APMA 0360: Homework 10

Milan Capoor

28 April 2023

Problem 1:

In 1 dimensions, suppose the Lagrangian $L = L(p, z, x)$ is given. Mimic the proof given in lecture to show that the Euler-Lagrange equation associated to the min problem

$$\min I[f] = \int_a^b L(f', f, x)\, dx$$

is

$$-(L_p(f', f, x))_x + L_z(f', f, x) = 0$$

Let g be an arbitrary function with $g(a) = g(b) = 0$. Consider

$$h(t) = I[f + tg] = \int_a^b L(f' + tg', f + tg, x)\, dx$$

Note that $h(0) = I[f]$ so $h(0)$ is a minimum which implies that $h'(0) = 0$.

Then,

$$\begin{aligned}
h'(t) &= \frac{d}{dt}\left[\int_a^b L(f'+tg', f+tg, x)\,dx\right] \\
&= \int_a^b \frac{d}{dt}L(f'+tg', f+tg, x)\,dx \\
&= \int_a^b L_p(f'+tg', f+tg, x)\left(\frac{d}{dt}(f'+tg')\right) \\
&\qquad + L_z(f'+tg', f+tg, x)\left(\frac{d}{dt}(f+tg)\right) \\
&\qquad + L_x(f'+tg', f+tg, x)\left(\frac{d}{dt}(x)\right)\,dx \\
&= \int_a^b g'L_p(f'+tg', f+tg, x) + gL_z(f'+tg', f+tg, x)\,dx \\
&= [gL_p(f'+tg', f+tg, x)]_a^b - \int_a^b g(L_p(f'+tg', f+tg, x))_x\,dx \\
&\qquad + \int_a^b gL_z(f'+tg', f+tg, x)\,dx \\
&= \int_a^b g[-(L_p(f'+tg', f+tg, x))_x + L_z(f'+tg', f+tg, x)]\,dx
\end{aligned}$$

Setting $t = 0$ we get

$$h'(0) = \int_a^b g[-(L_p(f, f, x))_x + L_z(f', f, x)]\,dx = 0$$

Because $g(a) = g(b) = 0$, by a Useful Fact, we know that

$$-(L_p(f, f, x))_x + L_z(f', f, x) = 0$$

Which is the Euler-Lagrange Equation we sought. ■

Problem 2:

Same setting as in above, and moreover assume that for all real numbers ξ and η we have

$$L_{pp}(f', f, x)\xi^2 + 2L_{pz}(f', f, x)\xi\eta + L_{zz}(f', f, x)\eta^2 \geq 0$$

This is sometimes called L is convex

Calculate $h''(t)$ where h is as in your proof of the E-L equation and set $t = 0$ to show that if L is convex, then

$$h''(0) \geq 0$$

This means that if L is convex and f solves the Euler-Lagrange equation (= critical point of h) then f has to be a minimizer.

From above,

$$h'(t) = \int_a^b g[-(L_p(f' + tg', f + tg, x))_x + L_z(f' + tg', f + tg, x)]\, dx$$

So taking the t-derivative:

$$\begin{aligned}
h''(t) &= \frac{d}{dt}\int_a^b g' L_p(f' + tg', f + tg, x) + gL_z(f' + tg', f + tg, x)\, dx \\
&= \int_a^b \frac{d}{dt} g' L_p(f' + tg', f + tg, x)\, dx \\
&\qquad + \int_a^b \frac{d}{dt} g L_z(f' + tg', f + tg, x)\, dx \\
&= \int_a^b g' \cdot \frac{d}{dt} L_p(f' + tg', f + tg, x)\, dx \\
&\qquad + \int_a^b g \cdot \frac{d}{dt} L_z(f' + tg', f + tg, x)\, dx
\end{aligned}$$

Looking at the first lagrangian derivative:

$$\begin{aligned}
\frac{d}{dt} L_p(f' + tg', f + tg, x) &= L_{pp}(f' + tg', f + tg, x)(\frac{d}{dt})(f' + tg') \\
&\qquad + L_{pz}(f' + tg', f + tg, x)(\frac{d}{dt})(f + tg) \\
&\qquad + L_{px}(f' + tg', f + tg, x)(\frac{d}{dt})(x) \\
&= g' L_{pp}(f' + tg', f + tg, x) + gL_{pz}(f' + tg', f + tg, x)
\end{aligned}$$

