t to $t + \tau$, each particle moves to the left or right with equal probability
8. $$hu(x, t + \tau) = hu(x, t) + \text{ change}$$
9. $$\text{change} = \text{in - out} = \begin{cases} \text{out} = \frac{1}{2}hu(x, t) + \frac{1}{2}hu(x, t) \\ \text{in} = \frac{1}{2}hu(x - h, t) + \frac{1}{2}hu(x + h, t) \end{cases}$$
   $$\implies \frac{1}{2}hu(x - h, t) + \frac{1}{2}hu(x + h, t) - hu(x, t)$$
10. $$hu(x, t + \tau) = hu(x, t) + \frac{1}{2}hu(x - h, t) + \frac{1}{2}hu(x + h, t) - hu(x, t)$$
11. $$hu(x, t + \tau) - hu(x, t) = \frac{h}{2}\left(u(x - h, t) - 2u(x, t) + u(x + h, t)\right)$$
12. Make some more transformations to get into the right form:
    $$\frac{u(x, t + \tau) - u(x, t)}{\tau} = \frac{h^2}{2\tau}\left(\frac{u(x - h, t) - 2u(x, t) + u(x + h, t)}{h^2}\right)$$

13. Limits:
$$\lim_{\tau \to 0} \frac{u(x,t+\tau) - u(x,t)}{\tau} = u_t(x,t)$$
Then by double l'Hopital's:
$$\lim_{h \to 0} \left(\frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2} \right) = u_{xx}$$

14.
$$u_t = \left(\lim_{tau,h \to 0} \frac{h^2}{2\tau} \right) u_{xx}$$

15. Then using the definition of tau:
$$\boxed{u_t = Du_{xx}}$$

6 Feb 6: Fourier Transform

Part I - Behavior of Solutions

The Heat Equation:
$$u_t = Du_{xx}$$
Where $u(x,t)$ is the temperature of a metal rod at x and t and $D > 0$ is a diffusivity constant dependent on material

Notice that if $u_{xx} > 0$, then $u_t = Du_{xx} > 0$ whenever u is concave up in x, u will increase in time and vice versa. In other words, over time the graph will "flatten out"

Part II - Interlude: The Gaussian Integral

Example:
$$\int_{-\infty}^{\infty} e^{-x^2} \, dx$$
Classically, e^{-x^2} does not have an antiderivative and yet we can take the integral with the following method:

1. Trick: Consider
$$I = \int_{-\infty}^{\infty} e^{-x^2}\ dx = \int_{-\infty}^{\infty} e^{-y^2}\ dy > 0$$
(The variable does not matter)

2. Multiply:
$$\begin{aligned}
I^2 &= (I)(I) \\
&= \left(\int_{-\infty}^{\infty} e^{-x^2}\ dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\ dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\ dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\ dx\,dy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\ r\,dr\,d\theta \\
&= 2\pi\int_{0}^{\infty} re^{-r^2}\ dr \\
&= 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} \qquad (u = -r^2) \\
&= 2\pi\left(-\frac{1}{2}e^{-\infty+\frac{1}{2}e^0}\right) \\
&= \pi
\end{aligned}$$

3. Therefore $I^2 = \pi$ and since $I > 0$, we get $I = \sqrt{\pi}$ and so:
$$I = \int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$$

Note: this same method can be used to calculate $\int_{-\infty}^{\infty}\sin(x^2)\ dx$

Part III - The Fourier Transform

The Fourier Transform functions in much the same way as the Laplace Transform of ODEs.
$$\widehat{f}(\kappa) = \int_{-\infty}^{\infty} f(x)e^{i\kappa x}\ dx$$

Notes:

- This is a function of κ as x is integrated out
- Interpretation: changes functions from phase space to frequency space
- Application: essential for signal processing and imaging
- Often represented with ξ instead of κ and $e^{-i\kappa x}$ rather than $e^{i\kappa x}$

Example: Calculate $\widehat{f}$ where $f(x) = e^{-x^2}$ Solution:

$$\widehat{f}(\kappa) = \int_{-\infty}^{\infty} e^{-x^2} e^{i\kappa x}\, dx$$

1. Find a differential equation for $\widehat{f}$

$$\begin{aligned}
\widehat{f}'(\kappa) &= \frac{d}{d\kappa} \int_{-\infty}^{\infty} e^{-x^2} e^{i\kappa x}\, dx \\
&= \int_{-\infty}^{\infty} e^{-x^2} e^{i\kappa x}(ix)\, dx \\
&= i \int_{-\infty}^{\infty} x e^{-x^2} e^{i\kappa x}\, dx
\end{aligned}$$

2. Integrate by parts with respect to x:

$$\begin{cases} du = xe^{-x^2} \implies u = -\frac{1}{2}e^{-x^2} \\ v = e^{i\kappa x} \implies dv = e^{i\kappa x}(i\kappa) \end{cases}$$

Integrating:

$$\begin{aligned}
&= i\left[-\frac{1}{2}e^{-x^2}e^{i\kappa x}\right]_{-\infty}^{\infty} - i\int_{-\infty}^{\infty} -\frac{1}{2}e^{-x^2}e^{i\kappa x}\, dx \\
&= 0 + \frac{i}{2}(i\kappa)\int_{-\infty}^{\infty} e^{-x^2}e^{i\kappa x}\, dx \\
&= -\frac{\kappa}{2}\widehat{f}(\kappa)
\end{aligned}$$

Giving us a new ODE to solve in the next lecture of

$$\widehat{f}'(\kappa) = -\frac{\kappa}{2}\widehat{f}(\kappa)$$

Part IV - The Schwartz Class

Notice that the infinite terms in the above example are 0 because e^{-x^2} goes to 0 very quickly.

This is the easiest class of functions to apply the Fourier transform to

Definition: f is *Schwartz* if it is infinitely differentiable and for every n

$$\lim_{x \to \pm\infty} \left| \frac{f(x)}{x^n} \right| = 0$$

And same for all derivatives of f.

In other words, f and its derivatives go to 0 at $\pm\infty$ faster than any power function x^n. This allows us to ignore the infinite terms in the Fourier integration

7 Feb 8: Fourier Transform and Heat Equation

Part I - Fourier Transform Example

$$\widehat{f}(\kappa) = \int_{-\infty}^{\infty} f(x)e^{i\kappa x}\, dx$$

Example: $\widehat{f}$ where $f(x) = e^{-x^2}$ Solution:

1. Find a Differential equation rather than try to solve directly

$$\widehat{f}'(\kappa) = -\frac{\kappa}{2} f(\kappa)$$

2. Solve the ODE

$$\widehat{f}' + \frac{\kappa}{2} f = 0$$

$$\left(\widehat{f} e^{\frac{\kappa^2}{4}}\right)' = 0$$

$$\widehat{f}(\kappa) = Ce^{-\frac{\kappa^2}{4}}$$

3. Find C

$$\kappa = 0 \implies \widehat{f}(\kappa) = Ce^0 = C$$

$$C = \widehat{f}(\kappa) = \int_{-\infty}^{\infty} e^{-x^2} e^{i0x}\, dx = \sqrt{\pi}$$

4. Answer

$$\boxed{\widehat{f}(\kappa) = \sqrt{\pi}e^{-\frac{\kappa^2}{4}}}$$

Note that if you apply the fourier to a gaussian, you get another gaussian!

More generally, The Fourier transform of $f(x) = e^{-ax^2}$ is

$$\boxed{\widehat{f}(\kappa) = \sqrt{\frac{\pi}{a}}e^{-\frac{\kappa^2}{4a}}}$$

Part II - Fourier Transform and Derivatives

Recall: The Laplace transform turns derivatives into products

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} = y(0)$$

Fact:

$$\boxed{\widehat{f'}(\kappa) = (-i\kappa)\widehat{f}(\kappa)}$$

Proof:

$$\begin{aligned}
\widehat{f'}(\kappa) &= \int_{-\infty}^{\infty} f'(x)e^{i\kappa x}\,dx \\
&\overset{\text{IBP}}{=} \left[f(x)e^{i\kappa x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\frac{d}{dx}e^{i\kappa x}\,dx \\
&= 0 - \int_{-\infty}^{\infty} f(x)\frac{d}{dx}e^{i\kappa x}\,dx = -i\kappa\int_{-\infty}^{\infty} f(x)\frac{d}{dx}e^{i\kappa x}\,dx \\
&= -i\kappa\widehat{f}(\kappa)
\end{aligned}$$

Part III - Fourier transform and the Heat Equation

Example: Solve

$$\begin{cases} u_t = Du_{xx} \\ u(x,0) = f(x) \quad \text{(given)} \end{cases}$$

Solution:

1. Apply the x fourier Transform

$$\widehat{u_t} = D\widehat{u_{xx}}$$

$$\widehat{u}(\kappa, t) = \int_{-\infty}^{\infty} u(x,t)e^{i\kappa x}\, dx$$

$$\begin{aligned}\widehat{u_{xx}}(\kappa, t) &\overset{\text{fact}}{=} (-i\kappa)\widehat{u_x}(\kappa, t)\\ &\overset{\text{fact}}{=} (-i\kappa)(-i\kappa)\widehat{u}(\kappa, t)\\ &= -\kappa^2\widehat{u}(\kappa, t)\end{aligned}$$

For u_t, do directly:

$$\begin{aligned}\widehat{u_t} &= \int_{-\infty}^{\infty} u_t(x,t)e^{i\kappa x}\, dx\\ &= \int_{-\infty}^{\infty} \frac{d}{dt}(u(x,t)e^{i\kappa x})\, dx\\ &= \frac{d}{dt} int_{-\infty}^{\infty} u(x,t)e^{i\kappa x}\, dx\\ &= \frac{d}{dt}\widehat{u}(\kappa, t)\end{aligned}$$

2. Solve the new ODE

$$\widehat{u_t} = D\widehat{u_{xx}} \implies \frac{d}{dt}\widehat{u}(\kappa, t) = -D\kappa^2\widehat{u}(\kappa, t)$$

Recall:

$$y' = ay \implies y = Ce^{at} = y(0)e^{at}$$

Similarly,

$$\widehat{u}(\kappa, t) = \widehat{u}(\kappa, 0)e^{-D\kappa^2 t}$$

Note:

$$u(x, 0) = f(x) \overset{\text{fourier}}{\Longrightarrow} \widehat{u}(\kappa, 0) = \widehat{f}(\kappa)$$

Therefore,

$$\boxed{\widehat{u}(\kappa, t) = \widehat{f}(\kappa)e^{-D\kappa^2 t}}$$

Problem: But how do we go from $\widehat{u}$ to u?

8 Feb 10: Convolution

Part I - Convolution

Definition:

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\; dy$$

Example: $(f \star g)(x)$ where $f(x) = e^x$ and

$$g(x) = \begin{cases} 1 & [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$\begin{aligned} (f \star g)(x) &= \int_{-\infty}^{\infty} f(x-y)\, g(y)\; dy \\ &= \int_0^1 e^{x-y}\; dy \\ &= e^x \int_0^1 e^{-y}\; dy \\ &= e^x[-e^{-y}]_0^1 = \boxed{(1-e^{-1})e^x} \end{aligned}$$

Fact:

$$\widehat{f \star g}(\kappa) = \widehat{f}(\kappa) \cdot \widehat{g}(\kappa)$$

Part II - Solving the Heat Equation

Example: Use the fourier transform to solve

$$\begin{cases} u_t = Du_{xx} \\ u(x,0) = f(x) \end{cases}$$

Solution: (Via ODEs)

$$\widehat{u}(\kappa, t) = \widehat{f}(\kappa) e^{-D\kappa^2 t}$$

Next, we wish to write $e^{-D\kappa^2 t}$ as a fourier transform. Note that for most equations this is impossible or VERY difficult but not for the Gaussian!

Recall:

$$\widehat{e^{-ax^2}} = \sqrt{\frac{\pi}{a}} e^{-\frac{\kappa^2}{4a}} \implies e^{-\frac{-\kappa^2}{4a}} = \widehat{\sqrt{\frac{a}{\pi}} e^{-ax^2}}$$

Therefore find a such that

$$e^{-\frac{\kappa^2}{4a}} = e^{-D\kappa^2 t}$$

$$\longrightarrow a = \frac{1}{4Dt}$$

So,

$$\sqrt{\frac{a}{\pi}} = \frac{1}{\sqrt{4\pi Dt}}$$

$$\longrightarrow e^{-\kappa^2 Dt} = \mathcal{F}(\frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}})$$

Or,

$$e^{-\kappa^2 Dt} = \widehat{g}(\kappa, t) \quad g(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

Grand Finale!

$$\widehat{u}(\kappa, t) = \widehat{f}(\kappa) e^{-\kappa^2 t} = \widehat{f}(\kappa)\widehat{g}(\kappa, t)$$

$$\widehat{u}(\kappa, t) = \mathcal{F}((f \star g)(\kappa, t))$$

$$u(x,t) = (f \star g) = \int_{-\infty}^{\infty} f(y)\ g(x-y,t)\ dy$$

where

$$g(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

Solving:

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} f(y)\, e^{-\frac{(x-y)^2}{4Dt}}\ dy} \quad (t > 0)$$

Part III - The Heat Kernel

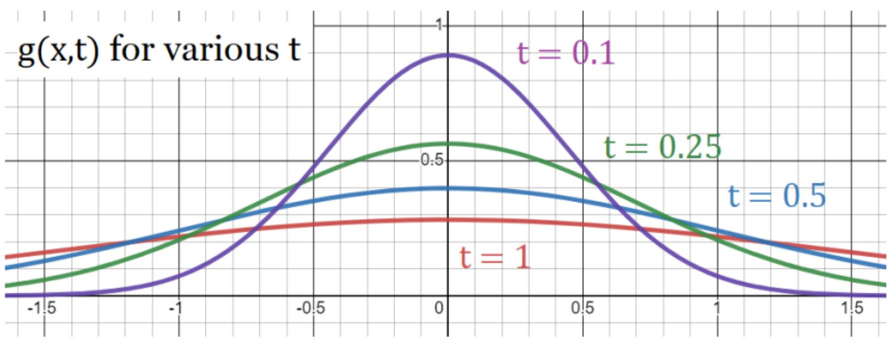
Definition: Heat kernel (AKA Fundamental sol of the heat equation)

$$g(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

Properties:

1. g itself solves $g_t = Dg_{xx}$
2. $\int_{-\infty}^{\infty} g(x,t)\, dx = 1$ for all t

Picture: For every t, $g(x,t)$ looks like a bell-curve e^{-x^2} but that gets more and more spread out as you increase t:



Note that as $t \to 0^+$, $g(x,t)$ is the Dirac delta at $x = 0$

Part V - Convolution Intuition

Example: What is the coefficient of x^2 in

$$(x^2 + 2x + 3)(2x^2 + 4x + 1)$$

Generally, the coeff of x^2 in $(a_2x^2 + a_1x + a_0)(b_2x^2 + b_1x + b_2)$ is

$$C_2 = a_0b_2 + a_1b_1 + a_2b_0$$

and more generally, the coefficient of x^k in $(a_nx^n + ...a_0)(b_nx^n + ... + b_0)$ is

$$C_k = \sum_{i=0}^{k} a_i b_{k-1}$$

Note the parallel to

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\; dy$$

9 Feb 13: Heat Equation Properties

Part I - Heat Equation Example

Example 1: Solve

$$\begin{cases} u_t = Du_{xx} \\ u(x,0) = e^{-x} \end{cases}$$

Solution:

$$u(x,t) = \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} e^{-y}\; dy$$

Looking at the exponent:

$$\frac{-(x-y)^2}{4Dt} - y = -\frac{(x-y)^2 + 4Dty}{4Dt}$$

Expand the numerator:

$$= -\frac{x^2 - 2xy - y^2 + 4Dty}{4Dt}$$

Note the numerator is a quadratic in y:

$$\begin{aligned} y^2 + (4Dt - 2x)y + x^2 &= (y + 2Dt - x)^2 - (2Dt - x)^2 + x^2 \\ &= (y + 2Dt - x)^2 - 4D^2t^2 + 4Dtx - x^2 + x^2 \\ &= (y + 2Dt - x)^2 + 4Dt(x - Dt) \end{aligned}$$

So the full numerator is

$$\frac{-(x-y)^2}{4Dt} = -\left(\frac{(y + 2Dt - x)^2 + 4Dt(x - Dt)}{4Dt}\right) = -\left(\frac{(y + 2Dt - x)^2}{4Dt} + (x - Dt)\right)$$

Substituting back in,

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\left(\frac{(y+2Dt-x)^2}{4Dt}+(x-Dt)\right)}\,dy \\
&= \frac{e^{Dt-x}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-frac(y+2Dt-x)^2 4Dt}\,dy \\
&= \frac{e^{Dt-x}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\left(\frac{y+2Dt-x}{\sqrt{4Dt}}\right)^2}\,dy
\end{aligned}
$$

Now use u-sub with

$$p = \frac{y+2Dt-x}{\sqrt{4Dt}}$$

so

$$dp = \frac{dy}{\sqrt{4Dt}} \implies dy = \sqrt{4Dt}\,dp$$

$$u(x,t) = \frac{e^{Dt-x}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-p^2}\sqrt{4Dt}\,dp = \frac{e^{Dt-x}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\,dp$$

$$\boxed{u(x,t) = e^{Dt-x}}$$

Part II - Infinite speed of propagation

Remember the heat equation solution is:

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy$$

with an initial condition $u(x,0) = f(x)$

Property 1: If $f \geq 0$ is positive somewhere and continuous, then $u(x,t)$ is positive everywhere.

This means that heat propagates at infinite speed because heat at one place affects heat everywhere else instantly. Note that the transport equation implies a finite speed of propagation.

Why? Suppose $f(x_0) > 0$ for some x_0. Then because f is continuous it is actually positive for all x in an interval around x_0 Also

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy$$

and we know the integrand is non-negative so we have

$$\frac{1}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy \geq \frac{1}{\sqrt{4}\pi Dt}\int_{a}^{b} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy$$

But the integrand of the second is also positive so

$$u(x,t) > 0$$

Part III - Smoothness

Property 2: $u(x,t)$ is infinitely differentiable (for $t > 0$) even if $f(x)$ might not be

Why? All the derivatives fall of $\exp(-\frac{(x-y)^2}{4Dt})$ and not on f:

$$\frac{d}{dx}u(x,t) = \frac{d}{dx}\frac{1}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy = \frac{1}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} \frac{d}{dx}e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy$$

But the term

$$e^{-\frac{(x-y)^2}{4Dt}}$$

is infinitely differentiable and

$$\frac{d}{dt}u(x,t) = Du_{xx}$$

but u_{xx} is also smooth

Part IV - Irreversibility

Property 3: The heat equation is irreversible ($u(x,0)$ cannot be determined from $u(x,1)$)

Why? "something something entropy"

Suppose $u(x,1) = |x|$ but by smoothness, $u(x,t)$ must be smooth for all t so $|x|$ must be smooth but this is a contradiction

10 Feb 15: Inverse Fourier Transform

Part I - Long-time behavior of the heat kernel

$$u(x,t) = \frac{1}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\,dy$$

Property 4:

$$\lim_{t\to\infty} u(x,t) = 0$$

"heat dissipates over time"

Why?

$$\begin{aligned}
|u(x,t)| &= \left|\frac{1}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\, dy\right| \\
&\le \frac{1}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} |f(y)| \underbrace{e^{-\frac{(x-y)^2}{4Dt}}}_{\le 1}\, dy \\
&\le \frac{1}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} |f(y)|\, dy \\
&= C \overset{t\to\infty}{\longrightarrow} 0
\end{aligned}$$

Part II - Boundedness

"u(x, t) does not blow up"

Property 5: If $|f(x) \le M$ for some M (and all x) then for all x and t we have

$$|u(x,t) \le M$$

Why?

$$\begin{aligned}
|u(x,t)| &= \left|\frac{1}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\, dy\right| \\
&\le \frac{1}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} \underbrace{|f(y)|}_{\le M} e^{-\frac{(x-y)^2}{4Dt}}\, dy \\
&\le \frac{M}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} e^{-\left(\frac{y-x}{\sqrt{4\pi Dt}}\right)^2}\, dy \quad u = \frac{y-x}{\sqrt{4Dt}} \implies du = \frac{dy}{\sqrt{4Dt}} \\
&= \frac{M}{\sqrt{4}\pi Dt}\int_{-\infty}^{\infty} e^{-u^2}\sqrt{4Dt}\, dy \qquad = \frac{M}{\sqrt{4}\pi Dt}\sqrt{4Dt}\sqrt{\pi} \\
&= M
\end{aligned}$$

Part III - Conservation of Mass

"The area under the curve of u – no matter its shape – is always the same"

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$$

Why?

