APMA 0360: Final Exam Review

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Contents

1 Things to know

1.1 Trig Identities

1. $\sin^2 x + \cos^2 x = 1$
2. $1 + \tan^2 x = \sec^2 x$
3. $\cos(-x) = \cos(x)$
4. $\sin(-x) = -\sin(x)$
5. $\cos(2x) = \cos^2 x - \sin^2 x$
6. $\sin(2x) = 2\cos x \sin x$
7. $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$
8. $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$
9. $\int_{-\infty}^{\infty} \cos(x^2)\ dx = \sqrt{\frac{\pi}{2}}$
10. $\int_{-\infty}^{\infty} \sin(x^2)\ dx = \sqrt{\frac{\pi}{2}}$

1.2 Integrations

1. $\int \tan x\ dx = \ln|\sec| + C$
2. $\int \sec c\ dx = \ln|\sec x + \tan x| + C$
3. $\int \frac{1}{1+x^2}\ dx = \tan^{-1} x + C$
4. $\int \sin^2(x)\ dx = \frac{x}{2} - \frac{1}{4}\sin(2x) + C$
5. $\int \cos^2(x)\ dx = \frac{x}{2} + \frac{1}{4}\sin(2x) + C$

Integration by parts:

$$\int f(x)\, g'(x)\ dx = f(x)g(x) - \int g(x)\, f'(x)\ dx$$

1.3 Ordinary Differential Equations

1.3.1 Separation of Variables

1.3.2 Integrating Factors

1.3.3 Auxiliary Equations

1.3.4 Undetermined Coefficients

2 Introduction

2.1 Check if a functions solves a PDE

To check if a given function solves a PDE, simply plug it in and differentiate to get an identity.

Example: Show that $u = f(x)g(y)$ solves $uu_{xy} = u_x u_y$

$$\begin{aligned} (fg)(fg)_{xy} &= (fg)_x(fg)_y \\ (fg)(f'g)_y &= (f'g)(fg') \\ &= (fg)(f'g') & &= f' \cdot f \cdot g' \cdot g \\ &= f' \cdot f \cdot g' \cdot g & &= f' \cdot f \cdot g' \cdot g \quad \checkmark \end{aligned}$$

2.2 Simple PDEs

- $u_x = 0 \implies u(x,y) = f(y)$
- $u_{xx} = 0 \implies u_x = f(y) \implies u(x,y) = xf(y) + g(y)$
- $u_{xx} + u = 0 \overset{y''+y=0}{\implies} u(x,y) = A(y)\cos x + B(y)\sin x$
- $u_{xy} = 0 \implies (u_x)_y = 0 \implies u_x = f(x) \implies u(x,y) = F(x) + G(y)$

2.3 Classification

Order: highest degree derivative

Example: The order of $2x^4u_{xxx} + 5yu_{xy} + 6u_{yyy} + 6u = x^4 + y^5$ is 3

Constant coefficient

Linear: coefficients depend on x, y but not u

- $L(u + v) = L(u) + L(v)$
- $L(cu) = cL(u)$

Example: The PDE $e^x u_{xx} + \sin(y)u_{yy} + \ln(xy)u = \cos(x^2 + y^2)$ is linear.

Homogeneous: RHS is 0

Example: The PDE $u_{xx} + 5u_{xy} = x^2 + y^2$ is Inhomogeneous

2.4 Conics

For a second order PDE of the canonical form

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y)$$

Let $D = b^2 - 4ac$. Then the PDE is

- elliptic if $D < 0$
- parabolic if $D = 0$
- hyperbolic if $D > 0$

Example: The type of the second-order PDE is $2u_{xx} + 3u_{xy} + 2u_{yy} = 0$ is elliptic because $(3)^2 - 4(2)(2) < 0$.

3 First Order-Linear PDE

3.1 Slope Method

1. Write as directional derivative
2. Show u is constant on characteristic lines

Example 1: $au_x + bu_y = 0$

$$\langle u_x, u_y \rangle \cdot \langle a, b \rangle = \nabla u \cdot \vec{v} = 0$$
$$m = \frac{b}{a} \implies y = \frac{b}{a}x + C \implies ay - bx = C$$
$$u(x, y) = f(ay - bx)$$

Example 2: $u_x + yu_y = 0$

$$\nabla u \cdot (1, y) = 0$$
$$\frac{y}{1} = y' \quad \text{(slope = derivative)}$$
$$y' = y \implies y = Ce^x \implies ye^{-x} = C$$
$$u(x, y) = f(ye^{-x})$$

Example 3:

$$\begin{cases} (2y)u_x + (3x^2 - 1)u_y = 0 \\ u(0, y) = \cos(y) \end{cases}$$

$$\nabla u \cdot (2y, 3x^2 - 1) = 0$$
$$\frac{3x^2 - 1}{2y} = y'$$
$$3x^2 - 1 \; dx = 2y \; dy$$
$$x^3 - x + C = y^2$$
$$y^2 - x^3 - x = C$$
$$u(x, y) = f(y^2 - x^3 - x)$$
$$u(0, y) = f(y^2) = \cos(y)$$
$$u(x, y) = \cos(\sqrt{y^2 - x^3 + x})$$

3.2 Coordinate Method

1. Define new variables ξ and η that are perpendicular
2. Rewrite the chain rule in ξ and η
3. Substitute definitions and solve

Example 1: $2u_x + 3u_y$

$$\{\xi = 2x + 3y \quad \text{(from equation)} \; \eta = -3x + 2y \quad \text{(perpendicular)}$$
$$u_x = u_\xi \cdot \xi_x + u_\eta \cdot \eta_x = 2u_\xi - 3u_\eta$$
$$u_y = u_\xi \cdot \xi_y + u_\eta \cdot \eta_y = 3u_\xi + 2u_\eta$$
$$2u_x + 3u_y = 2(2u_\xi - 3u_\eta) + 3(3u_\xi + 2u_\eta) = 13u_\xi = 0$$
$$u(\xi, \eta) = f(\eta)$$
$$u(x, y) = f(2y - 3x)$$

Example 2: $u_x + 2u_y + (2x - y)u = 0$

$$\begin{cases} \xi = x + 2y \\ \eta = -2x + y \end{cases}$$
$$u_x = u_\xi \cdot \xi_x + u_\eta \cdot \eta_x = u_\xi - 2u_\eta$$
$$u_y = 2u_\xi + u_\eta$$
$$(u_\xi - 2u_\eta) + 2(2u_\xi + u_\eta) - \eta u = 0$$
$$5u_\xi - \eta u = 0$$

Via ODEs,

$$u_\xi - \frac{\eta}{5}u = 0$$
$$(u\exp(\int -\frac{\eta}{5}\ d\xi))_\xi = 0$$
$$u\exp(-\frac{1}{5}\xi\eta) = f(\eta)$$
$$u(\xi,\eta) = f(\eta)\exp(\frac{1}{5}\xi\eta)$$
$$u(x,y) = f(-2x+y)\exp(\frac{1}{5}(x+2y)(-2x+y))$$

3.3 Transform Method

Rewrite the derivatives of u in terms of a new PDE v and solve.

Example 1: $au_x + bu_y + cu = 0$ where a, b, c are constants, $a \neq 0$ and $v(x,y) = u(x,y)e^{\frac{cx}{a}}$

$$u_x + \frac{c}{a}u = -\frac{b}{a}u_y$$
$$(u\exp(\frac{c}{a}x))_x = -\frac{b}{a}u_y \cdot \exp(\frac{c}{a}x)$$
$$v_x = -\frac{b}{a}u_y\exp(\frac{cx}{a})$$
$$v_x = -\frac{b}{a}v_y$$
$$v(x,y) = f(ay-bx) = u(x,y)\exp(\frac{cx}{a})$$
$$u(x,y) = f(ay-bx)\exp(-\frac{cx}{a})$$

4 The Transport Equation

4.1 Derivation

The mass on an interval is

$$M = \int_0^b u(x,t)\ dx$$

But mass is conserved so

$$
\begin{gathered}
M_1 = M_2 = \int_0^b u(x,t)\,dx = \int_{ch}^{b+ch} u(x,t+h)\,dx \\
\frac{d}{db}\int_0^b u(x,t)\,dx = \frac{d}{db}\int_{ch}^{b+ch} u(x,t+h)\,dx \\
u(b,t) = u(b+ch,t+h) \\
0 = u_x \cdot (b+ch)_h + u_t \cdot (t+h)_h = cu_x + u_t
\end{gathered}
$$

4.2 Solution

$$
u_t + cu_x = 0 \implies cu_x + u_t = 0 \tag{1}
$$

$$
u(x,t) = f(x-ct) \tag{2}
$$

5 Fourier Transform

5.1 The Gaussian

$$
I = \int_{-\infty}^{\infty} e^{-x^2}\,dx
$$

Derivation:

$$
\begin{aligned}
I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \\
&= 2\pi \int_0^{\infty} re^{-r^2}\,dt \\
&= \pi \\
I &= \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}
\end{aligned}
$$

