APMA 1360: Midterm II Practice Exam

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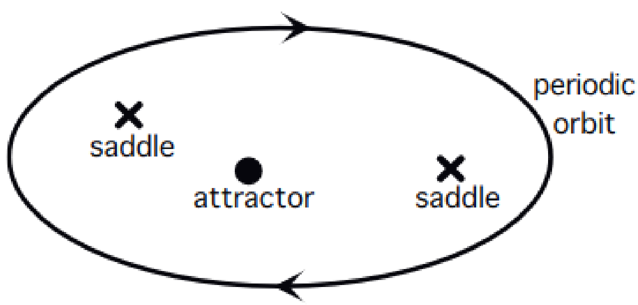
1 Problem 1

(a) Consider $u \in \mathbb{R}^n, \mu \in \mathbb{R}$, and $F(u, \mu) \in \mathbb{R}^n$. Assume $F(0,0) = 0$ and let $A := F_u(0,0)$. Which of the conditions below guarantee that we can solve the equation $F(u, \mu) = 0$ for u as a function of μ near $(u, \mu) = 0$? Circle all that apply:

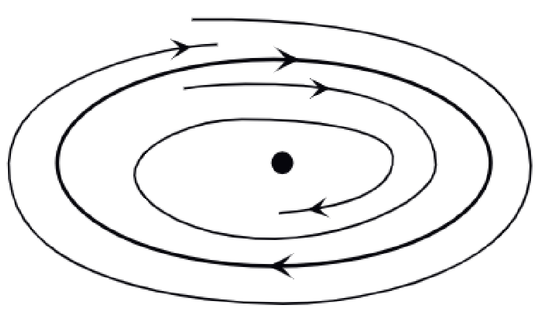
- $\boxed{\text{A is invertible}}$
- $\det A = 0$
- All eigenvalues of A have strictly negative real part
- $A \neq 0$

(b) Is the following phase diagram configuration possible or not? Support your answer.



No. $I(\text{saddle}) = -1$ and $I(\text{attractor}) = 1$ so $I(\text{orbit}) = -1 - 1 + 1 = -1$ but the index of a periodic orbit is always 1.

(c) Can the following phase diagram arise from a gradient system? Support your answer (in a few brief lines).



No. Since gradient systems have a Lyapunov functional (V for $\dot{u} = -\nabla V(u)$), by a Lemma, the system cannot have a (nontrivial) periodic orbit.

2 Problem 2

Compute the indices of all equilibria of the two systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x - 2y \\ -3y \end{pmatrix}, \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x^2 - y^2 \\ 0 \end{pmatrix}$$

First,

$$\begin{cases} 0 = -x - 2y \\ 0 = -3y \end{cases} \implies \begin{cases} y = 0 \\ x = 0 \end{cases} \implies (0,0) \text{ is only equilibrium}$$

Then,

$$J(x,y) = \begin{pmatrix} -1 & -2 \\ 0 & -3 \end{pmatrix} \implies \lambda = \{-1, -3\} \implies (0,0) \text{ attractor} \implies \boxed{I(0,0) = 1}$$

Similarly,

$$\begin{cases} 0 = -x^2 - y^2 \\ 0 = 0 \end{cases} \implies (0,0) \text{ only equilibrium}$$

and

$$J(x,y) = \begin{pmatrix} -2x & -2y \\ 0 & 0 \end{pmatrix} \implies J(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies \text{inconclusive}$$

