APMA 1360 - Homework 6

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14 March 2025

1 Indices of equilibria

Find the equilibria of each of the following systems and calculate their indices:

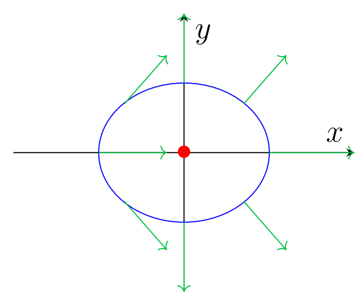
(i) $\dot{x} = x^2, \quad \dot{y} = y$

$$\begin{cases} \dot{x} = 0 \implies \implies x = 0 \\ \dot{y} = 0 \implies y = 0 \end{cases}$$

and

$$J(x,y) = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix} \implies J(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \implies \lambda = 0, 1$$

Since an eigenvalue is 0, we cannot determine anything from the eigenvalues alone. Let's consider the orbit:

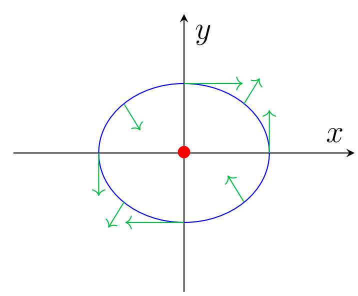


So $(0,0)$ has index 0.

(ii) $\dot{x} = y^3, \quad \dot{y} = x.$

$$J(x,y) = \begin{pmatrix} 0 & 3y^2 \\ 1 & 0 \end{pmatrix} \implies J(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \implies \lambda = 0$$

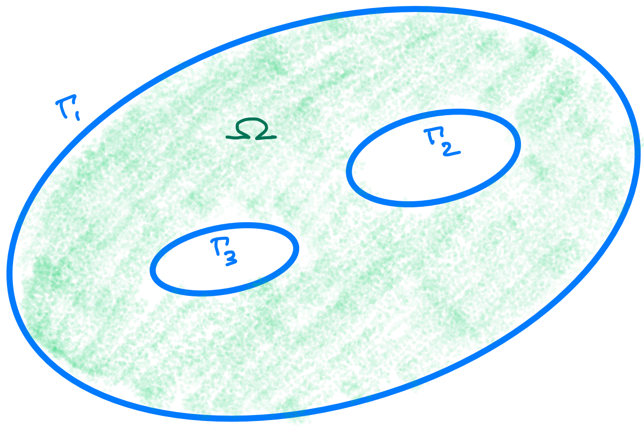
We can graph:



Which has index -1.

2 Index theory

We consider $\dot{u} = f(u)$ with $f \in C^1$ and $u \in \mathbb{R}^2$. Suppose that this differential equation has exactly three periodic orbits Γ_1, Γ_2, and Γ_3 that are aligned as indicated below. Assume that all equilibria are hyperbolic (and therefore also isolated). What can you say about their number and their type (attractor, repeller, or saddle) in Ω and inside Γ_2 and Γ_3?



By a Theorem from class, every periodic orbit contains at least one equilibrium. Further, the index of a periodic orbit is 1.

But since Γ_1 is a simple closed loop, $I(\Gamma_1) = \sum_{i=1}^{n} I(u_i) = 1$ for each equilibrium u_i in Γ_1.

In particular, this means that Γ_1 must contain at least one saddle which is not in Γ_2 or Γ_3. Since Ω can also contain other equilibria, we thus require that Ω contains one more saddle than attractors/repellers combined.

Meanwhile, Γ_2 and Γ_3 can each have:

- One attractor or one saddle
- n attractors + repellers and $n - 1$ saddles

3 Trapping regions

Consider the system

$$\dot{x} = a - x + x^2 y, \qquad \dot{y} = b - x^2 y$$

where $x, y \geq 0$ and $a, b > 0$.