5.2 The Fourier Transform

$$
\widehat{f}(\kappa) = \int_{-\infty}^{\infty} f(x)\,e^{i\kappa x}\,dx
$$

5.3 $\mathcal{F}(e^{-x^2})$

$$\begin{aligned}
\widehat{f}(\kappa) = \int_{-\infty}^{\infty} e^{-x^2}\ e^{i\kappa x}\ dx\ \widehat{f}'(\kappa) &= i\int_{-\infty}^{\infty} xe^{-x^2}\ e^{i\kappa x}\ dx \\
&= i\left[-\frac{1}{2}e^{-x^2}e^{i\kappa x}\right]_{-\infty}^{\infty} - i\int_{-\infty}^{\infty} -\frac{1}{2}e^{-x^2}\ e^{i\kappa x}\ dx \\
&= -\frac{\kappa}{2}\widehat{f}(\kappa)
\end{aligned}$$

$$\widehat{f}'(\kappa) = -\frac{\kappa}{2}\widehat{f}(\kappa) \implies C\exp(-\frac{\kappa^2}{4}) \implies \widehat{f}(\kappa) = \sqrt{\pi}\exp(-\frac{\kappa^2}{4})$$

General Form:

$$\mathcal{F}(e^{-ax^2}) = \sqrt{\frac{\pi}{a}}\exp(-\frac{\kappa^2}{4a})$$

5.4 Derivatives

$$\widehat{f}'(\kappa) = -i\kappa\widehat{f}(\kappa)$$

Proof:

$$\begin{aligned}
\widehat{f}'(\kappa) &= \int_{-\infty}^{\infty} f'(x)e^{i\kappa x}\ dx \\
&= [f(x)e^{i\kappa x}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\ \frac{d}{dx}e^{i\kappa x}\ dx \\
&= 0 - i\kappa\int_{-\infty}^{\infty} f(x)e^{i\kappa x}\ dx \\
&= -i\kappa\widehat{f}(\kappa)
\end{aligned}$$

5.5 Convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\ dy$$

$$\mathcal{F}((f * g)(\kappa)) = \widehat{f}(\kappa)\cdot\widehat{g}(\kappa)$$

5.6 Inverse Fourier

$$\mathcal{F}^{-1}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\kappa)e^{-i\kappa x}\ d\kappa$$

Example:

$$\begin{cases} u_t = -2u_{xxxx} \\ u(x,0) = f(x) \end{cases}$$

$$
\begin{aligned}
\mathcal{F}(u_t) &= \mathcal{F}(-2u_{xxxx}) \\
\frac{d}{dt}\mathcal{F}(u) &= -2(-i\kappa)^4\mathcal{F}(u) = -2\kappa^4\mathcal{F}(u) \\
\mathcal{F}(u) &= \widehat{f}(\kappa)e^{-2\kappa^4 t} \\
\mathcal{F}(u) &= \widehat{f}(\kappa)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2\kappa^4 t - i\kappa x}\ d\kappa\right) \\
u(x,t) &= \int_{-\infty}^{\infty} f(y)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2\kappa^4 t - i\kappa(x-y)}\ d\kappa\right)\ dy
\end{aligned}
$$

5.7 Shifting

If $g(x) = f(x-a)$ then

$$\widehat{g}(\kappa) = e^{i\kappa a}\widehat{f}(\kappa)$$

Proof:

$$
\begin{aligned}
\widehat{g}(\kappa) &= \mathcal{F}(f(x-a)) \\
&= \int_{-\infty}^{\infty} f(x-a)e^{i\kappa(x+a)}\ dx \\
&= e^{i\kappa a}\int_{-\infty}^{\infty} f(x-a)e^{i\kappa x}\ dx \\
&= e^{i\kappa a}\widehat{f}(\kappa)
\end{aligned}
$$

Example Application: Solve the transport PDE

$$
\begin{cases}
u_t + cu_x = 0 \\
u(x,0) = f(x)
\end{cases}
$$

$$
\begin{aligned}
\mathcal{F}(u_t) &= \mathcal{F}(-cu_x) \\
\frac{d}{dt}\mathcal{F}(u) &= i\kappa c\mathcal{F}(u) \\
\mathcal{F}(u) &= \mathcal{F}(u(x,0))e^{i\kappa ct} \\
\mathcal{F}(u) &= \widehat{f}(\kappa e^{i\kappa ct}) \\
&= \mathcal{F}(f(x-ct)) \\
u(x,t) &= f(x-ct)
\end{aligned}
$$

6 The Heat Equation

6.1 Solution

Example:

$$\begin{cases} u_t = Du_{xx} \\ u(x,0) = e^{3x} \end{cases}$$

$$\begin{gathered}
\mathcal{F}(u_t) = \mathcal{F}(Du_{xx}) \\
\frac{d}{dt}\mathcal{F}(u) = (-i\kappa)^2 D\mathcal{F}(u) \\
\mathcal{F}(u) = \mathcal{F}(e^{3x}) \exp(-\kappa^2 Dt) \\
\mathcal{F}(u) = \mathcal{F}(e^{3x})\mathcal{F}(\frac{1}{\sqrt{4\pi Dt}} \exp(-\frac{x^2}{4Dt})) \\
u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \exp(3y - \frac{(x-y)^2}{4Dt})\ dy
\end{gathered}$$

Looking at the exponent:

$$\begin{aligned}
3y - \frac{(x-y)^2}{4Dt} &= \frac{12Dty - x^2 + 2xy - y^2}{4Dt} \\
&= -\frac{y^2 - (12Dt + 2x)y + x^2}{4Dt} \\
&= -\frac{y^2 - (12Dt + 2x)y + (6Dt + x)^2 + x^2 - (6Dt + x)^2}{4Dt} \\
&= -\frac{(y - 6Dt - x)^2 + x^2 - x^2 + 12Dtx + 36D^2t^2}{4Dt} \\
&= -\frac{(y - x - 6Dt)^2 + 12Dt(x + 3Dt)}{4Dt} \\
&= -\frac{(y - x - 6Dt)^2}{4Dt} + 3(x + 3Dt)
\end{aligned}$$

Substituting this back in,

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \exp(-\frac{(y - x - 6Dt)^2}{4Dt} + 3(x + 3Dt))\ dy \\
&= \frac{e^{3(x+3Dt)}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \exp(-\left(\frac{y - x - 6Dt}{\sqrt{4Dt}}\right)^2)\ dy
\end{aligned}$$

$$p = \frac{y - x - 6Dt}{\sqrt{4Dt}} \implies dp = \frac{dy}{\sqrt{4Dt}} \implies dy = \sqrt{4Dt}\ dp$$

and

$$u(x,t) = \frac{e^{3(x+3Dt)}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-p^2} \sqrt{4Dt}\ dp = \frac{\exp(3(x + 3Dt))}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\ dp$$

$$u(x,t) = \exp(3x + 9Dt)$$

6.2 Properties

1. Infinite Speed of Propagation: if $f \geq 0$ is positive somewhere and continuous, it is positive everywhere

   Proof: the exponential function is positive and if f is continuous, it is positive around a region x_0 so the full infinite integrand is positive but that subset is strictly positive so u is strictly positive

2. Smoothness: u is infinitely differentiable

   Proof: the exponential term is infinitely differentiable

3. Irreversibility: $u(x, 0)$ cannot be determined from $u(x, 1)$

   Proof: $u(x, 1) = |x|$ can be given but is not smooth so this is a contradiction with the earlier property

4. Dissipation over time: $\lim_{t\to\infty} u(x, t) = 0$

   Proof:

$$\begin{aligned}
|u(x,t)| &= \left| \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \exp(-\frac{(x-y)^2}{4Dt}) f(y)\, dy \right| \\
&\leq \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} |f(y)| \underbrace{\exp(-\frac{(x-y)^2}{4Dt})}_{\leq 1}\, dy \\
&\leq \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} |f(y)|\, dy \\
&= C \overset{t\to\infty}{\longrightarrow} 0
\end{aligned}$$

5. Boundedness: if $|f(x)| \leq M$ then $|u(x,t)| \leq M$

   Proof:

$$|u(x,t)| \leq \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} |f(y)| \exp(-\frac{(x-y)^2}{4Dt})\, dy \leq M$$

6. Conservation of mass:

$$\int_{-\infty}^{\infty} u(x,t)\, dx = \int_{-\infty}^{\infty} f(x)\, dx$$

   Proof:

$$\begin{aligned}
&\frac{d}{dt} \int_{-\infty}^{\infty} u(x,t)\, dx = 0 \\
\implies &\int_{-\infty}^{\infty} u(x,t)\, dx = \int_{-\infty}^{\infty} u(x,0)\, dx = \int_{-\infty}^{\infty} f(x)\, dx
\end{aligned}$$

