APMA 0360: Homework 3

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Problem 1

Verify the following statements:

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

1. $u(x,t)$ solves the heat equation $u_t = Du_{xx}$

$$\begin{aligned}
u_t &= \frac{d}{dt}\frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \\
&= -\frac{2\pi D}{(4\pi Dt)^{\frac{3}{2}}} e^{-\frac{x^2}{4Dt}} + \left(\frac{1}{\sqrt{4\pi Dt}}\right)\left(\frac{x^2}{4Dt^2}\right) e^{-\frac{x^2}{4Dt}} \\
&= -\frac{2\pi D}{(4\pi Dt)^{\frac{3}{2}}} e^{-\frac{x^2}{4Dt}} + \frac{Dx^2}{8\sqrt{\pi}(Dt)^{5/2}} e^{-\frac{x^2}{4Dt}} \\
&= -\frac{2\pi D}{(4\pi Dt)^{\frac{3}{2}}} e^{-\frac{x^2}{4Dt}} + \frac{Dx^2}{8\sqrt{\pi}(Dt)^{5/2}} e^{-\frac{x^2}{4Dt}} \\
u_x &= \frac{d}{dx}\frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} = \left(\frac{1}{\sqrt{4\pi Dt}}\right)\left(\frac{-2x}{4Dt}\right) e^{-\frac{x^2}{4Dt}} \\
u_{xx} &= \left(\frac{1}{\sqrt{4\pi Dt}}\right)\left(\frac{-2}{4Dt}\right) e^{-\frac{x^2}{4Dt}} + \left(\frac{1}{\sqrt{4\pi Dt}}\right)\left(\frac{-2x}{4Dt}\right)^2 e^{-\frac{x^2}{4Dt}} \\
&= -\frac{1}{4\sqrt{\pi}(Dt)^{3/2}} e^{-\frac{x^2}{4Dt}} + \frac{x^2}{8\sqrt{\pi}(Dt)^{5/2}} e^{-\frac{x^2}{4Dt}}
\end{aligned}$$

$$u_t = Du_{xx}$$

$$-\frac{2\pi D}{(4\pi Dt)^{\frac{3}{2}}}e^{-\frac{x^2}{4Dt}}+\frac{Dx^2}{8\sqrt{\pi}(Dt)^{5/2}}e^{-\frac{x^2}{4Dt}} = D(-\frac{1}{4\sqrt{\pi}(Dt)^{3/2}}e^{-\frac{x^2}{4Dt}}+\frac{x^2}{8\sqrt{\pi}(Dt)^{5/2}}e^{-\frac{x^2}{4Dt}})$$

$$-\frac{2\pi D}{(4\pi Dt)^{\frac{3}{2}}}+\frac{Dx^2}{8\sqrt{\pi}(Dt)^{5/2}} = D(-\frac{1}{4\sqrt{\pi}(Dt)^{3/2}}+\frac{x^2}{8\sqrt{\pi}(Dt)^{5/2}})$$

$$D(-\frac{1}{4\sqrt{\pi}(Dt)^{3/2}}+\frac{x^2}{8\sqrt{\pi}(Dt)^{5/2}}) = D(-\frac{1}{4\sqrt{\pi}(Dt)^{3/2}}+\frac{x^2}{8\sqrt{\pi}(Dt)^{5/2}}) \quad \checkmark$$

2. $\int_{-\infty}^{\infty} u(x,t)\; dx = 1$ for all t

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}\; dx = 1$$

$$\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{4Dt}}\; dx = 1$$

Now let

$$I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{4Dt}}\; dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{4Dt}}\; dx$$

So

$$\begin{aligned}
I^2 &= \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{4Dt}}\; dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{4Dt}}\; dy\right)\\
&= \int_0^{\pi}\int_0^{\infty} re^{-\frac{r^2}{4Dt}}\; dr\, d\theta\\
&= 2\pi\int_0^{\infty} re^{-\frac{r^2}{4Dt}}\; dr\\
u &= -\frac{r^2}{4Dt} \implies du = -\frac{2r}{4Dt}\; dr\\
I^2 &= 2\pi\left[-2Dte^{-\frac{r^2}{4Dt}}\right]_0^{\infty}\\
&= 2\pi(0+2Dt) = 4\pi Dt
\end{aligned}$$

Therefore,

$$I = \sqrt{4\pi Dt}$$

and

$$\int_{-\infty}^{\infty} u(x,t)\; dx = \frac{\sqrt{4\pi Dt}}{\sqrt{4\pi Dt}} = 1 \quad \checkmark$$