Lemma:

$$\lim_{x\to\pm\infty} u_x(x,t) = 0$$

Then,

$$u_t = Du_{xx}$$

$$\int_{-\infty}^{\infty} u_t(x,t)\,dx = \int_{-\infty}^{\infty} Du_{xx}(x,t)\,dx$$

and by FTC

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,dx = D\left[u_x(xmt)\right]_{-\infty}^{\infty}$$

Thus by the lemma,

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,dx = D(0-0) = 0$$

So the integral is constant with respect to time:

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u(x,0)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$$

Part IV - Inverse Fourier Transform

Note that for the heat equation, we were very lucky to be able to write the Gaussian as a fourier transform

$$e^{-D\kappa^2 t} = \mathcal{F}\left(\frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}\right)$$

But what do we do in general?

Example: Solve

$$\begin{cases} u_t = -u_{xxxx} \\ u(x,0) = f(x) \end{cases}$$

Solution:

1. Fourier transform it

$$\mathcal{F}\left(u_t\right) = \mathcal{F}\left(-u_{xxxx}\right)$$

$$\frac{d}{dt}\widehat{u} = -(-i\kappa)^4\widehat{u} = -\kappa^4\widehat{u}$$

2. Solve the ODE

$$\widehat{u} = u(x,0)e^{-\kappa^4 t} = \widehat{f}(\kappa)e^{-\kappa^4 t}$$

3. Write the exponential term as a fourier transform

   Definition: *Inverse Fourier Transform*

$$\boxed{\check{f}(x) = \mathcal{F}^{-1}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\kappa)e^{-i\kappa x}\, d\kappa}$$

   So in this example,

$$e^{-\kappa^4 t} = \widehat{g}(\kappa) \quad g(x,t) = \mathcal{F}^{-1}\left(e^{-\kappa^4 t}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\kappa^4 t}e^{-i\kappa x}\, d\kappa$$

4. Convolution

5. So now we have

$$\begin{aligned}\widehat{u}(\kappa,t) &= \widehat{f}(\kappa)e^{-\kappa^4 t}\\ &= \widehat{f}(\kappa)\widehat{g}(\kappa,t)\\ &= \mathcal{F}\left(f \star g\right)(\kappa,t)\end{aligned}$$

   Therefore,

$$\boxed{\begin{cases} u(x,t) = \int_{-\infty}^{\infty} f(y)g(x-y)\, dy \\ \text{where} \quad g(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\kappa^4 t}e^{-i\kappa x}\, d\kappa \end{cases}}$$

11 Feb 17: Wave Equation Derivation

Part I - The Wave Equation

$$u_{tt} = c^2 u_{xx}$$

where $u = u(x,t)$ gives the displacement of a vibrating string at position x and time t and c is a constant giving the speed of the wave

Note: despite the only difference between this and the heat equation is an extra time derivative, the derivation and solution will be *completely* different

Part II - Derivation

1. Setting: start with a thin string of infinite length and consider a minute sub-piece from x to $x + \Delta x$

   Assumption: points on the string only move vertically

2. By Newton's second law of motion,

   $$F = ma$$

   By the assumption above and the definition of u, the displacement vector is

   $$s(x,t) = \langle 0, u(x,t) \rangle$$

   Therefore, acceleration is

   $$a(x,t) = s_{tt}(x,t) = \langle 0, u_{tt} \rangle$$

   Assumption: the string has constant density ρ

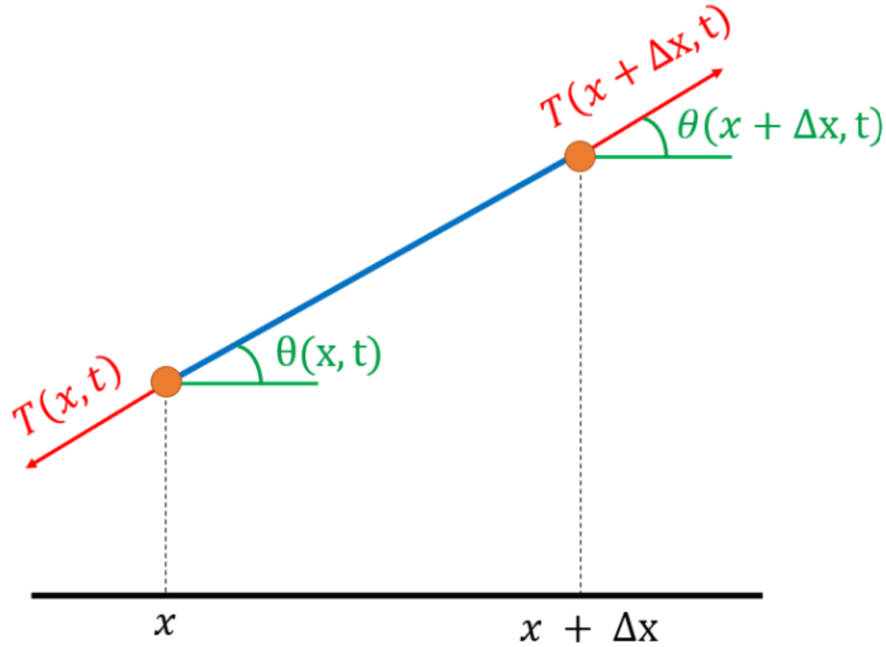
   Then, the mass of the string is density times length (which can be taken by assuming the length is the hypotenuse of a right triangle with legs Δx and Δu). Thus,

   $$m = \rho\sqrt{(\Delta x)^2 + (\Delta u)^2}$$

   So,

   $$F = ma = \langle 0, \rho\sqrt{(\Delta x)^2 + (\Delta u)^2} \rangle u_{tt}$$

3. Study of the Force: **Assumption: The only force acting on the string is the tension** So if $T(x,t)$ is the magnitude of the tension vector and $\theta(x,t)$ is the angle of the tension vector:



Then from trig, we can calculate the tension force via components of the resultant:

$$\begin{cases} x = T(x,t)\cos(\theta(x,t)) \\ y = T(x,t)\sin(\theta(x,t)) \end{cases} \implies -\langle T\cos(\theta), T\sin(\theta)\rangle(x,t)$$

Note: the minus comes from T pointing the opposite direction of the string

Then in the same way, the force at $(x + \Delta x)$ is

$$\langle T\cos(\theta), T\sin(\theta)\rangle(x+\Delta x, t)$$

so the net force is

$$F(x,t) = \langle T\cos(\theta), T\sin(\theta)\rangle(x+\Delta x, t) - \langle T\cos(\theta), T\sin(\theta)\rangle(x,t)$$

4. Then using $F = ma$ and comparing the components,

$$\begin{cases} T\cos(\theta)(x+\Delta x, t) - T\cos(\theta)(x,t) = 0 \\ T\sin(\theta)(x+\Delta x, t) - T\sin(\theta)(x,t) = \rho\sqrt{(\Delta x)^2 + (\Delta u)^2}\,u_{tt}(x,t) \end{cases}$$

Note, however, that both these LHS look like derivatives. Starting with the cos terms,

$$(T\cos(\theta))_x = 0$$

so $T(x,t)\cos(\theta(x,t))$ is constant in x. But $|\theta(x,t)| << 1$ so $\cos(\theta(x,t)) \approx 1$ and

$$T(x,t)\cos(\theta(x,t)) = T(x,t)$$

which is constant in x so $T(x,t) = T(t)$

Assumption: Tension is also constant in time $T(t) = T$

Then the sin terms,

$$\begin{aligned}(T\sin(\theta))_x &= \rho u_{tt}\left(\frac{\sqrt{(\Delta x)^2 + (\Delta u)^2}}{\Delta x}\right) \\ &= \rho u_{tt}\sqrt{\frac{(\Delta x)^2 + (\Delta u)^2}{\Delta x}} \\ &= \rho u_{tt}\sqrt{1 + \left(\frac{\Delta u}{\Delta x}\right)^2} \\ &= \rho u_{tt}\sqrt{1 + (u_x)^2}\end{aligned}$$

Assumption: if the displacements $\Delta u/\Delta x$ are small, then

$$\theta(x,t) = \tan^{-1}\frac{\Delta u}{\Delta x}$$

is small, proving the inequality above.

Then, as $\Delta x \to 0$, $|u_x| << 1$ so

$$\sqrt{1 + (u_x)^2} \approx 1$$

and

$$(T\sin(\theta))_x = \rho u_{tt}$$

but

$$\sin\theta = \tan\theta\cos\theta = \frac{\Delta u}{\Delta x}\cos\theta \to u_x$$

so

$$(Tu_x)_x = Tu_{xx} \quad \text{(assuming T is constant)}$$

and at last,

$$Tu_{xx} = \rho u_{tt} \longrightarrow u_{tt} = \frac{T}{\rho} u_{xx}$$

Set, $c = \sqrt{T/\rho} > 0$ and

$$\boxed{u_{tt} = c^2 u_{xx}}$$

12 Feb 22: Wave Equation Solution

Goal: Solve $u_{tt} = c^2 u_{xx}$

Part I - Factoring Method

But this kind of looks like

$$t^2 - c^2x^2 = (t - cx)(t + cx)$$

Definition: Differential operator

$$\frac{\partial}{\partial t} u = u_t$$

$$\left(\frac{\partial}{\partial t}\right)^2 u = u_{tt}$$

Using this operator we can more rigorously "factor" the PDE.

1. Apply the differential operator

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= \left[\left(\frac{\partial}{\partial t}\right)^2 - c^2 \left(\frac{\partial}{\partial x}\right)^2\right] u \\ &= \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) \end{aligned}$$

2. Solve the equation

$$u_{tt} - c^2 u_{xx} = 0$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) = 0$$

Let $v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u$ so

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0 \Longrightarrow v_t - cV_x = 0$$

3. Solve the transport PDE

$$v(x,t) = f(x+ct)$$

4. Solve for u

$$v := \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = u_t + cu_x$$

$$u_t + cu_x = f(x+ct)$$

But this is just an inhomogeneous transport equation! The homogeneous solution is just

$$u_0(x,t) = G(x-ct)$$

And a particular solution can be found using undetermined coefficients. Notice that the RHS is a function of $x+ct$ so we can guess

$$u_p = h(x+ct)$$

so

$$(h(x+ct))_t + c(h(x+ct))_x = f(x+ct)$$

$$ch'(x+ct) + ch'(x+ct) = f(x+ct)$$

$$2ch'(x+ct) = f(x+ct) \Longrightarrow h' = \frac{1}{2c}f'$$

$$h(x+ct) = \frac{1}{2c}F(x+ct)$$

where F is an antiderivative of f Thus giving the general solution

$$u(x,t) = G(x-ct) + \frac{1}{2c}F(x+ct)$$

$$\boxed{u(x,t) = G(x-ct) + F(x+ct)}$$

Interpretation: A wave is a sum of two functions, one moving to the left at speed c and the other to the right at speed c

Part II - Coordinate Method

1. Define variables
$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$

2. Chain rule
$$u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi + u_\eta$$
   and
$$\begin{aligned} u_{xx} = (u_x)_x &= \frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= u_{\xi_\xi} + u_{\eta_\eta} = u_{\xi\xi} + u_{\eta\xi} + u_{\xi\eta} + u_{\eta\eta} \\ &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \end{aligned}$$
   Similarly,
$$u_t t = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$

3. Plug into wave equation:
$$u_{tt} = c^2 u_{xx}$$
$$c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) = c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = 4u_{\xi\eta}$$
$$\boxed{u_{\xi\eta} = 0}$$

13 Feb 24: D'Alembert's Formula

Part I - Solving the wave equation (continued)

$$u_{tt} = c^2 u_{xx}$$

Using the coordinate method with the choices
$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$
we get the equation
$$u_{\xi\eta} = 0$$
so
$$u_\xi = f(\xi) \Longrightarrow u = F(\xi) + G(\eta)$$
thus
$$\boxed{u(x,t) = F(x - ct) + G(x + ct)}$$

Part II - D'Alembert's Formula

Example:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Solution:

1. General Solution
$$u(x,t) = F(x-ct) + G(x+ct)$$

2. Plug in the initial condition
$$u(x,0) = \phi(x) = F(x) + G(x)$$

3. Differentiate with t
$$u_t(x,t) = -cF'(x-ct) + cG(x+ct)$$
$$u_t(x,0) = \psi(x) = -cF'(x) + cG'(x)$$
$$-F'(x) + G'(x) = \frac{\psi(x)}{c}$$

4. Integrate over $[0,x]$
$$\int_0^x -F'(s) + G'(s)\ ds = \int_0^x \frac{\psi(s)}{c}\ ds$$
$$-F(x) + G(x) - (-F(0) + G(0)) = \frac{1}{c}\int_0^x \psi(s)\ ds$$

   This gives us the system of equations
$$\begin{cases} -F(x) + G(x) = A + \frac{1}{c}\int_0^x \psi(s)\ ds \\ F(x) + G(x) = \phi(x) \end{cases}$$

$$\implies \begin{cases} 2G(x) = \phi(x) + A + \frac{1}{c}\int_0^x \psi(s)\ ds \\ 2F(x) = \phi(x) - A - \frac{1}{c}\int_0^x \psi(s)\ ds \end{cases} \implies \begin{cases} F(X) = \frac{1}{2}\phi(x) - \frac{A}{2} - \frac{1}{2c}\int_0^x \psi(s)\ ds \\ G(X) = \frac{1}{2}\phi(x) + \frac{A}{2} + \frac{1}{2c}\int_0^x \psi(s)\ ds \end{cases}$$

5. Solution

$$\begin{aligned}
u(x,t) &= F(x-ct) + G(x+ct) \\
&= (\frac{1}{2}\phi(x-ct) - \frac{A}{2} - \frac{1}{2c}\int_0^{x-ct} \psi(s)\ ds) \\
&+ (\frac{1}{2}\phi(x+ct)\frac{A}{2} + \frac{1}{2c}\int_0^{x+ct} \psi(s)\ ds) \\
&= \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\left(\int_{x-ct}^{0} \psi(s)\ ds + \int_0^{x+ct} \psi(s)\ ds\right)
\end{aligned}$$

Which at last gives us d'Alembert's equation to solve the wave equation with initial conditions:

$$\boxed{u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds}$$

Part III - Example

$$\begin{cases} u_{tt} = u_{xx} \\ u(x,0) = 0 \\ u_t(x,0) = \cos(x) \end{cases} \implies \begin{cases} c = 1 \\ \phi(x) = 0 \\ \psi(x) = \cos(x) \end{cases}$$

Then using D'Alembert's:

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds \\
&= \frac{1}{2}(0+0) + \frac{1}{2}\int_{x-t}^{x+t} \cos(s)\ ds \\
&= \frac{1}{2}(\sin(x+t) - \sin(x-t)) \\
&= \frac{1}{2}(\sin x \cos t + \cos x \sin t - \sin x \cos -t - \cos x \sin -t) \\
&= \frac{1}{2}(2\cos x \sin t)
\end{aligned}$$

$$\boxed{u(x,t) = \sin(t)\cos(x)}$$

(Or, the wave takes the shape of cos with amplitude sin)

14 Feb 27: Midterm Review

Part I - First Order PDE

$$\begin{cases} (1+x^2)u_x + e^y u_y = 0 \\ u(0,y) = y^2 \end{cases}$$

$$\begin{gathered}
\nabla u \cdot (1+x^2, e^y) = 0 \\
y' = \frac{e^y}{1+x^2} \\
\frac{1}{e^y}\, dy = \frac{1}{1+x^2}\, dx \\
\tan^{-1} x = -e^{-y} + C \implies \tan^{-1} x + e^{-y} = C \\
u(x,y) = f(\tan^{-1}(x) + e^{-y}) \\
u(0,y) = f(\tan^{-1}(0) + e^{-y}) = y^2 = f(e^{-y}) \\
z := e^{-y} \implies -y = \ln z \implies y = -\ln z \\
f(z) = f(e^{-y}) = y^2 = (-\ln(z))^2 = (\ln z)^2 \\
\boxed{u(x,y) = \ln(\tan^{-1} x + e^{-y})^2}
\end{gathered}$$

Part II - Coordinate Method

$$au_x + bu_y + cu = 0$$

Solution:

$$\begin{cases} \xi = ax + by \\ \eta = ay - bx \end{cases}$$

$$\begin{cases} u_x = u_\xi \xi_x + u_\eta \eta_x = au_\xi - bu_\eta \\ u_y = u_\xi \xi_y + u_\eta \eta_y = bu_\xi + au_\eta \end{cases}$$

$$\begin{aligned}
au_x + bu_y + cu &= a(au_\xi - bu_\eta) + b(bu_\xi + au_\eta) + cu \\
&= a^2 u_\xi - abu_\eta + b^2 u_\xi + abu_\eta + cu \\
&= (a^2 + b^2)u_\xi + cu \\
&= u_\xi + \frac{c}{a^2 + b^2} u = 0
\end{aligned}$$

$$u = f(\eta)e^{-\frac{c}{a^2+b^2}\xi}$$

$$\boxed{u(x,y) = f(ay-bx)e^{-\left(\frac{c}{a^2+b^2}\right)(ax+by)}}$$

Part III - Fourier Transform

$$\begin{cases} au_x + bu_t + cu = 0 \\ u(x,0) = f(x) \end{cases}$$

Solution:

$$\mathcal{F}(au_x) + \mathcal{F}(bu_t) + \mathcal{F}(cu) \implies \mathcal{F}(bu_t) = -\mathcal{F}(au_x) - \mathcal{F}(cu)$$

$$\begin{aligned} b\frac{d}{dt}\widehat{u} &= -a(-i\kappa)\widehat{u} - c\widehat{u} \\ \frac{d}{dt}\widehat{u} &= \left(\frac{ai\kappa - c}{b}\right)\widehat{u} \\ \widehat{u} &= \widehat{f}(\kappa)e^{\frac{ai\kappa-c}{b}t} \\ &= \widehat{f}(\kappa)e^{i\kappa\left(\frac{a}{b}\right)t}e^{-\frac{ct}{b}} \\ &= e^{-\frac{ct}{b}}\mathcal{F}\left(f(x-\frac{a}{b}t)\right) \\ &= \mathcal{F}\left(e^{-\frac{ct}{b}}f(x-\frac{a}{b}t)\right) \end{aligned}$$

$$\boxed{u(x,t) = e^{-\frac{ct}{b}}f(x-\frac{a}{b}t)}$$

Part IV - Wave Equation Factoring Method

$$3u_{tt} + 10u_{xt} + 3u_{xx} = 0$$

Solution:

$$3t^2 + 10xt + 3x^2 \implies (x+3t)(3x+t)$$

$$\begin{aligned} 3u_{tt} + 10u_{xt} + 3u_{xx} &= 3\frac{\partial^2}{\partial t^2} + 10\frac{\partial^2}{\partial x\partial t} + 3\frac{\partial^2}{\partial x^2} \\ &= \left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial t}\right)\left(3\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u \end{aligned}$$

Let

$$v = \left(3\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) u$$

so

$$\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial t}\right) v = 0$$

$$v_x + 3v_t = 0 \implies v = f(x - \frac{t}{3})$$

Then

$$\left(3\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) u = 3u_x + u_t = f(x - \frac{t}{3})$$

Homogeneous solution:

$$3u_x + u_t = 0 \implies u_0(x,t) = G(x - 3t)$$

Particular solution:

$$f(x - 3t) \implies u_p(x,t) = h(x - \frac{t}{3})$$

$$3(h(x - \frac{t}{3}))_x + (h(x - \frac{t}{3}))_t = f(x - \frac{t}{3})$$

$$\implies h(s) = \frac{3}{8}F(S)$$

$$u_p = \frac{3}{8}F(x - \frac{t}{3})$$

$$u(x,t) = G(x - 3t) + \frac{3}{8}F(x - \frac{t}{3})$$

$$\boxed{u(x,t) = G(x - 3t) + F(x - \frac{t}{3})}$$

15 March 3: Energy Methods

Energy Method for Waves

Example 1: Conservation of Energy Suppose u solves the wave equation $u_{tt} = c^2 u_{xx}$. Then the following is constant

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u_t)^2 + c^2(u_x)^2 \, dx$$

where $(u_t)^2 = \frac{1}{2}mv^2$ is the kinetic energy and the second term $c^2(u_x)^2$ is the potential energy. Thus, for the wave equation, the total energy is conserved.

Method 1: Calculate $E'(t)$ and show it is 0 This is easier but requires you know E a priori.

