APMA 1360: Applied Dynamical Systems

Milan Capoor

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Chapter 1

Bifurcation Theory

1.1 Jan 22

Motivations - Applications + Phenomena

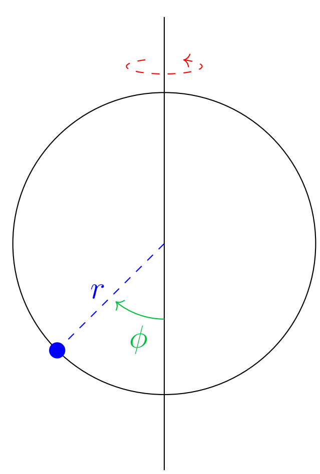
1. **Bifurcation theory:** How do systems change as parameters change?

   Examples:

   - Mechanical systems (e.g. what will happen to a bead as an apparatus is rotated at velocity ω?)
   - Chemical reactions (e.g. Belusov-Zhabotinsky reaction - oscillations in chemical reactions)
   - Tipping points (e.g. climate change, convection currents)
   - Population dynamics (e.g. predator-prey models, outbreaks)
   - Synchronization (e.g. firefly synchronous lighting, brain activity patterns)
   - Chaotic dynamics (e.g. double pendulum)
2. **Existence and Uniqueness**
3. **Dynamical theory**
4. **Chaotic dynamics**

Bifurcation Theory

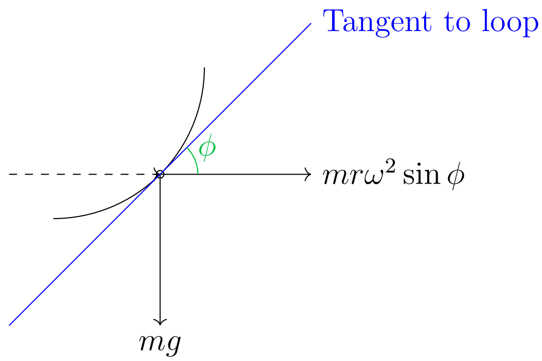
Example (Overdamped bead on loop)



Goal: What will happen to the bead as the loop is rotated at velocity ω?

We assume that the only forces on the bead are gravitation, friction, and centrifugal force.

This gives a force diagram:



From Newton's law,

$$\underbrace{mr\frac{d^2\phi}{dt^2}}_{\text{acceleration}} = -b\frac{d\phi}{dt} - mg\sin\phi + m\omega^2 r\sin\phi\cos\phi$$

Assuming $b \gg 1$, we can neglect the LHS so

$$\begin{aligned}\frac{d\phi}{dt} &= -\frac{mg}{b}\sin\phi + \frac{m\omega^2 r}{b}\sin\phi\cos\phi \\ &= \frac{mg}{b}\sin\phi\left(\frac{\omega^2 r}{g}\cos\phi - 1\right) \\ &= a\sin\phi(\mu\cos\phi - 1)\end{aligned}$$

1.2 Jan 24

Review

Definition: A function $u(t)$ is a solution of $\dot{u} = f(u)$ if $\frac{du(t)}{dt} = f(u(t))$ for all t in some open interval. In this case, we say "$u(t)$ satisfies $\dot{u} = f(u)$".

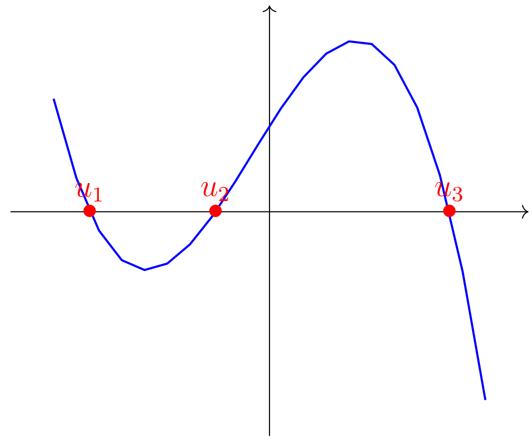
Theorem (Existence and Uniqueness): Assume $f \in C^1$ (class of continuously differentiable functions) and $u_0 \in \mathbb{R}$ is given. Then the differential equation $\dot{u} = f(u)$ with initial condition $u(0) = u_0$ has a unique solution $u(t)$ on some open interval containing $t = 0$.

