APMA 1360: Homework 2

Milan Capoor

7 February 2025

1 Logistic model of population dynamics

We consider a different fishing strategy for a fish population that consists of harvesting a fixed number of fish per unit time (often called constant-yield harvesting). This strategy is modeled by the differential equation

$$\frac{du}{dt} = ru\left(1 - \frac{u}{K}\right) - H$$

where we replaced the term μu in the model considered in #1.2 by the constant $H \geq 0$. As in #1.2, the variable $u(t)$ is the size of the population at time $t, r > 0$ is the growth rate of fish at small population levels, $K > 0$ is the carrying capacity, and $H \geq 0$ is the fixed number of fish caught per unit time interval.

(i) Nondimensionalize the system by changing the dependent variable u and the time variable t to reduce the number of parameters to just one.

Define $v = \frac{u}{K}$. Hence $du = K\ dv$ and we have

$$K\ dv = [rKv - rKv^2 - H]\ dt$$
$$dv = rv(1-v) - \frac{H}{K}\ dt$$

Then, if we let $\tau = rt$ then $d\tau = r\ dt$ so

$$\frac{dv}{d\tau} = v(1-v) - \frac{H}{Kr}$$

Let $C = \frac{H}{Kr} \geq 0$ be a constant parameter, so

$$\frac{dv}{d\tau} = v(1-v) - C$$

(ii) Analyse the resulting model mathematically: find all equilibria, determine their stability, and identify all bifurcation points (if any) at which the number of equilibria changes as a function of the fishing constant.

Let $f(v, C) = v(1-v) - C$. Then, we have

$$0 = v(1-v) - C \implies v = \frac{1 \mp \sqrt{1-4C}}{2}$$

Further,

$$f_v(v, C) = 1 - 2v$$

so

$$f_v\left(\frac{1}{2} - \frac{\sqrt{1-4C}}{2}\right) = \sqrt{1-4C} > 0$$

and

$$f_v\left(\frac{1}{2}+\frac{\sqrt{1-4C}}{2}\right) = -\sqrt{1-4C} < 0$$

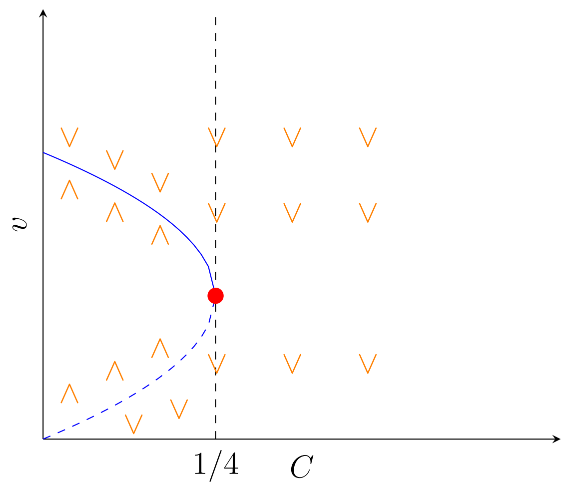
Hence, $v_1 = \frac{1}{2} + \frac{\sqrt{1-4C}}{2}$ is stable and $v_2 = \frac{1}{2} - \frac{\sqrt{1-4C}}{2}$ is unstable.

A birfucation point will occur when

$$v_1 = v_2 \implies \sqrt{1-4C} = -\sqrt{1-4C} \implies 1-4C = 0 \implies C = \frac{1}{4}$$

However, for $C > \frac{1}{4}$, v_1 and v_2 are complex so there are no equilibria. For $C < \frac{1}{4}$, meanwhile both v_1 and v_2 remain equilibria.

(iii) Draw the bifurcation diagram.



(iv) Discuss the implications of your analysis for the sustainability of this fishing strategy

If the fishing constant C exceeds 1/4 – i.e. if the number of fish caught per unit time exceeds 1/4 the number of fish born per unit time (at carrying capacity) – then the fishing rate will not be sustainable.

2 Transcritical bifurcations

Consider the differential equation

$$\frac{du}{dt} = uh(u, \mu) \tag{1}$$

where h is assumed to be infinitely often differentiable

1. Show that $u = 0$ is an equilibrium for all μ.

   $u = 0 \implies \frac{du}{dt} = 0h(0, \mu) = 0 \quad \forall \mu$

2. Show that $u = 0$ is not hyperbolic at $\mu = 0$ if and only if $h(0, 0) = 0$.

   Let $f(u, \mu) = uh(u, \mu)$.

