Periodic Orbits

Simple Closed Loop: curve with no self intersections that does not pass through any equilibria

Index: $I_\Gamma = \frac{1}{2\pi}(\phi_1 - \phi_0)$ is the net number of *counter-clockwise* rotations made by $F(u)$ traversing Γ counter-clockwise from ϕ_0 to ϕ_1.

- If $\Gamma \mapsto \tilde{\Gamma}$ is a continuous deformation wwithout passing through any equilibria, then $I_\Gamma = I_{\tilde{\Gamma}}$.
- If $F(u)$ is continuously deformed without creating equilibria on Γ, then I_Γ is invariant
- If Γ does not contain any equilibria, then $I_\Gamma = 0$.
- If Γ is a periodic orbit, then $I_\Gamma = 1$
- If we replace $F(u)$ by $-F(u)$ (time reversal), then the index is not changed

> **Theorem:** For $F \in C^1$, every periodic orbit of F contains at east one equilibrium

Isolated Equilibrium: $F(u) \neq 0$ for all $u \neq u_*$ near u_*

- The *index of an isolated equilibrium* is the index of a simple closed loop that encloses u_* but no other equilibria
- If u_* is an attractor or repeller, $I(u_*) = 1$
- If u_* is a saddle point, $I(u_*) = -1$

> **Theorem:** If Γ is a simple closed loop enclosing only n isolated equilibria, then $I_\Gamma = \sum_{i=1}^n I(u_i)$

> **Poincare-Bendixson Theorem:** For $\dot{u} = F(u)$ with $u \in \mathbb{R}^2$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$ in C^1 and R a closed bounded subset of $\mathbb{R}^2$ such that
>
> 1. Rdoes not contain any equilibria
> 2. $\exists u(0) \in \mathbb{R}$ so that $u(t) \in \mathbb{R}$ for all $t \geq 0$
>
> Then R contains a periodic orbit.

Hopf Bifurcation

Hopf Bifurcation: If

1. $\dot{u} = F(u, \mu)$ for $u \in \mathbb{R}^2$, $\mu \in \mathbb{R}$, $F \in C^3$
2. u_* is an equilibrium for all $\mu \approx \mu_*$
3. The Eigenvalues of $F_u(u_*, \mu)$ are of the form $\lambda_{1,2}(\mu) = \alpha(\mu) \pm i\beta(\mu)$, $\alpha(\mu_*) = 0$, $\beta(\mu_*) \neq 0$, and $\left.\frac{d\alpha}{d\mu}\right|_{\mu_*} \neq 0$

Then we have a *Hopf Bifucation* with perioud $\approx \frac{2\pi}{\beta(\mu_*)}$ and amplitude $\approx \sqrt{|\mu - \mu_*|}$.

- If the periodic orbit is stable, we say the bifurcation is *supercritical*. In this case, we have an attractor for $\mu \leq \mu_*$ and a repeller and stable periodic orbit for $\mu > \mu_*$.
- If the periodic orbit is unstable, we say the bifurcation is *subcritical*. In this case, we have an attractor and unstable periodic orbit for $\mu < \mu_*$ and a repeller for $\mu \geq \mu_*$

The paradigmatic example is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \alpha(x^2 + y^2) \begin{pmatrix} x \\ y \end{pmatrix}$$

Higher Dimensional Bifurcations

> **Multi-Dimensional Implicit Function Theorem:** Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a $C^K k$ function ($k \geq 1$) with $f(u_*, \mu_*) = 0$. If the Jacobian $f_u(u_*, \mu_*) \in \mathbb{R}^{n \times n}$ is invertible, then there exists a unique C^k function $g : \mathbb{R}^m \to \mathbb{R}^n$ with $u_* = g(\mu_*)$ such that $f(u, \mu) = 0 \iff u = g(\mu)$ for (u, μ) near (u_*, μ_*).

Expected Bifurcations:

1. If $\lambda_1 = 0$ and $\operatorname{Re}\lambda_j \neq 0$ for all $j \neq 1$,
   - If $F(-u, \mu) = -F(u, \mu)\ \forall u, \mu$, and $u_* = 0$, then we expect a pitchfork bifurcation.
   - If $F(u_*, \mu) = 0\ \forall \mu$, then we expect a transcritical bifurcation.
   - Otherwise, we expect a saddle-node
2. $\operatorname{Re}\lambda_{1,2} = 0$, $\operatorname{Im}\lambda_{1,2} \neq 0$ and $\operatorname{Re}\lambda_j \neq 0$ for all $j \neq 1, 2$,
   - We expect a Hopf bifurcations

Dissipitative Systems

Lemma: Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be C^2.