Now the second lagrangian:

$$\begin{aligned}\frac{d}{dt}L_z(f'+tg',f+tg,x) &= L_{pz}(f'+tg',f+tg,x)(\frac{d}{dt}f'+tg') \\ &\qquad + L_{zz}(f'+tg',f+tg,x)(\frac{d}{dt}f+tg) \\ &\qquad + L_{zx}(f'+tg',f+tg,x)(\frac{d}{dt}x) \\ &= g'L_{zp}(f'+tg',f+tg,x)+gL_{zz}(f'+tg',f+tg,x)\end{aligned}$$

Combining, we have

$$\begin{aligned}h''(t) = \int_a^b & g'(g'L_{pp}(f'+tg',f+tg,x)+gL_{pz}(f'+tg',f+tg,x)) \\ & + g(g'L_{zp}(f'+tg',f+tg,x)+gL_{zz}(f'+tg',f+tg,x))\,dx\end{aligned}$$

Setting $t=0$:

$$h''(0)=\int_a^b (g')^2L_{pp}(f',f,x)+2g'gL_{pz}(f',f,x)+(g)^2L_{zz}(f',f,x)\,dx$$

Then denote $\xi = g'(x)$ and $\eta = g(x)$ for all x. Thus,

$$h''(0)=\int_a^b L_{pp}(f',f,x)\xi^2+2L_{pz}(f',f,x)\xi\eta+L_{zz}(f',f,x)\eta^2\,dx$$

But if L is convex, then the integrand is non-negative by definition so

$$h''(0)\geq 0 \quad \blacksquare$$

Problem 3:

This time in 2 dimensions, suppose the Lagrangian $L = L(p, q, z, x, y)$ is given and consider the min problem

$$\min I[u] = \int_\Omega L(u_x, u_y, u, x, y)\, dx\, dy$$

Show that the Euler-Lagrange equation in that case is

$$-(L_p)_x - (L_q)_y + L_z = 0$$

Where the expression is evaluated at (u_x, u_y, u, x, y)

Hint: Here instead of g you take an arbitrary function v such that $v = 0$ in $\partial\Omega$. You will also need the following integration by parts formula, valid for all v with $v = 0$ on $\partial\Omega$:

$$\int_\Omega u_x v\, dx\, dy = -\int_\Omega u v_x\, dx\, dy$$

And similar for u_y. Assume the Useful Fact from lecture is still true in higher dimensions.

Let v be a function where $v = 0$ for all x in $\partial\Omega$. Then observe

$$h(t) = I[u + tv] = \int_\Omega L(u_x + tv_x, u_y + tv_y, u + tv, x, y)\, dx\, dy$$

Then as above, $h(0) = I[u]$ and $h'(0) = 0$. Taking the derivative:

$$\begin{aligned} h'(t) &= \frac{d}{dt}\int_\Omega L(u_x + tv_x, u_y + tv_y, u + tv, x, y)\, dx\, dy \\ &= \int_\Omega \frac{d}{dt} L(u_x + tv_x, u_y + tv_y, u + tv, x, y)\, dx\, dy \\ &= \int_\Omega v_x L_p(u_x + tv_x, u_y + tv_y, u + tv, x, y) \\ &\qquad + v_y L_q(u_x + tv_x, u_y + tv_y, u + tv, x, y) \\ &\qquad + v L_z(u_x + tv_x, u_y + tv_y, u + tv, x, y)\, dx\, dy \end{aligned}$$

Then as $v = 0$ on $\partial\Omega$,

$$\int_\Omega u_x v\, dx\, dy = -\int_\Omega u v_x\, dx\, dy$$

we have

$$h'(t) = \int_\Omega -v(L_p(u_x + tv_x, u_y + tv_y, u + tv, x, y))_x$$
$$- v(L_q(u_x + tv_x, u_y + tv_y, u + tv, x, y))_y$$
$$+ vL_z(u_x + tv_x, u_y + tv_y, u + tv, x, y)\, dx\, dy$$

With $t = 0$:

$$h'(0) = \int_\Omega v[(-L_p(u_x, u_y, u, x, y))_x + (-L_q(u_x, u_y, u, x, y))_y + (L_z(u_x, u_y, u, x, y))]\, dx\, dy = 0$$