6.3 Transform Method

Example 1:

$$\begin{cases} u_t = Du_{xx} + cu_x - au \\ u(x,0) = f(x) \end{cases}$$

Let $v(x,t) = u(x-ct,t)e^{at}$. Then,

$$u(x-ct,t) = v(x,t)e^{-at}$$
$$u(x,t) = v(x+ct,t)e^{-at}$$

$$\begin{cases} u_x = v_x(x+ct,t)e^{-at} \\ u_{xx} = v_{xx}(x+ct,t)e^{-at} \\ u_t = cv_x(x+ct,t)e^{-at} + v_t(x+ct,t)e^{-at} - av(x+ct,t)e^{-at} \end{cases}$$

$$cv_x(x+ct,t)e^{-at} + v_t(x+ct,t)e^{-at} - av(x+ct,t)e^{-at} = Dv_{xx}e^{-at} + cv(x,t)e^{-at} - av(x+ct,t)e^{-at}$$
$$v_t(x+ct,t)e^{-at} = Dv_{xx}(x+ct,t)e^{-at}$$
$$v_t(x+ct,t) = Dv_{xx}(x+ct,t)$$
$$v(x+ct,t) = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} f(y)\exp(-\frac{((x+ct)-y)^2}{4Dt})\,dy$$
$$u(x,t) = \frac{e^{-at}}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} f(y)\exp(-\frac{((x+ct)-y)^2}{4Dt})\,dy$$

7 Wave Equation

$$u_{tt} = c^2 u_{xx}$$

7.1 Factoring Method

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) = 0$$

Then let $v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)$ so

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = v_t - cv_x = 0 \implies v(x,t) = f(x+ct)$$
$$v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = u_t + cu_x = f(x+ct)$$
$$u_0 = G(x-ct)u_p = h(x+ct)$$
$$(h(x+ct))_t + c(h(x+ct))_x = f(x+ct)$$
$$ch'(x+ct) + ch'(x+ct) = f(x+ct)$$
$$h(x+ct) = \frac{1}{2c}F(x+ct)$$
$$u(x,t) = F(x+ct) + G(x-ct)$$

7.2 Coordinate Method

Define variables

$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$

Then

$$\begin{cases} u_x = u_\xi \cdot \xi_x + u_\eta \cdot \eta_x = u_\xi + u_\eta \\ u_{xx} = (u_\xi + u_\eta)_\xi \cdot \xi_x + (u_\xi + u_\eta)_\eta \cdot \eta_x = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \\ u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) \end{cases}$$

$$c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) = c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$
$$4u_{\xi\eta} = 0$$
$$u_{\xi\eta} = 0 \implies u_\xi = f(\xi)$$
$$u(x,t) = F(\xi) + G(\eta)$$
$$u(x,t) = F(x - ct) + G(x + ct)$$

Example: $u_{xx} + u_{xt} - 20u_{tt} = 0$ with $\xi = 5x - t$ and $\eta = 4x + t$.

$$u_x = u_\xi \xi_x + u_\eta \eta_x = 5u_\xi + 4u_\eta$$
$$u_{xx} = (u_x)_\xi \xi_x + (u_x)_\eta \eta_x = 25u_{\xi\xi} + 40u_{\eta\xi} + 16u_{\eta\eta}$$
$$u_{xt} = (u_x)_\xi \xi_t + (u_x)_\eta \eta_t = -5u_{\xi\xi} + u_{\eta\xi} + 4u_{\eta\eta}$$
$$u_t = u_\xi \xi_t + u_\eta \eta_t = -u_\xi + u_\eta$$
$$u_{tt} = (u_t)_\xi \xi_t + (u_t)_\eta \eta_t = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$$

$$25u_{\xi\xi} + 40u_{\eta\xi} + 16u_{\eta\eta} - 5u_{\xi\xi} + u_{\eta\xi} + 4u_{\eta\eta} - 20(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) = 0$$

$$u_{\xi\eta} = 0$$

$$u_\xi = f(\xi)$$

$$u = F(\xi) + G(\eta)$$

$$u(x,t) = F(5x - t) + G(4x + t)$$

7.3 Fourier Method

$$\mathcal{F}(u_{tt}) = \mathcal{F}(c^2 u_{xx})$$
$$\frac{d^2}{dt^2}\mathcal{F}(u) = (-i\kappa)^2 c^2 \mathcal{F}(u)$$
$$\frac{d^2}{dt^2}\mathcal{F}(u) = -(\kappa c)^2 \mathcal{F}(u)$$
$$\mathcal{F}(u) = \widehat{F}(\kappa)e^{i\kappa ct} + \widehat{G}(\kappa)e^{-i\kappa ct}$$
$$u(x,t) = F(x - ct) + G(x + ct)$$

7.4 D'Alembert's Formula

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

The general wave equation solution is

$$u(x,t) = F(x-ct) + G(x+ct)$$

so

$$u(x,0) = \phi(x) = F(x) + G(x)$$

$$u_t(x,0) = \psi(x) = -cF'(x) + cG(x) \implies -F'(x) + G'(x) = \frac{\psi(x)}{c}$$

$$\int_0^x -F'(s) + G'(s)\ ds = \int_0^x \frac{\psi(s)}{c}\ ds$$

$$-F(x) + G(x) - (-F(0) + G(0)) = \frac{1}{c}\int_0^x \psi(s)\ ds$$

Which gives system of equations

$$\begin{cases} -F(x) + G(x) = A + \frac{1}{c}\int_0^x \psi(s)\ ds \\ F(x) + G(x) = \phi(x) \end{cases} \implies \begin{cases} G(x) = \frac{\phi(x)}{2} + \frac{A}{2} + \frac{1}{2c}\int_0^x \psi(s)\ ds \\ F(x) = \frac{\phi(x)}{2} - \frac{A}{2} - \frac{1}{2c}\int_0^x \psi(s)\ ds \end{cases}$$

Which substituted back into the general solution give us D'Alembert's Formula:

$$\frac{1}{2}(\phi(x-ct) + \psi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds$$

8 Energy Methods

1. Multiply by a clever function (usually u or u_t)
2. Integrate WRT x

Example: Heat Equation

$$\begin{cases} u_t = Du_{xx} \\ u(x,0) = 0 \\ u(0,t) = 0 \\ u(l,t) = 0 \end{cases}$$

$$u_t u = D u_{xx} u$$
$$\int_0^l u_t u \; dx = \int_0^l D u_{xx} u \; dx$$
$$\frac{d}{dt}\left(\int_0^l u^2 \; dx\right) = D\left[u_x(l,t)u(l,t) - u_x(0,t) - \int_0^l u_x u_x \; dx\right]$$
$$\frac{d}{dt}\left(\int_0^l u^2 \; dx\right) = -D\int_0^l (u_x)^2 \; dx$$
$$-D\int_0^l (u_x)^2 \; dx \leq 0 \implies E(t) = \frac{1}{2}\int_0^l u^2 \; dx \leq 0$$
$$E(t) = \frac{1}{2}\int_0^l (u(x,t))^2 \; dx \leq E(0) = \frac{1}{2}(u(x,0))^2 \; dx = 0$$
$$0 \leq E(t) \leq E(0) = 0 \implies E(t) = 0 \quad \forall\, x, t$$

8.1 Uniqueness of Solutions

Wave Equation: There is at most one solution of

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Proof: Let u and v be solutions. Then let $w = u - v$. This also solves the PDE

$$w_{tt} = c^2 w_{xx}$$
$$w(x,0) = u(x,0) - v(x,0) = \phi(x) - \phi(x) = 0$$
$$w_t(x,0) = u_t(x,0) - v_t(x,0) = \psi(x) - \psi(x) = 0$$

Then with

$$w_{tt} = c^2 w_{xx}$$
$$w_{tt} w_t = c^2 w_{xx} w_t$$
$$\int_{-\infty}^{\infty} w_{tt} w_t \; dx = \int_{-\infty}^{\infty} w_{xx} w_t \; dx$$
$$\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2 \; dx\right) = [w_x w_t]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} w_x w_{xt} \; dx$$
$$\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2 \; dx\right) = -\int_{-\infty}^{\infty} \frac{d}{dt}\left(\frac{1}{2}(w_x)^2\right) \; dx$$
$$\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2 + c^2 (w_x)^2 \; dx\right) = 0$$
$$\frac{d}{dt}E(t) = 0$$

Therfore, $E(t) = E(0)$ and

$$\frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2 + c^2(w_x)^2 \; dx = \frac{1}{2}\int_{-\infty}^{\infty} (w_t(x,0))^2 + c^2(w_x(x,0))^2 \; dx$$

but $w(x,0) = 0 \implies (w(x,0))_x = 0 \implies w_x(x,0) = 0$ so

$$\frac{1}{2}\int_{-\infty}^{\infty} (w_t)^2 + c^2(w_x)^2 \; dx = 0$$

then as $(w_t)^2 + c^2(w_x)^2 \geq 0$ and the total integral is zero, by a Useful Hint $(w_t)^2 + c^2(w_x)^2 = 0$ wo

$$\begin{cases} w_t = 0 \\ w_x = 0 \end{cases} \implies w(x,t) = C$$

but $w(x,0) = 0 \implies w(x,t) = 0 = u - v \implies u = v$ and there is at most one solution.