Proof: Omitted

Example: $\dot{u} = au, u(0) = u_0$ has solution $u(t) = u_0 e^{at}$. Since au is continuous, $u(t) \in C^1$, hence the solution is unique.

Geometric Viewpoint

Example: Consider $\dot{u} = f(u)$,



For each point, $f(u_i) = 0 \implies u(t) = u_i$ is a solution for all t.

We can check:

$$\begin{cases} \frac{du}{dt}(t) = \frac{d}{dt}u_i = 0 \\ f(u(t)) = f(u_i) = 0 \end{cases}$$

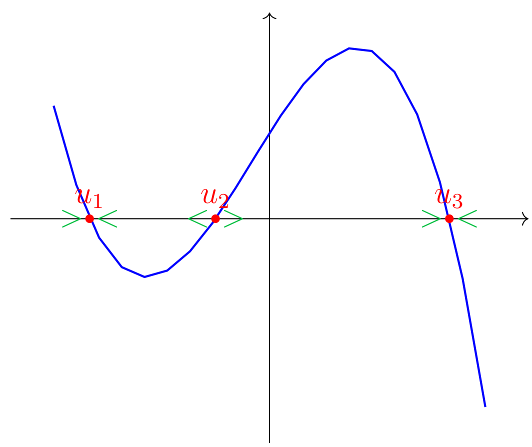
Hence, $u(t) = u_i$ is a solution.

We call the points u_1, u_2, u_3 *equilibrium points*, *rest states*, *steady states*, *fixed points*, or *stationary points.*

We can also consider the direction field of $\dot{u} = f(u(t))$:

$$\begin{cases} f(u) < 0 \implies u \text{ decreasing} \implies u \text{ moves left} \\ f(u) > 0 \implies u \text{ increasing} \implies u \text{ moves right} \end{cases}$$

So we can draw the phase diagram



In this case, we say that u_1, u_3 are stable but u_2 is unstable.

Stable: an equilibrium u_i is stable if all solutions for initial conditions near u_i converge to u_i as $t \to \infty$.

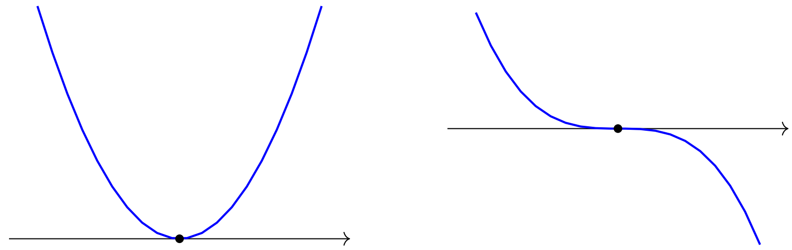
Unstable: an equilibrium u_i is unstable if there exists an initial condition near (but distinct from) u_i such that the solution moves away from u_i as $t \to \infty$.

Conditions for stability: Assuming u_i is an equilibrium,

- If $f'(u_i) < 0$, then u_i is stable.
- If $f'(u_i) > 0$, then u_i is unstable.
- If $f'(u_i) = 0$, then it is undetermined

What can $f'(u_i) = 0$ look like?

Examples:



Example 1 Revisited:

Recall

$$\dot{\phi} = a \sin\phi(\mu \cos\phi - 1) = f(\phi)$$

for $a, \mu > 0$ and $\mu \approx \omega^2$.

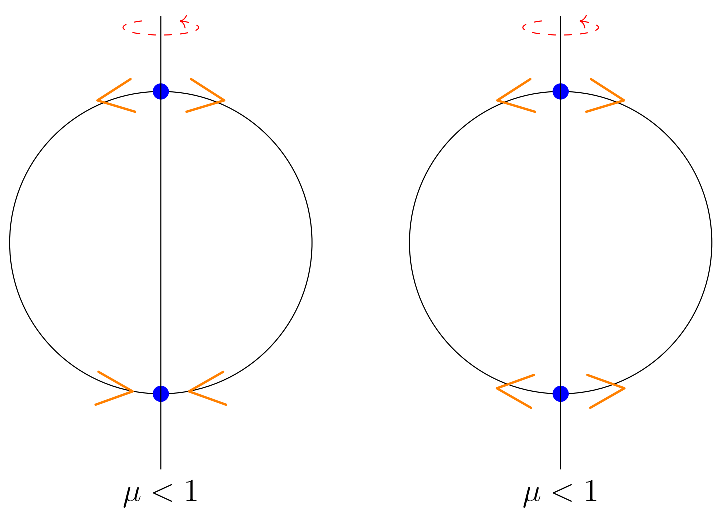
1. We can verify $f \in C^1$.
2. Find the equilibrium points:
$$a \sin\phi(\mu\cos\phi - 1) = 0 \implies \phi = \{0, \pi\}$$
3. Determine stability:
$$\begin{aligned} f'(\phi)\Big|_{\phi=0,\pi} &= [a\cos\phi(\mu\cos\phi - 1)]_{\phi=0,\pi} \\ &= \begin{cases} a(\mu - 1) & \phi = 0 \\ a(\mu + 1) & \phi = \pi \end{cases} \end{aligned}$$