Method 2: Energy Method

1. Start with $u_t = c^2u_{xx}$ **Trick:** Multiply the PDE by a clever function, here by u_t

$$u_{tt}u_t = c^2u_{xx}u_t$$

   Integrate with respect to x:

$$\int_{-\infty}^{\infty} u_{tt}u_t\ dx = c^2 \int_{-\infty}^{\infty} u_{xx}u_t\ dx$$

2. Study A From calculus,

$$y''y' = \left[\frac{1}{2}(y')^2\right]'$$

   Therefore,

$$u_{tt}u_t = \frac{d}{dt}\left[\frac{1}{2}(u_t)^2\right]$$

   so

$$A = \int_{-\infty}^{\infty} u_{tt}u_t\ dx = \frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty}(u_t)^2\ dx\right)$$

3. Study B

$$B = \int_{-\infty}^{\infty} u_{xx}u_t\ dx$$

Integrate by parts WRT x:

$$
\begin{aligned}
A &= \int_{-\infty}^{\infty} u_{xx} u_t \, dx \\
&\overset{\text{IBP}}{=} [u_x u_t]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_x u_{xt} \, dx \\
&= -\int_{-\infty}^{\infty} u_x u_{xt} \, dx \\
&= -\int_{-\infty}^{\infty} \frac{d}{dt}\left(\frac{1}{2}(u_x)^2\right) \, dx \\
&= \frac{d}{dt}\left(-\frac{1}{2}\int_{-\infty}^{\infty} (u_x)^2 \, dx\right)
\end{aligned}
$$

4. Then $A = c^2 B$ implies

$$
\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} (u_t)^2 \, dx\right) = c^2 \frac{d}{dt}\left(-\frac{1}{2}\int_{-\infty}^{\infty} (u_x)^2 \, dx\right)
$$

$$
\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} (u_t)^2 + c^2 (u_x)^2 \, dx\right) = 0
$$

$$
\frac{d}{dt} E(t) = 0
$$

so E is constant

Note: How do we know which function to multiply the PDE by? It's an art! Here we multiplied by u_t to make a time-derivative appear

Application: Uniqueness

Lemma: Suppose w solves the following PDE

$$
\begin{cases}
w_{tt} = c^2 w_{xx} \\
w(x,0) = 0 \\
w_t(x,0) = 0
\end{cases}
$$

Then $w(x,t) = 0$ for all x and t.

Why?

1. the energy $E(t)$ is constant so

$$E(t) = E(0)$$

$$\frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2 + c^2 (w_x)^2 \, dx = \frac{1}{2}\int_{-\infty}^{\infty} (w_t(x,0))^2 + c^2 (w_x(x,0))^2 \, dx$$

By assumption, we have $w_t(x,0) = 0$ and moreover

$$w(x,0) = 0 \implies (w(x,0))_x = 0 = 0_x \implies w_x(x,0) = 0$$

Therefore the above becomes

$$\frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2 + c^2 (w_x)^2 = 0$$

2. **Fact:** if $f \geq 0$ and $\int_{-\infty}^{\infty} f(x)\, dx = 0$ then f must be the zero-function

Therefore $(w_t)^2 + c^2(w_x)^2 = 0$ So $w_t = 0$ and $w_x = 0$ Hence $w(x,t) = C$ But plugging in $t = 0$ we get $C = w(x,0) = 0$ and $w(x,t) = 0$ $\square$

Application: There is at most one solution of

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Proof: Standard trick: suppose there are two solutions u and v and let $w = u - v$. Then we check that w solves

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = u(x,0) - v(x,0) = \phi(x) - \phi(x) = 0 \\ u_t(x,0) = u_t(x,0) - v_t(x,0) = \psi(x) - \psi(x) = 0 \end{cases}$$

By the lemma above, we get $w = 0$ so $u - v = 0$ so $u = v$ and thus there is exactly one solution of the above wave equation

Energy Method for Heat

Suppose we have a finite rod of length l with initial temperature 0 and insulated.

Example 2: Suppose u solves the PDE

$$\begin{cases} u_t = Du_{xx} \\ u(x,0) = 0 \quad \text{Initial} \\ u(0,t) = 0 \quad \text{Endpoint} \\ u(l,t) = 0 \quad \text{Endpoint} \end{cases}$$

Then $u(x,t) = 0$ for all x and t

Proof:

1. Start with $u_t = Du_{xx}$. Multiply by u

$$u_t u = Du_{xx} u$$

   Integrate with respect to x on $[0, l]$

$$\int_0^l u_t u \, dx = F \int_0^l u_{xx} u \, dx$$

2. Study A

$$A = \int_0^l \frac{d}{dt}\left(\frac{1}{2}u^2\right) \, dx = \frac{d}{dt}\left(\int_0^l u^2 \, dx\right)$$

3. Study B

$$\int_0^l u_{xx} u \, dx = u_x(l,t)u(l,t) - u_x(0,t) - \int_0^l u_x u_x \, dx = -\int_0^l (u_x)^2 \, dx$$

4. So $A = DB$ and

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^l u^2 \, dx\right) = -D\int_0^l (u_x)^2 \, dx \leq 0$$

   Then if you define

$$E(t) = \frac{1}{2}\int_0^l u^2 \, dx$$

you have $E'(t) \leq 0$ so the energy is decreasing

Interpretation: heat is dissipative. An insulated metal rod generally gets cooler with time

This also means that

$$E(t) \leq E(0)$$

so

$$E(t) = \frac{1}{2}\int_0^l (u(x,t)^2)\,dx \leq E(0) = \frac{1}{2}\int_0^l (u(x,0))^2\,dx = 0$$

5. Finale But $E(t) \geq 0$ by definition so $0 \leq E(t) \leq E(0) = 0 \implies E(t) = 0$ and

$$\frac{1}{2}\int_0^l (u(x,t))^2\,dx = 0$$

Therefore $(u(x,t))^2 = 0$ so $u(x,t) = 0$ for all x and t

16 March 6: Heat vs Wave Equations

Part I - Energy Method Application

Example:

$$\begin{cases} u_t = Du_{xx} + f(x,t) \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(l,t) = h(t) \end{cases}$$

Trick: $w := u - v$ and via the initial conditions, $w = 0$ so $u = v$ and there is only one solution

Part II - The Infinite Rod

Example:

$$\begin{cases} u_t = Du_{xx} \\ u(x,0) = f(x) \end{cases}$$

has many solutions including $u = f \star g$

Note: we have uniqueness if

$$|u(x,t)| \leq Ce^{ax^2}$$

for some $C > 0$ and $a > 0$ (meaning the PDE is in the Schwartz class)

Part III - Comparison of Waves and Diffusions

Heat Equation $\begin{cases} u_t = Du_{xx} \\ u(x,0) = f(x) \end{cases}$	Wave Equation $\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$
Existence via Fourier Transform $u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\, dy$	Existence via Factoring $u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$
Uniqueness (given u is Schwartz)	Uniqueness by energy method
Smooth (infinite differentiability of heat kernel)	Not smooth (initial condition splits but will not smooth)
Infinite speed of propagation	Finite speed of propagation
Not reversible	Reversible (if wave travels backwards)
Goes to zero over time (energy is decaying)	Does not go to zero (energy is conserved)

17 March 8: Separation of Variables I

This is the most important PDE technique in the course.

Part I - Boundary Value Problem

Example 1: Find all the values of λ for which the following ODE has a nonzero solution X

$$\begin{cases} X''(X) = \lambda X(x) \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

Case 1: $\lambda > 0$ Then $\lambda = \omega^2$ for positive ω and the aux equation is

$$r^2 = \omega^2 \implies r = \pm\omega$$

so

$$X(x) = Ae^{\omega x} + Be^{-\omega x}$$
$$X(0) = A + B = 0 \implies B = -A$$
$$X(x) = Ae^{\omega x} - Ae^{-\omega x}$$
$$X(\pi) = 0 \implies Ae^{\omega\pi} - Ae^{-\omega pi} = 0 \implies 2\pi\omega = 0$$

But then $\lambda = 0$ which contradicts $\lambda > 0$ so no nonzero solutions

Case 2: $\lambda = 0$ Aux: $r^2 = 0 \implies r = 0$

$$X(x) = A + Bx \implies X(0) = A = 0 \implies X(x) = Bx$$

$$X(\pi) = 0 \implies B\pi = 0 \implies B = 0$$

But then

$$X(x) = 0$$

So there are no nonzero solutions here either

Case 3: $\lambda < 0$

$$r^2 = \lambda = -\omega^2 \implies r = \pm\omega i$$
$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$
$$X(0) = A = 0 \implies X(x) = B\sin(\omega x)$$
$$X(\pi) = 0 \implies \sin(\omega\pi) = \implies \omega\pi = \pi m \implies \omega = m$$

Where m is any positive integer. So our eigenvalues are

$$\lambda = -m^2 \quad (m = 1, 2, ...)$$

and our eigenfunctions are

$$X(x) = \sin(mx) \quad (m = 1, 2, ...)$$

Part II - Separation of Variables

Example 2:

$$\begin{cases} u_t = Du_{xx} \\ u(0,t) = 0 \\ u(\pi,t) = 0 \\ u(x,0) = x^2 \end{cases}$$

Note that this is the same finite rod problem for which we proved uniqueness via energy methods. Then we just need to find one nonzero solution.

Solution:

1. Separation of variables Assume u is the form $u(x,t) = X(x)T(t)$ Plug into the PDE:

$$u_t = Du_{xx}$$
$$(X(x)T(t))_t = D(X(x)T(t))_{xx}$$
$$X(x)T'(t) = DX''(x)T(t)$$
$$\frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)}$$

   Note: always put extra terms on the T side to keep the X equation simple

2. Constant Note that

$$\left(\frac{X''(x)}{X(x)}\right)_t = 0$$

   and

$$\left(\frac{X''(x)}{X(x)}\right)_x = \left(\frac{T'(t)}{DT(t)}\right)_x = 0$$

   so

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{DT(t)} = \text{Constant} = \lambda$$

   Then

$$X''(x) = \lambda X(x)$$
$$T'(t) = D\lambda T(t)$$

   and instead of a PDE we have two ODEs!

Focusing on the X equation,

$$X''(x) = \lambda X(x)$$

we can use the boundary conditions

$$u(0,t) = 0 \overset{u=XT}{\Longleftarrow} X(0)T(t) = 0 \implies X(0) = 0$$

(you can cancel the T because fo $T = 0$ the problem is not interesting)

$$u(\pi,t) = 0 \implies X(pi) = 0$$

So

$$\begin{cases} X''(x) & = \lambda X(x) \\ X(0) & = 0 \\ X(\pi) & = 0 \end{cases}$$

which is the same ODE from part I!

$$\begin{cases} \lambda = -m^2 \quad (m = 1, 2, ...) \\ X(x) = \sin(mx) \quad (m = 1, 2, ...) \end{cases}$$

3. T equation

$$\begin{cases} \frac{T'}{DT} = \lambda = -m^2 \\ T' = -m^2 DT \\ T(t) = e^{-m^2 Dt} \end{cases}$$

Conclusion: For every $m = 1, 2, ...$

$$u(x,t) = X(x)T(t) = \sin(mx)e^{-m^2 Dt}$$

is a solution to the PDE

4. Linearity Since the PDE is linear, any linear combo of the above solution is also a solution!

$$u(x,t) = \sum_{m=1}^{\infty} A_m \sin(mx)e^{-m^2 Dt}$$

5. Initial Condition

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(mx) = x^2$$

Question: How do you find A_m (can you write x^2 as a linear combo of sines?)

Part III - Easier Problem

$$\begin{cases} u_t = Du_{xx} \\ u(0,t) = 0 \\ u(\pi,t) = 0 \\ u(x,0) = 3\sin(2x) + 4\sin(3x) \end{cases}$$

The steps above give

$$u(x,t) = \sum_{m=1}^{\infty} A_m \sin(mx) e^{-m^2 Dt}$$

so

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(mx) = 3\sin(2x) + 4\sin(3x)$$

and

$$\boxed{u(x,t) = 3\sin(2x)e^{-4Dt} + 4\sin(3x)e^{-9Dt}}$$

Interpretation: Initially, the solution starts out as the sum of sines but eventually dies down as the exponential terms go to 0

18 March 10: Separation of Variables II

Part I - Setting

Example 1:

$$\begin{cases} u_{tt} = c^2 u_{xx} \quad 0 < x < 1 \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = e^x \end{cases}$$

Note that you cannot use d'Alembert's formula because the bounds are not $-\infty < x < \infty$

Part II - Separation of variables

Assume that

$$u(x,t) = X(x)T(t)$$

Then plug this in

$$(X(x)T(t))_{tt} = c^2((X(x)T(t))_{xx})$$
$$XT'' = c^2X''T$$
$$\frac{X''}{X} = \frac{T''}{c^2T}$$

Note that

$$\left(\frac{X''}{X}\right)_t = 0$$
$$\left(\frac{X''}{X}\right)_x = \left(\frac{T''}{c^2T}\right)_x = 0$$

so

$$\frac{X''}{X} = \frac{T''}{c^2T} = \lambda$$

with λ constant. Then

$$X'' = \lambda X$$
$$T'' = c^2\lambda T$$

Using the boundary conditions

$$u(0,t) = 0 \implies X(0)T(t) = 0 \implies X(0) = 0$$
$$u(1,t) = 0 \implies X(1)T(t) = 0 \implies X(1) = 0$$

Then looking at the X ODE:

$$\begin{cases} X'' = \lambda X \\ X(0) = 0 \\ X(1) = 0 \end{cases}$$

$$r^2 = \lambda \implies r = \pm\omega$$

Case $\lambda > 0$:

$$X = Ae^{\omega x} + Be^{-\omega x}$$

Initial conditions show no nonzero solutions

Case $\lambda = 0$:

$$X(x) = A + Bx \longrightarrow X(0) = A = 0 \longrightarrow X(x) = 0$$

Case $\lambda < 0$:

$$r^2 = \lambda = -\omega^2 \implies r = \pm\omega i$$

$$X(x) = A\cos(\omega X) + B\sin(\omega X)$$

$$X(0) = A = 0 \implies X(x) = B\sin(\omega x)$$

$$X(1) = B\sin(\omega) = 0$$

$$\sin(\omega) = 0 \implies \omega = \pi m \quad (m = 1, 2, ...)$$

$$\lambda = -\omega^2 = -(\pi m)^2 \quad (eigenvalues)$$
$$X(x) = \sin(\pi m x) \quad (eigenfunction)$$

Now going all the way back to the T equation:

$$T'' = c^2 \lambda T$$

$$\frac{T''}{c^2 T} = \lambda = -(\pi m)^2$$

$$T'' = -(\pi m)^2 c^2 T$$

$$T'' = -(\pi m c)^2 T$$

$$T'' - (\pi m c)^2 T = 0$$

Auxiliary:

$$r^2 - (\pi m c)^2 = 0$$

$$r = \pm \pi m c i$$

$$T = A\cos(\pi m c t) + B\sin(\pi m c t)$$

Conclusion:

$$u(x,t) = X(x)T(t) = (A\cos(\pi m c t) + B\sin(\pi m c t))\sin(\pi m x)$$

General solution via linearity:

$$\boxed{u(x,t) = \sum_{m=1}^{\infty}(A_m\cos(\pi m c t) + B_m\sin(\pi m c t))\sin(\pi m x)}$$

Initial conditions:

$$u(x,0) = \sum_{m=1}^{\infty}(A_m \cos(0) + B_m \sin(0))\sin(\pi m x) = \sum_{m=1}^{\infty} A_m \sin(\pi m x) = x^2$$

The question of representing an arbitrary function as a sum of sin waves deals with Fourier series and will come up later.

$$u_t = \sum_{m=1}^{\infty}(-A_m \pi m c \sin(\pi m c t) + B_m \pi m c \cos(\pi m c t))\sin(\pi m x)$$

$$u_t(x,0) = \sum_{m=1}^{\infty} B_m \pi m c \sin(\pi m x) = e^x$$

Part III - Easier initial conditions

Example 2:

$$\begin{cases} u_{tt} = c^2 u_{xx} \quad 0 < x < 1 \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = 2\sin(2\pi x) + 3\sin(3\pi x) \\ u_t(x,0) = 4\sin(2\pi x) \end{cases}$$

General solution:

$$u(x,t) = \sum_{m=1}^{\infty}(A_m \cos(\pi m c t) + B_m \sin(\pi m c t))\sin(\pi m x)$$

Initials:

$$\begin{aligned} u(x,0) &= \sum_{m=1}^{\infty} A_m \sin(\pi m x) \\ &= A_1 \sin(\pi x) + A_2 \sin(2\pi x) + A_3 \sin(3\pi x) + ... \\ &= 0\sin(\pi x) + 1\sin(2\pi x) + 3\sin(3\pi x) + 0\sin(4\pi x) + ... \end{aligned}$$

therefore,

$$A_1 = 0 \quad A_2 = 1 \quad A_3 = 3 \quad A_m = 0\ (m \geq 4)$$

And from the other initial condition

$$u_t(x,0) = \sum_{m=1}^{\infty}(\pi mc)B_m \sin(\pi mx) = \pi c B_1 \sin(\pi x) + 2\pi c B_2 \sin(2\pi x) + 3\pi c B_3 \sin(3\pi x) = 4\sin(2\pi x)$$

so

$$B_1 = 0 \quad B_2 = \frac{2}{\pi c} \quad B_m = 0 \ (m \geq 3)$$

and

$$\begin{aligned} u(x,t) &= \sum_{m=1}^{\infty} = (A_m \cos(\pi mct) + B_m \sin(\pi mct)) \sin(\pi mx) \\ &= (A_2 \cos(2\pi ct) + B_2 \sin(2\pi ct)) \sin(2\pi x) + (A_3 \cos(3\pi ct) + B_3 \sin(3\pi ct)) \sin(3\pi x) \end{aligned}$$

At long last,

$$\boxed{u(x,t) = \left(\cos(2\pi ct) + \frac{2}{\pi c}\right) \sin(2\pi x) + (2\cos(3\pi ct)) \sin(3\pi x)}$$

19 March 13: Separation of Variables III

Part I - Setting

Heat equation but assuming velocity at the endpoints is 0 ("rate in = rate out")

$$\begin{cases} u_t = Du_{xx} \\ u_x(0,t) = 0 \\ u_x(\pi,t) = 0 \\ u(x,0) = x^2 \end{cases}$$

Part II - Separation of variables

Suppose $u(x,t) = X(x)T(t)$ Plug in

$$(X(x)T(t))_t = D(X(x)T(t))_{xx}$$

$$XT' = DX''T$$

$$\frac{T'}{DT} = \frac{X''}{X} = \lambda$$

$$\begin{cases} X'' = \lambda X \\ T' = \lambda D T \end{cases}$$

X boundary conditions:

$$\begin{aligned} u &= XT \\ u_x &= X'T \\ u_x(0,t) = 0 \implies X'(0) &= 0 \\ u_x(\pi,t) = 0 \implies X'(\pi) &= 0 \end{aligned}$$

So

$$\begin{cases} X'' = \lambda X \\ X'(0) = 0 \\ X'(\pi) = 0 \end{cases}$$

Boundary value problem: Case 1 $\lambda > 0$:

$$r^2 = \lambda = \omega^2 \implies r = \pm\omega$$

$$X = Ae^{\omega x} + Be^{-\omega x}$$

$$X'(0) = A - B = 0 \implies A = B$$

$$X = Ae^{\omega x} + Ae^{-\omega x}$$

$$X'(pi) = 0 \implies \omega\pi = -\omega\pi \implies \omega = 0$$

No nonzero solutions

Case 2 $\lambda = 0$

$$r^2 = 0 \implies X = A + Bx$$

$$X'(0) = 0 \implies B = 0 \implies X = A$$

So $\lambda = 0$ is an eigenvalue with eigenfunction $X(x) = A$

Case $\lambda < 0$:

$$r^2 = \lambda = -\omega^2 \implies r = \pm\omega i$$

$$X = A\cos(\omega x) + B\sin(\omega x)$$

$$X'(0) = B\omega = 0 \implies X = A\cos(\omega x)$$

$$X'(\pi) = 0 \implies \sin(\omega\pi) = 0 \implies \omega = m$$

$$\lambda = -\omega^2 = -m^2$$

So with the above case, the eigenvalues are

$$\lambda = -m^2 \quad (m = 0, 1, 2, ...)$$

and the eigenfunctions are

$$X(x) = \cos(mx) \quad (m = 0, 1, 2, ...)$$

Back to the T equation:

$$\begin{aligned}\frac{T'}{DT} &= \lambda = -m^2 \\ T' &= -m^2 DT \\ T(t) &= e^{-m^2 Dt}\end{aligned}$$

