APMA 0360: Midterm 1 Review Sheet

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Basic Math

Useful Trig:

- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $\cos(-x) = \cos x$
- $\sin(-x) = -\sin x$
- $\cos(2x) = \cos^2(x) - \sin^2(x)$
- $\sin(2x) = 2\sin x \cos x$
- $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$
- $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$
- $\int \tan x \; dx = \ln|\sec x| + C$
- $\int \sec x \; dx = \ln|\sec x + \tan x| + C$
- $\int \frac{1}{x^2+1} \; dx = \tan^{-1} x + C$
- $\int \sin^2 x \; dx = \frac{x}{2} - \frac{1}{4}\sin(2x)$
- $\int \cos^2 x \; dx = \frac{x}{2} + \frac{1}{4}\sin(2x)$

Introduction

Check if a function solves the PDE: plug the function in (differentiate) and see if you get an identity

Solve a simple PDE: Use ODEs Examples:

- $u_x = 0 \implies u(x, y) = f(y)$
- $u_{xx} = 0 \implies u_x = f(y) \implies u(x, y) = xf(y) + g(y)$
- $u_{xx} + u = 0 \overset{y''+y=0}{\implies} u(x, y) = A(y)\cos x + B(y)\sin x$
- $u_{xy} = 0 \implies (u_x)_y = 0 \implies u_x = f(x) \implies u(x, y) = F(X) + G(y)$

Classification:

Order: highest derivative

Constant coefficient

Linear: coefficients depend on x, y but not u

- $L(u + v) = L(u) + L(v)$
- $L(cu) = cL(u)$

Homogeneous: RHS is 0

Elliptic forms of Second Order PDEs:

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y)$$

Let $D = b^2 - 4ac$,

- elliptic if $D < 0$
- parabolic if $D = 0$
- hyperbolic if $D > 0$

First Order Linear

Directional Derivative:

1. Write as a directional derivative

2. Show that u is constant on characteristic lines

Example 1: $au_x + bu_y = 0$

$$\langle u_x, u_y \rangle \cdot \langle a, b \rangle = \nabla u \cdot \vec{v} = 0$$

$$m = \frac{b}{a} \implies y = \frac{b}{a}x + C \implies ay - bx = C$$
$$u(x,y) = f(ay - bx)$$

Example 2: $u_x + yu_y = 0$
$$\nabla u \cdot (1, y) = 0$$
$$\frac{y}{1} = y' \quad \text{slope = derivative}$$
$$y' = y \implies y = Ce^x \implies ye^{-x} = C \implies u(x,y) = f(ye^{-x})$$

Coordinate Method:

1. Define new variables ξ and η that are perpendicular
2. Rewrite the PDE in terms of ξ and η with the chain rule
3. Substitute definitions
4. Solve

Example: $2u_x + 3u_y = 0$

$$\begin{cases} \xi = 2x + 3y & \text{(from equation)} \\ \eta = -3x + 2y & \text{(perpendicular)} \end{cases}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = 2u_\xi - 3u_\eta$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = 3u_\xi + 2u_\eta$$

$$\begin{aligned} 2u_x + 3u_y &= 2(2u_\xi - 3u_\eta) + 3(3u_\xi + 2u_\eta) \\ &= 4u_\xi - 6u_\eta + 9u_\xi + 6u_\eta = 0 \\ &= 13u_\xi = 0 \end{aligned}$$

$$\implies u_\xi = 0 \implies u = f(\eta) \implies u(x,y) = f(2y-3x)$$

Transform Method: Rewrite the derivatives of u in terms of a new PDE v and solve

The Transport Equation

Derivation The mass on an interval is

$$M = \int_0^b u(x,t)\ dx$$

But as mass is conserved between time intervals,

$$M_1 = M_2 = \int_0^b u(x,t)\ dx = \int_{ch}^{b+ch} u(x,t+h)\ dx$$

where c is the speed of the fluid. Then,

$$\frac{d}{db}\int_0^b u(x,t)\ dx = \frac{d}{db}\int_{ch}^{b+ch} u(x,t+h)\ dx$$

By FTC,

$$u(b,t) = u(b+ch,t+h)$$

Differentiate WRT h,

$$0 = u_x \cdot (b+ch)_h + u_t \cdot (t+h)_h = cu_x + u_t$$

Solution

$$u_t + cu_x = 0 \implies cu_x - u_t = 0 \implies u(x,t) = f(x-ct)$$

The Fourier Transform

The Gaussian:

$$I = \int_{-\infty}^{\infty} e^{-x^2}\ dx$$

$$
\begin{aligned}
I^2 &= (I)(I) \\
&= \left(\int_{-\infty}^{\infty} e^{-x^2}\ dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\ dy\right) \quad \text{(trick)} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\ dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\ dx\,dy \\
&= 2\pi\int_{0}^{\infty} re^{-r^2}\ dr \\
&= \pi
\end{aligned}
$$

$$
\implies I = \int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}
$$

Application to sin and cos:

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} e^{ix^2}\ dx \\
I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(x^2+y^2)}\ dy\ dx \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} re^{ir^2}\ drd\theta \\
&= \int_{0}^{2\pi}\left[\frac{e^{ir^2}}{2i}\right]_{r=0}^{r=\infty}\ d\theta = \int_{0}^{2\pi} -\frac{1}{2i}\ d\theta \\
&= \pi i \\
I &= \int_{-\infty}^{\infty} e^{ix^2}\ dx = \sqrt{\pi i}
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{\pi i} &= \sqrt{\pi}\sqrt{i} \\
&= \sqrt{\pi}\sqrt{e^{i\frac{\pi}{2}}} \\
&= \sqrt{\pi}e^{i\frac{\pi}{4}} \\
&= \sqrt{\frac{\pi}{2}} + i\sqrt{\frac{\pi}{2}}
\end{aligned}
$$

$$\int_{-\infty}^{\infty} e^{ix^2}\,dx = \int_{-\infty}^{\infty} \cos(x^2) + i\sin(x^2)\,dx = \sqrt{\frac{\pi}{2}} + i\sqrt{\frac{\pi}{2}}$$

$$\begin{cases} \int_{-\infty}^{\infty} \cos(x^2)\,dx = \sqrt{\frac{\pi}{2}} \\ \int_{-\infty}^{\infty} \sin(x^2)\,dx = \sqrt{\frac{\pi}{2}} \end{cases}$$

The Fourier Transform:

$$\widehat{f}(\kappa) = \int_{-\infty}^{\infty} f(x)e^{i\kappa x}\,dx$$

Example: $\mathcal{F}(e^{-x^2})$

$$\begin{aligned} \widehat{f}(\kappa) &= \int_{-\infty}^{\infty} e^{-x^2}e^{i\kappa x}\,dx \\ \widehat{f}'(\kappa) &= i\int_{-\infty}^{\infty} xe^{-x^2}e^{i\kappa x}\,dx \\ &\overset{\text{IBP}}{=} i\left[-\frac{1}{2}e^{-x^2}e^{i\kappa x}\right]_{-\infty}^{\infty} - i\int_{-\infty}^{\infty} -\frac{1}{2}e^{-x^2}e^{i\kappa x}\,dx \\ &= -\frac{\kappa}{2}\widehat{f}(\kappa) \end{aligned}$$

$$\widehat{f}'(\kappa) = -\frac{\kappa}{2}\widehat{f}(\kappa) \implies \widehat{f}(\kappa) = Ce^{-\kappa^2/4} \implies \widehat{f}(\kappa) = \sqrt{\pi}e^{-\kappa^2/4}$$

Generally,

$$\mathcal{F}(e^{-ax^2}) = \sqrt{\frac{\pi}{a}}e^{-\frac{\kappa^2}{4a}}$$

Derivatives:

$$\widehat{f'}(\kappa) = (-i\kappa)\widehat{f}(\kappa)$$

Proof:

$$\begin{aligned} \widehat{f'}(\kappa) &= \int_{-\infty}^{\infty} f'(x)e^{i\kappa x}\,dx \\ &\overset{\text{IBP}}{=} \left[f(x)e^{i\kappa x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\frac{d}{dx}e^{i\kappa x}\,dx \\ &= 0 - i\kappa\int_{-\infty}^{\infty} f(x)e^{i\kappa x}\,dx \\ &= -i\kappa\widehat{f}(\kappa) \end{aligned}$$

Convolution:

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$$

$$\mathcal{F}((f \star g)(\kappa)) = \widehat{f}(\kappa) \cdot \widehat{g}(\kappa)$$

Inverse Fourier:

$$\mathcal{F}^{-1}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\kappa) e^{-i\kappa x}\,d\kappa$$

Shifting:

$$\mathcal{F}(f(x-a)) = e^{i\kappa a}\widehat{f}(\kappa)$$

The Heat Equation

Derivation Let $u = u(x,t)$ be the concentration of particles that move only left or right along a rod. Focus on the change in the concentration along an interval of length h around (x,t). Assume that as $t \to t + \tau$, each particle moves left or right with equal property,

$$hu(x, t+\tau) = hu(x,t) + \Delta u(x,t)$$

$$\Delta u(x,t) = \text{in - out} = \frac{1}{2}hu(x-h,t) + \frac{1}{2}hu(x+h,t) - hu(x,t)$$

(interpretation: all the original particles at (x,t) leave but half from $(x-h,t)$ and half from $(x+h,t)$ come) Therefore,

$$hu(x,t+\tau) - hu(x,t) = \frac{1}{2}hu(x-h,t) + \frac{1}{2}hu(x+h,t) - hu(x,t)$$

Make some transformations and write in the limit form of the derivative to get

$$\frac{u(x,t+\tau) - u(x,t)}{\tau} = \frac{h^2}{2\tau}\left(\frac{u(x-h,t) - 2u(x,t) + u(x+h,t)}{h^2}\right)$$

$$u_t = \frac{h^2}{2\tau}u_{xx}$$

Then define tau such that $h^2/2\tau = D$ giving

$$u_t = Du_{xx}$$

Solving

$$\begin{cases} u_t = Du_{xx} \\ u(x,0) = f(x) \end{cases}$$

Solution:

$$\begin{aligned} \mathcal{F}(u_t) = \mathcal{F}(Du_{xx}) \\ \frac{d}{dt}\widehat{u} &= D(-i\kappa)^2\widehat{u} \\ &= -D\kappa^2\widehat{u} \\ \widehat{u} &= \widehat{u}(\kappa,0)e^{-D\kappa^2 t} \\ &= \widehat{f}(\kappa)e^{-D\kappa^2 t} \end{aligned}$$