3 Problem 3

Find a value of a for which $E(u,v) = u^2 + av^2$ is a Lyapunov functional of

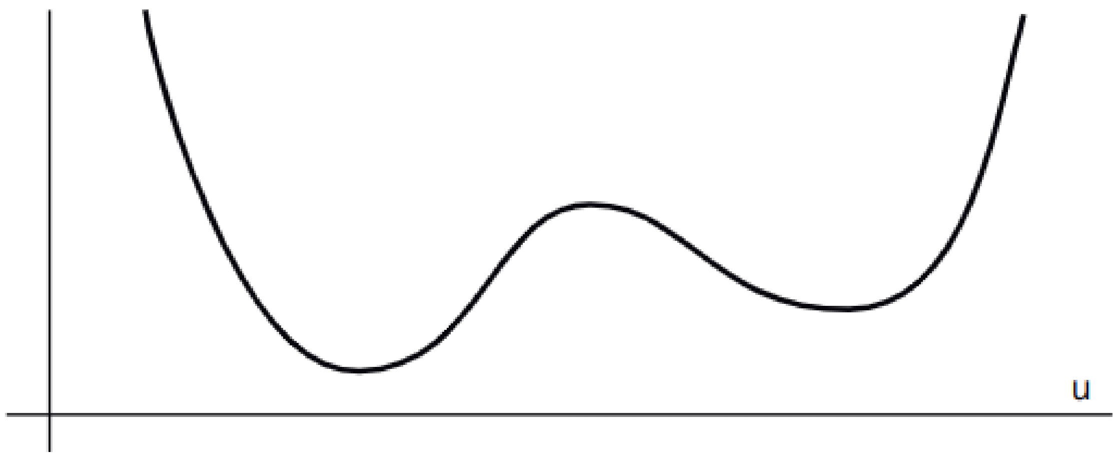
$$\begin{aligned}\dot{u} &= -u - 2v - 2u^3 \\ \dot{v} &= -3v + u - v^3\end{aligned}$$

$$\begin{aligned}\nabla E(u,v) &= \begin{pmatrix} 2u \\ 2av \end{pmatrix} \\ \langle \nabla E, F \rangle &= 2u(-u - 2v - 2u^3) + 2av(-3v + u - v^3) \\ &= -2u^2 - 4uv - 4u^4 - 6av^2 + 2auv - 2av^4 \\ &= (2a-4)uv - 6av^2 + \underbrace{(-2u^2 - 4u^4 - 2av^4)}_{<0}\end{aligned}$$

If $\operatorname{sign}(u) = \operatorname{sign}(v)$, then we need $0 < a \leq 2$ to ensure the function is negative. If $\operatorname{sign}(u) \neq \operatorname{sign}(v)$, then we need $a \geq 2$ to ensure the function is negative. Hence, for $\boxed{a = 2}$, E is a Lyapunov functional for all u, v.

4 Problem 4

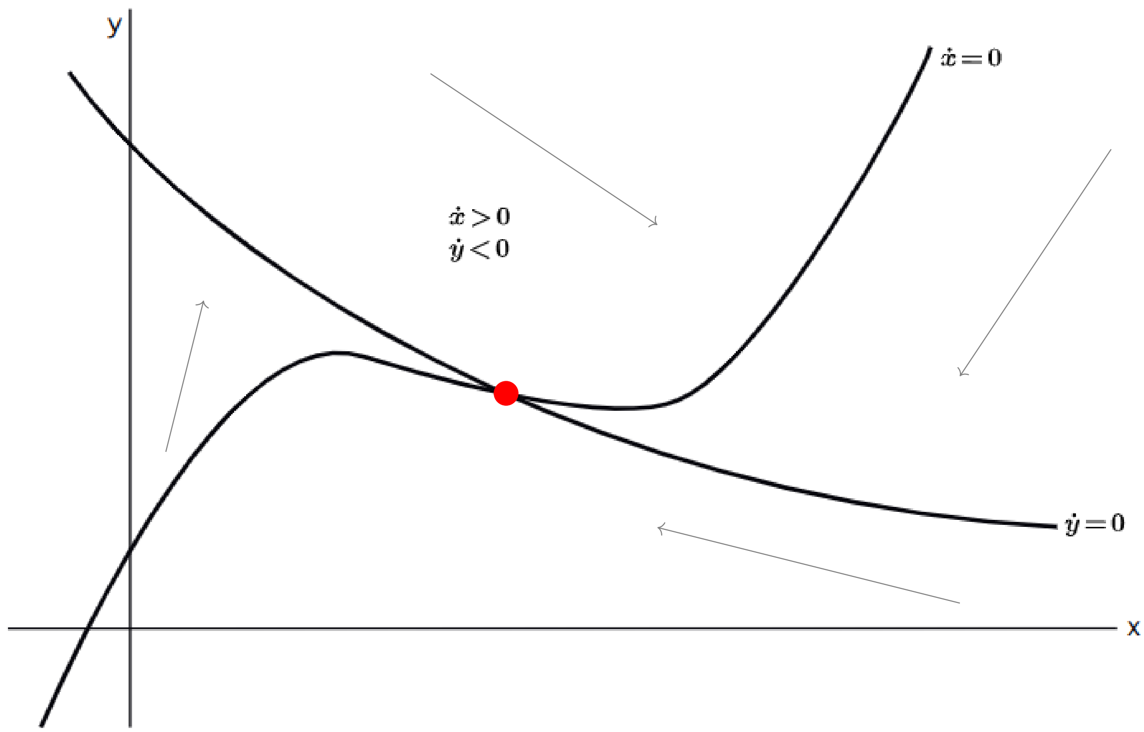
Sketch the phase diagram and classify all equilibria of the second-order equation $\ddot{u} = f(u)$ where the graph of the function $-\int^u f(x)\,dx$