Conclusion: for every $m = 0, 1, 2, ...$ the following solves the PDE:

$$u(x,t) = X(x)T(t) = e^{-m^2 Dt}\cos(mx)$$

Taking linear combinations we get

$$u(x,t) = \sum_{m=0\infty} A_m e^{-m^2 Dt}\cos(mx)$$

So with the initial condition

$$x^2 = \sum_{m=0}^{\infty} A_m \cos(mx)$$

Part III - Inhomogeneous Problems

Example 2:

$$\begin{cases} u_t = Du_{xx} \\ u(0,t) = 7 \\ u(\pi,t) = 7 \\ u(x,0) = x^2 \end{cases}$$

Trick: Let $v(x,t) = u(x,t) - 7$ Then

$$v_t = (u-7)_t = u_{tt} = Du_{xx} = D(v+7)_{xx} = Dv_{xx}$$

$$\begin{aligned} v(0,t) &= u(0,t) - 7 = 0 \\ v(\pi,t) &= u(\pi,t) - 7 = 0 \\ v(x,0) &= u(x,0) - 7 = x^2 - 7 \end{aligned}$$

Then solve the above system for v using separation of variables and use

$$u(x,t) = v(x,t) + 7$$

to solve for u

Example 3:

$$\begin{cases} u_t = Du_{xx} \\ u_x(0,t) = 7 \\ u_x(\pi,t) = 7 \\ u(x,0) = x^2 \end{cases}$$

Trick: Use $v(x,t) = u(x,t) - 7x$ Then

$$v_t = (u - 7x)_t = u_t = Du_{xx} = D(v + 7x)_{xx} = Dv_{xx}$$

$$\begin{aligned} v_t &= Dv_{xx} \\ v_x(0,t) &= u_x(0,t) - 7 = 0 \\ v_x(\pi,t) &= u_x(\pi,t) - 7 = 0 \\ v(x,0) &= u(x,0) - 7x = x^2 - 7x \end{aligned}$$

Solve for v using separation of variables and use

$$u(x,t) = v(x,t) + 7x$$

Example 4:

$$\begin{cases} u_t = Du_{xx} \\ u(0,t) = 1 \\ u(\pi,t) = 3 \\ u(x,0) = x^2 \end{cases}$$

Let $v(x,t) = u(x,t) - f(x)$ Where f is a linear function such that $f(0) = 1$ and $f(\pi) = 3$ Then

$$f(x) = \left(\frac{3-1}{\pi - 0}\right)(x - 0) + 1 = \frac{2}{\pi}x + 1$$

$$v(x,t) = u(x,t) - frac2\pi x - 1$$
$$v_t = (u - f(x))_t = u_t = D(v + f(x))_{xx} = Dv_{xx}$$

Here we used $f''(x)$ since f is linear

$$v(0,t) = u(0,t) - f(0) = 1 - 1 = 0$$
$$v(\pi,t) = u(\pi,t) - f(\pi) = 3 - 3 = 0$$
$$v(x,0) = u(x,0) - f(x) = x^2 - \frac{2}{\pi}x - 1$$

Then solve

$$\begin{cases} v_t = Dv_{xx} \\ v(0,t) = 0 \\ v(\pi,t) = 0 \\ v(x,0) = x^2 - \frac{2}{\pi}x - 1 \end{cases}$$

Then

$$u(x,t) = v(x,t) + f(x) = v(x,t) + \frac{2}{\pi}x + 1$$

Note: we could in theory also solve the case where $u_x(0,t) = 1$ and $u_x(\pi,t) = 3$ by subtracting a function whose *derivative* is $\frac{2}{\pi}x + 1$ except you would get an inhomogeneous wave equation for v

20 March 15: Fourier Series (I)

Part I - Prelude

Goal: given a function $f(x)$ on $(0,\pi)$ find A_m such that

$$f(x) = \sum_{m=1}^{\infty} A_m \sin(mx)$$

This is known as a Fourier sine series and is similar to a Taylor series

Part II - Orthogonality

Definition: Let u, v, w be vectors. Then u and v are orthogonal if

$$u \cdot v = 0$$

and $\{u, v, w\}$ is orthogonal if any two different vectors in that set are orthogonal

Fact: If a set is orthogonal and some vector x is in the span i.e.

$$x = au + bv + cw$$

for some a, b, c, then:

$$a = \frac{x \cdot u}{u \cdot u}, \quad b = \frac{x \cdot v}{v \cdot v}, \quad x = \frac{x \cdot w}{w \cdot w}$$

Proof:

$$x \cdot u = (au + bv + cu) \cdot u = a(u \cdot u) + b\underbrace{(v \cdot u)}_{0} + c\underbrace{(w \cdot u)}_{0} = x \cdot u = a(u \cdot u)$$

and the same for b and c.

Part III - Fourier Series

Definition:

$$\boxed{f \cdot g = \int_0^\pi f(x)g(x)\,dx}$$

Example: $(x^2) \cdot (x^3) = \int_0^\pi x^5\,dx = \frac{\pi^2}{6}$

Fact:

$$\{\sin(mx) | m = 1, 2, ...\} = \{\sin(x), \sin(2x), \sin(3x), ...\}$$

is orthogonal

Proof: Check that for $m \neq n$ then

$$\sin(mx) \cdot \sin(nx) = 0 \implies \int_0^\pi \sin(mx)\sin(nx)\,dx = 0$$

Consequence: In particular, if

$$f(x) = \sum_{m=1}^{\infty} A_m \sin(mx)$$

then

$$A_m = \frac{f \cdot \sin(mx)}{\sin(mx) \cdot \sin(mx)} = \frac{\int_0^\pi f(x)\sin(mx)\,dx}{\int_0^\pi \sin^2(mx)\,dx} = \frac{\int_0^\pi f(x)\ \sin(mx)\,dx}{\frac{\pi}{2}}$$

Fact:

1. If $f(x) = \sum_{m=1}^{\infty} A_m \sin(mx)$ on $(0, \pi)$ then

$$A_m = \frac{2}{\pi} \int_0^{\pi} f(x)\ \sin(mx\ dx)$$

2. If $f(x) = \sum_{m=1}^{\infty} A_m \sin(\frac{\pi m x}{L})$ on $(0, L)$ then

$$\boxed{A_m = \frac{2}{L} \int_0^{L} f(x) \sin(\frac{\pi m x}{L})\ dx}$$

Part IV - Examples

Example 2: Find A_m such that on $(0, \pi)$

$$x = \sum_{m=1}^{\infty} A_m \sin(mx)$$

$$\begin{aligned}
A_m &= \frac{2}{\pi} \int_0^{\pi} x \sin(mx)\ dx \\
&= \frac{2}{\pi} \left(\left[x(\frac{-\cos(mx)}{m}) \right]_0^{\pi} - \int_0^{\pi} (\frac{-\cos(mx)}{m})\ dx \right) \\
&= \frac{2}{\pi}(-\frac{\pi}{m} \cos(\pi m) + 0 + [\frac{1}{m^2} \sin(mx)]_0^{\pi}) \\
&= \frac{2}{\pi}(-\frac{\pi}{m}(-1)^m) \\
&= \frac{2}{m}(-1)^{m+1}
\end{aligned}$$

so

$$x = \sum_{m=1}^{\infty} (-1)^{m+1} \left(\frac{2}{m} \right) \sin(mx) = 2 \sin x - \sin(2x) + \frac{2}{3} \sin(3x) - \frac{1}{2} \sin(4x) + ...$$

Consequence 1: If you let $x = \frac{\pi}{2}$ in the above

$$\frac{\pi}{2} = 2 \sin(\frac{\pi}{2}) - \sin(\pi) + \frac{2}{3} \sin(\frac{3\pi}{2}) + ...$$

$$\frac{\pi}{2} = 2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + ...$$

$$\implies 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

Consequence 2: Example 3:

$$\begin{cases} u_t = Du_{xx} \\ u(0,t) = 0 \\ u(\pi,t) = 0 \\ u(x,0) = x \end{cases}$$

$$u(x,t) = \sum_{m=1}^{\infty} A_m e^{-m^2 Dt} \sin(mx)$$

$$u(x,0) = x = \sum_{m=1}^{\infty} A_m \sin(mx)$$

Then from the above:

$$\boxed{u(x,t) = \sum_{m=1}^{\infty} \frac{2}{m}(-1)^{m+1} e^{-m^2 Dt}}$$

21 March 17: Fourier Series (II)

Part I - Fourier Cosine Series

Goal: Find A_m such that on the interval $(0, \pi)$

$$f(x) = \sum_{m=0}^{\infty} A_m \cos(mx)$$

Application: This was needed to solve the Neumann problem

Luckily, this is practically the same problem as the cosine series so

$$A_m = \frac{f \cdot \cos(mx)}{\cos(mx) \cdot \cos(mx)} = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(mx)\, dx \quad (m = 1, 2, ...)$$

Note however, that A_0 corresponds to $\cos(0x)$ so

$$A_0 = \frac{f \cdot \cos(0)}{\cos(0) \cdot \cos(0)} = \frac{\int_0^{\pi} f(x) \cos(0)\, dx}{\int_0^{\pi} 1\, dx} = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx$$

So, the Fourier Cosine Series is:

$$\boxed{\begin{gathered} \text{If } f(x) = \sum_{m=0}^{\infty} A_m \cos(\frac{\pi m x}{L}) \quad 0 < x < L: \\ A_m = \frac{2}{L}\int_0^L f(x)\cos(\frac{\pi m x}{L})\,dx \\ A_0 = \frac{1}{L}\int_0^L f(x)\,dx \end{gathered}}$$

Part II - Tabular Integration

Example 1: Find A_m on $(0, 1)$ such that

$$x^3 = \sum_{m=0}^{\infty} A_m \cos(\pi m x)$$

First isolate $m = 0$:

$$A_0 = \int_0^1 x^3\,dx = \frac{1}{4}$$

$$A_m = 2\int_0^1 x^3 \cos(\pi m x)\,dx$$

Method:

1. Put x^3 on the left and $\cos(\pi m x)$ on the right
2. Differentiate x^3 to 0 and integrate $\cos(\pi m x)$
3. Cross multiply alternating signs

$$\begin{array}{ccc} x^3 & & \cos(\pi m x) \\ & \searrow & \\ -3x^2 & & \sin(\pi m x)/\pi m \\ & \searrow & \\ 6x & & -\cos(\pi m x)/(\pi m)^2 \\ & \searrow & \\ 6 & & -\sin(\pi m x)/(\pi m)^3 \\ & \searrow & \\ 0 & & \cos(\pi m x)/(\pi m)^4 \end{array}$$

so

$$
\begin{aligned}
A_m &= 2\left[x^3\left(\frac{\sin(\pi m x)}{\pi m}\right) - 3x^2\left(-\frac{\cos(\pi m x)}{(\pi m)^2}\right) + 6x\left(-\frac{\sin(\pi m x)}{(\pi m)^3}\right) - 6\left(\frac{\cos(\pi m x)}{(\pi m)^4}\right)\right]_0^1 \\
&= 6\left(\frac{(-1)^m}{(\pi m)^2}\right) - 12\left(\frac{(-1)^m}{(\pi m)^4}\right) + \left(\frac{12}{(\pi m)^2}\right) \\
&= 6\left(\frac{(-1)^m}{(\pi m)^2}\right) + \left(\frac{12}{(\pi m)^4}(\underbrace{(-1)^{m+1}+1}_{0 \text{ or } 2})\right)
\end{aligned}
$$

thus

$$
A_0 = \frac{1}{4} \qquad A_m = \begin{cases} 6/(\pi m)^2 & m \text{ even} \\ -6/(\pi m)^2 + 24/(\pi m)^4 & m \text{ odd} \end{cases}
$$

Part III - Full Fourier Series

Goal: Find A_m and B_m on $(-\pi, \pi)$ such that

$$
f(x) = \sum_{m=0}^{\infty} A_m \cos(mx) + B_m \sin(mx)
$$

Now the dot product is defined

$$
f \cdot g = \int_{-\pi}^{\pi} f(x)g(x)\ dx
$$

and

$$
\begin{aligned}
A_m &= \frac{f \cdot \cos(mx)}{\cos(mx) \cdot \cos(mx)} = \frac{\int_{-\pi}^{\pi} f(x)\cos(x)\ dx}{\int_{-\pi}^{\pi} \cos^2(mx)\ dx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\ dx \\
B_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\ dx
\end{aligned}
$$

Exception: $m = 0$

$$
\begin{aligned}
A_0 &= \frac{f \cdot \cos(0)}{\cos(0) \cdot \cos(0)} = \frac{\int_{-\pi}^{\pi} f(x)\ dx}{\int_{-\pi}^{\pi} 1\ dx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx \\
B_0 &= 0 \quad \text{(by convention because } \sin(0) = 0)
\end{aligned}
$$

Thus the full fourier series on $(-L, L)$ is:

$$\boxed{\begin{aligned} f(x) &= \sum_{m=0}^{\infty} A_m \cos(\frac{\pi m x}{L}) + B_m \sin(\frac{\pi m x}{L}) \\ A_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{\pi m x}{L})\, dx \\ B_m &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{\pi m x}{L})\, dx \\ A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\, dx \\ B_0 &= 0 \end{aligned}}$$

Example 2: Find A_m and B_m such that on $(-\pi, \pi)$ we have

$$x = \sum_{m=0}^{\infty} A_m \cos(mx) + B_m \sin(mx)$$

Solution:

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\, dx = 0$$

$$B_0 = 0$$

$$A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x\cos(mx)}_{\text{Odd}}\, dx = 0$$

$$B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x\sin(mx)}_{\text{Even}}\, dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin(mx)\, dx = \left(\frac{2}{\pi}\right)(-1)^{m+1}$$

$$x = \sum_{m=1}^{\infty} \left(\frac{2}{m}\right)(-1)^{m+1}\sin(mx)$$

Which is the sine series!

Note: In fact, this generalizes: If f is odd on $(-\pi, \pi)$ then the Full Fourier series is a sine series, and similar if f is even. But in practice you have to calculate both A_m and B_m

22 March 20: Fourier Series (III)

Part I - Complex Fourier Series

Goal: find C_m such that

$$f(x) = \sum_{m=-\infty}^{\infty} C_M e^{imx} \qquad -\pi < x < \pi$$

We define a new dot product

$$f \cdot g = \int_{-\pi}^{\pi} f(x)\overline{g(x)}$$

where

$$\overline{a+bi} = a - bi$$

then

$$C_m = \frac{f \cdot e^{imx}}{e^{imx} \cdot e^{imx}} = \frac{\int_{-\pi}^{\pi} f(x)\overline{e^{imx}}\, dx}{\int_{-\pi}^{\pi} e^{imx}\overline{e^{imx}}\, dx} = \frac{\int_{-\pi}^{\pi} f(x)e^{-imx}\, dx}{\int_{-\pi}^{\pi} e^{imx}e^{-imx}\, dx}$$

or

$$C_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}\, dx$$

Generally,

$$\boxed{\begin{array}{c} \text{If} \quad f(x) = \sum_{-\infty}^{\infty} C_m e^{i\left(\frac{\pi m x}{L}\right)} \quad \text{on } (-L, L): \\ C_m = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\left(\frac{\pi m x}{L}\right)\, dx} \end{array}}$$

Example: Find C_m such that on $(-\pi, \pi)$

$$e^x = \sum_{m=-\infty}^{\infty} C_m e^{imx}$$

$$
\begin{aligned}
C_m &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-imx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-im)x}\,dx \\
&= \frac{1}{2\pi}\left[\frac{e^{(1-im)x}}{1-im}\right]_{-\pi}^{\pi} \\
&= \frac{1}{(2\pi)(2-im)}\left(e^{(1-im)\pi} - e^{(1-im)(-\pi)}\right) \\
&= \frac{1}{(2\pi)(2-im)}\left(e^{\pi}e^{-im\pi} - e^{-\pi}e^{-im\pi}\right)
\end{aligned}
$$

Then notice that

$$
\begin{aligned}
e^{\pi mi} &= \cos(\pi m) + i\sin(\pi m) = (-1)^m \\
e^{-\pi mi} &= \cos(-\pi m) + i\sin(-\pi m) = (-1)^m
\end{aligned}
$$

So

$$C_m = \frac{1}{2\pi(1-im)}\left(e^{\pi}(-1)^m - e^{-\pi}(-1)^m\right) = \frac{(-1)^m}{\pi(1-\pi m)}\left(\frac{e^{\pi}-e^{-\pi}}{2}\right)$$

so

$$C_m = \frac{(-1)^m}{\pi(1-im)}\sinh(\pi)$$

and then

$$\boxed{e^x = \sum_{m=-\infty}^{\infty}\left(\frac{(-1)^m}{\pi(1-im)}\sinh(\pi)\right)e^{imx}}$$

Part II - Norms

Definition: $||u|| = \sqrt{u \cdot u}$ which is the length of u and $||cu|| = |c|||u||$ where c is any constant.

Pythagorean theorem: If u and v are orthogonal then

$$||u+v||^2 = ||u||^2 + ||v||^2$$

Consequence: If $\{u, v, w\}$ is orthogonal then

$$||u+v+w||^2 = ||u||^2 + ||v||^2 + ||w||^2$$

Part III - Parseval's Identity

Apply the Pythagorean theorem to Fourier series: Suppose that on $(0, \pi)$,

$$f(x) = \sum_{m=1}^{\infty} A_m \sin(mx)$$

then since $\{\sin(mx)\}$ is orthogonal

$$||f||^2 = \left|\left|\sum_{m=1}^{\infty} A_m \sin(mx)\right|\right|^2 = \sum_{m=1}^{\infty} ||A_m \sin(mx)||^2 = \sum_{m=1}^{\infty} |A_m|^2 \, ||\sin(mx)||^2$$

but remember that

$$||f||^2 = f \cdot f = \int_0^{|} pi(f(x))^2 \, dx$$

$$||\sin(mx)||^2 = \int_0^{\pi} \sin^2(mx) \, dx = \frac{\pi}{2}$$

So

$$\int_0^{\pi} (f(x))^2 = \frac{\pi}{2} \sum_{m=1}^{\infty} |A_m|^2$$

which gives us Parseval's identity:

$$\text{If} \quad f(x) = \sum_{m=1}^{\infty} A_m \sin(mx) \qquad \text{on } (0, \pi)$$

$$\text{Then} \quad \sum_{m=1}^{\infty} |A_m|^2 = \frac{2}{\pi} \int_0^{\pi} (f(x))^2 \, dx$$

Part IV- Examples

Example 1: $f(x) = x$ on $(0, \pi)$ From an earlier lecture,

$$x = \sum_{m=1}^{\infty} A_m \sin(mx) \qquad A_m = \frac{2(-1)^m}{m}$$

$$\int_0^\pi (f(x))^2 \, dx = \int_0^\pi x^2 \, dx = \frac{\pi^3}{3}$$

$$\sum_{m=1}^{|} infty|A_m|^2 = \sum_{m=1}^{\infty} \left| \frac{2}{m}(-1)^m \right|^2 = \sum_{m=1}^{|} infty \frac{4}{m^2}$$

By Parseval's,

$$4 \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{2}{\pi} \left(\frac{\pi^3}{3} \right) = \frac{2}{3}\pi^2$$

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{1}{4}(\frac{2}{3}\pi^2) = \frac{\pi^2}{6}$$

Example 2: Cosine Series This works exactly the same as sin except for the exception $m = 0$ where

$$|| \cos(0)||^2 = \int_0^\pi \cos^2(0) \, dx = \pi$$

so for

$$f(x) = \sum_{m=0}^{\infty} A_m \cos(mx)$$

on $(0, \pi)$, then Parseval's says

$$2||A_0||^2 + \sum_{m=1}^{\infty} |A_m|^2 = \frac{2}{\pi} \int_0^\pi (f(x))^2 \, dx$$

23 March 22: Laplace Equation (I)

Part I - Complex Parseval

Note that

$$||e^{imx}||^2 = e^{imx} \cdot e^{imx} = \int_{-\pi}^{\pi} e^{imx} e^{-imx} \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi$$

$$f \cdot f = \int_{-\pi}^{\pi} f(x)\overline{f(x)} \, dx = \int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

Therefore Parseval's identity becomes:

$$\boxed{\begin{aligned} &\text{If } f(x) = \sum_{-\infty}^{\infty} C_m e^{imx} in(-\pi, \pi) \text{ then} \\ &\sum_{-\infty}^{\infty} |C_m|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx \end{aligned}}$$

Example: $f(x) = e^x$ on $(-\pi, \pi)$ Before, we found

$$C_m = \frac{1}{\pi} \left(\frac{1}{1 - im} \right) (-1)^m \sinh(\pi)$$

So

$$|C_m|^2 = \left(\frac{1}{\pi^2} \right) \left(\frac{1}{|1 - im|^2} \right) |(-1)^m| |\sinh(\pi)|^2 = \frac{\sinh^2(\pi)}{\pi^2 (1 + m^2)}$$

Then by $|a + bi|^2 = a^2 + b^2$:

$$\sum_{m=-\infty}^{\infty} |C_m|^2 = \sum_{-\infty}^{\infty} \frac{\sinh^2(\pi)}{\pi^2 (m^2 + 1)} = \frac{\sinh^2(\pi)}{\pi^2} \sum_{-\infty}^{\infty} \frac{1}{m^2 + 1}$$

And

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\pi}^{|} pi |e^x|^2 \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2x} \, dx = \frac{1}{2\pi} \sinh(2\pi)$$

Therefore Parseval says,

$$\frac{\sinh^2(\pi)}{\pi} \int_{-\pi}^{\pi} \frac{1}{m^2 + 1} = \frac{1}{2\pi} \sinh(2\pi)$$

$$\int_{-\pi}^{\pi} \frac{1}{m^2 + 1} = \frac{\pi}{2} \left(\frac{\sinh(2\pi)}{\sinh^2(\pi)} \right)$$

Then notice that

$$\sum_{m=1}^{\infty} \frac{1}{m^2 + 1} = \frac{\pi}{4} \left(\frac{\sinh(2\pi)}{\sinh^2(\pi)} \right) - \frac{1}{2}$$

Part II - Convergence of Fourier Series

Question: When does a function f equal its Fourier series $\mathcal{F}$

Fact: There is a (finite) function f for which $\mathcal{F}(x) = \pm\infty$ *everywhere*

But

1) If f is continuous at x then $\mathcal{F}(x) = f(x)$
2) If f has a jump discontinuity at x then

$$\mathcal{F}(x) = \frac{f(x^-) + f(x^+)}{2} \quad \text{(Average of jumps)}$$

Where $f(x^-)$ and $f(x^+)$ are the left and right limits of f at x.

