> **Theorem (Existence and Uniqueness):** Consider $\dot{u} = f(u)$ with $u(0) = u_0$. Assume $u \in \mathbb{R}^2$ and $f \in C^1$. Then there exists a unique solution to the ODE on some interval around $t = 0$.

Equilibrium: If $\dot{u} = f(u) = 0$ at u_*, then u_* is an *equilibrium.*

- If $f'(u_*) < 0$, then u_* is *stable*
- If $f'(u_*) > 0$, then u_* is *unstable*

Phase diagrams:

1. Plot f
2. Zeros of f are equilibrium points
3. If f changes from positive to negative at the equilibrium, it is stable
4. If f changes from negative to positive at the equilibrium, it is unstable
5. Otherwise, it is a saddle

> **Implicit Function Theorem:** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^k $(k \geq 1)$ function. Assume $f(x_*, y_*) = 0$.
>
> Then:
>
> 1. If $f_x(x_*, y_*) \neq 0$, then $\exists! g : B_\varepsilon(y_*) \to \mathbb{R}$ with $x_* = g(y_*)$ and $g \in C^k$ such that $f(x, y) = 0 \iff x = g(y)$ for $(x, y) \in B_\varepsilon(x_*) \times B_\varepsilon(y_*)$
> 2. If $f_y(x_*, u_{)}) \neq 0$, then $\exists! h : B_\varepsilon(x_*) \to \mathbb{R}$ with $y_* = h(x_*)$ and $h \in C^k$ such that $f(x, y) = 0 \iff y = g(x)$ for $(x, y) \in B_\varepsilon(x_*) \times B_\varepsilon(y_*)$

Example: Find all zeros of $f(x, y) = y + y^2 e^x + (\sin x)^2 - xy$ near $(0, 0)$.

1. Check conditions:
   - $f \in C^\infty$
   - $f(0, 0) = 0$
   - $f_x(0, 0) = (0)^2(1) + 2(0)(1) - (0) = 0$
   - $f_y(0, 0) = 1 + 2(0)(1) - (0) \neq 0$
2. Apply IFT to get $f(x, y) = 0$ iff $x = g(y)$ with $g(0) = 0$
3. Taylor expand f:

   $$g(x) = g(0) + xg'(0) + O(x^2) = xg'(0) + O(x^2) = O(x)$$

Hyperbolic Equilibrium: If $f(u_*, \mu_*) = 0$ and $f_u(u_*, \mu_*) \neq 0$, we say (u_*, μ_*) is *hyperbolic*

0.1 Catelogue of bifurcations:

Saddle-node: $\dot{u} = \mu - u^2$

$$\begin{cases} f(u_*, \mu_*) = 0 \\ f_u(u_*, \mu_*) = 0 \\ f_\mu(u_*, \mu_*) \neq 0 \\ f_{uu}(u_*, \mu_*) \neq 0 \end{cases}$$

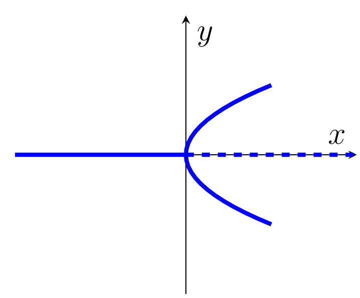
Transcritical: $\dot{u} = u(u - \mu)$.

$$\begin{cases} f(0, \mu) = 0 \\ f_u(0, 0) = 0 \\ f_{u\mu}(0, 0) \neq 0 \\ f_{uu}(0, 0) \neq 0 \end{cases}$$

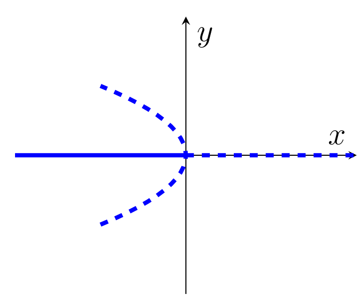
Pitchfork: $\dot{u} = \mu u - u^3$

$$\begin{cases} f(-u, \mu) = -f(u, \mu) \\ f_u(0, 0) = 0 \\ f_{u\mu}(0, 0) \neq 0 \\ f_{uuu}(0, 0) \neq 0 \end{cases}$$

Which has two forms:



Supercritical
$f_{uuu}(0,0)<0$



Subcritical
$f_{uuu}(0,0)>0$

0.2 Multidimensional Systems

For $\dot{u} = f(u)$ with $u \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$, we have two different methods for solving:

1. Find the eigenvalues of the Jacobian $Df(u_i)$ to determine stability for each equilibrium u_i

2. Plot nullclines and examine what the gradient does at each region of the phase plane. Notice the gradient ∇h is always perpendicular to the nullcline and pointing in the firection of increasing h.

Nullcline: $\{(x, y) : \dot{x} = 0\}$ or $\{(x, y) : \dot{y} = 0\}$. Where the nullclines intersect are the equilibrium points.