5 Problem 5

The figure shown below contains the two nullclines of the system

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$



(a) In the figure, label all equilibria.

(b) Use the information given below to construct a trapping region for this system.

(c) What do you need to assume about the fixed point(s) to be able to conclude the existence of a periodic orbit inside your trapping region?

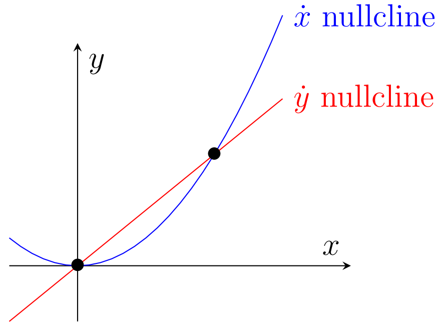
6 Problem 6

Consider the system

$$\dot{x} = -x^2 + y$$
$$\dot{y} = x - y$$

(a) Plot the nullclines of this system.

$$\begin{cases} 0 = -x^2 + y \\ 0 = x - y \end{cases} \implies \begin{cases} f : (x, x^2) \\ g : (x, x) \end{cases}$$



(b) Find and classify all equilibria (Hint: use the determinant and the trace).

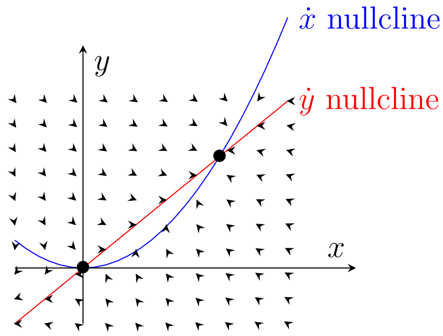
The nullclines give us equilibria at $(0, 0)$ and $(1, 1)$.

$$J(x, y) = \begin{pmatrix} -2x & 1 \\ 1 & -1 \end{pmatrix}$$

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \implies \begin{cases} \det J = -1 \\ \operatorname{tr} J = -1 \end{cases} \implies$$

$$J(1, 1) = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \implies \begin{cases} \det J = 1 \\ \operatorname{tr} J = -3 \end{cases}$$

(c) Sketch the phase portrait.



(d) Can this system have any periodic orbits? Support your answer.

7 Problem 7

Find all equilibria and classify their bifurcation points for the differential equation

$$\dot{x} = 1 - (b+1)x + ax^2y$$
$$\dot{y} = bx - ax^2y,$$

where $a, b > 0$ are strictly positive parameters.

$$\begin{array}{ll} 0 & = 1 - bx - x + ax^2y \\ 0 & = bx - ax^2y \\ \hline 0 & = 1 - x \end{array} \implies x = 1 \implies b = ay \implies (1, b/a)$$

and

$$J(x,y) = \begin{pmatrix} -(b+1) + 2axy & ax^2 \\ b - 2axy & -ax^2 \end{pmatrix}$$
$$J(1, b/a) = \begin{pmatrix} -(b+1) + 2b & a \\ b - 2b & -a \end{pmatrix} = \begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix}$$

And

$$\begin{cases} \det J = -a(b-1) + ab = a > 0 \\ \operatorname{tr} J = b - 1 - -a \end{cases} \implies \text{saddle}$$