APMA 1360 Homework 1

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31 Jan 2025

Problem 1 - ODEs on the Real Line

For each of (i)-(vi) below, find a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that the differential equation $\dot{u} = f(u)$ has the stated properties or, if there is no such function, explain in detail (with a proof or a precise argument) why not. If possible, give an explicit expression of the function; otherwise, sketch its graph.

(i) Every real number is an equilibrium. $\boxed{f = 0}$

(ii) Every integer is an equilibrium, and there are no other fixed points besides those. $\boxed{f(u) = \sin(\pi u)}$

(iii) There are exactly two equilibria, and both of them are stable.

Impossible. If there are exactly two equilibria, then f has two roots (say u_1, u_2 and WLOG assume $u_1 < u_2$).

As it is an equilibrium, $f(u_1) = 0$. By assumption, u_1 is stable so $f'(u_1) < 0$.

CASE 1: $f'(u) < 0$ for all u.

But then, f is strictly decreasing so $\forall \varepsilon > 0$, $f(u_1 + \varepsilon) < f(u_1) = 0$.

In particular, let $\varepsilon = u_2 - u_1 > 0$. then $f(u_1 + \varepsilon) = f(u_2) < f(u_1) = 0 \implies f(u_2) < 0$ which contradicts the fact that u_2 is an equilibrium.

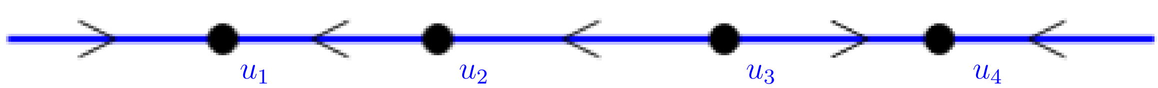
CASE 2: $\exists I \subseteq \mathbb{R}$ such that $f'(t) \geq 0$ for $t \in I$.

If $u_2 \in I$, then u_2 is not a stable equilibrium. Contradiction.

If $u_2 \notin I$ and f is not strictly decreasing everywhere (Case 1), then since $f \in C^1$, $u_1 < u_2$, $f(u_1 + \varepsilon) < 0$, and $f(u_2) = 0$, by the Intermediate Value Theorem, $\exists u_3 \in (u_1, u_2)$ such that $f(u_3) = 0$. Contradiction.

(iv) There are no equilibria. $\boxed{f = 1}$

(v) The phase diagram looks as shown in the picture below:



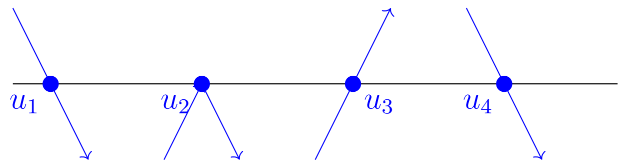
We label the equilibria as $u_1 < u_2 < u_3 < u_4$. Then,

- u_1, u_4 are stable
- u_2 is a saddle point
- u_3 is unstable

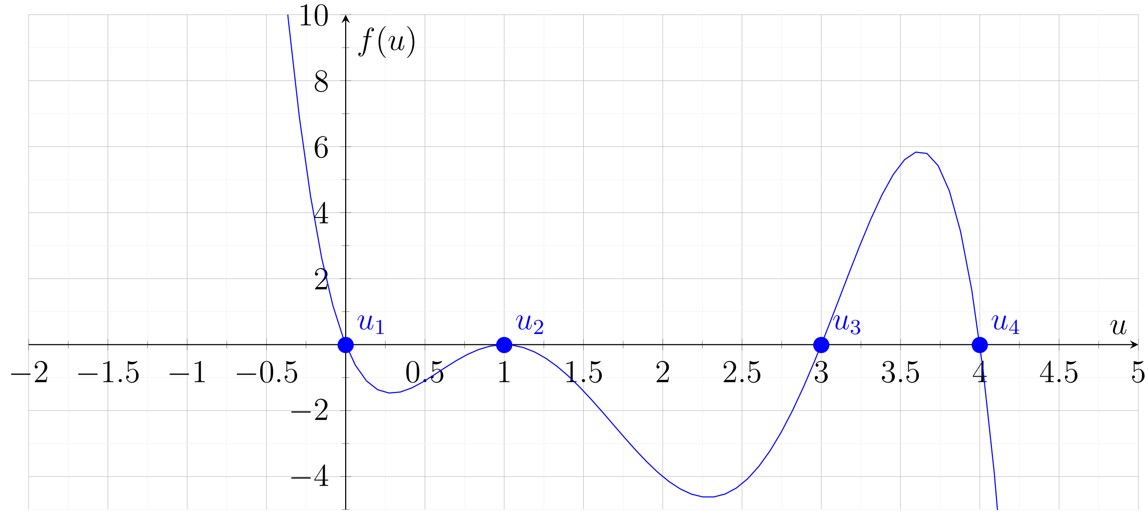
Hence,

$$f'(u_1) < 0, \quad f'(u_2) = 0, \quad f'(u_3) > 0, \quad f'(u_4) < 0$$

So, the graph of f must resemble



One possible function is $\boxed{f(u) = -u(u-1)^2(u-3)(u-4)}$:



(vi) The differential equation has a solution $u(t)$ that is periodic in t: there is a $T > 0$ with $u(t+T) = u(t)$ for all t, but $u(t)$ is not an equilibrium (that is, not a constant function).