Example: Let $f(x) = x$ on $(-\pi, \pi)$, draw the graph of F(x) on all of $\mathbb{R}$

Notice: $\mathcal{F}(x) = \sum A_m \cos(mx) + B_m \sin(mx)$ is periodic of period 2π so we first need to "repeat" f and then apply the rules above

Aside: There's a whole math subject called Harmonic Analysis just dedicated to the question of convergence of Fourier series, just to show how delicate this question is!

Part III - Laplace Equation

Laplace's equation is

$$\boxed{u_{xx} + u_{yy} = 0}$$

Note: This is sometimes written as $\Delta u = 0$ where $\Delta u = u_{xx} + u_{yy}$ Interpretation: $u(x, y)$ gives you the temperature of a metal plate Ω at (x, y) after a long time

Notice: There is no t in Laplace's equation, so no initial conditions. On the other hand, the boundary conditions are more complicated: you need to specify u on the whole boundary $\partial\Omega$ of the metal plate.

Part IV - Derivation

Setting: The temperature of a metal plate at (x, y) and any time t is given by the 2D heat equation, which is

$$u_t = D(u_{xx} + u_{yy})$$

Where$u = u(x, y, t)$

Derivation: See homework

After a long time, we assume u is becomes constant in t and so $u_t = 0$, Hence

$$0 = D(u_{xx} + u_{yy}) \implies u_{xx} + u_{yy} = 0$$

Part V - Applications

1. Physics - $u(x, y)$ is the temperature of a metal plate after a long time
2. Image processing - can convert a pixelated image into a smooth one
3. Music - Solutions are called harmonics

   Suppose you have a region Ω, think the surface of a drum and consider the eigenvalue problem: "For which λ does the following have a nonzero solution?"

   $$\begin{cases} -\Delta u = \lambda u \in \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

   Fact: There is an infinite sequence of eigenvalues λ_n with

   $$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ...$$

   For which the above has a nonzero solution. Then λ_1 is the *principal harmonic* and the others are *overtones*

24 March 24: Laplace Equation (II)

Part I - Application 3: Music

$$\begin{cases} -\Delta u = \lambda u & \in \Omega \\ u = 0 & \partial\Omega \end{cases}$$

Where Ω is some domain (for example a drum head) and $\partial\Omega$ is the boundary.

Note: The negative sign is to get positive eigenvalues

Fact: There is an infinite sequence of eigenvalues which give a nonzero solution

Famous question: "Can you hear the shape of a drum?" i.e. Given the eigenvalues can you determine Ω? *Answer:* Yes in 2D is Ω is smooth

Part II - "OMG Application:" Brownian Motion

Suppose that you take a Brownian motion random walk on (x, y) until you hit a boundary point (x^*, y^*) and pay a gain/loss $g(x^*, y^*)$ Let $u(x, y)$ be the expected value of the gain/loss starting at (x, y)

Then, *that expected value will solve*

$$\begin{cases} \Delta u = 0 & \in \Omega \\ u = g & \in \partial\Omega \end{cases}$$

Insane Consequence Suppose $g = 0$ everywhere except $g(x^*, y^*) > 0$ at some point (x^*, y^*) on the boundary (say some far away treasure).

Then by infinite speed of propagation, $u(x, y) > 0$ everywhere.

Note that because u is an expected value, this means that no matter how far away we are, there is always a positive chance of hitting (x^*, y^*)

Part III - Application 5: the Heat Equation

Consider the 1-dimension domain of the boundary. Start at point x at time t, do Brownian motion, stop at point x^* at time T, gain $g(x^*)$

Consider the average value $u(x, t)$ of the gain/loss function starting at (x, t). **Fact:** u solves

$$\begin{cases} u_t = u_x \\ u(x, T) = g(x) \end{cases}$$

Note: this is interesting because there is no guarantee that a terminal value problem has a solution.

Consequence: If $g(x^*)$ is positive somewhere than $u(x, t)$ is positive everywhere. This implies that it's always possible to reach any x^* (no matter how far) at any time T (no matter how small) no matter where you start

Part IV - Rotation Invariance

Goal: find the "fundamental solution" of Laplace's equation

Setting: Suppose $u(x, y)$ solves

$$u_{xx} + u_{yy} = 0$$

in $\mathbb{R}^2$ Notice that u is invariant under rotations

More precisely, let θ be constant and let

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So

$$\begin{cases} x' = \cos(\theta)x - \sin(\theta)y \\ y' = \sin(\theta)x + \cos(\theta)y \end{cases}$$

Fact:

$$u_{x'x'} + u_{y'y'} = u_{xx} + u_{yy} = 0$$

Proof:

$$\begin{aligned} u_x &= u_{x'} \cdot x'_x + u_{y'} \cdot y'_x \\ &= (u_{x'})\cos(\theta) + (u_{y'})\sin(\theta) \\ u_{xx} &= (u_x)_{x'} \cdot x'_x + (u_x)_{y'} \cdot y'_x \\ &= u_{x'x'}\cos^2(\theta) + 2u_{x'y'}\sin(\theta)\cos(\theta) + u_{y'y'}\sin^2(\theta) \end{aligned}$$

25 April 3: Laplace's Equation (III)

Part I - Review

The Laplace Equation in $\mathbb{R}^2$:

$$u_{xx} + u_{yy} = 0$$

Important Fact: u is invariant under rotations. This means that we can use the coordinate method in polar coordinates to get nice results.

Part II - Polar Coordinates

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \implies \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(\frac{y}{x}) \end{cases}$$

Goal: write $u_{xx} + u_{yy} = 0$ in terms of r and θ *Step 1:* Prep work

$$\frac{\partial r}{\partial x} = (\sqrt{x^2 + y^2})_x = \frac{x}{\sqrt{x^2 + y^2}} = \cos(\theta)$$

Similarly,

$$\frac{\partial r}{\partial y} = \sin\theta$$
$$\frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{r}$$
$$\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

Step 2: A very long coordinate method problem

$$\begin{aligned}
u_x &= u_r \cdot r_x + u_\theta \cdot \theta_x = u_r \cos\theta + u_\theta(-\frac{\sin\theta}{r}) \\
u_{xx} &= (u_x)_r \cdot r_x + (u_x)_\theta \cdot \theta_x \\
&= \left(u_r \cos\theta - u_\theta(\frac{\sin\theta}{r})\right)_r \cos(\theta) + \left(u_r\cos\theta - u_\theta\left(\frac{\sin\theta}{r}\right)\right)_\theta \left(-\frac{\sin\theta}{r}\right) \\
&= \left(u_{rr}\cos(\theta) - u_{\theta r}\left(\frac{\sin\theta}{r}\right) + u_\theta\left(\frac{\sin\theta}{r^2}\right)\right)\cos(\theta) \\
&+ \left(u_{r\theta}\cos\theta - u_r\sin\theta - u_{\theta\theta}\frac{\sin\theta}{r} - u_\theta\frac{\cos\theta}{r}\right)\left(\frac{-\sin\theta}{r}\right) \\
&= u_{rr}\cos^2\theta - u_{r\theta}\frac{\sin\theta\cos\theta}{r} + u_\theta\frac{\sin\theta\cos\theta}{r^2} - u_{r\theta}\frac{\cos\theta\sin\theta}{r} + u_r\frac{\sin^2\theta}{r} + u_{\theta\theta}\frac{\sin^2\theta}{r^2} + u_\theta\frac{\cos\theta\sin\theta}{r^2} \\
&= u_{rr}\cos^2\theta - 2u_{r\theta}\frac{\sin\theta\cos\theta}{r} + 2u_\theta\frac{\sin\theta\cos\theta}{r^2} + u_r\frac{\sin^2\theta}{r} + u_{\theta\theta}\frac{\sin^2\theta}{r^2}
\end{aligned}$$

Similarly,

$$u_{yy} = u_{rr}\sin^2\theta + 2u_{r\theta}\frac{\cos\theta\sin\theta}{r} - 2u_\theta\frac{\cos\theta\sin\theta}{r^2} + u_r\frac{\cos^2\theta}{r} + u_{\theta\theta}\frac{\cos^2\theta}{r^2}$$

Step 3: Combine

$$u_{xx} + u_{yy} = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}$$

So at long last, the **Polar Laplace** is:

$$\boxed{u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0}$$

Part II - Fundamental Solution

Because of rotational invariance, we can look for **radial solutions** (solutions that only depend on r) Note that there are many non-radial solutions but this gives a place to start.

If u is radial, then

$$u_{rr} + \frac{1}{r}u_r = 0$$

By integrating factors,

$$\exp(\int \frac{1}{r}\, dr) = e^{\ln r} = r$$

so

$$ru_{rr} + u_r = 0 \implies (ru_r)_r = 0$$

$$ru_r = A \implies u_r = \frac{A}{r}$$

$$u = \int \frac{A}{r}\, dr = A\ln r + B$$

Therefore, in 2 dimensions,

$$\boxed{u(x,y) = A\ln(\sqrt{x^2+y^2}) + B}$$

is a solution to laplace's equation.

Definition: the *Fundamental Solution* of $u_{xx} + u_{yy} = 0$ is

$$\boxed{\Phi(x,y) = -\frac{1}{2\pi}\ln(\sqrt{x^2+y^2})}$$

26 April 5: Laplace's Equation (IV)

Part I - Setting

Example 1:

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = 0 \\ u(\pi,y) = 0 \\ u(x,0) = x \\ u(x,1) = 3 \end{cases}$$

This gives us a rectangle of width π and height 1 with left and right edges at temperature 0, bottom edge at temperature x, and top edge at temperature 3.

Part II - Separation of Variables

Assume that u is of the form $u(x, y) = X(x)Y(y)$. So

$$u_{xx} + u_{yy} = X''Y + XY'' = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y}$$

Then note that as neither side of the equation depends on the other variable, both sides are constant and equal to some value λ. Cross multiplying,

$$\begin{cases} X'' = \lambda X \\ Y'' = -\lambda Y \end{cases}$$

Because X yields the zero boundary condition, we start there.

Looking at the boundary conditions,

$$u(0, y) = 0 \implies X(0) = 0$$

$$u(\pi, y) = 0 \implies X(\pi) = 0$$

This gives the ODE

$$\begin{cases} X'' = \lambda X \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

which has eigenfunction

$$X(xi) = \sin(mx) \quad (m = 1, 2, ...)$$

corresponding to eigenvalues $\lambda = -m^2 \quad (m = 1, 2, ...)$

Then for the Y equation,

$$\begin{aligned} Y''(y) &= -\lambda Y(y) \\ &= -(-m^2)Y \\ &= m^2 Y \end{aligned}$$

$$\implies Y = Ae^{my} + Be^{-my}$$

Analogy: For the wave equation,

$$T = A\cos(mct) + B\sin(mct)$$

so at $T(0) = A$

Recall:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\implies \begin{cases} e^x = \cosh(x) + \sinh(x) \\ e^{-x} = \cosh(x) - \sinh(x) \end{cases}$$

So

$$\begin{aligned} Y = Ae^{my} + Be^{-my} & \\ &= A(\cosh(my) + \sinh(my)) + B(\cosh(-my) + \sinh(-my)) \\ &= A\cosh(my) + A\sinh(my) + B\cosh(-my) + B\sinh(-my) \\ &= (A+B)\cosh(my) + (A-B)\sinh(my) \\ &= A\cosh(my) + b\sinh(my) \end{aligned}$$

and just like in the analogy, $Y(0) = A$!

Conclusion: For every $m = 1, 2, ...$

$$u(x, y) = X(x)Y(y) = (A\cosh(my) + B\sinh(my))\sin(mx)$$

26.1 Part III - Initial conditions

$$u(x, y) = \sum_{m=1}^{\infty} [A_m \cosh(my) + B_m \sinh(my)] \sin(mx)$$

For $u(x, 0) = x$,

$$u(x, 0) = \sum_{m=1}^{\infty} [A_m \underbrace{\cosh(0)}_{1} + B_m \underbrace{\sinh(0)}_{0}] \sin(mx)$$

$$x = \sum_{m=1}^{\infty} A_m \sin(mx) \quad (0 < x < \pi)$$

Using fourier series,

$$A_m = \frac{2}{\pi} \int_0^{\pi} x \sin(mx)\, dx = \frac{2}{m}(-1)^m$$

And for $u(x, 1) = 3$

$$u(x,1) = \sum_{m=1}^{\infty} \left[\frac{2}{m}(-1)^m \cosh(m) + B_m \sinh(m) \right] \sin(mx) = 3$$

Make the substitution

$$\tilde{B}_m = A_m \cosh(m) + B_m \sinh(m)$$

so

$$3 = \sum_{m=1}^{\infty} \tilde{B}_m \sin(mx)$$

and

$$\tilde{B}_m = \frac{2}{\pi} \int_0^{\pi} 3 \sin(mx)\, dx = \frac{2}{\pi} \left[\frac{-3\cos(mx)}{m} \right]_0^{\pi} = \frac{6}{\pi m}[(-1)^{m+1} + 1]$$

so from $\tilde{B}_m$,

$$A_m \cosh(m) + B_m \sinh(m) = \frac{6}{\pi m}[(-1)^{m+1} + 1]$$

$$B_m = \frac{\frac{6}{\pi m}[(-1)^{m+1} + 1] - (\frac{2}{m})(-1)^m \cosh(m)}{\sinh(m)}$$

Part IV - Conclusion

$$\boxed{u(x,y) = \sum_{m=1}^{\infty} \left[(\frac{2}{m})(-1)^m \cosh(my) + \frac{\frac{6}{\pi m}[(-1)^{m+1} + 1] - (\frac{2}{m})(-1)^m \cosh(m)}{\sinh(m)} \sinh(my) \right] \sin(mx)}$$

Part V - Variation

If one of the boundary conditions were not zero, you would not be able to solve the equation.

Trick: $u = v + w$ where v solves the situation where the left and right are 0 and where w solves the situation where the top and bottom are 0.

27 April 7: Laplace's Equation (V)

Fundamental Solution

Recall: Polar Laplace

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Searching for the radial solution gives the general solution

$$\boxed{u(x,y) = A\ln(\sqrt{x^2+y^2}) + B}$$

This also gives the *fundamental solution*

$$\Phi(x,y) = -\frac{1}{2\pi}\ln(\sqrt{x^2+y^2})$$

Note: $-1/2\pi$ is chosen so that $\Delta\Phi = -\delta(0,0)$ (the Dirac at 0)

Fact: A solution to Poisson's equation

$$u_{xx} + u_{yy} = -f(x,y)$$

is

$$\boxed{u(x,y) = \Phi * f = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(x-s, y-t)\, f(s,t)\, ds\, dt}$$

Part II - Two Dimensional Laplace

Suppose $u(x,y,z)$ solves $u_{xx} + u_{yy} + u_{zz} = 0$ Using spherical coordinates, the radial component is

$$u_{\rho\rho} + \frac{2}{\rho}u_\rho + \ \dots\ = 0$$

where $\rho = \sqrt{x^2+y^2+z^2}$ and the ellipses obscure terms that don't depend on ρ.

Solving for radial solutions gives

$$u = \frac{A}{\rho} + B$$

Fact: Laplace's Equation in 3 dimensions is solved by

$$\boxed{u(x,y,z) = \frac{A}{\sqrt{x^2+y^2+z^2}} + B}$$

Note: The fundamental n-dimensional solution is of the form

$$\Phi(x_1, ..., x_n) = \frac{C_n}{r^{n-2}}$$

Part III - Properties of Laplace's Equation

Property 1) Mean-Value Formula

Recall: The average value of f from a to b is

$$⨍ f(x)\,dx = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Notice: $b - a = |[a,b]| =$ the size/length of $[a,b]$

Definition: The average value of f on Ω is

$$⨍_\Omega f(x)\,dx$$

where $|\Omega|$ is the area/volume of Ω.

Notation: $B(x,r)$ is a ball centered at x with radius r.

Mean Value Formula: If $\Delta u = 0$ then for every x and every $r > 0$ we have

$$\boxed{⨍_{B(x,r)} u(y)\,dy = \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\,dy = u(x)}$$

Interpretation: The average value of u over the bal l is just the value at the center!

Note: Solutions to $\Delta u = 0$ are called *isotropic* (same from every direction)

Consequences:

1. Solutions to $\Delta u = 0$ are infinitely differentiable

   Proof: $u = ⨍_{B(x,r)} u(y)\,dy \longrightarrow$ one level smoother (because of integral)

2. *Liouville's Theorem:* If $\Delta u = 0$ and u is bounded ($|u| \leq c$) then u must be constant.

   Corollary: If u is not constant then it must blow up somewhere

27.0.1 Property 2) Strong Max Principle

Question: where is a metal plate hottest/coldest?

