APMA 0360: Homework 6

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Note: you do not need to show work for the three cases for boundary value problems except in problem 1.

Problem 1: Wave equation with mixed conditions

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u_x(0,t) = 0 \\ u(2,t) = 0 \end{cases}$$

Assume that $u(x,t) = X(x)T(t)$. Then

$$(X(x)T(t))_{tt} = c^2 (X(x)T(t))_{xx}$$

$$XT'' = c^2 X'' T$$

$$\frac{T''}{c^2 T} = \frac{X''}{X}$$

Then note that

$$\left(\frac{X''}{X}\right)_t = 0$$

and

$$\left(\frac{X''}{X}\right)_x = \left(\frac{T''}{c^2 T}\right)_x = 0$$

so

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$$

for some constant λ giving a system of ODE:

$$\begin{cases} X'' = \lambda X \\ T'' = c^2 T\lambda \end{cases}$$

Using the boundary conditions:

$$\begin{aligned} u(2,t) = 0 &\implies X(2)T(t) = 0 \implies X(2) = 0 \\ u_x(0,t) = 0 &\implies X'(0)T(t) = 0 \implies X'(0) = 0 \end{aligned}$$

Which gives an ODE for X:

$$\begin{cases} X'' = \lambda X \\ X(2) = 0 \\ X'(0) = 0 \end{cases}$$

Auxiliary equation:

$$r^2 = \lambda \implies r = \pm\omega$$

Case 1: $\lambda > 0$

$$X(x) = Ae^{\omega x} + Be^{-\omega x}$$

$$X'(0) = A\omega e^{\omega(0)} + B\omega e^{\omega(0)} = A\omega + B\omega = 0 \implies B = -A$$

$$X(x) = Ae^{\omega x} - Ae^{-\omega x}$$

$$X(2) = Ae^{2\omega} - Ae^{-2\omega} = 0 \implies e^{2\omega} = e^{2\omega} \implies -2\omega = 2\omega \implies \omega = 0$$

So there are no nonzero solutions

Case 2: $\lambda = 0$

$$X(x) = A + Bx$$

$$X(2) = A + 2B = 0 \implies A = -2B$$

$$X(x) = -2B + Bx$$

$$X'(x) = B \implies X'(0) = B = 0 \implies X(x) = 0$$

So there are no nonzero solutions here either

Case 3: $\lambda < 0$

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

$$X'(x) = -A\omega\sin(\omega x) + B\omega\cos(\omega x)$$

$$X'(0) = B\omega = 0 \implies B = 0$$

$$X(x) = A\cos(\omega x)$$

$$X(2) = A\cos(2\omega) = 0 \implies \cos(2\omega) = 0$$

So $\omega = \frac{\pi}{4} + \frac{\pi}{2}m \quad (m = 0, 1, 2, ...)$ and the eigenvalues are

$$\lambda = -(\frac{\pi}{4} + \frac{\pi}{2}m)^2$$

corresponding to the eigenfunctions

$$X(x) = \cos((\frac{\pi}{4} + \frac{\pi}{2}m)x)$$

So back to the T equation:

$$T'' = -c^2(\frac{\pi}{4} + \frac{\pi}{2}m)^2 T$$

$$z^2 + c^2(\frac{\pi}{4} + \frac{\pi}{2}m)^2 = 0$$

$$z = \pm i(c(\frac{\pi}{4} + \frac{\pi}{2}m))$$

$$T(t) = A\cos(c(\frac{\pi}{4} + \frac{\pi}{2}m)) + B\sin(c(\frac{\pi}{4} + \frac{\pi}{2}m))$$

so

$$\boxed{u(x,t) = \sum_{m=0}^{\infty} A_m \cos(c(\frac{\pi}{4} + \frac{\pi}{2}m)) + B_m \sin(c(\frac{\pi}{4} + \frac{\pi}{2}m))\cos((\frac{\pi}{4} + \frac{\pi}{2}m)x)}$$