   ($\implies$) Suppose $(u, \mu) = (0, 0)$ is not hyperbolic, i.e. $f(0, 0) = 0$ and $f_u(0, 0) = 0$.

   But $f_u = h(u, \mu) + uh_u(u, \mu)$ so $f_u(0, 0) = 0 \implies h(0, 0) = 0$.

   ($\impliedby$) Suppose $h(0, 0) = 0$. Then again,

   $$f_u(h, \mu) = h(u, \mu) + uh_u(u, \mu)$$

   so

   $$f_u(0, 0) = h(0, 0) + 0 \cdot h_u(0, 0) = 0 + 0 = 0 \implies (u, \mu) \text{ not hyperbolic} \quad \blacksquare$$

3. Assume that $h(0, 0) = 0$ and proceed as in class to analyse the "typical" bifurcation diagram of the differential equation (1). You can focus on the existence of equilibria: you do not need to analyse stability.

   Hint: The right-hand side of (1) vanishes if $u = 0$ or if $h(u, \mu) = 0$; argue why it therefore suffices to solve $h(u, \mu) = 0$ and focus initially on this equation using that $h(0, 0) = 0$. You can impose any assumptions on the Taylor coefficients of $h(u, \mu)$ as long as you argue why your assumptions are satisfied by a "typical" function $h(u, \mu)$

   Let $f(u, \mu) = uh(u, \mu)$. Clearly, $u = 0$ is an equilibrium for all μ.

   Hence, all other equilibria occur if $h(u, \mu) = 0$.

   CASE 1. Assume $h_u(0, 0) \neq 0$. Then by the IFT, $\exists g \in C^\infty$ such that $g(0) = 0$ and $h(u, \mu) = 0 \iff u = g(\mu)$.

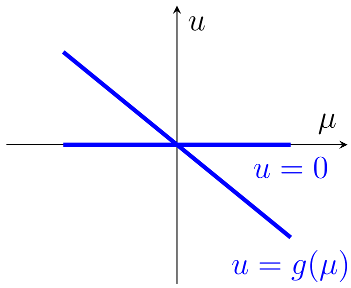
   We can differentiate:

   $$h_u(g(\mu), \mu)g'(\mu) + h_\mu(g(\mu), \mu) = 0$$
   $$h_u(g(0), 0)g'(0) + h_\mu(g(0), 0) = 0$$
   $$g'(0) = -\frac{h_\mu(0, 0)}{h_u(0, 0)}$$

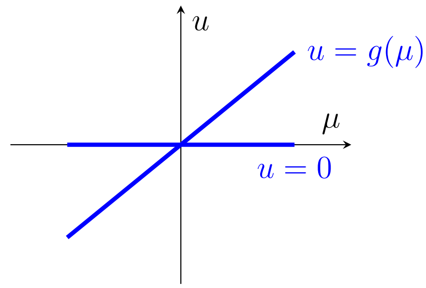
   CASE 1A. Further assume $h_\mu(0, 0) \neq 0$.

   Plugging these assumptions into the Taylor Expansion near $\mu = 0$,

   $$g(\mu) = g(0) + g'(0)\mu + O(\mu^2) \approx -\frac{h_\mu(0, 0)}{h_u(0, 0)}\mu$$



$$\begin{cases} h(0,0) = 0 \\ h_u(0,0) \neq 0 \\ h_\mu(0,0) \neq 0 \\ \text{sign}(h_\mu(0,0)) = \text{sign}(h_u(0,0)) \end{cases}$$



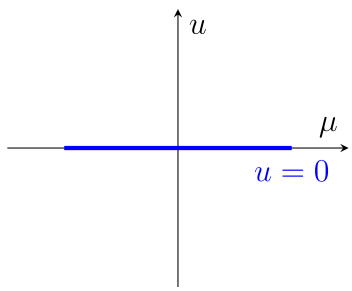
$$\begin{cases} h(0,0) = 0 \\ h_u(0,0) \neq 0 \\ h_\mu(0,0) \neq 0 \\ \text{sign}(h_\mu(0,0)) \neq \text{sign}(h_u(0,0)) \end{cases}$$

CASE 1B. Assume instead $h_\mu(0,0) = 0$.