As $v = 0$ everywhere on the boundary, the Useful Fact holds and we know that

$$(-L_p(u_x, u_y, u, x, y))_x + (-L_q(u_x, u_y, u, x, y))_y + (L_z(u_x, u_y, u, x, y)) = 0$$

which is simply the Euler-Lagrange equation

$$-(L_p)_x - (L_q)_y + L_z = 0$$

evaluated at (u_x, u_y, u, x, y) $\blacksquare$

Problem 4:

Find the Euler-Lagrange equations of the following minimization problems. You're allowed to directly use the Euler-Lagrange equations here, no need to reprove it!

1. $$\min I[u] = \int_\Omega \frac{1}{2}||\nabla u||^2 - F(u)\, dx\, dy$$

   Where F is an antiderivative of a given function f. Here $F(u)$ means "F of u" not "F times u"

$$\int_\Omega \frac{1}{2}||\nabla u||^2 - F(u)\, dx\, dy = \int_\Omega \frac{1}{2}(u_x)^2 + \frac{1}{2}(u_y)^2 - F(u)\, dx\, dy$$

So

$$L(p, q, z, x, y) = \frac{1}{2}p^2 + \frac{1}{2}q^2 - F(z)$$

Which means that

$$L_p = p$$
$$L_q = q$$
$$L_z = -f(z)$$

So by the Euler-Lagrange equation $-(L_p)_x - (L_q)_y + L_z = 0$

$$-(u_x)_x - (u_y)_y - f(u) = 0$$

or

$$\boxed{-(u_{xx} + u_{yy}) = f(u)}$$

2. Here w and f are given

$$\min I[u] = \int_\Omega e^{-w(x,y)} \left(\frac{1}{2} ||\nabla u||^2 - u f(x,y) \right) \, dx \, dy$$

$$\min I[u] = \int_\Omega e^{-w(x,y)} \left(\frac{1}{2} (u_x)^2 + \frac{1}{2} (u_y)^2 - u f(x,y) \right) \, dx \, dy$$

So

$$L(p,q,z,x,y) = L(u_x, u_y, u, x, y) = \frac{1}{2} u_x^2 e^{-w(x,y)} + \frac{1}{2} u_y^2 e^{-w(x,y)} - u f(x,y)$$

and as the Euler-Lagrange equation is $-(L_p)_x - (L_q)_y + L_z = 0$, we have

$$\begin{aligned} L_p &= u_x e^{-w(x,y)} \\ L_q &= u_y e^{-w(x,y)} \\ L_z &= -f(x,y) e^{-w(x,y)} \end{aligned}$$

so

$$-(u_x e^{-w(x,y)})_x - (u_y e^{-w(x,y)})_y - f(x,y) e^{-w(x,y)} = 0$$

$$-(u_{xx} e^{-w} - u_x w_x e^{-w}) - (u_{yy} e^{-w} - u_y w_y e^{-w}) = f(x,y) e^{-w(x,y)}$$

$$\boxed{-u_{xx} - u_{yy} + u_x w_x + u_y w_y = f(x,y)}$$

Problem 5:

Consider the following system called the Brusselator model

$$\begin{cases} u_t = D_1 u_{xx} + a - (b+1)u + u^2 v \\ v_t = D_2 v_{xx} + bu - u^2 v \end{cases}$$

Where a, b, D_1, D_2 are positive constants

1. Find all constants u_* and v_* for which

$$\begin{cases} u(x,t) = u_* \\ v(x,t) = v_* \end{cases}$$

solves the PDE above

For u and v having constant solutions, the PDE above becomes

$$\begin{cases} 0 = a - (b+1)u_* + u_*^2 v_* \\ 0 = bu_* - u_*^2 v_* \end{cases}$$

Taking the second equation:

$$(b - u_* v_*)u_* = 0 \implies u_* = \{0, \ \frac{b}{v_*}\}$$

Case 1: $u_* = 0$

The first equation becomes $a = 0$. But a is a strictly positive constant so this is a contradiction.