Impossible. Suppose there exists such a periodic solution u for such an f. Consider a point $u(t_0)$. By periodicity, $u(t_0) = u(t_0 + T)$.

Since $f \in C^1$, $u \in C^2$ and by Rolle's Theorem, $\exists t^* \in (t_0, t_0 + T)$ such that

$$f(u(t^*)) = u'(t^*) = \frac{u(t_0 + T) - u(t_0)}{T} = 0$$

If $f(u(t)) = 0$ for all $t \in (t_0, t_0 + T)$, hence $u(t)$ is an equilibrium. Contradiction of assumption.

Otherwise, $u(t^*)$ is an equilibrium. If f is decreasing at $u(t^*)$, then the equilibrium is stable, hence $u(t)$ is constant in t - Contradiction.

If f is increasing at $u(t^*)$, then since u is periodic, there must exist some other equilibrium point where f is decreasing, again leading to a stable equilibrium.

Problem 2 - Logistic model

The differential equation

$$\frac{du}{dt} = ru\left(1 - \frac{u}{K}\right) - \mu u$$

serves as a model for a population of fish with harvesting: $u(t)$ is the size of the population at time $t, r > 0$ is the growth rate of fish at small population levels, $K > 0$ is called the carrying capacity, and $\mu \geq 0$ is the percentage of fish caught per unit time interval.

1. Analyze this model mathematically: find all equilibria, determine their stability, and identify all bifurcation points (if any) at which the number of equilibria changes as a function of the fishing rate μ.

   Let $f(u) = ru\left(1 - \frac{u}{K}\right) - \mu u$. Then, the equilibria are given by

   $$u_1 = 0, \quad u_2 = K(1 - \frac{\mu}{r})$$

   (these are the only equilibria as f is quadratic in u.)

   Further,

   $$f'(u) = r - \mu - \frac{2r}{K}u$$

   and

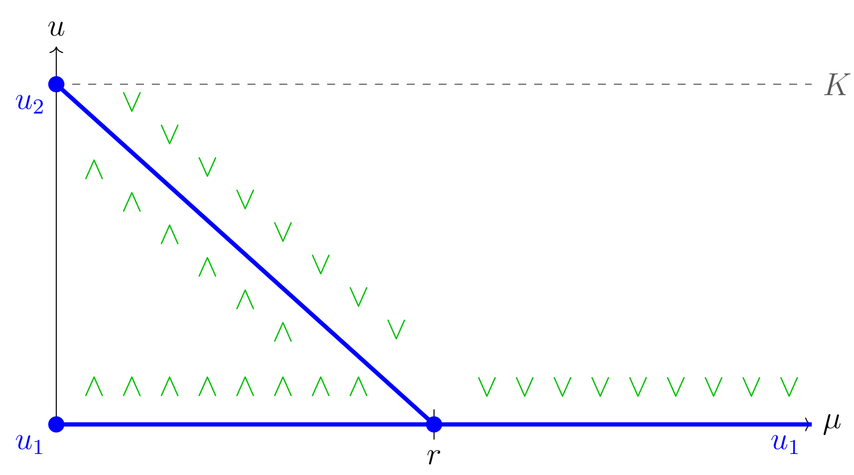
   $$f'(u_1) = r - \mu, \quad f'(u_2) = r - \mu - 2(r - \mu) = \mu - r$$

   Hence,

   - $u_1 = 0$ is stable for $\mu > r$, unstable for $\mu < r$, and undetermined for $\mu = r$.
   - $u_2 = \frac{K}{r}(r - \mu)$ is stable for $r > \mu$. For $\mu > r$, $u_2 = K(1 - \frac{\mu}{r}) < 0$ which is impossible. For $\mu = r$, $u_2 = 0$ and there exists only one equilibrium.

   By the above argument, $\mu = r$ is a bifurcation point since for $\mu < r$, there are two equilibria and for $\mu > r$, there is only one equilibrium.

2. Draw the bifurcation diagram with respect to (μ, u).



3. Discuss the meaning of the carrying capacity K and identify which equilibria are meaningful with respect to the underlying application, where $u(t)$ captures the size of the fish population. Discuss the implications of your analysis for the underlying application of fishing/harvesting of fish.

   If there were no fishing ($\mu = 0$), then since u_1 is unstable and u_2 is stable, the fish population would stabilize at $u = K$. As we introduce fishing ($\mu \nearrow$), the equilibrium solution $K - \frac{K}{r}\mu$, representing a stable fish population, decreases. Hence, if the fishing rate μ exceeds the growth rate r, then the fish population will die out. If, instead, $0 < \mu < r$, then the fish population will stabilize at a lower level than K and the fishing rate will be sustainable.

Problem 3 - Stability of Equilibria

In your own words, write down a concise definition of when we call an equilibrium stable.

For a differential equation of the form $\dot{u} = f(u)$, u^* is an equilibrium point if $f(u^*) = 0$. We say that u^* is a stable equilibrium if $\exists \varepsilon > 0$ such that $\forall u_i \in B_\varepsilon(u^*)$, $\lim_{t\to\infty} u(t) = u^*$ for all solutions $u(t)$ with $u(0) = u_i$.