APMA 1360: Homework 4

Milan Capoor

21 February 2025

1 Stability of linear systems

Consider a linear system of the form $\dot{u} = Au$, where A is a real $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. We classify the equilibrium $u = 0$ as

attractor	if $\operatorname{Re}\lambda_j < 0$ for all j,
repeller	if $\operatorname{Re}\lambda_j > 0$ for all j,
saddle	if $\operatorname{Re}\lambda_j \neq 0$ for all j, and there are two indices i, k with $\operatorname{Re}\lambda_i < 0 < \operatorname{Re}\lambda_k$
nonhyperbolic	if $\operatorname{Re}\lambda_j = 0$ for at least one index j.

(i) Argue that these four cases exhaust all possibilities and sketch three sample phase diagrams for planar systems where $u = 0$ is, respectively, an attractor, repeller, or saddle (you can choose A to be a diagonal matrix for these examples).

Suppose $n = 1$, i.e. the only eigenvalue is λ_1. In this case, we must have $\operatorname{Re}\lambda_1 < 0$, $\operatorname{Re}\lambda_1 > 0$, or $\operatorname{Re}\lambda_1 = 0$. All cases are covered by the above classification.

Now take $n = 2$, WLOG λ_1, λ_2. We have the following possible cases by sheer enumeration:

- $\operatorname{Re}\lambda_1 < 0$ and $\operatorname{Re}\lambda_2 < 0$: Attractor
- $\operatorname{Re}\lambda_1 > 0$ and $\operatorname{Re}\lambda_2 > 0$: Repeller
- $\operatorname{Re}\lambda_1 < 0$ and $\operatorname{Re}\lambda_2 > 0$: Saddle
- $\operatorname{Re}\lambda_1 > 0$ and $\operatorname{Re}\lambda_2 < 0$: Saddle
- $\operatorname{Re}\lambda_1 = 0$ and $\operatorname{Re}\lambda_2 = 0$: Nonhyperbolic
- $\operatorname{Re}\lambda_1 = 0$ and $\operatorname{Re}\lambda_2 < 0$: Nonhyperbolic
- $\operatorname{Re}\lambda_1 = 0$ and $\operatorname{Re}\lambda_2 > 0$: Nonhyperbolic
- $\operatorname{Re}\lambda_1 < 0$ and $\operatorname{Re}\lambda_2 = 0$: Nonhyperbolic
- $\operatorname{Re}\lambda_1 > 0$ and $\operatorname{Re}\lambda_2 = 0$: Nonhyperbolic

All cases are covered.

Let $n = m$ for some $m \geq 1$. Suppose the four cases are exhaustive for all $1 \leq k < m$. It suffices to show that the cases are exhaustive for $n = m$.

Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of A. By assumption, $\lambda_1, \dots, \lambda_{m-1}$ must

(a) All have $\operatorname{Re}\lambda_j < 0$ for some j

(b) All have $\operatorname{Re}\lambda_j > 0$ for some j

(c) Have at least one λ_i, λ_j such that $\operatorname{Re}\lambda_i < 0$ and $\operatorname{Re}\lambda_j > 0$ for some i, j

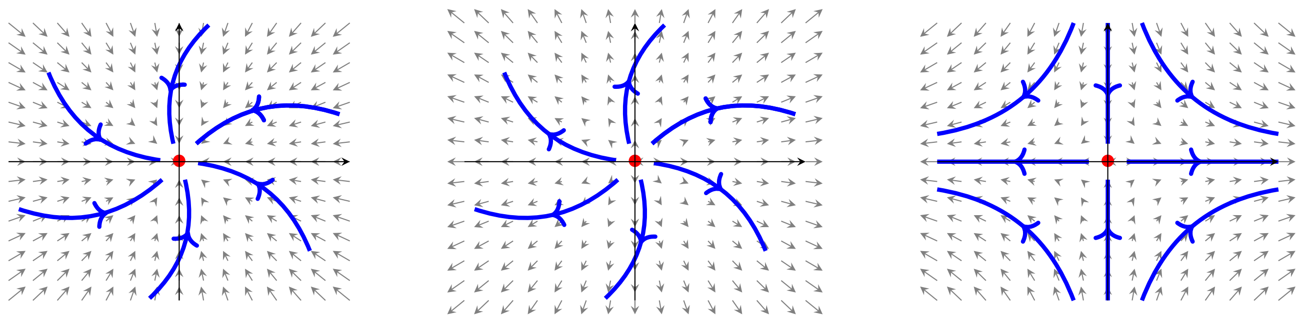
(d) Have at least one λ_i such that $\operatorname{Re}\lambda_i = 0$ for some i