(i) Find the unique equilibrium of this system.

$$\begin{array}{rl} 0 & = a - x + x^2 y \\ 0 & = b - x^2 y \\ \hline 0 & = a + b - x \end{array}$$

Hence,

$$x = a + b \implies y = \frac{b}{x^2} = \frac{b}{(a+b)^2}$$

so the only equilibrium is $(a+b, b/(a+b)^2)$.

(ii) Determine its stability properties and show that it is not hyperbolic precisely when (a, b) satisfies $b - a = (a+b)^3$.

We have Jacobian

$$J(x,y) = \begin{pmatrix} -1 + 2xy & x^2 \\ -2xy & -x^2 \end{pmatrix} \implies J(a+b, \frac{b}{(a+b)^2}) = \begin{pmatrix} -1 + \frac{2b}{a+b} & (a+b)^2 \\ -\frac{2b}{(a+b)} & -(a+b)^2 \end{pmatrix}$$

So,

$$\begin{aligned} \lambda_1 \lambda_2 &= \begin{vmatrix} -1 + \frac{2b}{a+b} & (a+b)^2 \\ -\frac{2b}{(a+b)} & -(a+b)^2 \end{vmatrix} \\ &= (a+b)^2 - 2b(a+b) + 2b(a+b) \\ &= (a+b)^2 \\ \lambda_1 + \lambda_2 &= -1 + \frac{2b}{a+b} - (a+b)^2 \\ &= \frac{a-b}{a+b} - (a+b)^2 \end{aligned}$$

We have $a, b > 0$ so $\det J > 0$ implies the equilibrium is not a saddle.

Now, we know the equilibrium is an:

- Attractor if $\frac{b-a}{a+b} > (a+b)^2$
- Repeller if $\frac{b-a}{a+b} < (a+b)^2$
- Non-hyperbolic if $\frac{b-a}{a+b} = (a+b)^2$

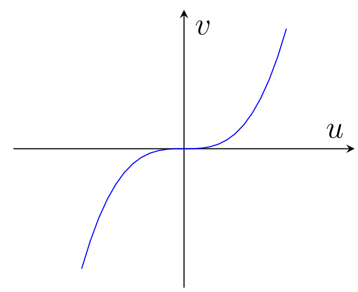
Suppose $b - a = (a+b)^3$. Then,

$$\begin{aligned} \frac{b-a}{a+b} - (a+b)^2 &= \frac{1}{a+b}\left(b - a - (a+b)^3\right) \\ &= b - a - (b - a) \\ &= 0 \end{aligned}$$

which is exactly the condition $\operatorname{tr} J = 0$, i.e where the equilibrium is non-hyperbolic.

(iii) Plot the curve determined by $b - a = (a+b)^3$ in the (a,b)-plane. *Hint:* Set $v = b - a$ and $u = a + b$, plot the curve in the (u,v)-space, and then see what these coordinates and the graph mean in the (a,b)-plane.

Let $v = b - a$ and $u = a + b$ so we are interested in the curve $v = u^3$

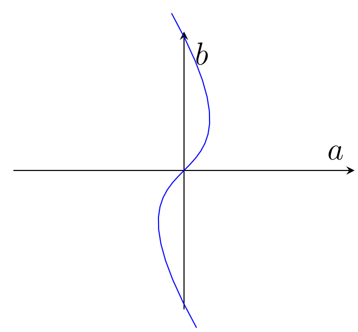


In (a,b)-space, this curve corresponds to the line $b - a = (a+b)^3$.

How do we transform from (u,v) to (a,b)? We have $u = a + b$ and $v = b - a$ so

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \implies \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

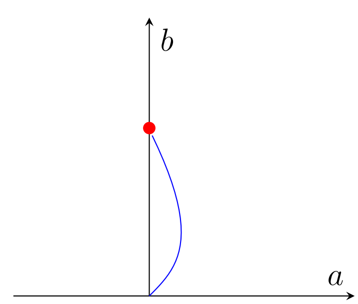
which happens to be the transformation associated with a rotation by $45°$ and a scaling of $1/\sqrt{2}$.



