APMA 0360: Homework 7

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Problem 1:

1. Use the trig identity $\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$ to show that if $m \neq n$ then
$$\int_0^\pi \sin(mx)\sin(nx)\ dx = 0$$
$$\begin{aligned}
&\int_0^\pi \sin(mx)\sin(nx)\ dx\\
&= \int_0^\pi \frac{1}{2}\cos(mx-nx) - \frac{1}{2}\cos(mx+nx)\ dx \qquad \text{(by identity)}\\
&= \int_0^\pi \frac{1}{2}\cos(mx-nx)\ dx - \int_0^\pi \frac{1}{2}\cos(mx+nx)\ dx\\
&= \frac{1}{2}\int_0^\pi \cos((m-n)x)\ dx - \frac{1}{2}\int_0^\pi \cos((m+n)x)\ dx\\
&= \frac{1}{2}\left(\left[\frac{1}{m-n}\sin((m-n)x)\right]_0^\pi - \left[\frac{1}{m+n}\sin((m+n)x)\right]_0^\pi\right)\\
&= \frac{1}{2}\left(\frac{\sin((m-n)\pi)}{m-n} - \frac{\sin((m+n)\pi)}{m+n}\right)\\
&= \frac{1}{2}\cdot\frac{(m+n)\sin(m\pi-n\pi) - (m-n)\sin(m\pi+n\pi)}{m^2-n^2}\\
&= \frac{1}{2}\cdot\frac{(m+n)(\sin(m\pi)\cos(n\pi) - \sin(n\pi)\cos(m\pi))}{m^2-n^2}\\
&\quad - \frac{1}{2}\frac{(m-n)(\sin(m\pi)\cos(n\pi) + \sin(n\pi)\cos(m\pi))}{m^2-n^2}
\end{aligned}$$
But as m and n are integers greater than , the functions $\sin(m\pi)$ and $\sin(n\pi)$ will be zero for all values of m and n so all terms will be 0 and thus the integral equals 0. $\blacksquare$

2. Show that

$$\int_0^\pi \sin^2(mx)\, dx = \frac{\pi}{2}$$

Using the same identity above but with $A = B = mx$,

$$\begin{aligned}
\int_0^\pi \sin^2(mx)\, dx &= \int_0^\pi \frac{1}{2}\cos(mx - mx) - \frac{1}{2}\cos(mx + mx)\, dx \\
&= \int_0^\pi \frac{1}{2} - \frac{1}{2}\cos(2mx)\, dx \\
&= \frac{\pi}{2} - \frac{1}{2}\int_0^\pi \cos(2mx)\, dx \\
&= \frac{\pi}{2} - \frac{1}{2}\left[\frac{\sin(2mx)}{2m}\right]_0^\pi \\
&= \frac{\pi}{2} - \frac{1}{2}\left(\frac{\sin(2\pi m)}{2m} - \frac{\sin(0)}{2m}\right)
\end{aligned}$$

and as $m = 1, 2, 3...$, $\sin(2\pi m) = 0$ for all m so

$$\int_0^\pi \sin^2(mx)\, dx = \frac{\pi}{2} \quad \blacksquare$$

3. Show that $\{e^{imx} | m \in \mathbb{Z}\}$ is orthogonal on $(-\pi, \pi)$ where

$$f \cdot g = \int_{-\pi}^{\pi} f(x)g(x)\, dx$$

The sequence is orthogonal if for all $m \neq n$,

$$e^{imx} \cdot e^{inx} = 0$$

where the function dot product is defined as

$$f \cdot g = \int_{-\pi}^{\pi} f(x)g(x)\, dx.$$

Then the proof amounts to showing that

$$\int_{-\pi}^{\pi} e^{imx}e^{inx}\, dx = 0$$

Which can be seen as follows:

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{imx} e^{inx}\, dx &= \int_{-\pi}^{\pi} e^{(m+n)ix}\, dx \\
&= \int_{-\pi}^{\pi} \cos((m+n)x) + i\sin((m+n)x)\, dx \\
&= \left[\frac{1}{m+n}\sin(mx+nx)\right]_{-\pi}^{\pi} - i\left[\frac{1}{m+n}\cos((m+n)x)\right]_{-\pi}^{\pi} \\
&= \left(\frac{\sin((m+n)\pi)}{m+n} - \frac{\sin(-(m+n)\pi)}{m+n}\right) - i\left(\frac{\cos((m+n)\pi)}{m+n} - \frac{\cos(-(m+n)\pi)}{m+n}\right) \\
&= \left(\frac{\sin((m+n)\pi)}{m+n} + \frac{\sin((m+n)\pi)}{m+n}\right) - i\left(\frac{\cos((m+n)\pi)}{m+n} - \frac{\cos((m+n)\pi)}{m+n}\right) \\
&= 2\left(\frac{\sin((m+n)\pi)}{m+n}\right)
\end{aligned}$$