If the system satisfies case 1 here, call the equilibrium an "almost-attractor". And so on for the other cases.

We then consider λ_m.

- CASE 1. $\operatorname{Re}\lambda_m < 0$. If the equilibrium is an almost-attractor, then all eigenvalues are negative and it is a true attractor. If the equilibrium is an almost-repeller, then λ_m makes it a saddle. If the equilibrium is almost-saddle or almost-nonhyperbolic, then λ_m has no effect and it remains a saddle or nonhyperbolic equilibrium respectively.
- CASE 2. $\operatorname{Re}\lambda_m = 0$. Automatically, the equilibrium is nonhyperbolic.
- CASE 3. $\operatorname{Re}\lambda_m > 0$. Analogously to case 1, the equilibrium is a repeller if it is an almost-repeller, a saddle if it is almost-attractor, and remains a saddle or nonhyperbolic equilibrium if it is almost-saddle or almost-nonhyperbolic respectively.

By the $n = 1$, this exhausts all possibilities. Hence, by induction, the four cases are exhaustive for all $n \in \mathbb{N}$.



$$\dot{u} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} u \qquad \dot{u} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u \qquad \dot{u} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u$$

(ii) For each of the two linear systems listed below, classify the equilibrium $u = 0$ as attractor, repeller, saddle, or nonhyperbolic depending on the value of $\mu \in \mathbb{R}$:

$$\dot{u} = \begin{pmatrix} -1 & 2 \\ 0 & \mu \end{pmatrix} u, \qquad \dot{u} = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} u$$

$$\dot{u} = \begin{pmatrix} -1 & 2 \\ 0 & \mu \end{pmatrix} u \implies J = \begin{pmatrix} -1 & 2 \\ 0 & \mu \end{pmatrix} \implies \lambda_1 = -1, \lambda_2 = \mu \implies \begin{cases} \text{Attractor} & \text{if } \mu < 0 \\ \text{Saddle} & \text{if } \mu > 0 \\ \text{Nonhyperbolic} & \text{if } \mu = 0 \end{cases}$$

$$\dot{u} = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} u \implies J = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} \implies \lambda_1 = \mu - i, \lambda_2 = \mu + i \implies \begin{cases} \text{Attractor} & \text{if } \mu < 0 \\ \text{Repeller} & \text{if } \mu > 0 \\ \text{Nonhyperbolic} & \text{if } \mu = 0 \end{cases}$$

2 Competing species model

We return to our model of two species that compete for food resources. The following system is the same model but I changed some of the values in the equation for the first species:

$$\begin{aligned} \dot{x} &= x(3-2x-y) \\ \dot{y} &= y(2-x-y). \end{aligned}$$

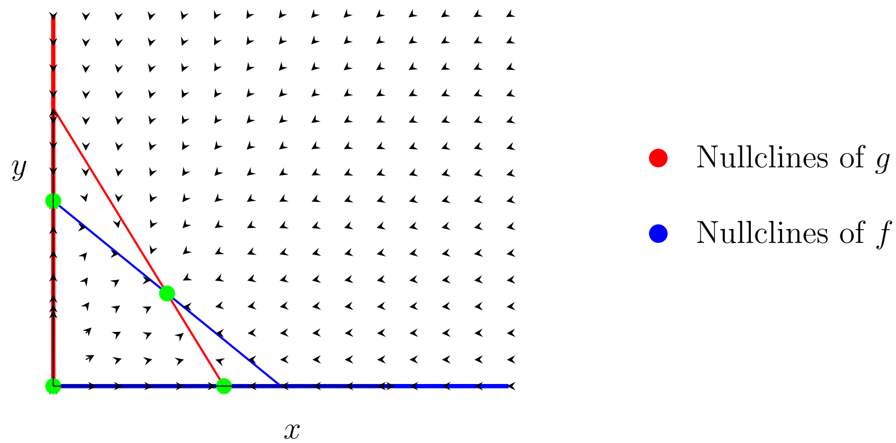
Find all equilibria, determine their stability using the eigenvalues of the Jacobian, find and plot the nullclines, and draw a phase portrait that contains representative solutions. Interpret your results in terms of our fictitious two populations.

Let $f(x,y) = x(3-2x-y)$ and $g(x,y) = y(2-x-y)$. Then, the nullclines of f are given by

$$\{(x,y):\ x(3-2x-y)=0\} = \{(0,y) : y \in \mathbb{R}\} \cup \left\{ \left(-\frac{y-3}{2}, y\right) : y \in \mathbb{R} \right\}$$

Similarly, the nullclines of g are given by

$$\{(x,y):\ y(2-x-y)=0\} = \{(x,0) : x \in \mathbb{R}\} \cup \{(x, 2-x) : x \in \mathbb{R}\}$$



Immediately, these intersections give us equilibria at $(0,0)$, $(0,2)$, $(1,1)$, and $(3/2,0)$.

We can take the Jacobian,

$$J(x,y) = \begin{pmatrix} 3-4x-y & -x \\ -y & 2-x-2y \end{pmatrix}$$

and evaluate:

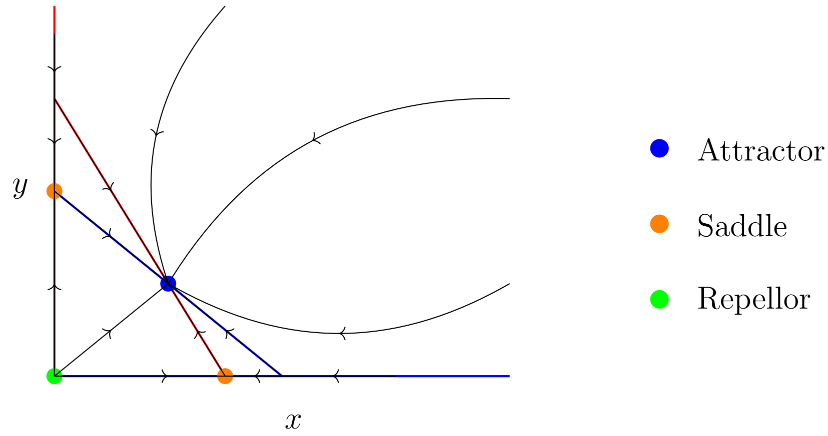
$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \implies \lambda = \{3,2\} \implies \text{repellor}$$

$$J(0,2) = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix} \implies \lambda = \{1,-2\} \implies \text{saddle}$$

$$J(1,1) = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \implies \lambda = \{-\frac{3}{2} - \frac{\sqrt{5}}{2}, -\frac{3}{2} + \frac{\sqrt{5}}{2}\} \implies \text{attractor}$$

$$J(3/2,0) = \begin{pmatrix} -3 & -3/2 \\ 0 & 1/2 \end{pmatrix} \implies \lambda = \{-3, 1/2\} \implies \text{saddle}$$

So our phase portrait should look something like this:



When we first saw this model in class, we had two competing species with no possible stable coexistence. Here, however, we can see that it is possible for a species to die out or for the two species to coexist at the equilibrium.