8.1.1 Midterm 2 Question

Suppose f is a function such that $f(0) = 0$ and for all x we have $xf(x) \geq 0$. Show that $u = 0$ is the only solution to the PDE

$$\begin{cases} u_t = -u_{xxxx} - f(u) \\ u(x,0) = 0 \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u_x(0,t) = 0 \\ u_x(L,t) = 0 \end{cases}$$

Solution: By energy methods,

$$\begin{aligned}
u_t u &= -u_{xxxx}u - f(u)u \\
\int_0^L \frac{d}{dt}\frac{1}{2}(u)^2 \; dx &= \int_0^L -u_{xxxx}u - \int_0^L f(u)u \; dx \\
\frac{d}{dt}\int_0^L \frac{1}{2}(u)^2 \; dx &= [-u_{xxx}u]_{x=0}^{x=L} - \int_0^L -u_{xxx}u_x \; dx - \int_0^L f(u)u \; dx \\
&= [u_{xx}u_x]_0^L - \int_0^L (u_{xx})^2 \; dx - \int_0^L f(u)u \; dx \\
&= -\int_0^L (u_{xx})^2 \; dx - \int_0^L f(u)u \; dx \\
E(t) &= \int_0^L (u)^2 \; dx \\
\frac{d}{dt}E(t) &\leq 0 \quad (u_{xx}, f(u)u \geq 0)
\end{aligned}$$

Which means that $E(t) \leq E(0)$:

$$0 \leq \int_0^L (u(x,t))^2 \, dx \leq \int_0^L (u(x,0))^2 \, dx = 0$$

so $u(x,t) = 0$ for all x and t. So there can only be one solution.

Now observe that for $u = 0$

$$(0)_t = -(0)_{xxxx} - f(0) \implies f(0) = 0$$

which is given so $u = 0$ is the only solution.

8.2 Monotony

Monotone: if $(f(x) - f(y))(x - y) \geq 0$ for all x and y.

Claim: if f is monotone, there is at most one solution to $u_{xx} = f(u)$ where $u(0) = 2$ and $-\infty < x < \infty$.

Proof: Let u and v be solutions to the PDE with $w = u - v$. Then

$$w_{xx} = u_{xx} - v_{xx} = f(u) - f(v)$$

Then using energy methods,

$$\begin{aligned} w_{xx}w &= (f(u) - f(v))w \\ \int_{-\infty}^{\infty} w_{xx}w \, dx &= \int_{-\infty}^{\infty} (f(u) - f(v))w \, dx \\ [w_x w]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (w_x)^2 \, dx &= \int_{-\infty}^{\infty} (f(u) - f(v))w \, dx \\ -\int_{-\infty}^{\infty} (w_x)^2 \, dx &= \int_{-\infty}^{\infty} (f(u) - f(v))(u - v) \, dx \end{aligned}$$

Then as f is monotone, the RHS is non-negative. But the LHS integrand is positive so the LHS is negative or zero. Hence, $w_x = 0$ and $w(x,t) = C$.

$$w(0) = u(0) - v(0) = 2 - 2 = 0 \implies w(x,t) = 0 = u - v \implies u = v$$

so there is only one solution.

8.3 Higher Dimensions

Example: n-dimensional Heat Equation

Show that there is at most one solution to

$$\begin{cases} u_t = D\Delta u + f(x,t) & \in \Omega \\ u(x,t) = g(x,t) & x \in \partial\Omega \\ u(x,0) = h(x) & x \in \Omega \end{cases}$$

Let u, v be solutions and $w = u - v$. Then

$$
\begin{gathered}
w_t = u_t - v_t = D\Delta u + f(x,t) - D\Delta v - f(x,t) = D\Delta w \\
w(x,t) = u - v = g - g = 0 \\
w(x,0) = u - v = h - h = 0
\end{gathered}
$$

So we can use energy methods on

$$
\begin{cases}
w_t = D\Delta w \\
w(x,t) = 0 \\
w(x,0) = 0
\end{cases}
$$

Using the general integration by parts formula $\int_\Omega (\Delta u)v\ dx = -\int_\Omega (\nabla u)\cdot(\nabla v)\ dx$,

$$
\begin{gathered}
w_t w = D\Delta w \cdot w \\
\frac{d}{dt}\int_\Omega \frac{1}{2}(w)^2\ dx = D\int_\Omega (\Delta w)(w)\ dx \\
\frac{d}{dt}\int_\Omega \frac{1}{2}(w)^2\ dx = -D\int_\Omega \underbrace{||\nabla w||^2}_{\geq 0}\ dx \\
\frac{d}{dt}\int_\Omega \frac{1}{2}(w)^2\ dx \leq 0
\end{gathered}
$$

so with $E(t) \geq 0$,

$$
E'(t) \leq 0 \implies 0 \leq E(t) \leq E(0) = \int_\Omega \frac{1}{2}(w(x,0))^2\ dx = 0 \implies E(t) = 0
$$

Hence, $w(x,t) = 0 \in \Omega$ and by the initial conditions, $w(x,t) = 0 \in \partial\Omega$. Then with $w = u - v = 0 \implies u = v$ and there is only one solution.

9 Separation of Variables

9.1 Heat

Example 2:

$$
\begin{cases}
tu_t = u_{xx} - u \\
u(0,t) = 0 \\
u(\pi,t) = 0 \\
u(x,1) = 1
\end{cases}
$$

$$u(x,t) = X(x)T(t)$$
$$tXT' = X''T - XT$$
$$\frac{tT'}{T} = \frac{X'' - X}{X}$$
$$\frac{tT'}{T} + 1 = \frac{X''}{X} = \lambda$$
$$u(\pi,t) = X(\pi)T(t) = 0 \implies X(\pi) = 0$$
$$u(0,t) = 0 \implies X(0) = 0$$

$$\begin{cases} X'' = \lambda X \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

$\lambda > 0$:

$$X = Ae^{\omega x} + Be^{-\omega x}$$
$$X(0) = A + B = 0 \implies X = Ae^{\omega x} - Ae^{-\omega x}$$
$$X(\pi) = Ae^{\omega\pi} - Ae^{-\omega\pi} = 0 \implies \omega = 0$$
$$X = 0$$

$\lambda = 0$:

$$X = Ax + B$$
$$X(0) = B = 0$$
$$X(\pi) = A\pi = 0 \implies A = 0$$
$$X = 0$$

$\lambda < 0$:

$$X = A\cos(\omega x) + B\sin(\omega x)$$
$$X(0) = A = 0$$
$$X(\pi) = B\sin(\omega\pi) = 0$$
$$\sin(\pi\omega) = 0$$
$$\omega = m = \{1, 2, \ ...\}$$

$\lambda = -m^2$ and $X(x) = \sin(mx)$

$$t\frac{T'}{T} = \lambda - 1 = -m^2 - 1$$
$$\frac{T'}{T} = \frac{-m^2-1}{t}$$
$$(\ln|T|)' = \frac{-m^2-1}{t}$$
$$\ln|T| = -(m^2+1)\ln|t| + C$$
$$T = Ct^{-(m^2+1)}$$
$$u = XT = Ct^{-(m^2+1)}\sin(mx)u = \sum_{m=1}^{\infty} A_m t^{-(m^2+1)}\sin(mx)$$
$$u(x,1) = \sum_{m=1}^{\infty} A_m \sin(mx)$$
$$A_m = \frac{2}{\pi}\int_0^{\pi} \sin(mx)\, dx = \frac{2}{\pi}(-\frac{\cos(\pi m)}{m} + \frac{\cos(0)}{m}]$$
$$A_m = \frac{2}{\pi m}[(-1)^{m+1} + 1]$$
$$u(x,t) = \sum_{m=1}^{\infty} \frac{2}{\pi m}[(-1)^{m+1} + 1]t^{-(m^2+1)}\sin(mx)$$

9.2 Wave

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = e^x \end{cases}$$

$$u(x,t) = X(x)T(t)$$
$$XT'' = c^2 X'' T$$
$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$$
$$\begin{cases} X'' = \lambda X \\ X(0) = 0 \\ X(1) = 0 \end{cases}$$

$\lambda > 0$:

$$X(0) = A + B = 0$$
$$X(1) = Ae^{\omega} - Ae^{-\omega} = 0 \implies \omega = -\omega \implies X = 0$$

