APMA 0360: Homework 9

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Problem 1:

Use the Laplacian in polar coordinates to find the radial solution of

$$u_{xx} + u_{yy} = 1$$

Such that $u = 0$ on the circles of radius $r = 1$ and $r = 2$

In polar coordinates, the Laplacian is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 1$$

As the solution is radial, $u_{\theta\theta} = 0$ so

$$u_{rr} + \frac{1}{r}u_r = 1$$

$$ru_{rr} + u_r = r$$

$$(ru_r)' = r$$

$$ru_r = \frac{1}{2}r^2 + A$$

$$u_r = \frac{1}{2}r + \frac{A}{r}$$

$$u(r) = \frac{r^2}{4} + A\ln r + B$$

Applying linearity and checking initial conditions:

$$\begin{cases} u(1) = \frac{1}{4} + A\ln(1) + B = 0 \implies B = -\frac{1}{4} \\ u(2) = 1 + A_m \ln(2) + B = 0 \implies A = -\frac{3}{4\ln 2} \end{cases}$$

so the radial solution of $\Delta u = 1$ such that $u = 0$ on $r^2 = 1$ and $r^2 = 4$ is

$$\boxed{u(x,y) = \frac{x^2+y^2}{4} - \frac{3}{4\ln 2}\ln(x^2+y^2) - \frac{1}{4}}$$

Problem 2:

Note: In the problem below, you may use without proof the fact that the Laplacian in spherical coordinates is

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{2}{r} u_r + \ \ldots$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and the extra terms don't depend on r

Find all radial solutions of

$$u_{xx} + u_{yy} + u_{zz} = k^2 u \quad k > 0$$

Hint: see above. To solve the ODE, use $v = ru(r)$

Let $v = ru(r)$. Then

$$v_r = u + r u_r$$

$$v_{rr} = 2u_r + r u_{rr}$$

So

$$\frac{v_{rr}}{r} = u_{rr} + \frac{2}{r} u_r = u_{xx} + u_{yy} + u_{zz} = k^2 u$$

$$\frac{v_{rr}}{r} = \frac{k^2 v}{r}$$

$$v_{rr} = k^2 v$$

$$v = Ae^{kr} + Be^{-kr}$$

$$ru = Ae^{kr} + Be^{-kr}$$

$$u = \frac{Ae^{kr}}{r} + \frac{Be^{-kr}}{r}$$

So

$$\boxed{u(x,y,z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left(Ae^{k\sqrt{x^2+y^2+z^2}} + Be^{-k\sqrt{x^2+y^2+z^2}} \right)}$$

Problem 3:

Solve using separation of Variables

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u_y(x,0) = 0 \\ u_y(x,\pi) = 0 \\ u(0,y) = 0 \\ u(\pi,y) = \cos^2 y = \frac{1}{2} + \frac{1}{2}\cos(2y) \end{cases}$$

Assume $u(x,y) = X(x)Y(y)$.

$$X''Y + XY'' = 0$$
$$-\frac{X''}{X} = \frac{Y''}{Y} = \lambda$$

Initial conditions:

$$u_y(x,0) = 0 \implies Y'(0) = 0$$
$$u_y(x,\pi) = 0 \implies Y'(\pi) = 0$$

Boundary Value:

$$Y'' = \lambda Y$$

$\lambda > 0$:

$$Y = Ae^{\omega x} + Be^{-\omega x}$$
$$Y' = A\omega e^{\omega y} + B\omega e^{-\omega y}$$
$$Y'(0) \implies A = -B$$
$$Y'(\pi) = A\omega e^{\pi\omega} - A\omega e^{-\pi\omega} \implies \omega = -\omega = 0 \implies Y = A - A = 0$$

$\lambda = 0$:

$$Y = A + By$$
$$Y'(0) = B = 0$$
$$Y'(\pi) = 0 = 0$$

So $\lambda = 0$ is an eigenvalue with $Y = A$. Which means that

$$X'' = 0 \implies X = A + Bx$$

so $u(x,y) = XY = A + Bx$ for some arbitrary constants.

$\lambda < 0$:

$$\begin{aligned}
Y(y) &= A\cos(\omega y) + B\sin(\omega y) \\
Y'(y) &= -A\omega\sin(\omega y) + B\omega\cos(\omega y) \\
Y'(0) &= B\omega = 0 \implies B = 0 \\
Y'(\pi) &= -A\omega\sin(\pi\omega) = 0 \implies \sin(\pi m) = 0 \quad (m = 1, 2, \ldots)
\end{aligned}$$

So $Y(y) = \cos(my)$ corresponding to $\lambda = -m^2$

Back to Laplace:

$$X''(X) = -\lambda X(x) = m^2 X(x)$$

$$\begin{aligned}
X(x) &= Ae^{mx} + Be^{-mx} \\
&= A(\cosh(mx) + \sinh(mx)) + B(\cosh(mx) - \sinh(mx)) \\
&= (A + B)\cosh(mx) + (A - B)\sinh(mx) \\
&= B\cosh(mx) + C\sinh(mx)
\end{aligned}$$