Problem 2

Consider the Korteweg-De Vries equation

$$\begin{cases} u_t = u_{xxx} \\ u(x,0) = f(x) \end{cases}$$

Find a formula for $\widehat{u}(\kappa, t)$ in terms of $\widehat{f}(\kappa)$

$$\mathcal{F}(u_t) = \mathcal{F}(u_{xxx})$$

By the derivative properties of the fourier transform,

$$\mathcal{F}(u_{xxx}) = (-i\kappa)^3 \widehat{u}(\kappa, t) = i\kappa^3 \widehat{u}(\kappa, t)$$

and then

$$\mathcal{F}(u_t) = \frac{d}{dt}\widehat{u}(\kappa, t)$$

Thus,

$$\frac{d}{dt}\widehat{u}(\kappa, t) = i\kappa^3 \widehat{u}(\kappa, t)$$

Solving by ODE analogy ($y' = ay$),

$$\widehat{u}(\kappa, t) = \widehat{u}(\kappa, 0)e^{i\kappa^3 t}$$

But from the initial condition

$$u(x,0) = f(x) \longrightarrow \widehat{u}(\kappa, 0) = \widehat{f}(\kappa)$$

So,

$$\boxed{\widehat{u}(\kappa, t) = \widehat{f}(\kappa)e^{i\kappa^3 t}}$$

Problem 3

In this problem, a and c are real numbers

1. Show that if $g(x) = f(x-a)$ then $\widehat{g}(\kappa) = e^{i\kappa a}\widehat{f}(\kappa)$

$$\begin{aligned}
\widehat{g}(\kappa) &= \widehat{f}(x-a) \\
&= \int_{-\infty}^{\infty} f(x-a)e^{i\kappa(x+a)}\,dx \\
&= e^{i\kappa a}\int_{-\infty}^{\infty} f(x-a)e^{i\kappa x}\,dx \\
&= e^{i\kappa a}\widehat{f}(\kappa) \quad \checkmark
\end{aligned}$$

2. Use the Fourier transform and (1) to solve the transport PDE

$$\begin{cases} u_t + cu_x = 0 \\ u(x,0) = f(x) \end{cases}$$

$$\widehat{u}(\kappa)_t = \mathcal{F}(-cu_x)$$

$$\frac{d}{dt}\widehat{u}(\kappa,t) = i\kappa c\widehat{u}(\kappa,t)$$

$$\begin{aligned}
\widehat{u}(\kappa,t) &= \mathcal{F}(u(x,0))e^{i\kappa c} \\
&= \widehat{f}(\kappa)e^{i\kappa ct} \\
&= \widehat{f}(x-ct)
\end{aligned}$$

$$\boxed{u(x,t) = f(x-ct)}$$

Problem 4

Use the Fourier transform to solve the PDE

$$\begin{cases} u_t = Du_{xx} - au \\ u(x,0) = f(x) \end{cases}$$

Here $D > 0$ and a is any real number.[1]

$$\widehat{u}_t = \mathcal{F}(Du_{xx}) - \mathcal{F}(au)$$

$$\frac{d}{dt}\widehat{u}(\kappa,t) = -D\kappa^2\widehat{u}(\kappa,t) - a\widehat{u}(\kappa,t) = -(D\kappa^2 + a)\widehat{u}(\kappa,t)$$

Via ODEs,

$$\widehat{u}(\kappa,t) = \widehat{u}(\kappa,0)e^{-(D\kappa^2+a)t} = e^{-at}\widehat{f}(\kappa)e^{-D\kappa^2 t} =$$

Now to set up the convolution:

$$\mathcal{F}(e^{-D\kappa^2 t}) = \widehat{g}(\kappa,t) \quad g(x,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}$$

Therefore,

$$\widehat{u}(\kappa,t) = e^{-at}\widehat{f}(\kappa)\widehat{g}(\kappa,t)$$

$$\widehat{u}(\kappa,t) = e^{-at}\mathcal{F}((f \star g)(\kappa,t))$$

$$\boxed{u(x,t) = \frac{e^{-at}}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} f(y)e^{-\frac{(x-y)^2}{4Dt}}\,dy}$$

[1] Write the answer in the form

$$u(x,t) = e^{-at} \times \text{ Some integral}$$

Problem 5

Solve the PDE

$$\begin{cases} u_t = Du_{xx} + cu_x - au \\ u(x,0) = f(x) \end{cases}$$

Here $D > 0$ and a, c are real numbers [2]

Let

$$v(x,t) = u(x-ct,t)e^{at}$$

so

$$u(x-ct,t) = v(x,t)e^{at}$$

and

$$u(x,t) = v(x+ct,t)e^{at}$$

Taking the partials we have:

$$\begin{cases} u_x = v_x(x+ct,t)e^{at} \\ u_{xx} = v_{xx}(x+ct,t)e^{at} \\ u_t = \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial t} = cv_x(x+ct,t)e^{at} + v_t(x+ct,t)e^{at} + av(x+ct,t)e^{at} \end{cases}$$

and substituting these into the original PDE,

$$cv_x(x+ct,t)e^{at} + v_t(x+ct,t)e^{at} + av(x+ct,t)e^{at} = Dv_{xx}(x+ct,t)e^{at} + cv_x(x+ct,t)e^{at} - av(x+ct,t)e^{at}$$

But this is just

$$v_t(x+ct,t) = Dv_{xx}(x+ct,t)$$

And this is simply the heat equation so

$$v(x+ct,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} f(y)\, e^{-\frac{((x+ct)-y)^2}{4Dt}}\, dy$$

and as

$$v(x+ct,t) = u(x,t)e^{-at}$$

we have

$$\boxed{u(x,t) = \frac{e^{at}}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} f(y)\, e^{-\frac{((x+ct)-y)^2}{4Dt}}\, dy}$$

[2] Hint: Let $v(x,t) = u(x-ct,t)e^{at}$ and find a PDE for v