$\lambda = 0$:

$$X(0) = B = 0$$
$$X(1) = A = 0$$
$$X = 0$$

$\lambda < 0$:

$$X = A\cos(\omega x) + B\sin(\omega x)$$
$$X(0) = A = 0$$
$$X(1) = B\sin(\omega) = 0 \implies \omega = \pi m \;\; \lambda = -(\pi m)^2$$
$$X = \sin(\pi m x)$$

$$T'' = \lambda c^2 T = -(\pi m c)^2 T$$
$$T = A\exp(\pi m c t) + B\sin(\pi m c t)$$
$$u(x,t) = \sum_{m=1}^{\infty} (A_m \cos(\pi m c t) + B_m \sin(\pi m c t))\sin(\pi m x)$$
$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(\pi m x) = x^2$$
$$A_m = \frac{2}{\pi}\int_0^1 x^2 \sin(\pi m x)\,dx = \frac{2}{\pi}\left[-x^2\frac{\cos(\pi m x)}{\pi m} + 2x\frac{\sin(\pi m x)}{(\pi m)^2} - 2\frac{\cos(\pi m x)}{(\pi m)^3}\right]_0^1$$
$$A_m = \frac{2}{\pi}\left[\frac{(-1)^{m+1}}{\pi m} + \frac{2(-1)^{m+1}}{(\pi m)^3} - \frac{2}{(\pi m)^3}\right] = \frac{2}{\pi^2 m}(-1)^{m+1} + \frac{4}{\pi^4 m^3}[(-1)^{m+1} - 1]$$
$$u(x,t) = \sum_{m=1}^{\infty}\left[\left(\frac{2}{\pi^2 m}(-1)^{m+1} + \frac{4}{\pi^4 m^3}[(-1)^{m+1}-1]\right)\cos(\pi m c t) + B_m \sin(\pi m c t)\right]\sin(\pi m x)$$
$$u_t(x,0) = \sum_{m=1}^{\infty} B_m \pi m c \sin(\pi m x) = e^x$$

$$\begin{aligned} B_m &= \frac{2}{\pi}\int_0^1 e^x \sin(\pi m x)\,dx \\ &= \frac{2}{\pi}\left[\frac{\pi m + e\sin(\pi m - e\pi m\cos(\pi m))}{\pi^2 m^2 + 1}\right] \\ B_m &= \frac{2}{\pi^2 m c}\left[\frac{\pi m + e\sin(\pi m - e\pi m\cos(\pi m))}{\pi^2 m^2 + 1}\right] \end{aligned}$$

$$\begin{aligned} u(x,t) = \sum_{m=1}^{\infty} &\left(\frac{2}{\pi^2 m}(-1)^{m+1} + \frac{4}{\pi^4 m^3}[(-1)^{m+1}-1]\right)\cos(\pi m c t)\sin(\pi m x) \\ &+ \frac{2}{\pi^2 m c}\left[\frac{\pi m + e\sin(\pi m - e\pi m\cos(\pi m))}{\pi^2 m^2 + 1}\right]\sin(\pi m c t)\sin(\pi m x) \end{aligned}$$

9.3 Laplace

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0, y) = 0 \\ u(\pi, y) = 0 \\ u(x, 0) = x \\ u(x, 1) = 3 \end{cases}$$

$$X''Y + XY'' = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$
$$\begin{cases} X'' = \lambda X \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

$\lambda > 0$:

$$X(0) = A + B = 0$$
$$X(\pi) = Ae^{\omega\pi} - Ae^{-\omega\pi} = 0 \implies \omega = -\omega$$
$$X = 0$$

$\lambda = 0$:

$$X(0) = B = 0$$
$$X(\pi) = A\pi = 0 \implies A = 0$$
$$X = 0$$

$\lambda < 0$:

$$X = A\cos(\omega x) + B\sin(\omega x)$$
$$X(0) = A = 0$$
$$X(\pi) = B\sin(\pi\omega) = 0$$
$$m = \{1, 2, ...\}$$
$$\lambda = -m^2$$
$$X = \sin(mx)$$

$$Y'' = m^2 Y \implies Y = Ae^{my} + Be^{-my} = A\cosh(my) + B\sinh(my)$$
$$u(x, y) = \sum_{m=1}^{\infty} (A\cosh(my) + B\sinh(my))\sin(mx)$$

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(mx) = x$$

$$A_m = \frac{2}{\pi}\int_0^{\pi} x\sin(mx)\,dx = \frac{2}{\pi}[-x\frac{\cos(mx)}{m} + \frac{\sin(mx)}{m^2}]_0^{\pi} = \frac{2}{\pi}(\frac{\pi}{m}(-1)^{m+1}) = \frac{2}{m}(-1)^{m+1}$$

$$u(x,1) = \sum_{m=1}^{\infty}\left(\frac{2}{m}(-1)^{m+1}\cosh(m) + B_m\sinh(m)\right)\sin(mx) = 3$$

$$\frac{2}{m}(-1)^{m+1}\cosh(m) + B_m\sinh(m) = \frac{2}{\pi}\int_0^{\pi} 3\sin(mx)\,dx = \frac{6}{\pi m}[(-1)^{m+1}+1]$$

$$B_m = \frac{\frac{6}{\pi m}[(-1)^{m+1}+1] - \frac{2}{m}(-1)^{m+1}\cosh(m)}{\sinh(m)}$$

$$u(x,y) = \sum_{m=1}^{\infty}\left(\frac{2}{m}(-1)^{m+1}\cosh(my) + \frac{\frac{6}{\pi m}[(-1)^{m+1}+1] - \frac{2}{m}(-1)^{m+1}\cosh(m)}{\sinh(m)}\sinh(my)\right)\sin(mx)$$

10 Fourier Series

10.1 Fourier Sine

Because $\{\sin(mx)|\ m = 1, 2, ...\}$ is orthogonal, for

$$f(x) = \sum_{m=1}^{\infty} A_m \sin(mx)$$

on $(0,\pi)$ we have

$$A_m = \frac{f\cdot\sin(mx)}{\sin(mx)\cdot\sin(mx)} = \frac{\int_0^{\pi} f(x)\sin(mx)\,dx}{\int_0^{\pi}\sin^2(mx)\,dx} = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)\,dx$$

More generally, for $f(x) = \sum_{m=1}^{\infty} A_m\sin(\frac{\pi m x}{L})$ on $(0,L)$

$$A_m = \frac{2}{L}\int_0^{L} f(x)\sin(\frac{\pi m x}{L})\,dx$$

10.2 Fourier Cosine

For

$$f(x) = \sum_{m=0}^{\infty} A_m\cos(\frac{\pi m x}{L})$$

on $(0,L)$

$$A_m = \frac{2}{L}\int_0^{L} f(x)\cos(\frac{\pi m x}{L})\,dx$$

$$A_0 = \frac{1}{L}\int_0^{L} f(x)\,dx$$

10.3 Full Fourier

We redefine the dot product to

$$f \cdot g = \int_{-L}^{L} f(x)g(x)\ dx$$

so on the interval $(-L, L)$, the coefficients of

$$f(x) = \sum_{m=0}^{\infty} A_m \cos(\frac{\pi m x}{L}) + B_m \sin(\frac{\pi m x}{L})$$

are

$$A_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{\pi m x}{L})\ dx$$
$$B_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{\pi m x}{L})\ dx$$
$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$
$$B_0 = 0$$

10.4 Complex Fourier

For complex numbers we redefine the dot product such that

$$f \cdot g = \int_{-L}^{L} f(x)\overline{g(x)}$$

where $\overline{a+bi} = a - bi$.

So on $(-L, L)$ the coefficients of

$$f(x) = \sum_{m=-\infty}^{\infty} C_m \exp(i(\frac{\pi m x}{L}))$$

are

$$C_m = \frac{1}{2L}\int_{-L}^{L} f(x)\exp(-i(\frac{\pi m x}{L}))\ dx$$

10.5 Parseval's Identity

Definition: $||u|| = \sqrt{u \cdot u}$ and $||cu|| = \text{abs}(c)\ ||u||$

Pythagorean Theorem: If $\{u, v, w\}$ is orthogonal,

$$||u + v + w||^2 = ||u||^2 + ||v||^2 + ||w||^2$$

Then on $(0, \pi)$ because $\{\sin(mx)\}$ is orthogonal,

$$\begin{aligned}
f(x) &= \sum_{m=1}^{\infty} A_m \sin(mx) \\
||f||^2 &= \left|\left|\sum_{m=1}^{\infty} A_m \sin(mx)\right|\right|^2 \\
&= \sum_{m=1}^{\infty} ||A_m \sin(mx)||^2 \\
&= \sum_{m=1}^{\infty} |A_m|^2 \; ||\sin(mx)||^2 \\
\int_0^{\pi} (f(x))^2 \; dx &= \sum_{m=1}^{\infty} |A_m|^2 \; \int_0^{\pi} \sin^2(mx) \; dx \\
&= \frac{\pi}{2} \sum_{m=1}^{\infty} |A_m|^2
\end{aligned}$$

This gives Parseval's identity:

$$\sum_{m=1}^{\infty} |A_m|^2 = \frac{2}{\pi} \int_0^{\pi} (f(x))^2 \; dx$$

10.5.1 Midterm 2 Question

Suppose $\{f_n\}_{n=1}^{\infty}$ is an orthogonal family of nonzero real functions on $(0, L)$ with the dot product $f \cdot g = \int_0^L f(x)g(x) \; dx$ such that for all $n = 1, 2, ...$

$$\int_0^L (f_n)^2 \; dx = 3L \quad \text{and} \quad \int_0^{:} x^3 f_n(x) \; dx = \frac{2L^4}{\sqrt{n}}$$

Derive Parseval's identity for $x^3 = \sum_{n=1}^{\infty} A_n f_n(x)$ on $(0, L)$ and calculate $\sum_{n=1}^{\infty} \frac{1}{n}$

