APMA 1360: Homework 9

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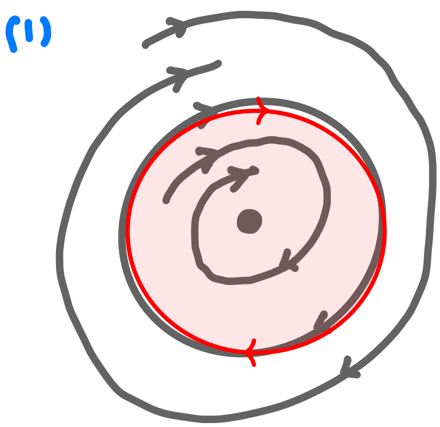
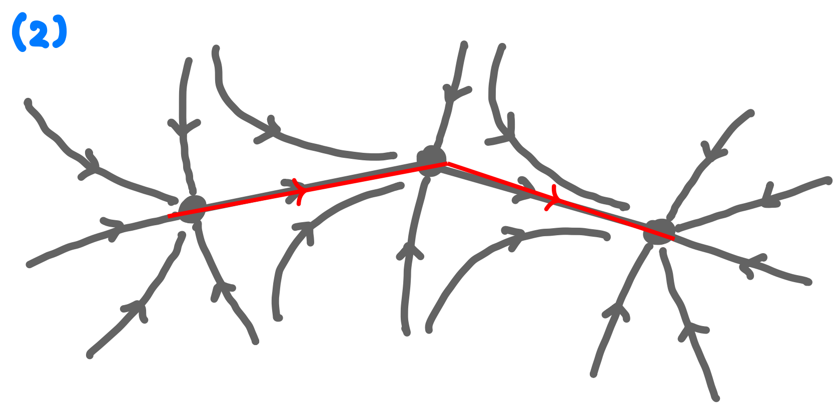
1 Attractors

Determine the attractor $\mathcal{A}$ for each of the two phase portraits drawn below, starting in each case with a large ball that contains the relevant phase portrait:





2 The Lorenz system has an attractor

Consider the Lorenz equations

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= \rho x - y - xz\\ \dot{z} &= -\beta z + xy\end{aligned}$$

for $\sigma, \beta, \rho > 0$ and define the function

$$E(x,y,z) = x^2 + y^2 + (z-\rho-\sigma)^2.$$

Show that there is a constant $R > 0$ such that $E(x,y,z)$ decreases strictly along each solution $(x,y,z)(t)$ of the Lorenz equations as long as $x(t)^2 + y(t)^2 + z(t)^2 \geq R^2$. In particular, solutions associated with any large initial condition eventually enter the ball of radius R, which implies that the Lorenz equations has a bounded attractor $\mathcal{A}$ inside the ball of radius R for all parameter values $\sigma, \beta, \rho > 0$.

Hint: Calculate $\frac{d}{dt}(E(x(t),y(t),z(t)))$ and try to combine as many of the terms as possible using completion of the square and whichever other methods you find useful.

It suffices to show that there $\exists R > 0$ such that $\frac{dE}{dt}(x,y,z) < 0$ for all $x^2+y^2+z^2 \geq R^2$.

$$\begin{aligned}\frac{d}{dt}E(x,y,z) &= 2x\dot{x} + 2y\dot{y} + 2(z-\rho-\sigma)\dot{z}\\ &= 2x\sigma(y-x) + 2y(\rho x - y - xz) + 2(z-\rho-\sigma)(-\beta z + xy)\\ &= 2x\sigma y - 2x^2\sigma + 2y\rho x - 2y^2 - 2yxz - 2\beta z^2 + 2zxy + 2\beta(\rho+\sigma)z - 2(\rho+\sigma)xy\\ &= -2\sigma x^2 - 2y^2 - 2\beta z^2 + 2\beta\rho z + 2\beta\sigma z\\ &= -2(\sigma x^2 + y^2) - 2\beta(z^2 - (\rho+\sigma)z)\\ &= -2(\sigma x^2 + y^2) - 2\beta\left(z - \frac{\rho+\sigma}{2}\right)^2 + \frac{\beta}{2}(\rho+\sigma)^2\end{aligned}$$

If we let $R_1 = \max\{1, \sigma, \beta\}$, then

$$-2(\sigma x^2 + y^2) - 2\beta\left(z - \frac{\rho+\sigma}{2}\right)^2 + \frac{\beta}{2}(\rho+\sigma)^2 \leq -2R_1\left(x^2 + y^2 + \left(z - \frac{\rho+\sigma}{2}\right)^2\right) + \frac{\beta}{2}(\rho+\sigma)^2$$