From the Gaussian,

$$e^{-\frac{\kappa^2}{4a}=e^{-D\kappa^2 t}} \implies a = \frac{1}{4Dt} \implies \sqrt{\frac{a}{\pi}} = \frac{1}{\sqrt{4\pi Dt}}$$

$$e^{-\kappa^2 Dt} = \mathcal{F}(\frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}})$$

$$\widehat{u} = \widehat{f}(\kappa) \cdot \mathcal{F}(\frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}) = \mathcal{F}((f \star g)(\kappa, t))$$

$$u(x,t) = \int_{-\infty}^{\infty} f(y)g(x-y,t)\ dy$$

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} f(y)e^{-\frac{(x-y)^2}{4Dt}}\ dy$$

If there is a specified exponential initial condition, simplify the exponent, complete the square, and u-sub to form the gaussian, then eliminate with $\sqrt{\pi}$

Properties

1. "Infinite speed of propagation:" If $f \geq 0$ is positive somewhere and continuous, u is positive everywhere *Proof:* the exponential function is positive and if f is continuous, it is positive around a region x_0 so the full infinite integrand is positive (so $\geq$ to a subset) but that subset is strictly positive so u is strictly positive

2. "Smoothness:" u is infinitely differentiable *Proof:* the exponential term is infinitely differentiable

3. "Irreversibility:" $u(x,0)$ cannot be determined from $u(x,1)$ *Proof:* $u(x,1) = |x|$ can be given but this is not smooth so violates the earlier property

4. "Heat dissipates over time:"

$$\lim_{t\to\infty} u(x,t) = 0$$

   Proof:

$$\begin{aligned}
|u(x,t)| &= \left| \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} f(y)\, dy \right| \\
&\leq \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} |f(y)| \underbrace{e^{-\frac{(x-y)^2}{4Dt}}}_{\leq 1}\, dy \\
&\leq \frac{1}{\sqrt{4}\pi Dt} \int_{-\infty}^{\infty} |f(y)|\, dy \\
&= C \overset{t\to\infty}{\longrightarrow} 0
\end{aligned}$$

5. "Boundedness:" if $|f(x)| \leq M$, then $|u(x,t)| \leq M$ *Proof:*

$$|u(x,t)| \leq \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} |f(y)| e^{-\frac{(x-y)^2}{4Dt}}\, dy \leq M$$

6. "Conservation of Mass:"

$$\int_{-\infty}^{\infty} u(x,t)\, dx = \int_{-\infty}^{\infty} f(x)\, dx$$

   Proof:

$$\frac{d}{dt} \int_{-\infty}^{\infty} u(x,t)\, dx = 0$$

$$\implies \int_{-\infty}^{\infty} u(x,t)\, dx = \int_{-\infty}^{\infty} u(x,0)\, dx = \int_{-\infty}^{\infty} f(x)\, dx$$

The Wave Equation

$$u_{tt} = c^2 u_{xx}$$

Factoring Method Apply the differential operator to the difference:

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) = 0$$

Let $v = (\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})$ so

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0 \implies v_t - cv_x = 0 \implies v(x,t) = f(x+ct)$$

Solve for u:

$$v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = u_t + cu_x \implies u_t + cu_x = f(x+ct)$$

Solve the inhomogeneous transport equation using undetermined coefficients:

$$u_p = h(x+ct) \overset{\text{plug in}}{\implies} h(x+ct) = \frac{1}{2c} F(x+ct)$$

where F is an antiderivative of f giving a general solution

$$u(x,t) = F(x+ct) + G(x-ct)$$

Coordinate Method Define variables

$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$

Chain rule:

$$\begin{cases} u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta \\ u_{xx} = (u_x)_\xi \xi_x + (u_x)_\eta \eta_x = (u_\xi)_\xi + (u_\eta)_\eta = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \\ u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) \end{cases}$$

$$c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) = c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \Longrightarrow 4u_{\xi\eta} = 0$$

$$u_{\xi\eta} = 0 \implies u_\xi = f(\xi) \implies u = F(\xi) + G(\eta)$$

$$u(x,t) = F(x-ct) + G(x+ct)$$

Fourier Transform

$$\mathcal{F}(u_{tt}) = c^2 \mathcal{F}(u_{xx})$$

$$\frac{d^2}{dt^2}\widehat{u} = c^2(-i\kappa)^2\widehat{u} = -\kappa^2 c^2 \widehat{u}$$

$$\xRightarrow{y''+ay=0} \widehat{u} = \widehat{F}(\kappa)e^{i\kappa ct} + \widehat{G}(\kappa)e^{-i\kappa ct}$$

$$u = F(x-ct) + G(x+ct)$$