Then putting this all together

$$u(x,y) = X(x)Y(y) = A_0 + B_0 x + (B\cosh(mx) + C\sinh(mx))\cos(my)$$

$$u(x,y) = A_0 + B_0 x + \sum_{m=1}^{\infty} (A_m\cosh(mx) + B_m\sinh(mx))\cos(my)$$

$$u(0,y) = A_0 + \sum_{m=1}^{\infty} A_m\cos(my) = 0 \implies \sum_{m=0}^{\infty} A_m\cos(my) = 0 \implies A_m = 0$$

$$u(\pi,y) = B_0\pi + \sum_{m=1}^{\infty} \underbrace{B_m\sinh(\pi m)}_{\tilde{B}_m}\cos(my) = \frac{1}{2} + \frac{1}{2}\cos(2y)$$

$$\begin{cases}
\tilde{B}_2 = \frac{1}{2} \implies B_2 = \frac{1}{2\sinh(2\pi)} \\
\tilde{B}_m = 0 \qquad (m \neq 2) \\
B_0\pi = \frac{1}{2} \implies B_0 = \frac{1}{2\pi}
\end{cases}$$

$$\boxed{u(x,y) = \frac{x}{2\pi} + \frac{1}{2\sinh(2\pi)}\sinh(2x)\cos(2y)}$$

Problem 4:

1. Use the energy method to show that if u solves the heat equation in n dimensions

$$\begin{cases} u_t = D\Delta u & \in \Omega \\ u(x,t) = 0 & x \in \partial\Omega \\ u(x,0) = 0 & x \in \Omega \end{cases}$$

Then u = 0

$$u_t u = D\Delta u \cdot u$$

$$\int_\Omega u_t u \; dx = D \int_\Omega \Delta u \cdot u \; dx$$

$$\frac{d}{dt} \int_\Omega \frac{1}{2}(u)^2 \; dx = -D \int_\Omega \underbrace{||\nabla u||^2}_{\geq 0} \; dx$$

$$\frac{d}{dt} \int_\Omega \frac{1}{2}(u)^2 \; dx \leq 0$$

So with $E(t) = \int_\Omega \frac{1}{2}(u)^2 \; dx \geq 0$,

$$E'(t) \leq 0 \implies 0 \leq E(t) \leq E(0) = \int_\Omega \frac{1}{2}(u(x,0))^2 \; dx = 0 \implies E(t) = 0$$

Hence,

$$\int_\Omega \frac{1}{2}(u(x,t))^2 \; dx = 0 \implies u(x,t) = 0 \quad \in \Omega$$

and because $u(x,t) = 0$ for $x \in \partial\Omega$, $u = 0$ $\blacksquare$.

2. Use (i) to show that there is at most one solution to

$$\begin{cases} u_t = D\Delta u + f(x,t) & \in \Omega \\ u(x,t) = g(x,t) & x \in \partial\Omega \\ u(x,0) = h(x) & x \in \Omega \end{cases}$$

Hint: Use the following integration by parts formula, valid for all $v(x,t)$ with $v(x,t) = 0$ for x on $\partial\Omega$

$$\int_\Omega (\Delta u) v \; dx = -\int_\Omega (\nabla u) \cdot (\nabla v) \; dx$$

Let $w = u - v$ where u and v are solutions. Then

$$\begin{aligned}
w_t &= (u - v)_t \\
&= D\Delta u + f(x,t) - D\Delta v - f(x,t) = D(\Delta u - \Delta v) \\
&= D((u_{xx} - v_{xx}) + (u_{yy} - v_{yy})) \\
&= D(w_{xx} + w_{yy}) \\
&= D\Delta w
\end{aligned}$$

and with the initial conditions,

$$w(x,t) = u(x,t) - v(x,t) = g(x,t) - g(x,t) = 0$$

$$w(x,0) = u(x,0) - v(x,0) = h(x) - h(x) = 0$$

Hence, w solves the system from part (i). Thus, $w = 0$ so $u - v = 0 \implies u = v$ and there is only one solution. $\blacksquare$

Problem 5:

Suppose u solves Laplace's equation on the disk $x^2 + y^2 \leq 4$ with $u = 3\sin(2\theta) + 1$ on $x^2 + y^2 = 4$.

Without finding the solution:

1. Find the maximum value of u on $x^2 + y^2 \leq 4$

   By the strong maximum principle, because $\Delta u = 0$ inside the boundary of $x^2 + y^2 = 4$, its maximum exists on that boundary. Thus it suffices to find the maximum of u on that circle.

   Thus with $0 \leq \theta \leq 2\pi$:

$$\frac{d}{d\theta}u = 6\cos(2\theta) = 0 \implies \theta = \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$$
$$u(\frac{\pi}{4}) = 4$$
$$u(\frac{3\pi}{4}) = -2$$
$$u(\frac{5\pi}{4}) = 4$$
$$u(\frac{7\pi}{4}) = -2$$

   So the maximum value of u on the disk is

$$\boxed{u(\sqrt{2}, \sqrt{2}) = u(-\sqrt{2}, -\sqrt{2}) = 4}$$

2. Find $u(0, 0)$

   Hint: For (b) the mean value formula also holds if you integrate u on circles/spheres, that is

$$\frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\, dy = u(x)$$

   Where $\partial B(x, r)$ is the circle/sphere centered at x and radius r. Integrating over a circle means integrating with respect to θ

By the Mean-Value Formula, as $\Delta u = 0$ then forall x and $r > 0$,

$$\frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\ dy = u(x)$$

Hence

$$\frac{1}{|x^2+y^2=4|}\int_0^{2\pi} 2u(y)\ d\theta = u(x)$$

$$\frac{1}{4\pi}\int_0^{2\pi} 6\sin(2\theta){+}2\ d\theta = \frac{1}{4\pi}[-3\cos(2\theta){+}2\theta]_0^{2\pi} = -\frac{3}{4\pi}{+}\frac{2\pi}{2\pi}{+}\frac{3}{4\pi}{-}0 = 1 = u(x)$$

Which means that the value at the center of the circle is 1. Or:

$$\boxed{u(0,0) = 1}$$