Now choose $R_2 = \sqrt{\frac{\beta}{4R_1}(\rho+\sigma)^2}$ so

$$\begin{aligned}-2R_1\left(x^2 + y^2 + \left(z - \frac{\rho+\sigma}{2}\right)^2\right) + \frac{\beta}{2}(\rho+\sigma)^2 &= -2R_1\left(x^2 + y^2 + \left(z - \frac{\rho+\sigma}{2}\right)^2\right) + 2R_1R_2^2\\ &= -2R_1\left(x^2 + y^2 + \left(z - \frac{\rho+\sigma}{2}\right)^2 - R_2^2\right)\end{aligned}$$

Finally, $R_1 > 0$ so we just need

$$x^2 + y^2 + \left(z - \frac{\rho+\sigma}{2}\right)^2 - R_2^2 \geq 0 \implies x^2 + y^2 + \left(z - \frac{\rho+\sigma}{2}\right)^2 \geq R_2^2$$

which is the equation for a sphere of radius R_2 centered at $(0, 0, \frac{\rho+\sigma}{2})$.

Let

$$B = \left\{ (x,y,z) : x^2 + y^2 + \left(z - \frac{\rho+\sigma}{2} \right)^2 > R_2^2 \right\}$$

then from geometry,

$$B \subseteq \left\{ (x,y,z) : x^2 + y^2 + z^2 > \left(\frac{\rho+\sigma}{2} + R_2 \right)^2 \right\}$$

In particular, this means that if we let

$$\boxed{R > \frac{\rho+\sigma}{2} + \sqrt{\frac{\beta}{4\max\{1,\beta,\sigma\}}(\rho+\sigma)^2}}$$

we will have $\frac{d}{dt}E < 0$ for all $x^2 + y^2 + z^2 \geq R^2$, which implies that E decreases strictly on solutions of the Lorenz equations and there exists a bounded attractor $\mathcal{A}$ inside the ball of radius R for all parameter values $\sigma, \beta, \rho > 0$.

3 Volume of the attractor of the Lorenz system

Consider a differential equation $\dot{u} = f(u)$ with $u = (x, y, z) \in \mathbb{R}^3$ and $f = (f_1, f_2, f_3)$, and denote its solutions by $u(t) = \Phi_t(u_0)$. Let B_0 be a ball in $\mathbb{R}^3$ and set $B(t) := \Phi_t(B_0)$. We denote the volume of the set $B(t)$ by $V(t)$. We have the following theorem (you do not need to prove this result): if

$$\operatorname{div} f(x, y, z) := \frac{\partial f_1}{dx}(x, y, z) + \frac{\partial f_2}{dy}(x, y, z) + \frac{\partial f_3}{dz}(x, y, z)$$

does not depend on (x, y, z), then $V(t) = V(0)\mathrm{e}^{\operatorname{div}(f)t}$.

Can you apply this result to the Lorenz equations, what is the limit of $V(t)$ as $t \to \infty$, and what does this imply for the volume of the attractor $\mathcal{A}$ of the Lorenz equations?

First, we calculate the divergence of the Lorenz equations:

$$\begin{aligned}
\operatorname{div} f(x, y, z) &= \frac{\partial f_1}{dx}(x, y, z) + \frac{\partial f_2}{dy}(x, y, z) + \frac{\partial f_3}{dz}(x, y, z) \\
&= \frac{\partial}{\partial x}(\sigma(y - x)) + \frac{\partial}{\partial y}(\rho x - y - xz) + \frac{\partial}{\partial z}(-\beta z + xy) \\
&= -\sigma - 1 - \beta
\end{aligned}$$

which is indeed independent of (x, y, z). Therefore, we can apply the result to the Lorenz equations.

We have

$$V(t) = V(0)e^{\operatorname{div}(f)t} = V(0)e^{-([\sigma+\beta]t+t)}$$

Since $\sigma, \beta > 0 \implies \sigma + \beta > 0$, we have that $e^{-([\sigma+\beta]t+t)}$ is positive and decreasing for all t.

Hence, $V(t) \to 0$. Since V represents the volume of the set containing the orbit of B_0, we conclude that $\mathcal{A}$ is a point (or at least a measure zero set).