Strong Max Principle: If $\Delta u = 0$ in Ω, then $\max u$ and $\min u$ are attained on $\partial\Omega$ and ONLY on $\partial\Omega$ (except if u is constant)

Proof: Suppose u has a max M at some point x in Ω. But by mean value formula we get

$$\fint_{B(x,r)} u(y)\, dy = u(x) = M$$

(the highest value is equal to the average value so u is constant)

28 April 10: Midterm 2 Review

Part I - Separation of Variables

Example 1:

$$\begin{cases} tu_t = u_{xx} - u \\ u(0,t) = 0 \\ u(\pi, t) = 0 \\ u(x,1) = 1 \end{cases}$$

Step 1: Separation of Variables

$$\begin{gathered} u(x,t) = X(x)T(t) \\ tXT' = X''T - XT \\ \frac{tT'}{T} = \frac{X''}{X} - 1 \\ \frac{tT'}{T} + 1 = \frac{X''}{X} = \lambda \end{gathered}$$

Step 2: Boundary Value

$$u(0,t) = X(0)T(t) = 0 \implies X(0) = 0$$
$$u(\pi,t) = 0 \implies X(\pi) = 0X'' = \lambda X$$

(skipping the three cases)

$$\{X(x) = \sin(mx),\ \lambda = -m^2 \quad |m = 1, 2, ...\}$$

Step 3: T Equation

$$t\frac{T'}{T} = (-m^2 - 1)$$
$$\frac{T'}{T} = -\frac{m^2+1}{t}$$
$$(\ln|T|)' = -\frac{m^2+1}{t}$$
$$\ln|T| = -(m^2+1)\ln|t| + C$$
$$|T| = t^{-(m^2+1)}e^C$$
$$T(t) = Ct^{-(m^2+1)}$$

Step 4: Final u

$$u = XT$$
$$u(x,t) = Ct^{-(m^2+1)}\sin(mx)$$
$$u(x,t) = \sum_{m=1}^{\infty} A_m t^{-(m^2+1)}\sin(mx)$$
$$u(x,1) = \sum_{m=1}^{\infty} A_m \sin(mx) = 1$$

$$\begin{aligned} A_m &= \frac{2}{\pi}\int_0^{\pi} \sin(mx)\ dx \\ &= \frac{2}{\pi}\left[-\frac{\cos(mx)}{m}\right]_0^{\pi} \\ &= \frac{2}{\pi m}\left[-\cos(\pi m) + 1\right] \\ &= \frac{2}{\pi m}[(-1)^{m+1} + 1] \end{aligned}$$

$$\boxed{u(x,t) = \sum_{m=1}^{\infty} \frac{2}{\pi m} \left[(-1)^{m+1} + 1\right] t^{-(m^2+1)} \sin(mx)}$$

Part II - Parseval

Ex 2: Derive Parseval's identity om $(0, \pi)$ for

$$x = \sum_{m=0}^{\infty} A_m \sin\left(\left(\frac{2m+1}{2}\right) x\right)$$

using only the Orthogonality of the sine functions. Use this to calculate

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^4} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \ ...$$

Taking the norms:

$$\begin{aligned}
||x||^2 &= \left|\left|\sum_{m=0}^{\infty} A_m \sin\left(\left(\frac{2m+1}{2}\right) x\right)\right|\right|^2 \\
&= \sum_{m=0}^{\infty} \left|\left| A_m \sin\left(\left(\frac{2m+1}{2}\right) x\right)\right|\right|^2 \\
&= \sum_{m=0}^{\infty} |A_m|^2 \left|\left| \sin\left(\left(\frac{2m+1}{2}\right) x\right)\right|\right|^2
\end{aligned}$$

LHS:

$$||x||^2 = \int_0^{\pi} x^2 \ dx = \frac{\pi^3}{3}$$

RHS:

$$\begin{aligned}
\left\|\sin\left(\left(\frac{2m+1}{2}\right)x\right)\right\|^2 &= \int_0^\pi \sin^2\left(\left(\frac{2m+1}{2}\right)x\right)\,dx \\
&= \int_0^\pi \frac{1}{2} - \frac{1}{2}\cos((2m+1)x)\,dx \\
&= \frac{\pi}{2} - \frac{1}{2}\int_0^\pi \cos((2m+1)x)\,dx \\
&= \frac{\pi}{2} - \frac{1}{2}\left[\frac{\sin(2m+1)x}{2m+1}x\right]_0^\pi \\
&= \frac{\pi}{2}
\end{aligned}$$

Putting together,

$$\frac{\pi^3}{3} = \sum_{m=0}^{\infty} |A_m|^2 \frac{\pi}{2}$$

$$\sum_{m=0}^{\infty} |A_m|^2 = \frac{2\pi^2}{3}$$

Calculating the coefficients,

$$\begin{aligned}
A_m &= \frac{x \cdot \sin\left(\left(\frac{2m+1}{2}\right)x\right)}{\sin\left(\left(\frac{2m+1}{2}\right)x\right)\cdot \sin\left(\left(\frac{2m+1}{2}\right)x\right)} \\
&= \frac{2}{\pi}\left[\frac{\pi\cos(2m+1)}{2m+1} - \frac{\sin(2m+1)}{(2m+1)^2}\right] \\
&\overset{IBP}{=} \frac{8}{\pi}\frac{1}{(2m+1)^2}(-1)^m
\end{aligned}$$

Then,

$$\sum_{m=0}^{\infty} |A_m|^2 = \frac{2\pi^2}{3}$$

$$\sum_{m=0}^{\infty} \frac{64}{\pi^2} \frac{1}{(2m+1)^4} |(-1)^m|^2 = \frac{2\pi^2}{3}$$

$$\frac{64}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^4} = \frac{2\pi^2}{3}$$

$$\boxed{\sum_{m=0}^{\infty} \frac{1}{(2m+1)^4} = \frac{\pi^4}{96}}$$

Part III - Energy Method + Laplace

Ex 3: Use energy methods to show that if

$$\begin{cases} \Delta u = 0 & \in \Omega \\ u = 0 & \in \partial\Omega \end{cases}$$

Then $u = 0$ for all x and t.

Hint: use that for all v with $v = 0$ on $\partial\Omega$,

$$\int_{\Omega} (\Delta u) v \, dx = -\int_{\Omega} (\nabla u) \cdot (\nabla v) \, dx$$

Starting with energy methods,

$$\Delta u = 0$$

$$(\Delta u) u = 0$$

$$\int_{\Omega} (\Delta u) u \, dx = 0$$

$$-\int_{\Omega} (\nabla u) \cdot (\nabla u) \, dx = 0$$

$$-\int_{\Omega} \underbrace{||\nabla u||^2}_{\geq 0} \, dx = 0$$

So

$$||\nabla u||^2 = 0 \implies \nabla u = 0 \implies u = C \implies u = 0 \quad \blacksquare$$

29 April 14: Calculus of Variations (I)

Part I: Laplace Equation Property 2

Recall: *Strong Maximum Principle* If $\Delta u = 0$ in Ω then $\max u$ is attained on $\partial\Omega$ and only on $\partial\Omega$ except in the trivial constant case. Same with $\min u$

Consequence: Uniqueness

Suppose u and v both solve

$$\begin{cases} \Delta u = f & \in \Omega \\ u = g & \in \partial\Omega \end{cases}$$

Then $u = v$

Proof: Let $w = u - v$. then w solves

$$\begin{cases} \Delta w = \Delta u - \Delta v = f - f = 0 \quad \in \Omega \\ w = g - g = 0 \quad \in \partial\Omega \end{cases}$$

$$\begin{cases} \Delta w = 0 & \in \Omega \\ w = 0 & \in \partial\Omega \end{cases}$$

But then

$$\max_{\Omega} w = \max_{\partial\Omega} w = 0$$

$$\min_{\Omega} w = \min_{\partial\Omega} w = 0$$

SO $w = 0 \implies u - v = 0 \implies u = v \quad \blacksquare$

Property 3: Positivity

Suppose u solves

$$\begin{cases} \Delta u = 0 & \in \Omega \\ u = g & \in \partial\Omega \end{cases}$$

When $g \geq 0$ and $g(x_0) > 0$ for some x_0 on $\partial\Omega$

Then $u > 0$ on all of Ω

Proof:

$$\min_{\Omega} w = \min_{\partial\Omega} w = \min_{\partial\Omega} g \geq 0$$

so $u \geq 0 \quad \in \Omega$. Now suppose $u(x) = 0$ for some x in Ω. But this is a contradiction because this new minimum will not be on $\partial\Omega$. Hence $u > 0 \quad \in \Omega$

Part III - Calculus of Variations

The Calculus of variation helps turn minimization problems into differential equations and vice verse.

Goal: Find a function $f(x)$ such that

$$I[f] = \frac{1}{2}\int_0^1 (f'(x))^2 \, dx$$

is smallest among all functions f on $[0,1]$ such that $f(0) = 0$ and $f(1) = 1$.

Note that this is similar to a kinetic energy $\frac{1}{2}mv^2$

Example: $f(x) = x^3$ gives

$$I[f] = \frac{1}{2}\int_0^1 (3x^2)^2 \, dx = \frac{1}{2}\int_0^1 9x^4 \, dx = \frac{9}{10}$$

Part IV - A useful trick

Fact: If

$$\int_0^1 f(x)\, g(x) \, dx = 0$$

for all g with $g(0) = g(1) = 0$ then $f = 0$ (when f and g are continuous).

Proof:

Suppose $f(x_0) \neq 0$ for some x_0 in $(0,1)$ "WLOG" $f(x_0) > 0$. Since f is continuous, we have $f > 0$ on some interval $(x_0 - r, x_0 + r)$. Let g be any continuous function such that $g > 0$ on $(x_0 - r, x_0 + r)$ and $g = 0$ outside that interval. On the one hand, by assumption, we have

$$\int_0^1 f(x)\, g(x) \, dx = 0$$

On the other hand, since $g = 0$ outside $(x_0 - r, x_0 + r)$ we get

$$\int_0^1 f(x)\, g(x) \, dx = \int_{x_0 - r}^{x_0 + r} \underbrace{f(x)}_{>0} \, \underbrace{g(x)}_{>0} \, dx > 0$$

Which is a contradiction, so $f = 0$ on $(0,1)$ and therefore also on $[0,1]$ by continuity
■

Part V - Euler-Lagrange Equation

Back to the minimization problem:

$$\text{Suppose f minimizes } I[f] = \tfrac{1}{2}\int_0^1 (f'(x))^2 \, dx$$

Let g be arbitrary with $g(0) = g(1) = 0$

Step 1: Given f, g and $t \in \mathbb{R}$, consider

$$h(t) = I[f + tg] = \frac{1}{2}\int_0^1 (f'(x) + tg'(x))^2 \, dx$$

Notice in particular that $h(0) = I[f]$ so h has a min at $t = 0$

In particular, from calculus, we have $h'(0) = 0$

Note: It's precisely here that we move from minimization to a differential equation

Step 2:

$$\begin{aligned}
h'(t) &= \frac{d}{dt}\left[\frac{1}{2}\int_0^1 (f'(x) + tg'(x))^2 \, dx\right] \\
&= \frac{1}{2}\int_0^1 \frac{d}{dt}(f'(x) + tg'(x))^2 \, dx \\
&= \frac{1}{2}\int_0^1 2(f'(x) + tg'(x))g'(x) \, dx \\
&= \int_0^1 (f' + tg')g' \, dx \\
&= [(f'(x) + tg'(x))g(x)]_{x=0}^{x=1} - \int_0^1 (f' + tg')'g \, dx \\
&= -\int_0^1 (f'' + tg'')g \, dx \quad (g(0) = g(1) = 0)
\end{aligned}$$

Step 3:

Setting $t = 0$ we get

$$h'(0) = -\int_0^1 (g'' + 0g'')g \, dx = -\int_0^1 f''g \, dx$$

Since $h'(0) = 0$ by assumption, we have

$$\int_0^1 (-f''(x))g(x)\,dx = 0 \quad \forall g$$

Since g was arbitrary, by the fact above we get

$$-f''(x) = 0$$

Step 4:

Solving, we get

$$f(x) = Ax + B$$

Using $f(0) = 0$ and $f(1) = 1$ we get $A = 1$ and $B = 0$ so

$$f(x) = x$$

Notice how we were able to solve this minimization problem by turning it into an ODE. This ODE is so important it has its own special name: **The Euler-Lagrange Equation:**

$$\boxed{\min I[f] = \frac{1}{2}\int_0^1 (f')^2\,dx \Longrightarrow -f'' = 0}$$

More generally suppose the Lagrangian $L = L(p, z, x)$ is given. Then

$$\boxed{\min I[f] = \int_a^b L(f', f, x)\,dx \Longrightarrow -(L_p(f', f, x))_x + L_z(f', f, x) = 0}$$

Part VI - Higher Dimensions

In higher dimensions the Euler-Lagrange becomes a PDE!

Two Dimensional Case: Suppose $L = L(p, q, z, x, y)$ is given and consider

$$\min I[u] = \int_\Omega L(u_x, u_y, u, x, y)\,dx\,dy$$

with $u = g$ on $\partial\Omega$.

Fact: The Euler-Lagrange Equation in that case is

$$-(L_p)_x - (L_q)_y + L_z = 0$$

(evaluated at (u_x, u_y, u, x, y) with $u = g$ on $\partial\Omega$)

Example 1: Dirichlet Energy

$$\min I[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx\, dy = \frac{1}{2}\int_\Omega (u_x)^2 + (u_y)^2 \, dx\, dy$$

This is the analog of $\int (f')^2 \, dx$ from before. In that case,

$$L(p, q, u, x, y) = \frac{1}{2}(p)^2 + \frac{1}{2}(q)^2$$

So $L_p = p$ and $L_q = q$ and $L_z = 0$ and in that case the E-L equation becomes

$$-(u_x)_x - (u_y)_y + 0 = 0 \implies -(u_{xx} + u_{yy}) = 0$$

which is Laplace's Equation!!

More precisely,

$$\begin{cases} -\Delta u = 0 & \in \Omega \\ u = g & \in \partial\Omega \end{cases}$$

Example 2:

$$\min I[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 - fu \, dx\, dy$$

Then the E-L equation becomes

$$\begin{cases} -\Delta u = f & \in \Omega \\ u = g & \in \partial\Omega \end{cases}$$

Which is Poisson's!

Example 3: Minimal Surface Equation

$$\min I[u] = \int_\Omega \sqrt{1 + |\nabla u|^2} \, dx\, dy = \int_\Omega \sqrt{1 + (u_x)^2 + (u_y)^2} \, dx\, dy$$

In that case E-L equation becomes

$$-\left(\frac{u_x}{\sqrt{1 + (\nabla u)^2}}\right)_x - \left(\frac{u_y}{\sqrt{1 + (\nabla u)^2}}\right)_y = 0$$

This is very interesting because in practice it is hard to solve the PDE but easier to solve the minimization problem! Because in fact

$$\int_\Omega \sqrt{1+|\nabla u|^2}\, dx\, dy = \text{Surface area of f over } \Omega$$

So the minimization question just becomes: Among all the graphs u with $u = g$ on $\partial\Omega$, which is the one with smallest surface area?

Application: This is interestingly related to how soap bubbles and soap films behave!

30 April 17: Calculus of Variations (II)

Part I - Calculus of Variations

Goal: Find a function f such that

$$I[f] = \frac{1}{2}\int_0^1 (f'(x))^2\, dx$$

is smallest among all functions f on $[0,1]$ such that $f(0) = 0$ and $f(1) = 1$.

Part II - Euler-Lagrange Equation

Suppose f minimizes

$$I[f] = \frac{1}{2}\int_0^1 (f'(x))^2\, dx$$

Let f be arbitrary with $g(0) = g(1) = 0$.

Step 1: Given f, g and $t \in \mathbb{R}$, consider

$$h(t) = I[f+tg] = \frac{1}{2}\int_0^1 (f'(x) + tg'(x))^2\, dx$$

Note that $h(0) = I[f]$ so h has a min at $t = 0$.

From calculus $h'(0) = 0$

Step 2:

$$
\begin{aligned}
h'(t) &= \frac{d}{dt}\left[\frac{1}{2}\int_0^1 (f'(x)+tg'(x))^2\,dx\right] \\
&= \frac{1}{2}\int_0^1 \frac{d}{dt}(f'(x)+tg'(x))^2\,dx \\
&= \frac{1}{2}\int_0^1 2(f'(x)+tg'(x))g'(x)\,dx \\
&= \int_0^1 (f'+tg')g'\,dx \\
&= [(f'+tg')g]_{x=0}^{x=1} - \int_0^1 (g'+tg')'g\,dx \\
&= -\int_0^1 (f''+tg'')g\,dx
\end{aligned}
$$

Step 3: Setting $t=0$ we get

$$
h'(0) = -\int_0^1 (f''+0g'')g\,dx = -\int_0^1 f''g\,dx
$$

Since $h'(0)=0$ by assumption, we have

$$
\int_0^1 (-f''(X))g(x)\,dx = 0 \quad \forall g
$$

Since g was arbitrary, by a useful fact,

$$
-f''(x) = 0
$$

Step 3: Solving, we get

$$
f(x) = Ax + B
$$

Then using $f(0)=0$ and $f(1)=1$ we get $A=1$ and $B=0$ so

$$
\boxed{f(x) = x}
$$

This ODE has its own name. **Definition:** The *Euler-Lagrange Equation* associated to the min problem

$$
\min I[f] = \frac{1}{2}\int_0^1 (f')^2\,dx \implies -f'' = 0
$$

More generally suppose the Lagrangian $L = L(p, z, x)$ is given then

Fact:

$$\min I[f] = \int_a^b L(f', f, x)\, dx \Longrightarrow -(L_p(f', f, x))_x + L_z(f', f, x) = 0$$

31 April 19: Ecology Application

31.1 Part I - Calculus of Variations Redux

Example: Minimal surface equation

$$\min I[u] = \int_\Omega \sqrt{1 + ||\nabla u||^2}\, dx\, dy = \int_\Omega \sqrt{1 + (u_x)^2 + (u_y)^2}\, dx\, dy$$

So $L(p, q, z, x, y) = \sqrt{1 + p^2 + q^2}$ and the Euler Lagrange equation is

$$-\left(\frac{u_x}{\sqrt{1 + ||\nabla u||^2}}\right)_x - \left(\frac{u_y}{\sqrt{1 + ||\nabla u||^2}}\right)_y = 0$$

This is insanely difficult to solve but it is much easier to find $\min I[u]$ because $I[u]$ is nothing other than the surface area of the graph of u.

Application: this equation actually describes how soap films behave

Part II - Ecology Application

Imagine an ecosystem of bunnies and foxes. Let $u(x, t)$ be the density of bunnies at position x and time t. Let $v(x, t)$ be the density of foxes.

Model 1: Diffusion model

$$\begin{cases} u_t = u_{xx} \\ v_t = 50 v_{xx} \end{cases}$$

(the species move left and right but the foxes move much faster)

If u solves the heat equation, $u(x, t) \to 0$ as $t \to \infty$, so even with no interactions both populations die out.

Note: This is a bad model because they don't interact!

Model 2: Reaction model ODE model: $u = u(t)$ and $v = v(t)$

$$\begin{cases} u_t = u - v & \text{(reproduction and eaten by foxes)} \\ v_t = 8u - 5v & \text{(eat bunnies \& compete)} \end{cases}$$

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 8 & -5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

A has eigenvalues/vectors $\lambda = -1 \longrightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\lambda = -3 \longrightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ so

$$\begin{bmatrix} u \\ v \end{bmatrix} = C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

So as $t \to \infty$, the species also die out!

Note: this is a bad model because it doesn't take into account x at all.

Model 3: Reaction-Diffusion Model (invented by Alan Turing)

$$\begin{cases} u_t = u_{xx} + u - v \\ v_t = \underbrace{50 v_{xx}}_{\text{diffusion}} + \underbrace{8u - 5v}_{\text{reaction}} \end{cases}$$

First guess: Assume

$$\begin{bmatrix} u \\ v \end{bmatrix} = e^{\lambda t} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \implies \begin{cases} u(x,t) = u_0 e^{\lambda t} \\ v(x,t) = v_0 e^{\lambda t} \end{cases}$$

Plugging this into the PDE, we get

$$\begin{cases} \lambda u_0 = u_0 - v_0 \\ \lambda v_0 = 8u_0 - 5v_0 \end{cases} \implies \begin{bmatrix} 1 & -1 \\ 8 & -5 \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \lambda \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

Which is really just an eigenvalue problem!

So λ is an eigenvalue of $A = \begin{bmatrix} 1 & -1 \\ 8 & -5 \end{bmatrix}$ which gives us the same solution as before:

$$\begin{bmatrix} u \\ v \end{bmatrix} = C_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

But this is no better than model two.

Better guess:

$$\begin{bmatrix} u \\ v \end{bmatrix} = e^{\lambda t}\cos(kx)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \implies \begin{cases} u = e^{\lambda t}\cos(kx)u_0 \\ v = e^{\lambda t}\cos(kx)v_0 \end{cases}$$

Plugging into PDE,

$$\lambda \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \left(\begin{bmatrix} -k^2 & 0 \\ 0 & -50k^2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 8 & -5 \end{bmatrix} \right) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

Then with $k = \frac{1}{2}$ we have the coefficient matrix become $B = \begin{bmatrix} 3/4 & -1 \\ 8 & -35/2 \end{bmatrix}$ which has a positive eigenvalue $\lambda \approx 0.3$! and a solution

$$e^{\lambda t}\cos(kx)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = e^{0.3t}\cos(0.5x)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

which grows exponentially! So the populations blow up.

32 April 21: Covid Model

Part I - Setting

- There are two types of people: healthy/susceptible (S) and infected (I)
- There is no recovery; if you are infected, you die with rate γ
- An infected person infects a fraction of b people per day
- Healthy people stay in their own place (no diffusion)
- Infected people spread/diffuse randomly with constant D

Unknowns: $S(x,t)$ is the number of susceptible people at position (x,t)

$I(x,t)$ is the number of infected people at position (x,t)

Part II - The PDE

The more people that are infected, the higher the turnover rate (S to I) so the rate depends on I as well as the infection rate b.

The rate of death does not depend on the number of infected people so it is constant.

