APMA 1360 - Homework 7

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1 Brusselator model

Consider the system

$$\dot{x} = 1 - (b+1)x + ax^2y, \qquad \dot{y} = bx - ax^2y,$$

which is a model for autocatalytic chemical reactions, where $x, y \geq 0$ and $a, b > 0$.

(i) Find all equilibria and use the Jacobian to classify them.

$$\begin{array}{rl} 0 & = 1 - (b+1)x + ax^2y \\ 0 & = bx - ax^2y \\ \hline 0 & = 1 - x \end{array}$$

Substituting $x = 1$,

$$0 = b - ay \implies y = \frac{b}{a}$$

so our only equilibrium is $\boxed{(1, b/a)}$. The Jacobian is

$$J(x,y) = \begin{pmatrix} -b-1+2axy & ax^2 \\ b-2axy & -ax^2 \end{pmatrix} \implies J(1, b/a) = \begin{pmatrix} -b-1+2b & a \\ b-2b & -a \end{pmatrix} = \begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix}$$

We know

$$\begin{aligned} \lambda_1\lambda_2 &= -ab + a + ab = a \\ \lambda_1 + \lambda_2 &= b - 1 - a \end{aligned}$$

Since $a > 0$, $\det J > 0$ and we know there cannot be a saddle. $\operatorname{tr} J = b - a - 1$ is

$$\begin{cases} > 0 & \text{if } b - a > 1 \\ < 0 & \text{if } b - a < 1 \end{cases} \implies \boxed{\begin{cases} \text{repeller if } b - a > 1 \\ \text{attractor if } b - a < 1 \end{cases}}$$

(ii) Show that a Hopf bifurcation occurs at some value of the parameter b (this value will depend on a).

Let

$$F(x,y) = \begin{pmatrix} 1 - (b+1)x + ax^2y \\ bx - ax^2y \end{pmatrix}$$

Clearly $F \in C^3$.

In part i, we showed that the Jacobian at the equilibrium is

$$J(1, b/a) = \begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix}$$

Notice that we could manually calculate the eigenvalues by

$$\begin{aligned}
0 &= (b-1-\lambda)(-a-\lambda)+ab \\
&= \lambda^2+(a-b+1)\lambda+a \\
\lambda &= \frac{-(a-b+1)\pm\sqrt{(a-b+1)^2-4a}}{2} \\
\lambda &= \alpha(a,b)\pm i\beta(a,b)
\end{aligned}$$

In order for a Hopf Bifurcation to occur, we need

(a) $\alpha(a,b)=0$

(b) $\beta(a,b)\neq 0$

(c) $\frac{\partial\alpha}{\partial a}\neq 0$

(d) $\frac{\partial\alpha}{\partial b}\neq 0$

This gives us a system of equations

$$-(a-b+1)=0 \tag{1}$$

$$(a-b+1)^2-4a\neq 0 \tag{2}$$

$$\frac{\partial\alpha}{\partial a}=-\frac{1}{2}\neq 0 \tag{3}$$

$$\frac{\partial\alpha}{\partial b}=1\neq 0 \tag{4}$$

Equation (1) implies $b=a+1$. Substituting this into (2) gives

$$(a-(a+1)+1)^2-4a=-4a<0$$

and equations (3) and (4) are trivially satisfied.

Hence, for $\boxed{b=a+1}$, a Hopf Bifurcation occurs around the equilibrium $(1,1+1/a)$.

(iii) Find the approximate period of the periodic orbit that bifurcates from the equilibrium at this Hopf bifurcation.

We know that the period of the periodic orbit is given by $\approx\frac{2\pi}{\beta(a,b)}$. Letting, $b=a+1$,

$$\lambda^2+(a-b+1)\lambda+a=0 \implies \lambda^2+a=0 \implies \lambda=\pm i\sqrt{a} \implies \beta(a,b)=\sqrt{a}$$

so the period is given by

$$\boxed{\approx\frac{2\pi}{\sqrt{a}}}$$

2 Bead on a rotating hoop revisited

Consider the second-order equation

$$\frac{d^2\phi}{dt^2} = -a\frac{d\phi}{dt} + (\gamma\cos\phi - 1)\sin\phi \tag{5}$$

that we discussed as a model for a bead that slides on a rotating hoop subject to gravitational and friction forces. The parameter a and γ are both strictly larger than zero: we keep $a > 0$ fixed and vary $\gamma > 0$.