And because m and n are both integers, the product $(m+n)\pi$ will always be an integer multiple of π so sin will be zero and thus

$$\int_{-\pi}^{\pi} e^{imx} e^{inx}\, dx = 2\left(\frac{\sin((m+n)\pi)}{m+n}\right) = 0$$

for all m and n. ■

Problem 2:

By showing all your steps, including the 3 cases, solve the following wave equation with Neumann boundary conditions

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u_x(0,t) = 0 \\ u_x(\pi,t) = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = \cos(3x) \end{cases}$$

Note: answer in video

Assume that $u(x,t) = X(x)T(t)$ so

$$(X(x)T(t))_{tt} = c^2 (X(x)T(t))_{xx}$$
$$XT'' = c^2 X''T$$
$$\frac{T''}{c^2 T} = \frac{X''}{X}$$

Then notice that

$$\left(\frac{T''}{c^2T}\right)_x = 0$$
$$\left(\frac{T''}{c^2T}\right)_t = \left(\frac{X''}{X}\right)_t = 0$$

so

$$\frac{T''}{c^2T} = \frac{X''}{X} = \lambda$$

Looking at the X terms, we have the ODE

$$\begin{cases} X'' = \lambda X \\ u_x(0,t) = 0 \implies X'(0) = 0 \\ u_x(\pi,t) = 0 \implies X'(\pi) = 0 \end{cases}$$

so

$$r^2 = \lambda \implies r = \pm\omega$$

Case 1: $\lambda > 0$

$$X = Ae^{\omega x} + Be^{-\omega x}$$
$$X' = A\omega e^{\omega x} - B\omega e^{-\omega x}$$
$$X'(0) = A\omega - B\omega = 0 \implies A - B = 0 \implies A = B$$
$$X' = A\omega e^{\omega x} - A\omega e^{-\omega x}$$
$$X'(\pi) = A\omega e^{\pi\omega} - A\omega e^{-\pi\omega} = 0 \implies 2\pi\omega = 0$$

But then $\lambda = 0$ which contradicts $\lambda > 0$ so there are no nonzero solutions

Case 2: $\lambda = 0$

$$r = 0 \implies X(x) = A + Bx \implies X'(0) = B = 0 \implies X(x) = A$$

$$X'(\pi) = 0$$

So $\lambda = 0$ is an eigenvalue with eigenfunction $X(x) = A$

Case 3: $\lambda < 0$

$$r = \pm\omega i \implies X(x) = A\cos(\omega x) + B\sin(\omega x)$$
$$X'(x) = -A\omega\sin(\omega x) + B\omega\cos(\omega x)$$
$$X'(0) = B\omega = 0 \implies B = 0$$
$$X'(\pi) = -A\omega\sin(\omega\pi) = 0 \implies \sin(\omega\pi) = 0 \implies \omega = m$$

So the eigenvalues are $\lambda = \{-m^2 | m = 0, 1, 2...\}$ corresponding to eigenfunction $X(x) = \cos(mx) \quad (m = 0, 1, 2...)$

Going back to the T equation,

$$T'' = -c^2m^2T$$

$$r^2 = -c^2m^2 \implies r = \pm cmi$$

so

$$T(t) = A\cos(cmt) + B\sin(cmt)$$

and

$$u(x,t) = (A\cos(cmt) + B\sin(cmt))\cos(mx)$$

which by linearity gives

$$u(x,t) = At + B + \sum_{m=1}^{\infty}(A_m\cos(cmt) + B_m\sin(cmt))\cos(mx)$$

Then with initial conditions:

$$u(x,0) = B + \sum_{m=1}^{\infty} A_m \cos(mx) = x^2$$

Which amounts to finding the cosine series of x^2 on $(0, \pi)$: For

$$x^2 = \sum_{m=0}^{\infty} A_m \cos(\pi m x)$$

$$B = A_0 = \frac{1}{\pi} \int_0^{\pi} x^2 \, dx = \frac{\pi^2}{3}$$

and

$$A_m = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(mx) \, dx$$

which via tabular integration:

$$\begin{array}{ccc} x^2 & & \cos(mx) \\ & \searrow & \\ -2x & & \sin(mx)/(m) \\ & \searrow & \\ 2 & & -\cos(mx)/(m)^2 \\ & \searrow & \\ -0 & & -\sin(mx)/(m)^3 \end{array}$$

is

$$\begin{aligned} A_m &= \frac{2}{\pi} \left[x^2 \left(\frac{\sin(mx)}{m} \right) + 2x \left(\frac{\cos(mx)}{m^2} \right) - 2 \left(\frac{\sin(mx)}{m^3} \right) \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\pi^2 \left(\frac{\sin(m\pi)}{m} \right) + 2\pi \left(\frac{\cos(m\pi)}{m^2} \right) - 2 \left(\frac{\sin(m\pi)}{m^3} \right) \right] \\ &= \frac{2}{\pi} \left(\frac{2\pi}{m^2} \right) (-1)^m \\ &= \frac{4}{m^2} (-1)^m \end{aligned}$$

and then with the other initial condition,

$$u_t(x,0) = \sum_{m=1}^{\infty} (-A_m cm \sin(cm(0)) + B_m cm \cos(cm(0))) \cos(mx) = \sum_{m=1}^{\infty} B_m cm \cos(mx) = \cos(3x)$$

which tells us that

$$3B_3c = 1 \implies B_3 = \frac{1}{3c}$$

and all other $B_m = 0$. Then, after an eternity,

$$\boxed{\begin{aligned} u(x,t) = \frac{\pi^2}{3} &+ \sum_{m=1}^{2} \left(\frac{4}{m^2}(-1)^m \cos(mct) \right) \cos(mx) \\ &- \left(\frac{4}{9}\cos(3ct) + \frac{1}{3c}\sin(3ct) \right) \cos(3x) \\ &+ \sum_{m=4}^{\infty} \left(\frac{4}{m^2}(-1)^m \cos(mct) \right) \cos(mx) \end{aligned}}$$

Problem 3:

1. Find the Fourier sine series of $f(x) = x^2$ on $(0, 1)$

For

$$x^2 = \sum_{m=0}^{\infty} A_m \sin(mx)$$

on $(0, 1)$,

$$A_m = 2\int_0^1 x^2 \sin(\pi m x)\, dx.$$

Then using tabular integration,

$$\begin{array}{ccc} x^2 & & \sin(\pi m x) \\ & \searrow & \\ -2x & & -\cos(\pi m x)/(\pi m) \\ & \searrow & \\ 2 & & -\sin(\pi m x)/(\pi m)^2 \\ & \searrow & \\ -0 & & \cos(\pi m x)/(\pi m)^3 \end{array}$$

$$\begin{aligned} A_m &= 2\left[-x^2\left(\frac{\cos(\pi m x)}{\pi m}\right) + 2x\left(\frac{\sin(\pi m x)}{(\pi m)^2}\right) + 2\left(\frac{\cos(\pi m x)}{(\pi m)^3}\right)\right]_0^1 \\ &= 2\left[-\frac{\cos(\pi m)}{\pi m} + 2\frac{\sin(\pi m)}{(\pi m)^2} + 2\frac{\cos(\pi m)}{(\pi m)^3} - \frac{2}{(\pi m)^3}\right] \\ &= -\frac{2}{\pi m}(-1)^m + \frac{4}{(\pi m)^3}((-1)^m - 1) \end{aligned}$$

The second term is -2 for odd m and 0 for even so the fourier series is

$$A_m = \begin{cases} \frac{2}{\pi m}(-1)^{m+1} & m \text{ even} \\ \frac{2}{\pi m}(-1)^{m+1} - \frac{4}{(\pi m)^3} & m \text{ odd} \end{cases}$$

so

$$\boxed{x^2 = \sum_{m=1}^{\infty}\left(\frac{2}{\pi m}(-1)^{m+1} + \frac{4}{(\pi m)^3}((-1)^m - 1)\right)\sin(\pi m x)}$$