Solution:

$$
\begin{aligned}
x^3 &= \sum_{n=1}^{\infty} A_n f_n(x) \\
||x^3||^2 &= \left|\left|\sum_{n=1}^{\infty} A_n f_n(x)\right|\right|^2 \\
\int_0^L x^6 \, dx &= \sum_{n=1}^{\infty} |A_n|^2 \, ||f_n(x)||^2 \quad \text{(by orthogonality)} \\
\frac{L^7}{7} &= \sum_{n=1}^{\infty} |A_n|^2 \, \int_0^L (f_n)^2 \, dx \\
&= \sum_{n=1}^{\infty} 3L \, |A_n|^2 \\
&= 3L \sum_{n=1}^{\infty} |A_n|^2 \\
\frac{L^6}{21} &= \sum_{n=1}^{\infty} |A_n|^2 \\
A_n &= \frac{\int_0^L x^3 f_n \, dx}{\int_0^L (f_n)^2 \, dx} = \frac{1}{3L} \cdot \frac{2L^4}{\sqrt{n}} = \frac{2L^3}{3\sqrt{n}} \\
|A_n|^2 &= \frac{4L^6}{9n} \\
\sum_{n=1}^{\infty} \frac{4L^6}{9n} &= \frac{L^6}{21} \\
\sum_{n=1}^{\infty} \frac{1}{n} &= \frac{9}{4(21)} = \frac{3}{28}
\end{aligned}
$$

11 Laplace Equation

11.1 Derivation

From the 2D heat equation, $u_t = D(u_{xx} + u_{yy})$, we assume that $\lim_{t\to\infty} = 0$ so

$$u_{xx} + u_{yy} = 0$$

11.2 Rotational Invariance

Theorem: for some constant θ where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$u_{x'x'} + u_{y'y'} = u_{xx} + u_{yy} = 0$$

Proof:

$$\begin{cases} x' = \cos(\theta)x - \sin(\theta)y \\ y' = \sin(\theta)x + \cos(\theta)y \end{cases}$$

$$\begin{aligned}
u_x &= u_{x'} \cdot x'_x + u_{y'} \cdot y'_x = (u_{x'})\cos(\theta) + (u_{y'})\sin(\theta) \\
u_{xx} &= (u_x)_{x'} \cdot x'_x + (u_y)_{y'} \cdot y'_x = u_{x'x'}\cos^2(\theta) + 2u_{y'x'}\sin(\theta)\cos(\theta) + (u_{y'y'})\sin^2(\theta) \\
u_y &= u_{x'} \cdot x'_y + u_{y'} \cdot y'_y = -(u_{x'})\sin(\theta) + (u_{y'})\cos(\theta) \\
u_{yy} &= (u_x)_{x'} \cdot x'_y x + (u_y)_{y'} \cdot y'_y = u_{x'x'}\sin^2(\theta) - 2u_{y'x'}\sin(\theta)\cos(\theta) + (u_{y'y'})\cos^2(\theta)
\end{aligned}$$

$$u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'} \quad \blacksquare$$

11.3 Polar Laplace

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies r = \sqrt{x^2 + y^2}$$

$$r_x = \frac{x}{\sqrt{x^2 + y^2}} = \cos(\theta)$$

$$r_y = \sin\theta$$

$$\theta_x = -\frac{\sin\theta}{r}$$

$$\theta_y = \frac{\cos\theta}{r}$$

$$\begin{aligned}
u_x &= u_r \cdot r_x + u_\theta \cdot \theta_x = u_r\cos\theta + u_\theta\left(-\frac{\sin\theta}{r}\right) \\
u_{xx} &= (u_x)_r \cdot r_x + (u_x)_\theta \cdot \theta_x \\
&= u_{rr}\cos^2\theta - 2u_{r\theta}\frac{\sin\theta\cos\theta}{r} + 2u_\theta\frac{\sin\theta\cos\theta}{r^2} + u_r\frac{\sin^2\theta}{r + u_{\theta\theta}}\frac{\sin^2\theta}{r^2} \\
u_{yy} &= u_{rr}\sin^2\theta + 2u_{r\theta}\frac{\sin\theta\cos\theta}{r} - 2u_\theta\frac{\sin\theta\cos\theta}{r^2} - u_r\frac{\sin^2\theta}{r + u_{\theta\theta}}\frac{\sin^2\theta}{r^2}
\end{aligned}$$

$$u_{xx} + u_{yy} = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}$$

so the polar laplace is

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0$$

11.4 Fundamental Solution

We look for radial solutions such that $u_\theta = 0$, so the polar laplace equation takes the form

$$u_{rr} + \frac{1}{r}u_r = 0$$

with constants $A = -\frac{1}{2\pi}$ and $B = 0$

By integrating factors,

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r &= 0 \\ ru_{rr} + u_r &= 0 \\ (ru_r)_r &= 0 \\ ru_r &= A \\ u_r &= \frac{A}{r} \\ u &= A\ln r + B \\ u &= A\ln(\sqrt{x^2+y^2}) \end{aligned}$$

$$\Phi(x,y) = -\frac{1}{2\pi}\ln(\sqrt{x^2+y^2})$$

11.5 Subharmonics

Definition: $u(x,y)$ is subharmonic if

$$-(u_{xx} + u_{yy}) \le 0$$

Example 1: Suppose u is harmonic and $f'' \ge 0$. Let $v = f(u)$. Show that v is subharmonic.

$$\begin{cases} v_x = f'(u)\cdot u_x \\ v_{xx} = f''(u)\cdot u_x + f'(u)\cdot u_{xx} \\ v_y = f'(u)\cdot u_y \\ v_{yy} = f''(u)\cdot u_y + f'(u)\cdot u_{yy} \end{cases}$$

$$\begin{aligned} v_{xx} + v_{yy} &= f''(u)\cdot u_x + f'(u)\cdot u_{xx} + f''(u)\cdot u_y + f'(u)\cdot u_{yy} \\ &= f''(u)(u_x + u_y) + f'(u)(u_{xx} + u_{yy}) \end{aligned}$$

If u is harmonic, $\Delta u = 0$ so

$$v_{xx} + v_{yy} = \underbrace{f''(u)(u_x + u_y)}_{\ge 0}$$

so

$$-(v_{xx} + v_{yy}) \le 0$$

and v is subharmonic. ■

Example 2: Suppose u is harmonic and let $w = (u_x)^2 + (u_y)$. Show w is subharmonic.

$$\begin{aligned} w_x &= 2u_x \cdot u_{xx} + u_{xy} \\ w_{xx} &= 2u_{xx}^2 + 2u_x \cdot u_{xxx} + u_{xxy} \\ w_y &= 2u_x \cdot u_{xy} + u_{yy} \\ w_{yy} &= 2u_{xy}^2 + 2u_x \cdot u_{xyy} + u_{yyy} \end{aligned}$$

$$\begin{aligned} w_{xx} + w_{yy} &= 2u_{xx}^2 + 2u_x \cdot u_{xxx} + u_{xxy} + 2u_{xy}^2 + 2u_x \cdot u_{xyy} + u_{yyy} \\ &= 2u_{xx}^2 + 2u_{xy}^2 + 2u_x(u_{xxx} + u_{xyy}) + (u_{xxy} + u_{yyy}) \\ &= 2u_{xx}^2 + 2u_{xy}^2 + 2u_x(u_{xx} + u_{yy})_x + (u_{xx} + u_{yy})_y \\ &= 2u_{xx}^2 + 2u_{xy}^2 \end{aligned}$$

Both of these are non-negative so the RHS is non-nonnegative. Thus,

$$-(w_{xx} + w_{yy}) \leq 0$$

and w is sub-harmonic. ■

11.6 Mean-Value Formula

If $\Delta u = 0$ then for every x and every $r > 0$ we have

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy \ = u(x)$$

which means that the average value over the ball B centered at (x, r) is just the value at the center.

Consequences:

1. Solutions to $\Delta u = 0$ are infinitely differentiable (this integral just gets one level smoother)
2. *Liouville's theorem:* If $\Delta u = 0$ and $|u| \leq c$ then u must be constant
3. Corollary: if u is not constant, it must blow up somewhere

General form: the mean value formula holds if you integrate on circles/spheres:

$$\frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dy = u(x)$$

where $\partial B(x, r)$ is the circle/sphere centered at (x, r).

Example: Suppose u solves Laplace's equation on the disk $x^2 + y^2 \leq 4$ with $u = 3\sin(2\theta) + 1$ on $x^2 + y^2 = 4$. Find $u(0, 0)$.

Because $\Delta u = 0$,

$$\frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\ dy = u(x)$$
$$\frac{1}{|x^2+y^2=4|}\int_0^{2\pi} 2u(y)\ d\theta = u(x)$$
$$\frac{1}{4\pi}\int_0^{2\pi} 6\sin(2\theta)+2\ d\theta = \frac{1}{4\pi}[-3\cos(2\theta)+2\theta]_0^{2\pi} = u(x)$$
$$-\frac{3}{4\pi}+1+\frac{3}{4\pi}-0 = 1 = u(x)$$

So the value at the centre is 1: $u(0,0) = 1$.

11.7 Strong Maximum Principle

If $\Delta u = 0$ in Ω then $\max u$ and $\min u$ are attained on $\partial\Omega$ and only on $\partial\Omega$ (unless u is constant).