   Hence, $\phi = 0$ is always unstable since $a(\mu+1) > 0$. $\phi = \pi$ is stable $\mu < 1$, unstable $\mu > 1$ and undetermined for $\mu = 1$.

   In fact, this makes sense. μ is the ratio of the centrifugal force to the gravitational force. If $\mu < 1$, the gravitational force is stronger and the bead will fall to the bottom. If $\mu > 1$, the centrifugal force is stronger and the bead will move outwards.

1.3 Jan 27

Recall: We return one more time to the example of the bead on a loop. Last time, we determined the system has equilibria

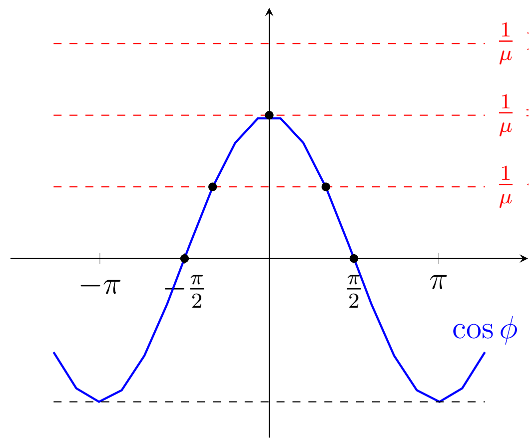


In the case on the right, the equilibria are not consistent. Therefore, there need to be additional equilibria.

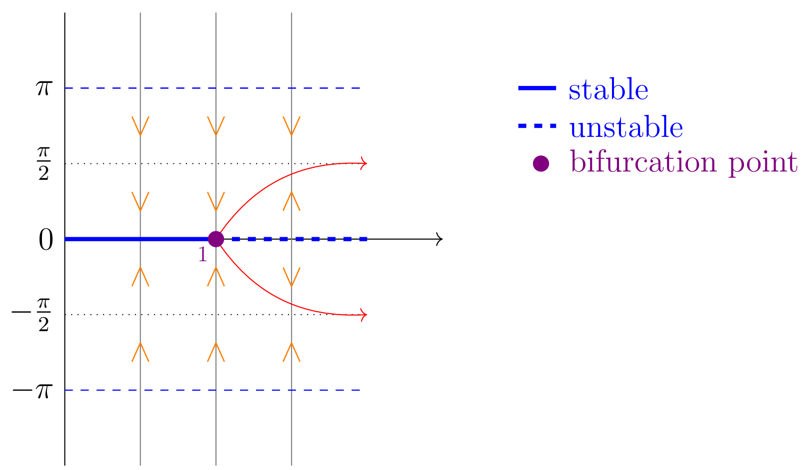
We can check:

$$f(\phi) - a \sin \phi(\mu \cos \phi - 1)$$

Setting $a \sin \phi = 0$ gives $\phi = \{0, \pi\}$. Taking $\mu \cos \phi - 1 = 0$ gives $\phi = \arccos \frac{1}{\mu}$:



This gives us the bifurcation diagram:



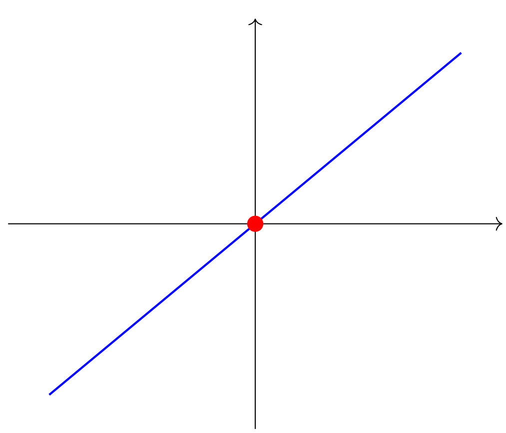
where the curve is given by $\mu = \frac{r\omega^2}{g} \approx \frac{\text{centrifugal}}{\text{gravitational}}$.

