# APMA 1360: Homework 8

Milan Capoor

4 April 2025

## 1 Stability

Consider the system $\dot{u} = F(u)$ with $u \in \mathbb{R}^n$ and $F \in C^2$. We assume that $u = 0$ is an equilibrium and that its Jacobian $A = F_u(0)$ is a diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$ on the diagonal and zeros everywhere else.

I stated in class that if all eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A$ have strictly negative real part, then the origin is stable. The goal of this problem is to prove this statement.

(i) Assume that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the diagonal matrix $A$ are strictly negative. Show that there is a $\delta > 0$ so that

$$V(u) = \sum_{j=1}^{n} u_j^2 = |u|^2$$

is a Lyapunov functional for all $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ with $|u| < \delta$.

$$\nabla V(u) = \begin{pmatrix} 2u_1 \\ \vdots \\ 2u_n \end{pmatrix} = 2u$$

From the Taylor series, if $|u| < 1$,

$$|F(u) - F(0) - F_u(0)u| \leq C\,|u|^2$$

but $F(0) = 0$ so we have

$$|F(u) - F_u(0)u| = |F(u) - Au| \leq C\,|u|^2$$

Hence, it suffices to show

$$\langle V(u), F(u)\rangle < 0 \iff \langle V(u), Au + F(u) - Au\rangle < 0 \iff 2u \cdot Au + 2u \cdot (F(u) - Au) < 0$$

First consider $2u \cdot Au$:

$$\begin{aligned} 2u \cdot Au &= 2\sum_{j=1}^{n} u_j \lambda_j u_j \\ &= 2\sum_{j=1}^{n} \lambda_j u_j^2 \\ &= 2\sum_{j=1}^{n} \tilde{\lambda} u_j^2 < 0 \qquad (\tilde{\lambda} := \max(\lambda_{i=1:n})) \\ &= 2\tilde{\lambda}\,|u|^2 \\ &< 2\tilde{\lambda}\delta^2 \end{aligned}$$

Now consider $2u \cdot (F(u) - Au) < 0$. Again by the Taylor expansion,

$$\begin{aligned}
2u \cdot (F(u) - Au) &\leq 2\left|u \cdot (F(u) - Au)\right| \\
&= 2\left|u\right| \left|F(u) - Au\right| \\
&\leq 2C\left|u\right|\left|u\right|^2 \\
&\leq 2C\delta^3
\end{aligned}$$

So it suffices to have

$$2\tilde{\lambda}\delta^2 + 2C\delta^3 < 0$$

Since we need $0 < |u| < 1$, assume $\delta < 1$, giving

$$2\tilde{\lambda}\delta^2 + 2C\delta^3 < 2\tilde{\lambda}\delta + 2C\delta^2$$

Requiring this to be less than zero,

$$2\tilde{\lambda}\delta + 2C\delta^2 < 0 \implies \delta < -\frac{\tilde{\lambda}}{C}$$

By construction, for $\delta = \min\left(1, -\frac{\max\{\lambda_1, \dots, \lambda_n\}}{C}\right)$, $V$ is a Lyapunov functional for all $u$ with $|u| < \delta$.

(ii) In the case described in (ii), prove that the origin is stable.

We wish to show that $\exists \delta > 0$ such that $|u(0)| < \delta \implies u(t) \to 0$.

Let $\delta$ be the same as in (i). Then, since $V$ is a Lyapunov functional, by a Lemma from class,

$$\frac{d}{dt}V(u(t)) < 0 \implies \frac{d}{dt}\left|u(t)\right|^2 < 0$$

But this implies that $\left|u(t)\right|^2 < \left|u(0)\right|^2 < \delta^2$ for all $t > 0$. Since $\left|u(t)\right|^2 \geq 0$ by definition of the norm, we have that $\left|u(t)\right|^2 \to 0 \implies \left|u(t)\right| \to 0$ as $t \to \infty$, exactly as desired.