2. Find the Fourier cosine series of $f(x) = x^2$ on $(0, 1)$

For

$$x^2 = \sum_{m=0}^{\infty} A_m \cos(\pi m x)$$

$$A_0 = \int_0^1 x^2 \, dx = \frac{1}{3}$$

and

$$A_m = 2\int_0^1 x^2 \cos(\pi m x) \, dx$$

which via tabular integration:

$$\begin{array}{ccc} x^2 & & \cos(\pi m x) \\ & \searrow & \\ -2x & & \sin(\pi m x)/(\pi m) \\ & \searrow & \\ 2 & & -\cos(\pi m x)/(\pi m)^2 \\ & \searrow & \\ -0 & & -\sin(\pi m x)/(\pi m)^3 \end{array}$$

is

$$\begin{aligned} A_m &= 2\left[x^2\left(\frac{\sin(\pi m x)}{\pi m}\right) + 2x\left(\frac{\cos(\pi m x)}{(\pi m)^2}\right) - 2\left(\frac{\sin(\pi m x)}{(\pi m)^3}\right)\right]_0^1 \\ &= 2\left(\frac{\sin(\pi m)}{\pi m}\right) + 4\left(\frac{\cos(\pi m)}{(\pi m)^2}\right) - 4\left(\frac{\sin(\pi m)}{(\pi m^3)}\right) \\ &= \frac{4}{(\pi m)^2}(-1)^m \end{aligned}$$

so

$$\boxed{x^2 = \frac{1}{3} + \sum_{m=1}^{\infty} \frac{4}{(\pi m)^2}(-1)^m \cos(\pi m x)}$$

Problem 4:

1. Find the Fourier sine series of $f(x) = x$ on $(0, L)$

   For
$$x = \sum_{m=1}^{\infty} A_m \sin(mx)$$

$$\begin{aligned} A_m &= \frac{2}{L}\int_0^L x\sin(\frac{\pi m x}{L})\,dx \\ &= \frac{2}{L}\left[x\left(-\frac{L}{\pi m}\cos(\frac{\pi m x}{L})\right) - \left(-(\frac{L}{\pi m})^2\sin(\frac{\pi m x}{L})\right)\right]_0^L \\ &= \frac{2}{L}\left[-\frac{L^2}{\pi m}\cos(\pi m) + \frac{L^2}{(\pi m)^2}\sin(\pi m)\right] \\ &= \frac{2L}{\pi m}(-1)^{m+1} \end{aligned}$$

   so
$$\boxed{x = \frac{2L}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}}\sin(\frac{\pi m x}{L})$$

2. Integrate the series in (1) term-by-term (assume this is allowed) to find the Fourier cosine series of x^2 Note: For the A_0 term you have to do it directly using the definition, because of the constant of integration.

   First doing A_0:
$$A_0 = \int_0^L x\,dx = \frac{L^2}{2}$$

   then for all other m,
$$\int \frac{2L(-1)^{m+1}}{\pi m}\sin(mx)\,dx = -\frac{2L(-1)^{m+1}}{\pi m^2}\cos(mx)$$

   so
$$\boxed{x^2 = \frac{L^2}{2} - \frac{2L}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^2}\cos(mx)}$$

3. Plug in $x = 0$ in your result from (2) to find

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - ...$$

$$f(x) = x^2 = \frac{L^2}{2} - \frac{2L}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2} \cos(mx)$$

$$f(0) = 0 = \frac{L^2}{2} - \frac{2L}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2}$$

so

$$\frac{2L}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2} = \frac{L^2}{2}$$

and

$$\boxed{\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2} = \frac{\pi}{4} L}$$

Problem 5:

Find the complex Fourier series of e^{ax} on $(-\pi, \pi)$ where $a > 0$

If

$$f(x) = e^{ax} = \sum_{-\infty}^{\infty} C_m e^{imx}$$

then

$$\begin{aligned} C_m &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} e^{-imx}\, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-im)x} \\ &= \frac{1}{2\pi} \left[\frac{e^{(a-im)x}}{a - im} \right]_{-\pi}^{\pi} \\ &= \frac{1}{(2\pi)(a - im)} \left(e^{\pi a} e^{-im\pi} - e^{-\pi a} e^{im\pi} \right) \end{aligned}$$

But notice that

$$\begin{aligned} e^{\pi mi} &= \cos(\pi m) + i \sin(\pi m) = (-1)^m \\ e^{-\pi mi} &= \cos(-\pi m) + i \sin(-\pi m) = -\cos(\pi m) = (-1)^m \end{aligned}$$

so

$$C_m = \frac{1}{\pi(a - im)} \left(\frac{e^{\pi a} - e^{-\pi a}}{2} \right) (-1)^m = \frac{(-1)^m \sinh(\pi a)}{\pi(a - im)}$$

and

$$\boxed{e^{ax} = \sum_{m=-\infty}^{\infty} \left(\frac{(-1)^m}{\pi(1 - \pi m)} \sinh(\pi a) \right) e^{imx}}$$