1 Practice Problems

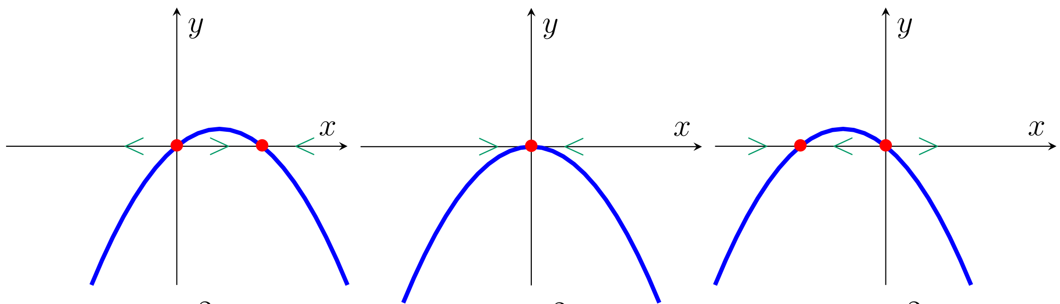
1. Consider the system $\dot{x} = x(r - 2 - x)$

(a) Determine all fixed points

$x = 0$ and $x = r - 2$.

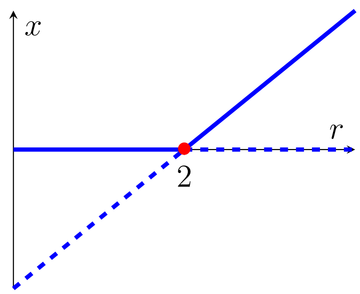
(b) Classify the stability as a function of r $f(x) = x(r - 2 - x)$. $f'(x) = r - 2 - 2x$ so $f'(0) = r - 2$ and $f'(r - 2) = -r + 2$. Thus, $x = 0$ is stable for $r < 2$ and $x = r - 2$ is stable for $r > 2$.

(c) Sketch all qualitatively different phase portraits



$r > 2$ $\qquad$ $r = 2$ $\qquad$ $r < 2$

(d) Sketch the bifurcation diagram and identify the type of bifurcation



Transcritical

2. Make a phase portrait and classify all equilibria of the following system

$$\begin{cases} \dot{x} = x(3 - x) - 2xy \\ \dot{y} = y(2 - y) - xy \end{cases}$$

We have Jacobian,

$$J(x, y) = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - 2y - x \end{pmatrix}$$

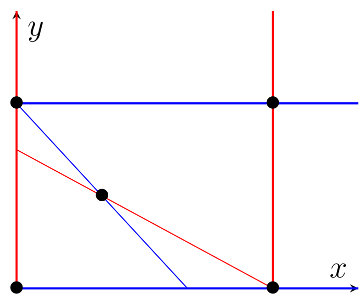
and nullclines

$$\{(x, y) : \dot{x} = 0\} = \{(0, y)\} \cup \{(3, y)\} \cup \{(x, \frac{3 - x}{2})\}$$
$$\{(x, y) : \dot{y} = 0\} = \{(x, 0)\} \cup \{(x, 2)\} \cup \{(2 - y, y)\}$$

which give us equilibria at $(0, 0)$, $(3, 0)$, $(0, 2)$, $(3, 2)$, and $(1, 1)$.

Hence, we know stabilities:

$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \implies \text{repeller}$$
$$J(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -3 \end{pmatrix} \implies \text{attractor}$$
$$J(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \implies \text{attractor}$$
$$J(3,2) = \begin{pmatrix} -7 & -6 \\ -2 & -5 \end{pmatrix} \implies \text{attractor}$$
$$J(1,1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \implies \text{saddle}$$



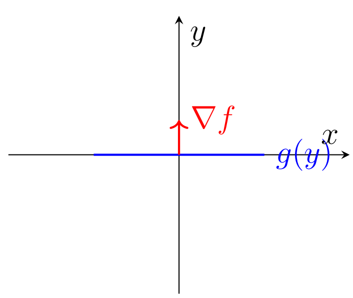
$\dot{x} = 0$, $\dot{y} = 0$

3. Which of the following functions can we apply IFT at $(x, y) = (0, 0)$ and what does IFT give us?

(a) $y^2 + x^2 + e^x - 1 = 0$

- $f(0,0) = 0$
- $f_x(0,0) = 1 \neq 0$
- $f_y(0,0) = 0$

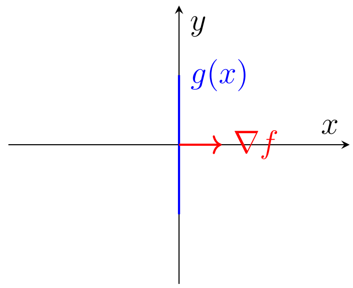
Hence we can apply IFT to get $g \in C^\infty$ with $f(x,y) = 0 \iff x = g(y)$ around $(0,0)$.



(b) $ye^x = 0$

- $f(0,0) = 0$
- $f_x(0,0) = 0$
- $f_y(0,0) = 1 \neq 0$

so we can apply IFT to get $f(x,y) = 0 \iff y = g(x)$.



(c) $\sin x + \sin y = 0$

- $f(0,0) = 0$
- $f_x(0,0) = 1$
- $f_y(0,0) = 1$

so we can apply IFT in either variable.

Further, $\nabla f = (1,1)$ so we look something like