(i) Define the angular velocity $v := d\phi/dt$ and write (5) as a first-order system, which we refer to as equation (2), for the variables (ϕ, v).

Let $v = d\phi/dt$. Then, we have

$$\begin{cases} \dot\phi = v \\ \dot v = -av + (\gamma\cos\phi - 1)\sin\phi \end{cases}$$

(ii) Find the equilibria of the system (2) and determine their stability properties. Also, identify all bifurcations in this system.

We want $(\dot\phi, \dot v) = (0,0)$ which immediately suggests $v = 0$, so

$$(\gamma\cos\phi - 1)\sin\phi = 0 \implies \begin{cases} \cos\phi = 1/\gamma \\ \sin\phi = 0 \end{cases} \implies \phi = \{\arccos\frac{1}{\gamma}, 0, \pi\}$$

where $\gamma \geq 1$ (else $\cos\phi = 1/\gamma$ has no solutions).

We have Jacobian

$$\begin{aligned} J(\phi, v) &= \begin{pmatrix} 0 & 1 \\ \gamma\cos 2\phi - \cos\phi & -a \end{pmatrix} \\ J(\arccos\frac{1}{\gamma}, 0) &= \begin{pmatrix} 0 & 1 \\ 2 - \frac{1}{\gamma} & -a \end{pmatrix} \\ J(\pi, 0) &= \begin{pmatrix} 0 & 1 \\ \gamma + 1 & -a \end{pmatrix} \\ J(0,0) &= \begin{pmatrix} 0 & 1 \\ \gamma - 1 & -a \end{pmatrix} \end{aligned}$$

Further,

$$\det J(\arccos\frac{1}{\gamma}, 0) = 2 - \frac{1}{\gamma} > 0 \quad \forall\gamma$$

hence we have ad an attractor or repeller. We can check

$$\operatorname{tr} J(\arccos\frac{1}{\gamma}, 0) = -a < 0 \quad \forall a > 0$$

which means that $\boxed{\phi = \arccos\frac{1}{\gamma} \text{ is an attractor}}$

Similarly,

$$\det J(\pi, 0) = -(\gamma + 1) < 0 \quad \forall\gamma$$

so $\boxed{\phi = \pi \text{ is a saddle for all } \gamma}$

Finally,

$$\det J(0,0) = -(\gamma - 1)$$

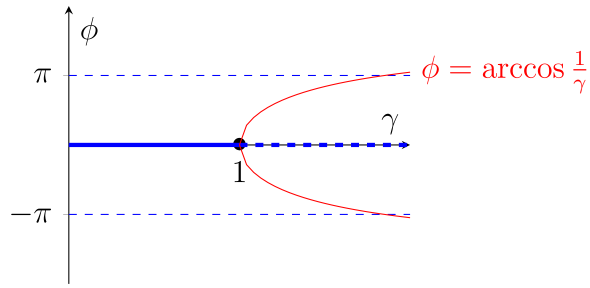
is

$$\begin{cases} < 0 & \text{if } \gamma > 1 \\ > 0 & \text{if } \gamma < 1 \end{cases}$$

and

$$\text{tr } J(0,0) = -a < 0 \quad \forall \gamma$$

so $\boxed{\phi = 0 \text{ is a saddle for } \gamma > 1 \text{ and an attractor } \gamma < 1}$



Clearly, we have a $\boxed{\text{supercritical pitchfork bifurcation}}$ at $\gamma = 1$.

(iii) Compare your findings to those we obtained for the one-dimensional system where we omitted the second-order derivative $d^2\phi/dt^2$.

In the case where we omitted the second-order derivative, we had equilibria at $\phi = 0, \pi$, with

- $\phi = 0$ stable if $\gamma < 1$ and unstable if $\gamma > 1$
- $\phi = \pi$ is always unstable

and a pitchfork bifurcation at $\gamma = 1$.

In the case where we included the second-order derivative, we again have a pitchfork bifurcation at $\gamma = 1$, but now

- $\phi = 0$ is an attractor for $\gamma < 1$ and a saddle for $\gamma > 1$
- $\phi = \pi$ is a saddle
- $\phi = \arccos \frac{1}{\gamma}$ is an attractor

And this makes sense! In a sense, this is exactly the same result but in the higher dimension, we have saddles instead of unstable equilibria.