Now, $g'(0) = 0$ and the Taylor Expansion near $\mu = 0$ is

$$g(\mu) = g(0) + g'(0)\mu + O(\mu^2) \approx 0$$

so we have



$$\begin{cases} h(0,0) = 0 \\ h_u(0,0) \neq 0 \\ h_\mu(0,0) = 0 \end{cases}$$

CASE 2. Rather than letting $h_u(0,0) \neq 0$, suppose $h_u(0,0) = 0$.

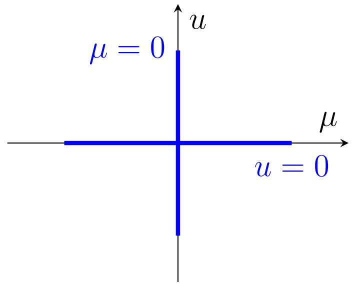
Now, just apply the IFT to the other variable to get $\mu = g(u)$ and

$$\begin{gathered} h_u(u, g(u)) + h_\mu(u, g(u))g'(0) = 0 \\ h_u(0,0) + h_\mu(0,0)g'(0) = 0 \\ h_\mu(0,0)g'(0) = 0 \end{gathered}$$

But assuming $h_\mu(0,0) = 0$,

$$g(u) = g(0) + g'(0)u + O(u^2) = 0$$

and we get the bifurcation diagram



$$\begin{cases} h(0,0) = 0 \\ h_u(0,0) = 0 \\ h_\mu(0,0) \neq 0 \end{cases}$$

If, instead, both $h_u(0,0) = 0$ and $h_\mu(0,0) = 0$, then we would need to consider higher order terms.

3 Checking the conditions for saddle-node bifurcations

Show that the differential equation

$$\frac{du}{dt} = \sin\mu + (1+\mu)u\sin u - u^3 e^u$$

udergoes a saddle-node bifurcation at $(u,\mu) = (0,0)$. Sketch the resulting bifurcation diagram and indicate the stability of the equilibria in the (μ, u)-plane near $(0,0)$.

Hint: Use the results we derived in class

Let $f(u,\mu) = \sin\mu + (1+\mu)u\sin u - u^3 e^u$. It suffices to show

$$\begin{cases} f(0,0) = 0 \\ f_u(0,0) = 0 \\ f_{uu}(0,0) \neq 0 \\ f_\mu(0,0) \neq 0 \end{cases}$$

Consider:

$$\begin{aligned} f_u(u,\mu) &= (1+\mu)\sin u + (1+\mu)u\cos u - 3u^2 e^u - u^3 e^u \\ f_{uu}(u,\mu) &= 2(1+\mu)\cos u + (1+\mu)u\sin u - ue^u(u^2+6u+6) \\ f_\mu(u,\mu) &= \cos\mu + u\sin u \end{aligned}$$

So

$$\begin{aligned} f(0,0) &= \sin 0 + (1+0)(0)s\in 0 - 0^3 e^0 = 0 && \checkmark \\ f_u(0,0) &= (1+0)\sin 0 + (1+0)(0)\cos 0 - 3(0)^2 e^0 - (0)^3 e^0 = 0 && \checkmark \\ f_{uu}(0,0) &= 2(1+0)\cos 0 + (1+0)(0)\sin 0 - 0e^0(0^2 + 6(0) + 6) = 2 \neq 0 && \checkmark \\ f_\mu(0,0) &= \cos 0 + 0\sin 0 = 1 \neq 0 && \checkmark \end{aligned}$$

Hence, $(u,\mu) = (0,0)$ is a saddle-node bifurcation.

By the Saddle-node bifurcation Theorem, $\exists g \in C^2$ with $f(u,\mu) = 0 \iff \mu = g(u)$ near $(0,0)$. Further, $g(0) = 0$ and

$$g(u) = -\frac{1}{2}\frac{f_{uu}(0,0)}{f_\mu(0,0)}u^2 + O(u^3) = -\frac{1}{2}\cdot\frac{2}{1}\cdot u^2 = -u^2 + O(u^3)$$

Hence,

$$f_u(u, g(u)) = f_{uu}(0,0)u + O(u^2) = 2u + O(u^2)$$

which means that u is stable for $u < 0$ and unstable for $u > 0$.