Problem 2: Wave equation with friction

$$\begin{cases} u_{tt} + ru_t = c^2 u_{xx} \\ u(0,t) = 0 \\ u(\pi, t) = 0 \end{cases}$$

Note: Assume $0 < r < 2c$

Assume $u(x,t) = X(x)T(t)$ so

$$(X(x)T(t))_{tt} + r(X(x)T(t))_t = c^2(X(x)T(t))_{xx}$$

$$XT'' + rXT' = c^2 X''T$$
$$\frac{T'' + rT'}{c^2 T} = \frac{X''}{X}$$
$$\frac{T''}{c^2 T} + \frac{rT'}{c^2 T} = \frac{X''}{X}$$
$$\left(\frac{T''}{c^2 T} + \frac{rT'}{c^2 T}\right)_x + \left(\frac{X''}{X}\right)_t = 0$$
$$\frac{T''}{c^2 T} + \frac{rT'}{c^2 T} = \frac{X''}{X} = \lambda$$

Looking at the X equation:

$$X'' = \lambda X$$

$$u(0,t) = X(0)T(t) = 0 \implies X(0) = 0$$

$$u(\pi, t) = 0 \implies X(\pi) = 0$$

$$r^2 = \lambda \implies r = \pm\omega$$

Looking at $\lambda < 0$:

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

$$X(0) = A = 0 \implies X(x) = B\sin(\omega x)$$

$$X(\pi) = B\sin(\pi\omega) = 0 \implies \sin(\pi\omega) = 0$$

so $\omega = m \quad (m = 0, 1, 2, ...)$ and the eigenvalues are

$$\lambda = -m^2 \quad (m = 0, 1, 2, ...)$$

corresponding to eigenfunction

$$X(x) = \sin(mx)$$

Going back to the T equation:

$$T'' + rT' = c^2 T\lambda$$

$$T'' + rT' = -c^2 m^2 T$$

So we have auxiliary equation

$$s^2 + rs + c^2 m^2 = 0 \implies s = \frac{-r \pm \sqrt{r^2 - 4c^2m^2}}{2}$$

However, as $r < 2c$ we know that $r^2 < 4c^2 \implies r^2 - 4c^2m^2 < 0$ so the solutions are complex. Then,

$$s = \frac{-r \pm i\sqrt{4c^2m^2 - r^2}}{2}$$

and

$$T(t) = e^{-rt}\left(A\cos\left(\frac{\sqrt{4c^2m^2 - r^2}}{2}x\right) + B\sin\left(\frac{\sqrt{4c^2m^2 - r^2}}{2}x\right)\right)$$

so

$$u(x,t) = e^{-rt}\left(A\cos\left(\frac{\sqrt{4c^2m^2 - r^2}}{2}x\right) + B\sin\left(\frac{\sqrt{4c^2m^2 - r^2}}{2}x\right)\right)\sin(mx)$$

which by linearity gives us

$$\boxed{u(x,t) = \sum_{m=0}^{\infty} e^{-rt}\left(A_m\cos\left(\frac{\sqrt{4c^2m^2 - r^2}}{2}x\right) + B_m\sin\left(\frac{\sqrt{4c^2m^2 - r^2}}{2}x\right)\right)\sin(mx)}$$

Problem 3:

$$\begin{cases} tu_t = u_{xx} + 2u \\ u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$$

Note: Define $\lambda' = \lambda - 2$

Let $u(x,t) = X(x)T(t)$. Then

$$t(X(x)T(t))_t = (X(x)T(t))_{xx} + 2(X(x)T(t))$$

$$tXT' = X''T + 2XT$$
$$\frac{tT'}{T} = \frac{X''}{X} + 2$$
$$\frac{X''}{X} = \frac{tT'}{T} + 2 = \lambda$$

for some constant λ or with $\lambda' = \lambda - 2$,

$$\frac{X''}{X} = \frac{tT'}{T} = \lambda'$$

Then

$$\begin{cases} X'' = \lambda' X \\ tT' = \lambda' T \end{cases}$$

Then,

$$u(\pi,t) = 0 \implies X(\pi)T(t) = 0 \implies X(\pi) = 0$$
$$u(0,t) = 0 \implies X(0)T(t) = 0 \implies X(0) = 0$$