Case 2: $u_* = \frac{b}{v_*}$

Here, the first equation becomes

$$0 = a - \frac{b(b+1)}{v_*} + \frac{b^2}{v_*} = a + \frac{-b^2 - b + b^2}{v_*} = a - \frac{b}{v_*} \implies b = av_* \implies v_* = \frac{b}{a}$$

Substituting this back into the form of u_* then implies that $u_* = a$ Thus for all values of a, b, D_1, D_2:

$$\boxed{\begin{cases} u(x,t) = u_* = a \\ v(x,t) = v_* = \frac{b}{a} \end{cases}}$$

2. The linearization of the PDE about (u_*, v_*) is

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} D_1 u_{xx} \\ D_2 v_{xx} \end{bmatrix} + \begin{bmatrix} -(b+1) + 2u_* v_* & (u_*)^2 \\ b - 2u_* v_* & -(u_*)^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Where u_* and v_* are the values you found in (a) (don't prove this, but it's basically the Jacobian of the above PDE evaluated at u_* and v_*)

Suppose the solutions $u(x,t)$ and $v(x,t)$ are of the form

$$\begin{cases} u(x,t) = e^{\lambda t}\cos(\kappa x) u_0 \\ v(x,t) = e^{\lambda t}\cos(\kappa x) v_0 \end{cases}$$

Plug u and v into the linearized PDE and find a matrix A depending on κ such that

$$A \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \lambda \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} D_1 u_{xx} \\ D_2 v_{xx} \end{bmatrix} + \begin{bmatrix} -(b+1) + 2(a)(\frac{b}{a}) & (a)^2 \\ b - 2(a)(\frac{b}{a}) & -(a)^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} D_1 u_{xx} \\ D_2 v_{xx} \end{bmatrix} + \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{cases} u(x,t) = e^{\lambda t}\cos(\kappa x) u_0 \\ v(x,t) = e^{\lambda t}\cos(\kappa x) v_0 \end{cases} \implies \begin{cases} u_t = \lambda e^{\lambda t}\cos(\kappa x) u_0 \\ v_t = \lambda e^{\lambda t}\cos(\kappa x) v_0 \end{cases} \begin{cases} u_{xx} = -\kappa^2 e^{\lambda t}\cos(\kappa x) u_0 \\ v_{xx} = -\kappa^2 e^{\lambda t}\cos(\kappa x) v_0 \end{cases}$$

$$\begin{bmatrix} \lambda e^{\lambda t}\cos(\kappa x) u_0 \\ \lambda e^{\lambda t}\cos(\kappa x) v_0 \end{bmatrix} = \begin{bmatrix} -D_1 \kappa^2 e^{\lambda t}\cos(\kappa x) u_0 \\ -D_2 \kappa^2 e^{\lambda t}\cos(\kappa x) v_0 \end{bmatrix} + \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix} \begin{bmatrix} e^{\lambda t}\cos(\kappa x) u_0 \\ e^{\lambda t}\cos(\kappa x) v_0 \end{bmatrix}$$

$$\lambda e^{\lambda t}\cos(\kappa x) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = e^{\lambda t}\cos(\kappa x) \left(\begin{bmatrix} -D_1 \kappa^2 u_0 \\ -D_2 \kappa^2 v_0 \end{bmatrix} + \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right)$$

For $u, v \neq 0$:

$$\begin{aligned}
\lambda \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} &= \begin{bmatrix} -D_1\kappa^2 u_0 \\ -D_2\kappa^2 v_0 \end{bmatrix} + \begin{bmatrix} (b-1)u_0 + a^2 v_0 \\ -bu_0 - a^2 v_0 \end{bmatrix} \\
&= \begin{bmatrix} -D_1\kappa^2 u_0 + (b-1)u_0 + a^2 v_0 \\ -D_2\kappa^2 v_0 - bu_0 - a^2 v_0 \end{bmatrix} \\
&= \begin{bmatrix} (-D_1\kappa^2 + b - 1)u_0 + a^2 v_0 \\ -bu_0 + (-D_2\kappa^2 - a^2)v_0 \end{bmatrix} \\
&= \begin{bmatrix} -D_1\kappa^2 + b - 1 & a^2 \\ -b & -D_2\kappa^2 - a^2 \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}
\end{aligned}$$

Thus

$$A \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \lambda \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

for

$$\boxed{A = \begin{bmatrix} -D_1\kappa^2 + b - 1 & a^2 \\ -b & -D_2\kappa^2 - a^2 \end{bmatrix}}$$