11.7.1 Proof

Suppose u has a max M at some point x in Ω. But then the mean value formula gives

$$⨍_{B(x,r)} u(y)\ dy = u(x) = M$$

which means that the highest value is also the average value and u is constant.

11.7.2 Finding the maximum

Example: Suppose u solves Laplace's equation on the disk $x^2+y^2 \leq 4$ with $u = 3\sin(2\theta)+1$ on $x^2+y^2 = 4$. Find the maximum of u on the disk.

By the strong-max principle, because $\Delta u = 0$ inside the boundary, the max exists on the boundary. Thus we just need to find the max of $u = 3\sin(2\theta+1)$ which is 4 by the range of sin.

11.7.3 Uniqueness of Poisson's Equation

Suppose u and v both solve

$$\begin{cases}\Delta u = f & \in \Omega\\ u = g & \in \partial\Omega\end{cases}$$

Let $w = u - v$. Then w solves

$$\begin{cases}\Delta w = \Delta u - \Delta v = f - f = 0 & \in \Omega\\ w = u - v = g - g = 0 & \in \partial\Omega\end{cases} \implies \begin{cases}\Delta w = 0 & \in \Omega\\ w = 0 & \in \partial\Omega\end{cases}$$

But by the strong-max principle,

$$\max_{\Omega} w = \max_{\partial\Omega} w = 0$$

similarly,

$$\min \Omega w = \min \partial\Omega w = 0$$

so

$$w = u - v = 0 \implies u = v \quad \blacksquare$$

11.7.4 Positivity of Solutions

Suppose u solves

$$\begin{cases} \Delta u = 0 & \in \Omega \\ u = g & \in \partial\Omega \end{cases}$$

When $g \geq 0$ and $g(x_0) > 0$ for some x_0 on $\partial\Omega$. Then $u > 0$ on all of Ω.

Proof:

$$\min_{\Omega} u = \min_{\partial\Omega} u = \min_{\partial\Omega} g \geq 0$$

but for some x_0, $g > 0$ so if $u = 0$ at some point in Ω, the new minimum will not be on $\partial\Omega$, leading to a contradiction. Hence, $u > 0 \in \Omega$.

11.8 Midterm 2 Problem

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = 0 \\ u(x,\pi) = 0 \\ u(0,y) = 2y \\ u(2,y) = 0 \end{cases}$$

$$u = X(x)Y(y)$$
$$X''Y + XY'' = 0$$
$$\frac{Y''}{Y} = -\frac{X''}{X} = \lambda$$
$$\begin{cases} Y'' = \lambda Y \\ Y(0) = 0 \\ Y(\pi) = 0 \end{cases}$$

$\lambda < 0$:

$$Y = A\cos(\omega y) + B\sin(\omega y)$$
$$Y(0) = A = 0$$
$$Y(\pi) = B\sin(\pi\omega) = 0 \implies \omega = m = \{1, 2, ...\}$$
$$\lambda = -m^2$$
$$Y = \sin(my)$$

$$X'' = m^2 X \implies X = Ae^{mx} + Be^{-mx} = A\cosh(mx) + B\sinh(mx)$$
$$u(x,y) = \sum_{m=1}^{\infty} (A_m \cosh(mx) + B_m \sinh(mx)) \sin(my)$$

$$\begin{aligned} u(0,y) &= \sum_{m=1}^{\infty} A_m \sin(my) \\ A_m &= \frac{2}{\pi} \int_0^{\pi} 2y \sin(my)\, dy \\ &= \frac{2}{\pi} \left[-2y \frac{\cos(my)}{m} - 2\frac{\sin(my)}{m} \right]_0^{\pi} \\ &= \frac{4}{m}(-1)^{m+1} \end{aligned}$$

$$\begin{aligned} u(x,y) &= \sum_{m=1}^{\infty} (\frac{4}{m}(-1)^{m+1} \cosh(mx) + B_m \sinh(mx)) \sin(my) \\ u(2,y) &= \sum_{m=1}^{\infty} (\frac{4}{m}(-1)^{m+1} \cosh(2m) + B_m \sinh(2m)) \sin(my) \\ &\frac{4}{m}(-1)^{m+1} \cosh(2m) + B_m \sinh(2m) = 0 \end{aligned}$$

$$\begin{aligned} B_m &= -\frac{\frac{4}{m}(-1)^{m+1}\cosh(2m)}{\sinh(2m)} \\ &= \frac{4}{m}(-1)^m \coth(2m) \end{aligned}$$

$$u(x,y) = \sum_{m=1}^{\infty} (\frac{4}{m}(-1)^{m+1} \cosh(mx) + \frac{4}{m}(-1)^m \coth(2m) \sinh(mx)) \cos(my)$$

12 Calculus of Variations

Calculus of Variations turns minimization problems into differential equations

Trick: If
$$\int_0^1 f(x)\, g(x)\, dx = 0$$
for all g with $g(0) = g(1) = 0$ then $f = 0$ (when f and g are continuous)

12.1 Derivation of the Euler-Lagrange Equations

Suppose f minimizes

$$I[f] = \frac{1}{2}\int_0^1 (f'(x))^2\, dx$$

Let f be arbitrary with $g(0) = g(1) = 0$ and $f(0) = 0,\ f(1) = 1$.

Consider

$$h(t) = I[f + tg] = \frac{1}{2}\int_0^1 (f'(x) + tg'(x))^2\, dx$$

Note that $h(0) = I[f]$ so h has a min at $t = 0$. Thus $h'(0) = 0$.

$$\begin{aligned} h'(t) = \frac{d}{dt}\left[\frac{1}{2}\int_0^1 (f'(x) + tg'(x))^2\, dx\right] \\ &= \frac{1}{2}\int_0^1 \frac{d}{dt}(f'(x) + tg'(x))^2\, dx \\ &= \frac{1}{2}\int_0^1 2(f'(x) + tg'(x))g'(x)\, dx \\ &= \int_0^1 (f' + tg')'\, dx \\ &= [(f' + tg')]_0^1 - \int_0^1 (g' + tg')'g\, dx \\ &= -\int_0^1 (f'' + tg'')g\, dx \end{aligned}$$

$$\begin{aligned} h'(0) = -\int_0^1 f''g\, dx = 0 \implies -f''(x) \quad \text{(by the trick above)} \\ f(0) = 0 \implies B = 0 \\ f(1) = 1 \implies A = 1 \end{aligned}$$

So the Euler-Lagrange associated to this min problem is $f(x) = x$

General Lagrangian: $L = L(p, z, x)$

$$\min I[f] = \int_a^b L(f', f, x)\, dx = -(L_p(f', f, x))_x + L_z(f', f, x) = 0$$

Higher Dimensional Lagrangian: In 2D, $L = l(p, q, z, x, y)$

$$\min I[u] = \int_\Omega L(u_x, u_y, u, x, y)\, dx\, dy$$

with $u = g$ on $\partial\Omega$ corresponds to the Euler-Lagrange equation

$$-(L_p)_x - (L_q)_y + L_z = 0$$

(evaluated at (u_x, u_y, u, x, y))

12.2 Apply the Euler-Lagrange Equations

Example 1:

$$\min I[u] = \int_\Omega \frac{1}{2}||\nabla u||^2 - F(u)\; dx\, dy$$

where G is an antiderivative of a given function f

$$\begin{gathered}
\min I[u] = \int_\Omega \frac{1}{2}(u_x)^2 + \frac{1}{2}(u_y)^2 - F(u)\; dx\, dy \\
L(p,q,z,x,y) = \frac{1}{2}p^2 + \frac{1}{2}q^2 - F(z) \\
L_p = p, \quad L_q = q, \quad L_z = -f(z)
\end{gathered}$$

So by the Euler-Lagrange equation $-(L_p)_x - (L_q)_y + L_z = 0$,

$$-(u_x)_x - (u_y)_y - f(u) = 0 \implies -(u_{xx} + u_{yy}) = f(u)$$

Example 2:

$$\min I[u] = \int_\Omega \exp(-w(x,y)) \left(\frac{1}{2}||\nabla u||^2 - uf(x,y)\right)\; dx\, dy$$

$$\begin{gathered}
\min I[u] = \int_\Omega \exp(-w(x,y)) \left(\frac{1}{2}(u_x)^2 + \frac{1}{2}(u_y)^2 - uf(x,y)\right)\; dx\, dy \\
L(p,q,z,x,y) = L(u_x,u_y,u,x,y) = \frac{1}{2}u_x^2 \exp(-w(x,y)) + \frac{1}{2}u_y^2 \exp(-w(x,y)) - uf(x,y)\exp(-w(x,y)) \\
L_p = u_x \exp(-w(x,y)) \\
L_q = u_y \exp(-w(x,y)) \\
L_z = -f(x,y)\exp(-w(x,y))
\end{gathered}$$