However, we are only interested in the region where $a, b > 0$. What is the b-intercept?

$$a = 0 \implies b = b^3 \implies b = 1 \ (b > 0)$$

So we have

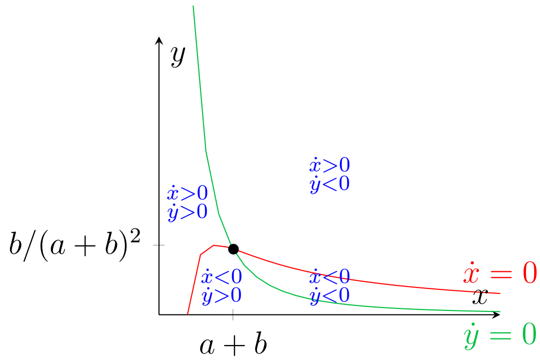


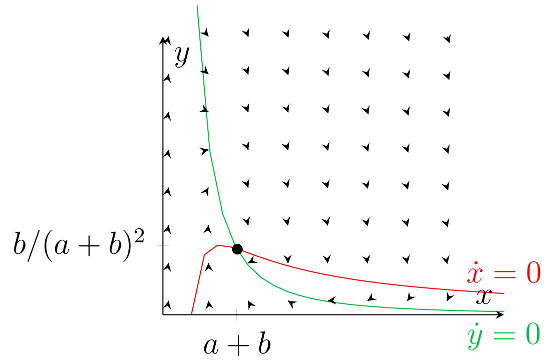
(iv) Show that this system has, for appropriate values of (a,b), at least one periodic orbit by constructing a trapping region. This is very similar to (but slightly more involved than) the example we went through in class: be careful and go step-by-step.

We would like for this region (call it Ω) to be a repeller, hence for $(a,b) \in \Omega$, require $\frac{b-a}{a+b} < (a+b)^2$.

Let's plot nullclines

$$\begin{cases} \dot{x} = 0 \implies a - x + x^2 y = 0 \implies y = \frac{x-a}{x^2} \\ \dot{y} = 0 \implies b - x^2 y = 0 \implies y = \frac{b}{x^2} \end{cases}$$

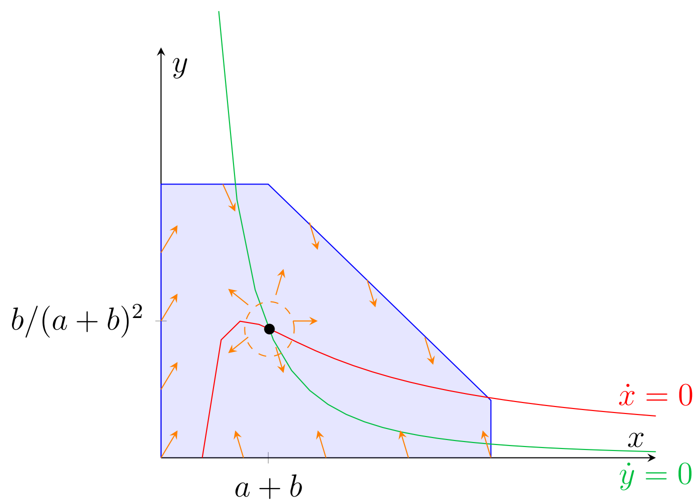




Clearly, the x and y axes will work as two bounds on our trapping region. Now we just need to find lines where the vector field points inwards.

From part (i), we need $\dot{x}+\dot{y}=a+b-x$. So let's try a slope of -1:

$$\frac{\dot{y}}{\dot{x}}<-1 \implies \dot{y}<-\dot{x} \implies \frac{b}{x^2}<\frac{x-a}{x^2} \implies b<x-a \implies x>a+b \quad \checkmark$$



By construction, this region is

- Closed and bounded
- Does not contain any equilibria
- For all $u(0)\in\Omega$, $u(t)\in\Omega$ for all $t\geq 0$

Hence, by the Poincaré-Bendixson theorem, there must be at least one periodic orbit in Ω.