Giving us an ODE for X:

$$\begin{cases} X'' = \lambda' X \\ X(\pi) = 0 \\ X(0) = 0 \end{cases}$$

From the auxiliary equation, with $\lambda' < 0$

$$r^2 = \lambda \implies r = \pm\omega i$$

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$
$$X(0) = A = 0 \implies X(x) = B\sin(\omega x)$$

$$X(\pi) = B\sin(\pi\omega) = 0 \implies \sin(\pi\omega) = 0 \implies \omega = m \quad (m = 0, 1, 2, ...)$$

So

$$\lambda' = -m^2 \quad (m = 0, 1, ...)$$

$$X(x) = \sin(mx)$$

Then

$$tT' = -m^2 T$$

$$T = e^{\int \frac{m^2}{t}\, dt} = t^{-m^2}$$

and

$$u(x,t) = \sum_{m=0}^{\infty} A_m \sin(mx) t^{-m^2}$$

Then making the going back to λ from λ':

$$\boxed{u(x,t) = \sum_{m=1}^{\infty} A_m \sin((m - \sqrt{2})x) t^{2-m^2}}$$

Problem 4:

1. Solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = 3\sin(2\pi x) \\ u_t(x,0) = 0 \end{cases}$$

Note: you are allowed to directly use the formula from lecture

From lecture,

$$u(x,t) = \sum_{m=1}^{\infty} (A_m \cos(\pi mct) + B_m \sin(\pi mct)) \sin(\pi mx)$$

Then using the new initial conditions,

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(\pi mx) = 3\sin(2\pi x)$$

Which suggests that $A_2 = 3$ and $A_m = 0$ for all other values of m. And from the other initial condition,

$$u_t(x,t) = \sum_{m=1}^{\infty} (-A_m(\pi mc)\sin(\pi mct) + B_m(\pi mc)\cos(\pi mct)) \sin(\pi mx)$$

$$u_t(x,0) = \sum_{m=1}^{\infty} B_m(\pi mc) \sin(\pi mx) = 0$$

which implies that B_m equals 0 for all m. So at last, we have

$$\boxed{u(x,t) = 3\cos(2\pi x)\sin(2\pi x)}$$

2. Use (1) to solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 5 \\ u(1,t) = 5 \\ u(x,0) = 3\sin(2\pi x) + 5 \\ u_t(x,0) = 0 \end{cases}$$

Let $v(x,t) = u(x,t) - 5$. Then

$$v_{tt} = (u-5)_{tt} = u_{tt} = c^2 u_{xx} = c^2(v+5)_{xx} = c^2 v_{xx}$$

So

$$\begin{cases} v_{tt} = c^2 v_{xx} v(0,t) = u(0,t) - 5 = 0 \\ v(1,t) = u(1,t) - 5 = 0 \\ v(x,0) = u(x,0) - 5 = 3\sin(2\pi x) \\ v_t(x,0) = u_t(x,0) = 0 \end{cases}$$

But this system is exactly the same as from part 1! So

$$v(x,t) = 3\cos(2\pi x)\sin(2\pi x)$$

And as, $u = v + 5$,

$$\boxed{u(x,t) = 3\cos(2\pi x)\sin(2\pi x) + 5}$$

3. Use (1) to solve

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 3 \\ u(1,t) = 5 \\ u(x,0) = 3\sin(2\pi x) + (2x+3) \\ u_t(x,0) = 0 \end{cases}$$