Thus:

$$\boxed{S} \xrightarrow{bI} \boxed{I} \xrightarrow{\gamma}$$

Which gives us:

$$\begin{aligned} \frac{dS}{dt} &= \text{In} - \text{Out} = 0 - (bI)S = -bIS \\ \frac{dI}{dt} &= \text{In} - \text{Out} + \text{Diffusion} = (bI)S - \gamma I + DI_{xx} \end{aligned}$$

Or

$$\begin{cases} S_t = -bSI \\ I_t = bSI - \gamma I + DI_{xx} \end{cases}$$

Part III - Rescaling

Goal: Rescale time variables and our unknowns so that the mean mortality rate is 1.

$$\tau = 1 \iff t = \frac{1}{\gamma}$$

$$\boxed{\tau = \gamma t}$$

Then we rescale the unknowns by a constant. Let N be the total population. So

$$\begin{cases} s = \frac{S}{N} \\ i = \frac{I}{n} \end{cases}$$

$$s(x,\tau) = \frac{S(x,t)}{N} = \frac{S(x,\frac{\tau}{\gamma})}{N} \implies S(x,t) = Ns(x,\tau) = Ns(x,\gamma t)$$
$$I(x,t) = Ni(x,\gamma t)$$

So by chain rule

$$S_t = (Ns(x,\gamma t))_t = N\gamma s_\tau(x,\gamma t)$$

But by the PDE

$$S_t = -bS(x,t)I(x,t)$$

so

$$N\gamma s_\tau(x,\gamma t) = -bNs(x,\gamma t)Ni(x,\gamma t)$$

Which at long last gives

$$s_\tau = -\frac{bN}{\gamma}s_i$$

And following the same process for i_τ:

$$i_\tau = \frac{bN}{\gamma}s_i - i + \frac{D}{\gamma}i_{xx}$$

Now let $R_0 = \frac{bN}{\gamma}$ and $d = \frac{D}{\gamma}$ then the system simplifies to

$$\boxed{\begin{cases} s_\tau = -R_0 s_i \\ i_\tau = R_0 s_i - i + d i_{xx} \end{cases}}$$

where d is the rescaled diffusion constant and R_0 is similar to the number of people an infected person infects

Part IV - Traveling Waves

Goal: Convert this PDE system into an ODE by looking for special solutions

Definition: A traveling wave is a solution of the form

$$u(x,t) = f(x-ct)$$

where $f = f(z)$.

Then we assume that the PDE is of the special form

$$\begin{aligned} s(x,\tau) &= f(x-c\tau) \\ i(x,\tau) &= g(x-c\tau) \end{aligned}$$

So

$$\begin{aligned} s_\tau &= -R_0 s_i \\ &= (f(x-ct))_\tau \\ &= f'(x-c\tau)(-c) \\ &= -cf'(x-c\tau) \end{aligned}$$

$$\implies -cf'(x-c\tau) = -R_0 f(x-c\tau)g(x-c\tau)$$
$$\implies -cf'(z) = -R_0 f(z)g(z)$$

Similarly with $i_\tau = R_0 si - i + di_{xx}$

which at last gives us the system of ODE:

$$\boxed{\begin{cases} cf'(z) = R_0 f(z)g(z) \\ cg'(z) = -R_0 f(z)g(z) + g(z) - dg''(z) \end{cases}}$$

33 April 24: SIR Model (II)

Part I - Review

- $S(x,t)$ is the number of healthy of people
- $I(x,t)$ is the number of infected people

This gives the PDE
$$\begin{cases} S - t = -bSI \\ I_t = bSI - \gamma I + DI_{xx} \end{cases}$$

Where b is the infection rate, γ is the death rate, D is the diffusion rate.

By rescaling time such that $\tau = \gamma t$, we get new unknowns:

$$s = \frac{S}{N} \qquad i = \frac{I}{n}$$

where N is the total population before the disease (assumed constant)

Which gives new functions

$$s(x,\tau) = \frac{S(x,t)}{N} \qquad i(x,\tau) = \frac{I(x,t)}{N}$$

So our rescaled PDE is
$$\begin{cases} s_\tau = -R_0 si \\ i_\tau = R_0 si - i + di_{xx} \end{cases}$$

where $R_0 = bN/\gamma$ (which basically represents the number of people a person infects) and $d = D/\gamma$ (a rescaled diffusion constant)

We look for solutions of a special form (traveling waves) such that

$$\begin{cases} s(x,\tau) = f(x-ct) \\ i(x,\tau) = g(x-ct) \end{cases}$$

Substituting into the PDE,

$$\begin{cases} cf'(z) = R_0 f(z) g(z) \\ cg'(z) = -R_0 f(z) g(z) + g(z) - d g''(z) \end{cases}$$

Part II - Left Front

Assumptions on f and g:

1. $\lim_{z\to\pm\infty} g(z) = 0$ and $\lim_{z\to\pm\infty} g'(z) = 0$
2. $\lim_{z\to+\infty} f(z) = 1$
3. $\lim_{z\to-\infty} f(z) = k \quad (0 < k < 1)$

Goal: Find an equation for k (the number of healthy people after the pandemic)

back to the ODE:

$$\begin{cases} cf' = R_0 f g \\ cg' = -R_0 f g + g - d g'' \end{cases}$$

Step 1: Solve for g

$$g = \frac{c}{R_0}\left(\frac{f'}{f}\right) = \frac{c}{R_0}(\ln f)'$$

Step 2:

$$\begin{aligned} cg' &= -R_0 f g + g - d g'' \\ &= -cf' + \frac{c}{R_0}(\ln f)' - d g'' \\ (cg)' &= (-cf + \frac{c}{R_0}\ln f - d g')' \\ cg &= -cf + \frac{c}{R_0}\ln f - d g' + \underbrace{L}_{\text{constant}} \end{aligned}$$

Step 3:

$$cg(z) = -cf(z) + \frac{c}{R_0}\ln(f(z)) - dg'(z) + L$$

for all z.

Plugging in $z = +\infty$:

$$0 = -c(1) + \frac{c}{R_0}\ln(1) - 0 + L \implies L = c$$

Then at $z = -\infty$:

$$0 = -ck + \frac{c}{R_0}\ln k - 0 + c$$

$$ck - \frac{c}{R_0}\ln k = c \implies k - \frac{1}{R_0}\ln k = 1 \implies \ln k = -R_0(1-k) = R_0(k-1)$$

So at last,

$$\boxed{\ln k = R_0(k-1)}$$

Part III - Two Cases

Does the equation above have a nontrivial solution?

Case 1: $R_0 < 1$ (Good case, no outbreak)

Here, there are no other solutions ($k = 1$ is the only solution and no one was ever sick)

Proof:

$$f(k) = \ln k - R_0(k-1) \implies f' = \frac{1}{k} - R_0 = 0 \implies k = \frac{1}{R_0}$$

Case 2: $R_0 > 1$ (bad case, outbreak!) There is another solution because the average person is infecting multiple people.

34 April 26: Method of Characteristics

Part I - Method of Characteristics Revisited

Example 1:

$$\begin{cases} u_t + x^2 u_x = 0 \\ u(x,0) = f(x) \end{cases}$$

You *could* solve this using the slope and the derivative like at the beginning of the course but there is a more effective wave

Step 1:

Motivation: Take a curve $(x(t), t)$. How does the solution $u(x, t)$ change on this curve. Consider $h(t) = u(x(t), t)$. Then

$$\begin{aligned} h'(t) = \frac{d}{dt} u(x(t), t) \\ &= u_x(x(t), t)x'(t) + u_t(x(t), t) \\ &= u_x(x(t), t)x'(t) - (x(t))^2 u_x(x(t), t) \quad \text{(from the PDE)} \\ &= u_x(x(t), t)[x'(t) - (x(t))^2] \end{aligned}$$

So if $x' = x^2$ then $h'(t) = 0$ and $u(x(t), t)$ would be constant!

Definition: The ODE $x' = x^2$ is *characteristic ODE*. The curve $x(t)$ is the *characteristic curve*

Step 2: Solve $x' = x^2$

Using separation of variables,

$$\begin{aligned} \frac{dx}{dt} &= x^2 \\ \frac{1}{x^2}\, dx &= dt \\ -\frac{1}{x} &= t + C \\ x(t) = -\frac{1}{t + C} \end{aligned}$$

Note: $x = 0$ is also a solution because $0' = 0^2$

Step 3: Since $h(t)$ is constant

$$\begin{aligned} h(t) &= h(0) \\ u(x(t), u) &= u(x(0), 0) \\ u\left(-\frac{1}{t + C}, t\right) &= u\left(-\frac{1}{C}, 0\right) \\ u\left(-\frac{1}{t + C}, t\right) &= f(-\frac{1}{C}) \end{aligned}$$

Step 4:

$$\begin{aligned}
x &= -\frac{1}{t+C} \\
t+C &= -\frac{1}{x} \\
C &= -t - \frac{1}{x} = -\left(\frac{tx+1}{x}\right) \\
-\frac{1}{C} &= \frac{x}{1+xt}
\end{aligned}$$

Therefore,

$$u(x,t) = u\left(-\frac{1}{t+C}, t\right) = f(-\frac{1}{C}) = \boxed{f\left(\frac{x}{1+xt}\right)}$$

Application: It is possible that characteristic curves intersect. This is quite a problem because how can u have one value on one curve and a different value on another curve, unless u is constant? This phenomenon is called a *shock* and is useful in modeling traffic flow

Part II - More General

Example 2:

$$\begin{cases} u_t + xu_x = 2u \\ u(x,0) = f(x) \end{cases}$$

Step 1: Let $h(t) = u(x(t),t)$. Then

$$\begin{aligned}
h'(t) &= u_x x'(t) + u_t \\
&= u_x \cdot x' + 2u - xu_x \\
&= 2u(x(t),t) + u_x(x(t),t)[x'(t) - x(t)] \\
&= 2h(t) + u_x(x(t),t)[x' - x(t)]
\end{aligned}$$

Step 2: One characteristic ODE is

$$x'(t) - x(t) = 0 \implies x'(t) = x(t) \implies x(t) = Ce^t$$

If $x' - x = 0$ then

$$h'(t) = 2h(t) \implies h(t) = h(0)e^{2t}$$

which is another characteristic ODE.

Step 3: Now u isn't constant on the curve but we still have an explicit formula:

$$\begin{aligned} h(t) &= h(0)e^{2t} \\ u(x(t),t) &= u(x(0),0)e^{2t} \\ u(Ce^t,t) &= u(Ce^0,0)e^{2t} \\ u(Ce^t,t) &= g(C)e^{2t} \end{aligned}$$

Step 4: If $x = Ce^t$ then $C = xe^{-t}$ so

$$u(x,t) = f(C)e^{2t} = \boxed{f(xe^{-t})e^{2t}}$$

35 April 28: Fluid Flow Application (I)

Part I - Vector Calculus Overview

Note: this will be presented in two dimensions but can be generalized to higher dimensions as needed.

Definitions:

1. *Vector Field:* a function F with two components. Usually represented by arrows on a field. Ex.
$$F(x,t) = \langle 2x - 3y, x^2 + y^2 \rangle$$
2. *Normal Vector:* a vector $\vec{n}$ that is perpendicular to $\partial\Omega$ and points outwards
3. Net Flux:
$$\int_{\partial\Omega} F \cdot d\vec{S} = \int_{\partial\Omega} F \cdot n \; ds$$
Note: In 2D this is a line integral. In 3D it would be a surface integral.

   Interpretation: If $F(x,y)$ is water flow, then $\int_{\partial\Omega} F \cdot d\vec{S}$ is the total water in and out of $\partial\Omega$.
4. *Divergence:*
$$\text{If } F = \langle P, Q \rangle \text{ then } \operatorname{div}(F) = P_x + Q_y$$
Interpretation: $\operatorname{div}(F)$ represents how much F spreads out.

5. *Divergence Theorem:*

$$\int_{\partial\Omega} F \cdot d\vec{S} = \int\int_{\Omega} \mathrm{div}(F)\, dx\, dy$$

Part II - Incompressibility

Goal: derive the Navier-Stokes PDE modeling fluid dynamics

Assumption: Suppose you have an incompressible fluid with constant density ρ.

We want to find a PDE for the velocity field $F = \langle u(x,y,t), v(x,y,t)\rangle$ of the fluid.

Definition: *Incompressibility* means that the net number of particles entering or exiting Ω is 0. i.e.,

$$\int_{\partial\Omega} (\rho F) \cdot d\vec{S} = \rho \int_{\partial\Omega} F \cdot n\ ds = 0 \overset{div}{\Longrightarrow} \int_{\Omega} F\ dx\, dy = 0$$

If $\int_{\Omega} f = 0$ for all Ω then f is the zero function. Otherwise, assume $f > 0$ somewhere and take $\Omega = \{f > 0\}$ in the identity above to get a contradiction. Thus, we get

$$\mathrm{div}(F) = 0$$

so with $F = \langle u, v\rangle$,

$$\boxed{u_x + v_y = 0}$$

Part III - Fluid flow

Definition: Given a vector field F, the *flow* is the solution $\vec{x}(t)$ of the ODE

$$\vec{x}'(t) = F(\vec{x}(t))$$

Example: If $F(x,y) = \langle 2x + 3y, -x^2 + 3y^2\rangle$, then the flow is $\vec{x}(t) = \langle x(t), y(t)\rangle$ where

$$\begin{cases} x'(t) = 2x(t) + 3y(t) \\ y'(t) = -(x(t))^2 + 3(y(t))^2 \end{cases}$$

Interpretation: $\vec{x}(t)$ is the trajectory of a particle that follows the velocity field F

Part IV - The Navier Stokes Equation

Given the incompressible field F above, consider the flow $\vec{x}' = F(\vec{x})$. This gives the flow equations

$$\begin{cases} x'(t) = u(x(t), y(t), t) \\ y'(t) = v(x(t), y(t), t) \end{cases}$$

We solve via $F = ma$!

Acceleration:

$$\begin{aligned} x''(t) &= (x'(t))' \\ &= (u(x(t), y(t), t))' \\ &= u_x(x'(t)) + u_y(y'(t)) + u_t \\ &= u_x u + u_y v + u_t \quad \text{(by the flow equations)} \end{aligned}$$

Similarly for $y''(t)$.

Force: We assume the only force is the pressure from outside the fluid. Let $P = P(x, y)$ be the pressure at a given point.

In 3D, Pressure $= \frac{\text{Force}}{\text{Area}}$ so in 2D, Force $=$ Pressure $\times$ Length

Assumption: The outside fluid exerts a force $-Pn\ ds$ onto the fluid (a force of magnitude P applied opposite the normal vector across a small area ds).

Thus the total force is

$$\int_{\partial\Omega} -Pn\ ds = \int_{\partial\Omega} -P(x, y)\langle n_1, n_2\rangle\ ds = \int_{\partial\Omega} \langle -Pn_1, -Pn_2\rangle\ ds$$

The first component is

$$\begin{aligned} \int_{\partial\Omega} -Pn_1\ ds &= \int_{\partial\Omega} -Pn_1 + 0n_2\ ds \\ &= \int_{\partial\Omega} \langle -P, 0\rangle \cdot \langle n_1, n_2\rangle\ ds \\ &= \int_{\partial\Omega} \langle -P, 0\rangle \cdot n\ ds \\ &= \int_{\partial\Omega} \operatorname{div}\langle -P, 0\rangle\ dx\, dy \quad \text{(by divergence theorem)} \\ &= \int_{\partial\Omega} -P_x\ dx\, dy \end{aligned}$$

Similarly for the second component so the total force on Ω is

$$\int_{\Omega} -\langle P_x, P_y \rangle \, dx \, dy$$

Then rearranging some physics equations:

$$\begin{aligned} F &= ma \\ F &= \rho V a \\ A\rho &= \frac{F}{V} \end{aligned}$$

So

$$\begin{gathered} \rho x''(t) = -P_x \\ u_t + u u_x + v u_y = -\frac{1}{\rho} P_x \\ \boxed{u_t + u u_x + v u_y + \frac{1}{\rho} P_x = 0} \end{gathered}$$

36 May 1: Fluid Flow Applications (II)

Part I - Recap: Fluid Flow

Setting: Suppose an incompressible fluid with constant density ρ

Goal: Find a PDE for the velocity field $F = \langle u(x, y, t),\ v(x, y, t) \rangle$ of the fluid

From incompressibility we get the PDE

$$\operatorname{div}(F) = 0 \implies \boxed{u_x + v_y = 0}$$

Considering the fluid flow of F, $\vec{x}' = F(\vec{x})$, and looking at the components, we get the flow equations

$$\begin{cases} x'(t) = u(x(t), y(t), t) \\ y'(t) = v(x(t), y(t), t) \end{cases}$$

Then using $F = ma$ we get

$$\begin{cases} x''(t) = u_t + u u_x + v u_y \\ y''(t) = v_t + u v_x + v v_y \end{cases}$$

Then assuming that the only force is the external pressure $P = P(x, y)$ from outside the fluid on $\partial\Omega$. From the fact that in 1D, pressure is force-by-length, we have

$$\text{Force} = \text{Pressure} \times \text{Length}$$

Assuming that the outside fluid exerts a force opposite the normal vector at a point across a small length,

$$F = \int_{\partial\Omega} -P\vec{N}\, ds = \int_{\partial\Omega} \langle -PN_1,\ -PN_2 \rangle\, ds = \left\langle \int_{\partial\Omega} -PN_1\, ds,\ \int_{\partial\Omega} -PN_2\, ds \right\rangle$$

The components can each be written

$$\int_{\partial\Omega} -PN_1\, ds = \int_{\partial\Omega} \langle -P, 0 \rangle \cdot \langle N_1, N_2 \rangle\, ds$$

which allows us to use the divergence theorem:

$$\begin{aligned} F &= \left\langle \int_{\Omega} \operatorname{div}\langle -P, 0 \rangle\, dx\, dy,\ \int_{\Omega} \operatorname{div}\langle 0, -P \rangle\, dx\, dy \right\rangle \\ &= \left\langle \int_{\Omega} -P_x\, dx\, dy,\ \int_{\Omega} -P_y\, dx\, dy \right\rangle \\ &= \int_{\Omega} -\langle P_x, P_y \rangle\, dx\, dy \end{aligned}$$

Now we only need the mass to complete $F = ma$ but we only know the density. Notice:

$$\begin{aligned} \text{Force} &= \text{Mass} \times \text{Acceleration} \\ &= \frac{\text{Mass}}{\text{Area}} \times \text{Area} \times \text{Acceleration} \\ &= \text{Density} \times \text{Area} \times \text{Acceleration} \\ \Longrightarrow \text{Density} \times \text{Acceleration} &= \frac{\text{Force}}{\text{Area}} \end{aligned}$$

So

$$\rho \langle x''(t), y''(t) \rangle = \langle -P_x, -P_y \rangle$$

Looking at components,

$$\begin{cases} \rho x''(t) = -P_x \\ \rho y''(t) = -P_y \end{cases}$$

which by the flow equations gives the inviscid Navier-Stokes equations:

$$\begin{cases} u_t + uu_x + vu_u + \frac{1}{\rho}P_x = 0 \\ v_t + uv_x + vv_y + \frac{1}{\rho}P_y = 0 \end{cases}$$

However we can also add a diffusion-like term $D\Delta u$ and $D\Delta v$ to each equation to account for the viscosity (here friction will cause slow particles to speed up and fast particles to slow down).

At long last, we have the Navier-Stokes Equations:

$$\begin{cases} u_t + uu_x + vu_u + \frac{1}{\rho}P_x = D\Delta u \\ v_t + uv_x + vv_y + \frac{1}{\rho}P_y = D\Delta v \\ u_x + v_y = 0 \end{cases}$$

Finally, note that we can rewrite this as

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} + \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \frac{1}{\rho} \begin{bmatrix} P_x \\ P_y \end{bmatrix} = D \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}$$

so

$$\boxed{F_t + (\nabla F)F + \frac{1}{\rho}\nabla P = D\Delta F}$$

37 May 3: Wealth Distribution

Part I - Setting

Suppose you have $N = 2$ people, Beff Jezos and Melon Usk, who exchange wealth with each other. Let $W_1(t)$ be the wealth of Beff Jezos at time t and $W_2(t)$ be the wealth of Melon Usk. Assume they exchange a fraction $k/2$ of their wealth each day.

Part II - Models

Chemical Tank (Compartmental) Model:

$$\boxed{W_1(t)} \underset{k/2}{\overset{k/2}{\rightleftarrows}} \boxed{W_2(t)}$$

Then we can set up the ODE as the difference of in flow and out flow:

$$\begin{cases} W_1'(t) = \frac{k}{2}(W_2(t) - W_1(t)) \\ W_2'(t) = \frac{k}{2}(W_1(t) - W_2(t)) \end{cases}$$

Remarks:

1. Here we use $k/2$ because $N = 2$. In general, this would be k/N
2. We can use unequal exchanges but that would be more complicated

Problem: In real life, wealth fluctuates randomly rather than deterministically (a la the stock market)

Stochastic Model:

Motivation: If things were not random we would expect W_1 to behave like

$$W_1'(t) = \mu W_1(t) \implies W_1 = Ce^{\mu t}$$

where $\mu > 0$ is the mean growth rate of individual wealth.

In this model we assume much the same but with random fluctuations:

$$\begin{cases} W_1'(t) = (\mu + \sigma B(t))W_1(t) \\ W_2'(t) = (\mu + \sigma B(t))W_2(t) \end{cases}$$

where $B(t)$ is Brownian motion.

