APMA 0360: Homework 5

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Problem 1

Solve

$$\begin{cases} u_{xx} - 3u_{xt} - 4u_{tt} = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = e^x \end{cases}$$

Note: you may assume without proof that the general solution is

$$u(x,t) = F(4x+t) + G(x-t)$$

$$u(x,0) = F(4x) + G(x) = x^2$$
$$u_t(x,t) = F'(4x+t) - G'(x-t)$$
$$u_t(x,0) = F'(4x) - G'(x) = e^x$$

Integrating with respect to x and incorporating the constant into arbitrary function G,

$$\frac{1}{4}F(4x) = e^x + G(x) + C$$

Giving a system of equations for F and G:

$$\begin{cases} F(4x) + G(x) = x^2 \\ \frac{1}{4}F(4x) - G(x) = e^x + C \end{cases}$$

$$\frac{5}{4}F(4x) = x^2 + e^x + C \implies F(4x) = \frac{4}{5}x^2 + \frac{4}{5}e^x + C \implies F(x) = \frac{4}{5}\left(\frac{x}{4}\right)^2 + \frac{4}{5}e^{x/4} + C$$

$$\frac{4}{5}x^2 + \frac{4}{5}e^x + C + G(x) = x^2 \implies G(x) = \frac{1}{5}x^2 - \frac{4}{5}e^x - C$$

Then looking at the general solution

$$\begin{aligned} u(x,t) &= F(4x+t) + G(x-t) \\ &= \frac{4}{5}\left(\frac{4x+t}{4}\right)^2 + \frac{4}{5}e^{\frac{4x+t}{4}} + C + \frac{1}{5}(x-t)^2 - \frac{4}{5}e^{x-t} - C \\ &= \end{aligned}$$

$$\boxed{u(x,t) = \frac{4}{5}\left(\frac{4x+t}{4}\right)^2 + \frac{4}{5}e^{\frac{4x+t}{4}} + \frac{1}{5}(x-t)^2 - \frac{4}{5}e^{x-t}}$$

Problem 2:

Check by differentiating that

$$u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$

solves

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Note: it may help to write the integral as

$$\Psi(x+ct) - \Psi(x-ct)$$

where Ψ is the antiderivative of ψ

$$u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}(\Psi(x+ct) - \Psi(x-ct))$$

Derivatives:

$$\begin{aligned}
u_t &= -\frac{1}{2}c\phi'(x-ct) + \frac{1}{2}c\phi'(x+ct) + \frac{1}{2}\psi(x+ct) + \frac{1}{2}\psi(x-ct) \\
u_{tt} &= \frac{1}{2}c^2\phi''(x-ct) + \frac{1}{2}c^2\phi''(x+ct) + \frac{1}{2}c\psi'(x+ct) - \frac{1}{2}c\psi'(x-ct) \\
u_x &= \frac{1}{2}\phi'(x-ct) + \frac{1}{2}\phi'(x+ct) + \frac{1}{2c}\psi(x+ct) - \frac{1}{2c}\psi(x-ct) \\
u_{xx} &= \frac{1}{2}\phi''(x-ct) + \frac{1}{2}\phi''(x+ct) + \frac{1}{2c}\psi'(x+ct) - \frac{1}{2c}\psi'(x-ct)
\end{aligned}$$

PDE:

$$u_{tt} = c^2 u_{xx}$$

$$c^2 u_{xx} = \frac{1}{2}c^2\phi''(x-ct) + \frac{1}{2}c^2\phi''(x+ct) + \frac{1}{2}c\psi'(x+ct) - \frac{1}{2}c\psi'(x-ct) = u_{tt} \quad \checkmark$$

First initial condition:

$$u(x,0) = \frac{1}{2}(\phi(x) + \phi(x)) + \frac{1}{2c}(\Psi(x) - \Psi(x)) = \frac{1}{2}(2\phi(x)) = \phi(x) \quad \checkmark$$

Second initial condition:

$$u_t(x,0) = -\frac{1}{2}c\phi'(x) + \frac{1}{2}c\phi'(x) + \frac{1}{2}\psi(x) + \frac{1}{2}\psi(x) = \psi(x) \quad \checkmark$$

Problem 3:

Show there is at most one solution to the following wave equation, where $0 < x < l$

$$\begin{cases} u_{tt} = c^2 u_{xx} + f(x,t) \\ u_x(0,t) = g(t) \\ u_x(l,t) = h(t) \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Let u and v be solutions to the PDE such that $w = u - v$. Then

$$\begin{aligned} w_{tt} &= u_{tt} - v_{tt} \\ &= (c^2 u_{xx} + f(x,t)) - (c^2 v_{xx} + f(x,t)) \\ &= c^2 u_{xx} - c^2 v_{xx} \\ &= c^2 w_{xx} \end{aligned}$$

Multiply by u_t and integrate WRT x:

$$\int_0^l w_{tt} w_t \, dx = c^2 \int_0^l w_{xx} w_t \, dx$$

LHS:

$$\int_0^l w_{tt} w_t \, dx = \frac{d}{dt}\left(\frac{1}{2}\int_0^l (u_t)^2 \, dx\right)$$

RHS:

$$
\begin{aligned}
c^2 \int_0^l w_{xx} w_t \, dx &= [w_x w_t]_0^l - \int_0^l w_x w_{xt} \, dx \\
&= (w_x(l,t) w_t(l,t) - w_x(0) w_l(0)) - c^2 \int_0^l w_x w_{xt} \, dx \\
&= (u_x(l,t) - v_x(l,t))(u_t(l,t) - v_t(l,t)) - (u_x(0,t) - v_x(0,t))(u_t(0,t) - v_t(0,t)) \\
&\quad - c^2 \int_0^l \frac{d}{dt} \left(\frac{1}{2} (w_x)^2 \right) \, dx \\
&= (h(t) - h(t))(u_t(l,t) - v_t(l,t)) - (g(t) - g(t))(u_t(0,t) - v_t(0,t)) \\
&\quad - c^2 \frac{d}{dt} \left(-\frac{1}{2} \int_0^l (w_x)^2 \, dx \right) \\
&= -c^2 \frac{d}{dt} \left(-\frac{1}{2} \int_0^l (w_x)^2 \, dx \right)
\end{aligned}
$$

Then

$$
\frac{d}{dt} \left(\frac{1}{2} \int_0^l (w_t)^2 \, dx \right) = -c^2 \frac{d}{dt} \left(-\frac{1}{2} \int_0^l (w_x)^2 \, dx \right)
$$

$$
\frac{d}{dt} \left(\frac{1}{2} \int_0^l (w_t)^2 + c^2 (w_x)^2 \, dx \right) = 0
$$

where energy is constant for

$$
E(t) = \frac{1}{2} \int_0^l (w_t)^2 + c^2 (w_x)^2 \, dx
$$

As the energy is constant, $E(t) = E(0)$ and

$$
\begin{aligned}
\frac{1}{2} \int_0^l (w_t)^2 + c^2 (w_x)^2 \, dx &= \frac{1}{2} \int_0^l (w_t(x,0))^2 + c^2 (w_x(x,0))^2 \, dx \\
&= \frac{1}{2} \int_0^l (u_t(x,0) - v_t(x,0))^2 + c^2 (u_t(x,0) - v_t(x,0))^2 \, dx \\
&= \frac{1}{2} \int_0^l (\phi(x) - \phi(x))^2 + c^2 (u_x(x,0) - v_x(x,0))^2 \, dx \\
&= \frac{1}{2} \int_0^l c^2 (u_x(x,0) - v_x(x,0))^2 \, dx
\end{aligned}
$$

Then,

$$\begin{aligned}
\frac{1}{2}\int_0^l c^2(u_x(x,0)-v_x(x,0))^2\,dx &= \frac{1}{2}\left[\frac{1}{3}c^2(u(x,0)-v(x,0))^3\right]_0^l \\
&= \frac{c^2}{6}((u(l,0)-v(l,0))-(u(0,0)-v(0,0))) \\
&= \frac{c^2}{6}((\phi(l)-\phi(l))-(\phi(0)-\phi(0))) \\
&= 0
\end{aligned}$$

So at long last

$$E(t) = E(0) = 0$$

which means that

$$\frac{1}{2}\int_0^l (w_t)^2 + c^2(w_x)^2\,dx = 0$$

Deriving WRT x,

$$\frac{1}{2}(w_t)^2 + \frac{1}{2}c^2(w_x)^2 = 0$$

so w_t and w_x are zero. Hence $w(x,t) = C$ but at $t = 0$,

$$w(x,0) = u(x,0) - v(x,0) = 0 - 0 = 0$$

so w is 0 for all x, t and

$$u = v$$

showing there is only one solution. ■

Problem 4:

Suppose u solves the following heat-like PDE where $0 < x < l$

$$\begin{cases} u_t = Du_{xx} - u^3 \\ u(0,t) = u(l,t) \\ u_x(0,t) = u_x(l,t) \\ u(x,0) = 0 \end{cases}$$

Show that $u(x,t) = 0$ for all x and t

Multiply by u to get

$$u_t u = Du_{xx}u - u^4$$

Integrate with respect to x on $[0,l]$

$$\int_0^l u_t u \, dx = \int_0^l Du_{xx}u - u^4 \, dx$$

Looking at the LHS,

$$\int_0^l u_t u \, dx = \int_0^l \frac{d}{dt}\left(\frac{1}{2}u^2\right) \, dx = \frac{1}{2}\frac{d}{dt}\int_0^l u^2 \, dx$$

Then the RHS:

$$\int_0^l Du_{xx}u - u^4 \, dx = D\int_0^l u_{xx}u \, dx - \int_0^l u^4 \, dx$$

$$\begin{aligned} D\int_0^l u_{xx}u \, dx &\overset{\text{IBP}}{=} u_x(l,t)u(l,t) - u_x(0,t)u(0,t) - \int_0^l u_x u_x \, dx \\ &= u_x(l,t)u(l,t) - u_x(l,t)u(l,t) - \int_0^l (u_x)^2 \, dx \\ &= -\int_0^l (u_x)^2 \, dx \end{aligned}$$

So

$$\frac{d}{dt}\frac{1}{2}\int_0^l u^2 \, dx = -\int_0^l (u_x)^2 \, dx - \int_0^l u^4 \, dx$$

Then define an energy function E such that

$$E(t) = \frac{1}{2}\int_0^l u^2\,dx$$

so

$$E'(t) = \frac{d}{dt}\frac{1}{2}\int_0^l u^2\,dx = -\int_0^l (u_x)^2\,dx - \int_0^l u^4\,dx$$

which is negative because both integrands are positive, showing that energy is decreasing. But then $E(t) \leq E(0)$ so

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^l u(x,t)^2\,dx &\leq \frac{1}{2}\int_0^l u(x,0)^2\,dx \\ &= \frac{1}{2}\int_0^l 0^2\,dx \\ &= 0 \end{aligned}$$

so because $E(t) \geq 0$ by definition

$$0 \leq E(t) \leq E(0) = 0 \implies E(t) = 0$$

or

$$\frac{1}{2}\int_0^l u(x,t)^2\,dx = 0$$

so

$$u(x,t)^2 = 0 \implies u(x,t) = 0$$

for all x and t. $\blacksquare$

Problem 5:

Definition: f is *monotone* if $(f(x) - f(y))(x - y) \geq 0$ for all x and y

Show that if f is monotone, then there is at most one solution to the following PDE (ODE) where $u = u(x)$ and $-\infty < x < \infty$. Assume any terms are $\pm\infty$ are 0

$$\begin{cases} u_{xx} = f(u) \\ u(0) = 2 \end{cases}$$

Note: Do the usual subtraction trick. The definition above should give you an idea what to multiply your PDE by

If f is monotone Let u and v be solutions to the PDE and $w = u - v$. Then

$$w_{xx} = u_{xx} - v_{xx} = f(u) - f(v)$$

Using energy methods,

$$w_{xx}w = (f(u) - f(v))w$$

$$\int_{-\infty}^{\infty} w_{xx}w \, dx = \int_{-\infty}^{\infty} (f(u) - f(v))(u - v) \, dx$$

Looking at the LHS,

$$\begin{aligned} \int_{-\infty}^{\infty} w_{xx}w \, dx &= [w_x w]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} w_x w_x \, dx \\ &= -\int_{-\infty}^{\infty} (w_x)^2 \, dx \end{aligned}$$

So

$$-\int_{-\infty}^{\infty} (w_x)^2 \, dx = \int_{-\infty}^{\infty} (f(u) - f(v))(u - v) \, dx$$

Then because f is monotone, the integrand of the RHS is greater than or equal to zero, and so must be the RHS integral. However, $(w_x)^2$ is positive so the LHS must be negative or zero. Thus w_x must be equal 0 for all x and w is then constant. Then from the initial conditions,

$$w(0) = u(0) - v(0) = 2 - 2 = 0$$

so w is 0 for all x and t and

$$u = v$$

showing that there is only one solution. ■