Let $v(x,t) = u(x,t) - f(x)$ where f is a linear function such that $f(0) = 3$ and $f(1) = 5$. Then

$$f(x) = \frac{5-3}{1-0}(x-0) + 3 = 2x + 3$$

and

$$v(x,t) = u(x,t) - 2x - 3$$

Which gives us a new PDE

$$\begin{cases} v_{tt} = c^2 v_{tt} \\ v(0,t) = 0 \\ v(l,t) = 0 \\ v(x,0) = 3\sin(2\pi x) + (2x+3) - (2x+3) = 3\sin(2\pi x) \\ v_t(x,0) = u_t(x,0) - 2(0) = 0 \end{cases}$$

Once again, this corresponds to the solution from part 1 so

$$v(x,t) = 3\cos(2\pi x)\sin(2\pi x)$$

and

$$\boxed{u(x,t) = 3\cos(2\pi x)\sin(2\pi x) + 2x + 3}$$

Problem 5: The Infinity Laplacian

$$(u_t)^2(u_{tt}) + 2(u_x)(u_t)(u_{xt}) + (u_x)^2(u_{xx}) = 0$$

1. For separation of variables use $u(x,t) = X(x) + T(t)$ to show that

$$(T')^2 T'' = -(X')^2 X''$$

Let $u(x,t) = X(x) + T(t)$ so

$$\begin{aligned} &((X(x)+T(t))_t)^2((X(x)+T(t))_{tt}) \\ &\quad + 2((X(x)+T(t))_x)((X(x)+T(t))_t)((X(x)+T(t))_{xt}) \\ &\quad + ((X(x)+T(t))_x)^2((X(x)+T(t))_{xx}) = 0 \end{aligned}$$

$$\begin{aligned} (T')^2(T'') + 2(X')(T')(0) + (X')^2(X'') &= 0 \\ (T')^2 T'' + (X')^2(X'') &= 0 \\ (T')^2 T'' &= -(X')^2(X'') \quad \blacksquare \end{aligned}$$

2. Show both sides are constant, equal to λ. For simplicity assume $\lambda = 1/3$. Use this to solve for X and T and then for u(x, t) Assume that any constants of integration are 0.

$$\begin{aligned} ((T')^2 T'')_x &= 0 \\ ((T')^2 T'')_t = (-(X')^2(X''))_t &= 0 \end{aligned}$$

So both sides are constant and

$$(T')^2 T'' = -(X')^2(X'') = \lambda = \frac{1}{3}$$

This gives us a system of ODEs

$$\begin{cases} (T')^2 T'' = \frac{1}{3} \\ -(X')^2(X'') = \frac{1}{3} \end{cases}$$

Starting with T, let $v(t) = T'(t)$ so

$$\begin{aligned} v^2 v' &= \frac{1}{3} \\ v^2 \frac{dv}{dt} &= \frac{1}{3} \\ v^2 \, dv &= \frac{1}{3} \, dt \\ \frac{1}{3} v^3 &= \frac{1}{3} t + C_1 \\ v &= \sqrt[3]{t + C_1} \end{aligned}$$

So

$$T' = (t + C_2)^{\frac{1}{3}}$$

$$T = \frac{3}{4}(t + C_1)^{\frac{4}{3}} + C_2$$

but assuming that all the constants of integration are 0,

$$T = \frac{3}{4} t^{\frac{4}{3}}$$

Similarly, for x let $w(x) = X'(x)$ so

$$\begin{aligned} w^2 w' &= -\frac{1}{3} \\ \frac{w^3}{3} &= -\frac{1}{3} x + C_1 \\ w &= \sqrt[3]{-x + C_1} \end{aligned}$$

$$X' = (-x + C_1)^{\frac{1}{3}} \implies X = \frac{3}{4}(-x + C_1)^{\frac{4}{3}} + C_2 \implies X = \frac{3}{4} x^{\frac{4}{3}}$$

so

$$\boxed{u(x,t) = \frac{3}{4}(t^{\frac{4}{3}} + x^{\frac{4}{3}})}$$