Using the E-L equation $-(L_p)_x - (L_q)_y + L_z = 0$,

$$\begin{gathered}
-(u_x \exp(-w(x,y)))_x - (u_y \exp(-w(x,y)))_y - f(x,y)\exp(-w(x,y)) = 0 \\
-(u_{xx}e^{-w} - u_x w_x e^{-w}) - (u_{yy}e^{-w} - u_y w_y e^{-w}) = f(x,y)e^{-w} \\
-u_{xx} - u_{yy} + u_x w_x + u_y w_y = f(x,y)
\end{gathered}$$

13 Ecology Application

Example: Given a linearized PDE of the form

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} D_1 u_{xx} \\ D_2 v_{xx} \end{bmatrix} + \begin{bmatrix} -(b+1) + 2u_* v_* & (u_*)^2 \\ b - 2u_* v_* & -(u_*)^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

plug in $u(x,t) = e^{\lambda t}\cos(\kappa x) u_0$ and $v(x,t) = e^{\lambda t}\cos(\kappa x) v_0$ to get an equation of the form $B \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \lambda \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$

Given $u_* = a$ and $v_* = b/a$, the system reduces to

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} D_1 u_{xx} \\ D_2 v_{xx} \end{bmatrix} + \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Taking the derivatives of the solutions

$$\begin{cases} u_t = \lambda e^{\lambda t}\cos(\kappa x) u_0 \\ v_t = \lambda e^{\lambda t}\cos(\kappa x) v_0 \\ u_{xx} = -\kappa^2 e^{\lambda t}\cos(\kappa x) u_0 \\ v_{xx} = -\kappa^2 e^{\lambda t}\cos(\kappa x) v_0 \end{cases}$$

$$\begin{aligned} \begin{bmatrix} \lambda e^{\lambda t}\cos(\kappa x) u_0 \\ \lambda e^{\lambda t}\cos(\kappa x) v_0 \end{bmatrix} &= \begin{bmatrix} -D_1\kappa^2 e^{\lambda t}\cos(\kappa x) u_0 \\ -D_2\kappa^2 e^{\lambda t}\cos(\kappa x) v_0 \end{bmatrix} + \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix} \begin{bmatrix} e^{\lambda t}\cos(\kappa x) u_0 \\ e^{\lambda t}\cos(\kappa x) v_0 \end{bmatrix} \\ \lambda \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} &= \begin{bmatrix} -D_1\kappa^2 u_0 \\ -D_2\kappa^2 v_0 \end{bmatrix} + \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \\ &= \begin{bmatrix} -D_1\kappa^2 u_0 \\ -D_2\kappa^2 v_0 \end{bmatrix} + \begin{bmatrix} (b-1)u_0 & a^2 v_0 \\ -bu_0 & -a^2 v_0 \end{bmatrix} \\ &= \begin{bmatrix} (-D_1\kappa^2 + b - 1)u_0 + a^2 v_0 \\ -bu_0 + (-D_2\kappa^2 - a^2)v_0 \end{bmatrix} \\ &= \begin{bmatrix} -D_1\kappa^2 + b - 1 & a^2 \\ -b & -D_2\kappa^2 - a^2 \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \end{aligned}$$

14 COVID Application

Reaction-Diffusion Model:

$$\begin{cases} S_t = -bSI \\ I_t = bSI - \gamma I + DI_{xx} \end{cases}$$

where:

- S is the number of susceptible people
- I is the number of infected people

- γ is the death rate
- b is the infection rate
- D is the diffusion rate

14.1 Rescaling

Goal: Rescale the time variables and unknowns so the mean mortality rate (τ) is 1

$$\tau = \gamma t$$

We rescale by the total population so

$$\begin{cases} s = \frac{S}{N} \\ i = \frac{I}{N} \end{cases}$$

$$s(x,\tau) = \frac{S(x,t)}{N} = \frac{S(x,\frac{\tau}{\gamma})}{N} \implies S(x,t) = Ns(x,\gamma t)$$

$$I(x,t) = Ni(x,\gamma t)$$

By chain rule,

$$S_t = (Ns(x,\gamma t))_t = N\gamma s_\tau(\gamma t)$$

Which by the PDE,

$$N\gamma s_\tau(\gamma t) = -bNs(x,\gamma t)Ni(x,\gamma t)$$

so

$$s_\tau = -\frac{bN}{\gamma}s_i$$

similarly,

$$i_\tau = \frac{bN}{\gamma}s_i - i + \frac{D}{\gamma}i_{xx}$$

Let $R_0 = \frac{bN}{\gamma}$ and $d = \frac{D}{\gamma}$ so

$$\begin{cases} s_\tau = -R_0 s_i \\ i_\tau = R_0 s_i - i + d i_{xx} \end{cases}$$

14.2 Traveling Waves

We assume the PDE is of the special form

$$s(x,\tau) = f(x - ct)$$
$$i(x,\tau) = g(x - ct)$$

So

$$\begin{aligned}
s_\tau &= -R_0 s_i \\
&= (f(x-ct))_\tau \\
&= -cf'(x-ct) \\
&= -R_0 f(x-c\tau)g(c-c\tau) \\
&= -cf'(z) = -R_0 f(z)g(z)
\end{aligned}$$

Similarly with $i_\tau = R_0 si - i + di_{xx}$,

$$\begin{cases} c'f(z) = R_0 f(z)g(z) \\ cg'(z) = -R_0 f(z)g(z) + g(z) - dg''(z) \end{cases}$$

15 Method of Characteristics

1. Let $h(t) = u(x(t), t)$
2. Calculate h'
3. Solve for x
4. Plug in to get the second characteristic ODE in h'
5. Solve for h
6. Substitute for u
7. Initial Condition

Example 1:

$$\begin{cases} u_t + u_x + u = \exp(x+2t) \\ u(x,0) = 0 \end{cases}$$

$$\begin{aligned}
h(t) &= u(x(t), t) \\
h'(t) &= u_x \cdot x'(t) + u_t \\
&= u_x \cdot x'(t) + \exp(x+2t) - u_x - u \\
&= u_x(x'-1) - u + \exp(x+2t)
\end{aligned}$$

$x'(t) - 1 = 0 \implies x'(t) = 1$ is our characteristic ODE with $x(t) = t + C$ so

$$\begin{aligned}
h'(t) &= -u(x(t),t) + \exp(3t + C) \\
&= -h(t) + \exp(3t + C) \\
h'(t) + h(t) &= \exp(3t + C) \\
(he^t)' &= \exp 4t + C \\
h(t)e^t &= \frac{1}{4}\exp(4t + C) + B \\
h(t) &= \frac{1}{4} exp(3t + C) + Be^{-t} \\
u(x(t),t) &= \frac{1}{4}\exp(3t + C) + Be^{-t} \\
u(x(0),0) &= \frac{e^C}{4} + B = 0 \implies B = -\frac{e^C}{4} \\
u(x(t),t) &= \frac{1}{4}\exp(3t + C) - \frac{1}{4}e^{C-t} \\
u(t + C,t) &= \frac{1}{4}\exp(3t + C) - \frac{1}{4}e^{C-t} \\
u(x,t) &= \frac{1}{4}\exp(x + 2t) - \frac{1}{4}e^{x-2t} \\
&= \frac{1}{4}e^x(e^{2t} - e^{-2t}) \\
&= \frac{1}{2}e^x \sinh(2t)
\end{aligned}$$

Example 2:

$$\begin{cases} u_t + x^2 u_x = 0 \\ u(x,0) = f(x) \end{cases}$$

$$\begin{aligned}
h(t) &= u(x(t),t) \\
h' &= u_x \cdot x' + u_t \\
&= u_x \cdot x' - x^2 u_x \\
&= (x' - x^2)u_x \\
&= 0
\end{aligned}$$

$$\begin{aligned}
x' &= x^2 \\
\frac{1}{x^2}\,dx &= dt \\
-\frac{1}{x} &= t + C \\
x(t) &= -\frac{1}{t + C}
\end{aligned}$$

$$h'(t) = 0 \implies h(t) = C$$
$$h(t) = h(0)$$
$$u(x(t), t) = u(x(0), 0)$$
$$u(-\frac{1}{t+C}, t) = u(-\frac{1}{C}, 0)$$
$$= f(-\frac{1}{C})$$

$$x = -\frac{1}{t+C} \implies t + C = -\frac{1}{x}$$
$$C = -t - \frac{1}{x} = -(\frac{tx+1}{x})$$
$$-\frac{1}{C} = \frac{x}{1+xt}$$
$$u(x,t) = f(-\frac{1}{C}) = f(\frac{x}{1+xt})$$

Example 3:

$$\begin{cases} u_t + xu_x = 2u \\ u(x,0) = f(x) \end{cases}$$

$$h(t) = u(x(t), t)$$
$$h'(t) = u_x x' + 2u - xu_x$$
$$= (x' - x)u_x + 2u$$
$$= 2u$$
$$= 2h(t)$$

$$x' = x \implies x = Ae^t$$
$$h'(t) = 2h(t) \implies h(t) = h(0)e^{2t}$$

$$u(x(t), t) = u(x(0), 0)e^{2t}$$
$$u(Ae^t, t) = u(A, 0)e^{2t}$$
$$u(Ae^t, t) = f(A)e^{2t} u(x,t) = f(xe^{-t})e^{2t}$$