Interpretation: W_1 behaves like $Ce^{\mu t}$ but fluctuates randomly with standard deviation σ

Real Model:

We choose to combine both prior models

$$\begin{cases} W_1'(t) = (\mu + \sigma B(t))W_1(t) + \frac{k}{2}(W_2(t) - W_1(t)) \\ W_2'(t) = (\mu + \sigma B(t))W_2(t) + \frac{k}{2}(W_1(t) - W_2(t)) \end{cases}$$

Part III - Simplifying the model

We define $\overline{W}(t)$ as the average wealth

$$\overline{W}(t) = \frac{W_1(t) + W_2(t)}{2}$$

Which lets us rewrite the model in terms of $\overline{W}(t)$:

$$\begin{aligned}
\frac{k}{2}(W_2 - W_1) &= k\left(\frac{W_2 - W_1}{2}\right) \\
&= k\left(\frac{W_2 + W_1 - W_1 - W_1}{2}\right) \\
&= k\left(\frac{W_1 + W_2}{2} - \frac{2W_1}{2}\right) \\
&= k(\overline{W} - W_1)
\end{aligned}$$

Similarly,

$$\frac{k}{2}(W_1 - W_2) = k(\overline{W} - W_2)$$

So the model becomes

$$\begin{cases} W_1'(t) = (\mu + \sigma B(t))W_1(t) + k(\overline{W} - W_1) \\ W_2'(t) = (\mu + \sigma B(t))W_2(t) + k(\overline{W} - W_2) \end{cases}$$

Note: This is not decoupled as $\overline{W}$ depends on W_1 and W_2

Now we define

$$x_1(t) = \frac{W_1(t)}{W(t)} \qquad x_2(t) = \frac{W_2(t)}{W(t)}$$

which is like the normalization of the wealth and

$$x_1(t) = 1 \implies W_1(t) = \overline{W}(t)$$

Then using the ODE and $W_1 = x_1(t)\overline{W}(t)$ and $W_2 = x_2(t)\overline{W}(t)$ one can show that $x_1(t)$ and $x_2(t)$ satisfy the system

$$\begin{cases} x_1'(t) = \sigma B(t)x_1(t) + k(1 - x_1(t)) \\ x_2'(t) = \sigma B(t)x_2(t) + k(1 - x_2(t)) \end{cases}$$

Which is good because there is no μ term and no coupling!

Part IV - Macroeconomic Model

Notice that x_1 and x_2 solve the same equation, namely the general Macroeconomic model measuring the total population wealth:

$$\boxed{X'(t) = \sigma B(t) X(t) + k(1 - X(t))}$$

where $X(t)$ is a random variable representing the wealth of the general population at time t.

Let $u(x,t)$ be the probability density function of $X(t)$ so that $\int_a^b u(x,t)\ dx$ is the fraction of population whose wealth $X(t)$ at time t is in $[a,b]$, i.e.

$$\int_a^b u(x,t)\ dt = \mathbb{P}(a \le X(t) \le b)$$

so $u(x,t)$ is the distribution of wealth $X(t)$ in the population

Fact: $u(x,t)$ solves the Fokker-Planck equation

$$\boxed{u_t + k((1-x)u)_x = \frac{\sigma^2}{2}(x^2 u)_{xx}}$$

Interpretation: This is a hybrid of transport and heat equations. : $k((1-x)u)_x$ is a transport term, it comes from the deterministic part $k(1-X(t))$ of the Macro Model, whereas $\frac{\sigma^2}{2}(x^2u)_{xx}$ is a diffusion term coming from the stochastic part $\sigma B(t) X(t)$

Part V - Steady State Solutions

We want to find time-invariant solutions $u = u(x)$ corresponding to people whose wealth doesn't change over time.

$$u_t = 0 \implies k((1-x)u)_x = \frac{\sigma^2}{2}(x^2 u)_{xx}$$

This can be solved via integrating and integrating factors to get

$$u(x) = \frac{1}{x^{1+m}} e^{-\frac{m-1}{x}}$$

where $m = 1 + \frac{2k}{\sigma^2}$ is the *Pareto Index*

Part VI - Solving our goal

Question: How is the wealth among the super rich distributed? i.e.,

$$\int_x^\infty u(y)\,dy = ? \quad \text{for large } x$$

Fact: For x large enough

$$\int_x^\infty u(y)\,dy \approx \frac{1}{x^m}$$

where $m = 1 + \frac{2k}{\sigma^2}$ (the Pareto index)

Interpretation: If m is large, the distribution is narrower (a smaller fraction of people has wealth $\geq x$ when x is large). If m is small, the distribution is broader and more people have more wealth.

38 May 5: Peyam's PhD PDE

Part I - The Great Debate

Consider a chemical reaction $A \rightleftharpoons B$ where a molecule A transforms into B and vice-versa.

Note: This is not a reaction like $C + O_2 \rightleftharpoons CO_2$ because this is already too complicated. This is just one molecule into another.

There are two ways of viewing this reaction.

Macroscopic Level

$$\begin{cases} \alpha = \alpha(t) = \text{concentration of A at time t} \\ \beta = \beta(t) = \text{concentration of B at time t} \end{cases}$$

These then satisfy the ODE

Reaction-Diffusion System:

$$\boxed{\begin{cases} \alpha_t = k(\beta - \alpha) \\ \beta_t = k(\alpha - \beta) \end{cases}}$$

where $k > 0$ is the *reaction-rate constant* and determining it is a big deal in chemistry.

Note: We could add a Laplacian term to this to get $\alpha_t - \Delta\alpha$ to have the diffusion aspect.

Intuitively, if $\alpha > \beta$ A decreases and B increases until the this switches and the two reach equilibrium.

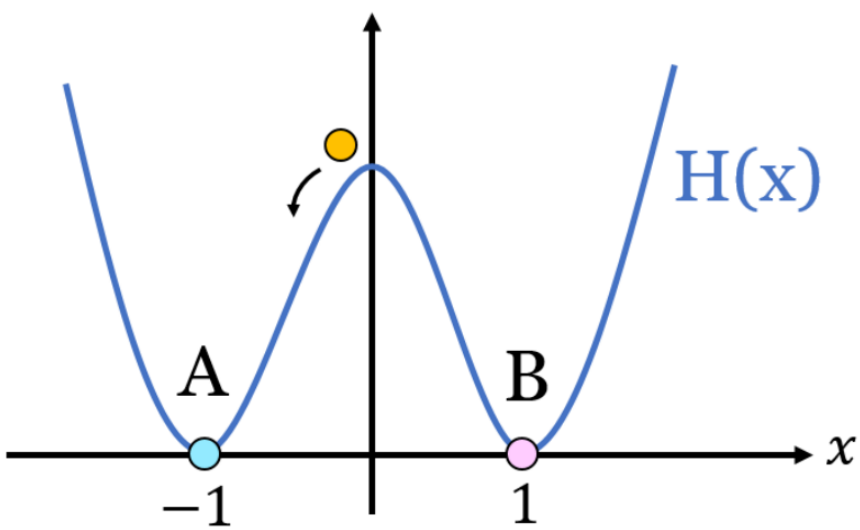
Microscopic view

Instead of just having two states A and B, think of the molecule as having a continuum of states where on the left endpoint you have A and on the right endpoint you have B, with infinitely many hybrid states in between

We can parameterize those states with a chemical variable x where $x = -1 \to A$ and $x = 1 \to B$

In an example where this describes protein folding, this x could represent the angle of twist.

Now we can represent the state via a probabilistic double-well function H:



On the potential, this movement between the states performs Brownian Motion so we can model it with the *Kramers-Smoluchowski PDE* (K-S):

$$\boxed{\sigma u_t = (\sigma u_x)_x}$$

where

- $u = u(x,t)$ is the density of the particle
- Note: though it has not been included, everything depends on a "zooming factor" where $\epsilon = 0$ is the macroscropic view and $\epsilon > 0$ is the microscopic state,
- $\sigma = \sigma(x) = \exp(-\frac{H(x)}{\epsilon})$

Note: Technically you need to add some constants so the total mass of σ is 1 and divide the RHS by τ. If you ignore σ and the constants, you get $u_t = u_{xx}$ so this is really just a scaled version of the heat equation!

Those two models stirred a GREAT debate in the chemical world, because some chemists say that the macroscopic model is better, while others claim that the microscopic one is more accurate.

The paper claims that there shouldn't be a debate at all, because it turns out that the two models are just two different sides of the same coin!

Part II - The main result

It turns out that if you take the limit of the Kramers-Smoluchowski PDE as $\epsilon \to 0$ you get the reaction-diffusion system!

Problem: How does this possibly make sense? In K-S you have one function $u(x,t)$ and in R-D you have two function $\alpha(t)$ and $\beta(t)$

Main Theorem 1: As $\epsilon \to 0$,

$$\boxed{\sigma(x)u(x,t) \to \alpha(t)\delta_{-1} + \beta(t)\delta_1}$$

where δ_i is the Dirac Delta at $x = i$.

Interpretation: As the zooming factor goes to 0, the solution to K-S (multiplied by σ) converges to two spikes of height α and β

Proof: Energy methods! Multiply K-S by u and IBP to get one identity, then multiply the PDE by u_t to get another identity then use some analysis results.

Main Theoream 2: α and β solve R-D with

$$\boxed{k = \frac{\sqrt{-H''(0)H''(1)}}{2\pi}}$$

This is AMAZING because it tells us the PDEs are related and solves the major chemistry problem of finding k because H is easy to approximate with numerical methods.

Part III - Sketch of the proof of Main Theorem 2

Step 1: Start with K-S:

$$\sigma u_t = (\sigma u_x)_x$$

The σ term blows up near ± 1 so we use energy methods to get rid of it.

We choose a function whose derivative is $1/\sigma$:

$$\phi(x) = \int_0^x \frac{1}{\sigma(y)}\, dy$$

Step 2: Energy methods:

$$\underbrace{\int_{-2}^{2} \sigma u_t \phi \, dx}_{A} = \underbrace{\int_{-2}^{2} (\sigma u_x)_x \phi \, dx}_{B}$$

We choose ± 2 because all the terms behave there. Any number bigger than 1 would suffice.

Step 3: Study A

Remember that

$$\sigma u \to \alpha \delta_{-1} + \beta \delta_1$$

But it turns out that

$$sigma u_t \to \alpha_t \delta_{-1} + \beta_t \delta_1$$

too. This means that σu_t concentrates at ± 1 with spikes α_t and β_t. So

$$A = \int_{-2}^{2} \sigma u_t \phi \, dx \to \alpha_t \phi(-1) + \beta_t \phi(1)$$

But

$$\phi(1) = \int_0^1 \frac{1}{\sigma} \to \frac{1}{k}$$

which can be shown with the Maclaurin series and a Laplace expansion near 1 (this is where $H''(0)$ and $H''(1)$ come in). By symmetry,

$$\phi(-1) = \int_0^{-1} \frac{1}{\sigma}\, dx \to -\frac{1}{k}$$

Hence,

$$A \to \alpha_t \left(-\frac{1}{k}\right) + \beta_t \left(\frac{1}{k}\right) = \frac{1}{k}(-\alpha_t + \beta_t)$$

Step 4: Study B

$$\begin{aligned}
B &= \int_{-2}^{2} (\sigma u_x)_x \phi \, dx \\
&\overset{IBP}{=} -\int_{-2}^{2} \sigma u_x \phi_x \, dx \\
&= -\int_{-2}^{2} \sigma u_x \frac{1}{\sigma} \, dx \quad (\phi \text{ is an antiderivative of } 1/\sigma) \qquad = -\int_{-2}^{2} u_x \, dx \\
&= u(-2,t) - u(2,t) \\
&\to 2\alpha - 2\beta \quad \text{(via estimates)}
\end{aligned}$$

Step 5: Combining

$$\begin{aligned}
& A = B \\
\implies & \frac{1}{k}(-\alpha_t + \beta_t) = 2\alpha - 2\beta \\
\implies & \alpha_t - \beta_t = 2k(\beta - \alpha)
\end{aligned}$$

On the other hand, if you just integrate K-S you get

$$\alpha_t + \beta_t = 0$$

So treating these as a system, we get:

$$\begin{cases} \alpha_t = k(\beta - \alpha) \\ \beta_t = k(\alpha - \beta) \end{cases}$$

Which is R-D!!! ■

Part IV - Variations

Triple Wells: If H has three wells corresponding to α, β, γ we get

$$\sigma u \to \alpha\delta_{-2} + \beta\delta_0 + \gamma\delta_2$$

where

$$\begin{cases} \alpha_t = k(\beta - \alpha) \\ \beta_t = k(\alpha - 2\beta + \gamma) \\ \gamma = k(\beta - \gamma) \end{cases}$$

In other words, each well interacts with its nearest neighbor and β has two neighbors.

Infinitely Many Wells: If H has infinitely many wells α_m then

$$\sigma u \to \sum_m \alpha_m \delta_{2m}$$

where

$$\begin{cases} (\alpha_m)_t = k(\alpha_{m-1} - 2\alpha_m + \alpha_{m+1}) \end{cases}$$

Higher Dimensions: Suppose H has wells at $(0, -1)$ and $(0, 1)$ and a saddle at $(0, 0)$

Then K-S becomes:

$$\sigma u_t = (\sigma u_x)_x + (\sigma u_y)_y$$

and we get the same result:

$$\sigma u \to \alpha\delta_{(0,-1)} + \beta\delta_{(0,1)}$$

where

$$\begin{cases} \alpha_t = k(\beta - \alpha) \\ \beta_t = k(\alpha - \beta) \end{cases}$$

This time k is more complicated and depends on the eigenvalues of D^2H at $(0, 0)$

Interestingly, this time ϕ is defined with a PDE so you get to use a PDE to solve a PDE.

Part V - Open Projects

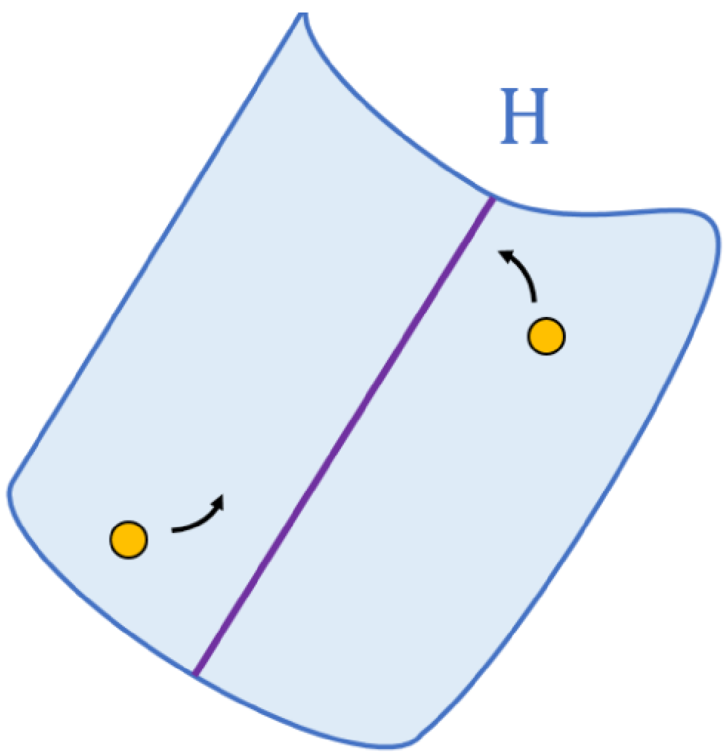
In higher dimensional cases with three symmetric potentials we can just adapt the methods above.

If the wells are at random locations (not symmetric), though, we need to do a lot more work to prove the estimates and convergence. But this can be solved with some probability theory (interpret the wells as states of a Markov process and use some results about metastable random processes)

Tooth Problem: What if you take H, but spin it around the z-axis to look like a tooth?

Now there is a circle of wells, and you can show that σu concentrates on that circle, but the question is: What is the resulting PDE?

Beaver Dam: What if H looks like a (version) of a beaver dam



Now σu concentrates on a line but the PDE is uncertain because the particle can go left, right, forward, or backward.

Conclusion: For any H, you have a new problem!

39 Lecture May 8: Final Exam Review

Part I - Fourier Transform

Example 1:

1. Let $h(x)$ be a function with the following property: for all $f(x)$ we have

$$\int_{-\infty}^{\infty} h(x)f(x)\,dx = f(0)$$

Find $\widehat{h}(\kappa)$.

Solution:

$$\begin{aligned}\widehat{h}(\kappa) &= \int_{-\infty}^{\infty} h(x) \underbrace{e^{i\kappa x}}_{f(x)}\,dx \\ &= f(0) = e^{ik(0)} = \boxed{1}\end{aligned}$$

2. With h above, solve the PDE

$$u_{tt} - 4u = -h(x)$$

Solution:

$$\begin{aligned}\mathcal{F}(u_{tt}) - 4\mathcal{F}(u) &= -\mathcal{F}h(x) \\ \frac{d^2}{dt^2}\mathcal{F}(u) - 4\mathcal{F}(u) &= -1\end{aligned}$$

Homogeneous solution:

$$y'' - 4y = 0 \implies r = \pm 2 \implies y_0 = Ae^{2t} + Be^{-2t}$$

Particular Solution:

$$y_p = C$$

$$(C'') - 4(C) = -1 \implies C = \frac{1}{4}$$
$$y_p = \frac{1}{4}$$

General Solution:

$$y = y_0 + y_p$$
$$\mathcal{F}(u(\kappa, t)) = A(\kappa)e^{2t} + B(\kappa)e^{-2t} + \frac{1}{4}$$

Since $A(\kappa)$ and $B(\kappa)$ are arbitrary, $A(\kappa) = \widehat{f}(\kappa)$ for some f and $B(\kappa) = \widehat{g}(\kappa)$ for some g.

Then

$$\frac{1}{4} = \frac{1}{4} \cdot 1 = \frac{1}{4}\widehat{h}$$

so

$$\begin{aligned}\mathcal{F}(u(\kappa, t)) &= \widehat{f}(\kappa) + \widehat{g}(\kappa) + \frac{1}{4}\widehat{h} \\ &= \mathcal{F}(f(x)e^{2t} + g(x)e^{-2t} + \frac{1}{4}h(x))\end{aligned}$$

and

$$\boxed{u(x, t) = f(x)e^{2t} + g(x)e^{-2t} + \frac{1}{4}h(x)}$$

with f, g arbitrary.

Part II - Method of Characteristics

Example 2: Solve using the characteristic equations

$$\begin{cases} u_t + u_x + u = \exp(x + 2t) \\ u(x, 0) = 0 \end{cases}$$

Let $h(t) = u(x(t), t)$. Then

$$\begin{aligned}h' &= u_x \cdot x'(t) + u_t \\ &= u_x \cdot x'(t) - u_x - u + \exp(x + 2t) \qquad = u_x(x' - 1) - u + \exp(x + 2t)\end{aligned}$$

We denote $x'(t) - 1 = 0 \implies x'(t) = 1$ as our characteristic ODE. Then

$$x(t) = t + C$$

so

$$\begin{aligned} h'(t) &= u_x \cdot 0 - u + \exp((t + C) + 2t) \\ &= -u(x(t), t) + \exp(3t + C) \\ &= -h(t) + \exp(3t + C) \end{aligned}$$

So our second characteristic ODE is $h'(t) + h(t) = \exp(3t + C)$. Thus, by integrating factors,

$$\begin{aligned} (h(t)e^t)' &= e^t e^{3t+C} = e^{4t+C} \\ h(t)e^t &= \frac{1}{4}e^{4t+C} + B \\ h(t) &= \frac{1}{4}e^{3t+C} + Be^{-t} \end{aligned}$$

Then by the definition of h:

$$\begin{aligned} u(x(t), t) &= \frac{1}{4}\exp(3t + C) + B\exp(-t) \\ &\overset{t=0}{=} u(x(0), 0) = \frac{1}{4}e^C + B = 0 \implies B = -\frac{1}{4}e^C \\ &= \frac{1}{4}\exp(3t + C) - \frac{1}{4}e^C e^{-t} \\ u(t + C, t) &= \frac{1}{4}\exp(3t + C) - \frac{1}{4}e^{C-t} \quad (x = t + C) \\ u(x, t) &= \frac{1}{4}e^{3t+(x-t)} - \frac{1}{4}e^{(x-t)-t} \quad (C = x - t) \\ &= \frac{1}{4}e^{2t+x} - \frac{1}{4}e^{x-2t} \\ &= \boxed{\frac{1}{4}e^x(e^{2t} - e^{-2t})} \end{aligned}$$

Summary of the Method:

1. Let $h(t) = u(x(t), t)$
2. Calculate $h'(t)$

3. Find the characteristic ODE for x'
4. Plug in to get the second characteristic ODE in h'
5. Solve for h
6. Resub back into definition of h
7. Use initial condition