If $\langle \nabla E(u), F(u) \rangle < 0$ for all $u \in \mathbb{R}^n$ for which $F(u) \neq 0$, then $E(u(t))$ decreases strictly in t for each solution $u(t)$ which is not an equilibrium

Lyapunov Functional: $E : \mathbb{R}^n \to \mathbb{R}^n$ in C^1 is a *Lyapunov Functional of* $\dot{u} = F(u)$ if $\langle E(u(t)), F(u(t)) \rangle < 0$ for all $u \in \mathbb{R}^n$ for which $F(u) \neq 0$

Lemma: Assume that E is a Lyapunov functional $\dot{u} = F(u)$. Then

1. The system cannot have any nontrivial periodic orbits
2. If $E(u) \to \infty$ and $|u| \to \infty$ and $\dot{u} = F(u)$ only has isolated equilibria, then for each $u(t)$, $\exists! u_*$ s.t. $u(t) \to u_*$.

Omega-limit set: $\omega(u(0)) = \{v \in \mathbb{R}^n : \exists t_k \nearrow \infty \text{ s.t. } u_{t_k} \to v\}$

Corollary: The ω-limit set $\omega(v(0)) = \{v \in \mathbb{R}^n : \exists t_k \nearrow \infty \text{ s.t. } u_{t_k} \to v\}$ is invariant: if $v(0) \in \omega(v_0)$, then $v(t) \in \omega(v(0))$ for all $t \in \mathbb{R}$.

Conservative Systems

Conservative Systems: Consider $\dot{u} = F(u)$ with $u \in \mathbb{R}^n$ and $F \in C^1$. Let $H : \mathbb{R}^n \to \mathbb{R}$ be C^2. We say H is *conserved* if $\langle \nabla H(u), F(u) \rangle = 0$ for all $u \in \mathbb{R}^n$

Lemma: Assume H is conserved. Then $\frac{d}{dt}(H(u(t))) = 0$ for each solution $u(t)$ and the level set $H^{-1}(c) = \{u \in \mathbb{R}^n : H(u) = c\}$ is invariant for each fixed $c \in \mathbb{R}$.

Corollary: If H is conserved and if $\nabla H(u)$ vanishes only at isolated points, then $\dot{u} = F(u)$ cannot have any attractors or repellers.

Attractors:

Attractor: Consider $\dot{u} = f(u)$ with $u \in \mathbb{R}^n$ and denote the solution $u(t)$ with $u(0) = u_0$ by $u(t; u_0) = \phi_t(u_0)$ where $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$. We assume there is a ball $B \subseteq \mathbb{R}^n$ with $\phi_t(B) \subseteq B$ for all $t \geq 0$ so that B is forward invariant. The *attractor* $\mathcal{A}$ of $\dot{u} = f(u)$ in B is $\mathcal{A} = \bigcap_{t \geq 0} \phi_t(B)$.

Lemma: Let $\mathcal{A}$ be the attractor in B. Then

- $\mathcal{A}$ is not empty
- $\mathcal{A}$ is invariant ($u_0 \in \mathcal{A} \implies \phi_t(u_0) \in \mathcal{A}$)
- $\mathcal{A}$ is the largest invariant set in B ($u_0 \in B \wedge \phi_t(u_0) \in B \implies u_0 \in \mathcal{A}$)
- $\forall u_0 \in B, \text{dist}(\phi_t(u_0), \mathcal{A}) = \min_{u \in A} |\phi_t(u_0) - u| \to 0$

Sensitive Dependence on Initial Conditions: We say $\dot{u} = f(u)$ with $u \in \mathbb{R}^n$ has *sensitive dependence on initial conditions* on a set $\mathcal{A}$ if $\forall u_0 \in \mathcal{A}$, $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists \tilde{u}_0 \in \mathcal{A}$ and $T > 0$ with $|u_0 - \tilde{u}_0| < \delta$ and $|\phi_T(u_0) - \phi_T(\tilde{u}_0)| \geq \varepsilon$

Poincare maps: Consider $\dot{u} = f(u)$ with $u \in \mathbb{R}^3$. Assume we can find a two-dimensional section $\Sigma \subseteq \mathbb{R}^3$ so that each solution starting in Σ immediately leaves Σ but eventually returns.

For $u_0 \in \Sigma$, let $T(u_0)$ be the time of first return of $\phi_t(u_0)$ to Σ. We define $\pi(u_0) = \phi_{T(u_0)}(u_0)$ the *Poincare map* of Σ.

We can iterate π defining $u_0 \in \Sigma, u_n = \pi(u_{n-1}) = \pi^n(u_0)$ and obtain a discrete dyamical system.