Here is additional information that may be useful:

- Recall that we say that an equilibrium $u_*$ is stable if there is a $\delta > 0$ so that $u(t) \to u_*$ as $t \to \infty$ for all solutions $u(t)$ of $\dot{u} = F(u)$ for which $|u(0) - u_*| < \delta$. Here, $|u|$ denotes the norm of a vector in $\mathbb{R}^n$.
- You can use without proof that for each function $F \in C^2$ there is a constant $C > 0$ so that

$$|F(u) - F(0) - F_u(0)u| \leq C|u|^2$$

for all $u$ with $|u| \leq 1$. This follows from the Taylor series expansion for $F$.

# 2 Lyapunov functionals

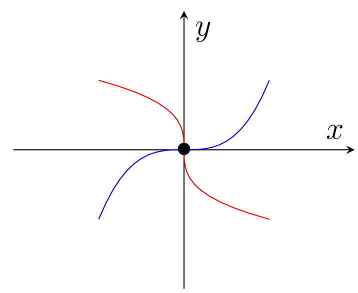
Consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - x^3 \\ -x - y^3 \end{pmatrix}$$

(i) Show that this system has a Lyapunov functional of the form $V(x, y) = ax^2 + by^2$ for appropriate values of $a, b$.

Let $F(x, y) = \begin{pmatrix} y - x^3 \\ -x - y^3 \end{pmatrix}$.

Trivially, we have $F(x, y) = 0$ where $(x, y) = (x, x^3)$ and $(x, y) = (-y^3, y)$




That is, only at $(x, y) = (0, 0)$.

We want to show that $V(x, y) = ax^2 + by^2$ is a Lyapunov functional for $F$ for all $(x, y) \neq (0, 0)$:

$$\begin{aligned} \nabla V(x, y) &= \begin{pmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \end{pmatrix} = \begin{pmatrix} 2ax \\ 2by \end{pmatrix} \\ \langle \nabla V(x, y), F(x, y) \rangle &= \begin{pmatrix} 2ax \\ 2by \end{pmatrix} \cdot \begin{pmatrix} y - x^3 \\ -x - y^3 \end{pmatrix} \\ &= 2a(xy - x^4) - 2b(xy + y^4) \end{aligned}$$

Hence, we need

$$2a(xy - x^4) < 2b(xy + y^4)$$

If $a = b > 0$,

$$2a(xy - x^4) - 2a(xy + y^4) = -2ax^4 - 2ay^4 < 0 \quad \forall x, y$$

Hence $V$ is a Lyapunov functional for $F$ if $a = b > 0$.

(ii) Use this result to show that this system cannot have any periodic orbits.

By a Lemma from class, since $V$ is a Lyapunov functional, the system cannot have any (non-trivial) periodic orbits.

# 3 Conserved systems

Consider the second-order equation $\ddot{x} = x - x^2$.

(i) Write this equation as a second-order system for $(x, \dot{x})$.

Let $y = \dot{x}$. Then,

$$\ddot{x} = \dot{y} = x - x^2$$

so we may write

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ x - x^2 \end{pmatrix} = F(x, y)$$

(ii) Find all equilibria.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = 0 \implies \begin{cases} y = 0 \\ x - x^2 = 0 \end{cases} \implies \begin{cases} y = 0 \\ x = \{0, 1\} \end{cases}$$

which means we have equilibria at $(0, 0)$ and $(1, 0)$.

(iii) Find a conserved quantity.

As in class, we suspect

$$H(x, y) = \frac{y^2}{2} - \int_0^x t - t^2 \, dt = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3}$$

will be conserved.

It suffices to check $\langle \nabla H(x, y), F(x, y) \rangle = 0$:

$$\nabla H(x, y) = \begin{pmatrix} -x + x^2 \\ y \end{pmatrix}$$

$$\langle \nabla H(x, y), F(x, y) \rangle = -(x - x^2)y + y(x - x^2) = 0$$

as desired.

(iv) Sketch the phase portrait.