Notice if $f'(\phi_*) \neq 0$, then the equilibrium ϕ_* varies continuously with μ. If $f'(\phi_*) = 0$, then new equilibria emerge and dynamics change.

Parameter-Dependent Differential Equations:

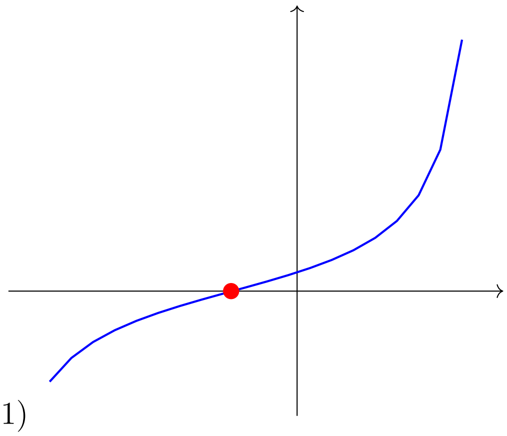
Consider $\dot{u} = f(u, \mu)$ for $u, \mu \in \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$.

Example: $f(u, 0) = u$

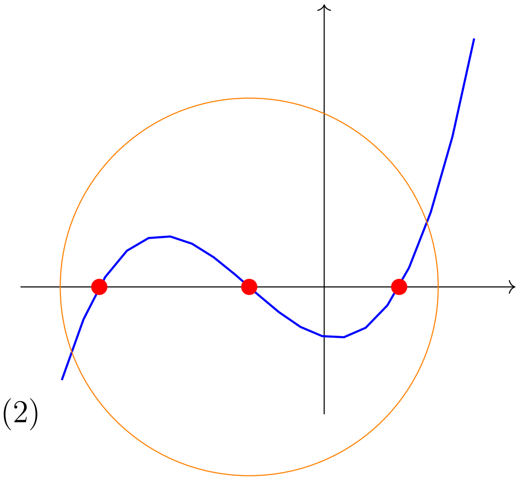


Here, $u = 0$ is an unstable equilibrium. ($f(0,0) = 0$ and $f_u(0,0) = 1 > 0$).

What happens if we change μ slightly? Choose $\mu \approx 0$:



(1)



(2)

On the left, the equilibrium moves but is unique and still unstable. On the right, we have three equilibria and we can shrink the ball as $\mu \to 0$.

For (2), say

$$f(u,\mu) = \begin{cases} u + \mu & u \leq -\mu \\ \frac{u}{2}(\frac{u^2}{\mu^2} - 1) & -\mu \leq u \leq \mu \\ u - \mu & u = \mu \end{cases}$$

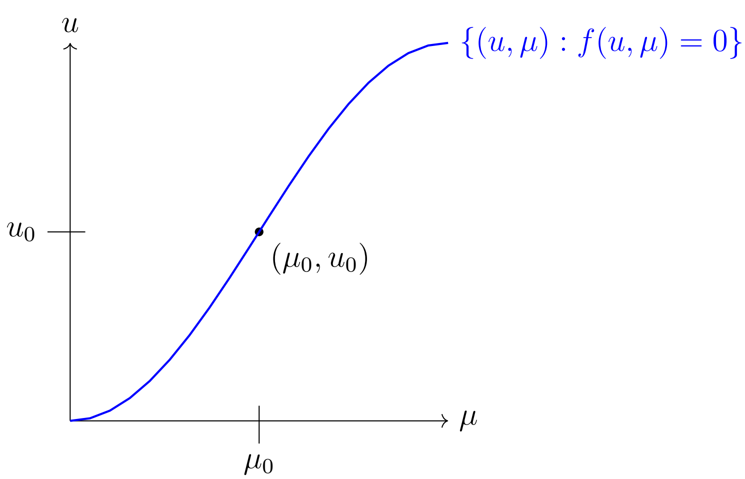
with $|f(u,\mu)| \leq$ const. uniformly in μ, u

Properties of (2):

- $f(u,\mu)$ is continuous in u, μ.
- $f(u,\mu)$ is differentiable in u for all (u,μ)
- $f_u(u,\mu)$ is not continuous

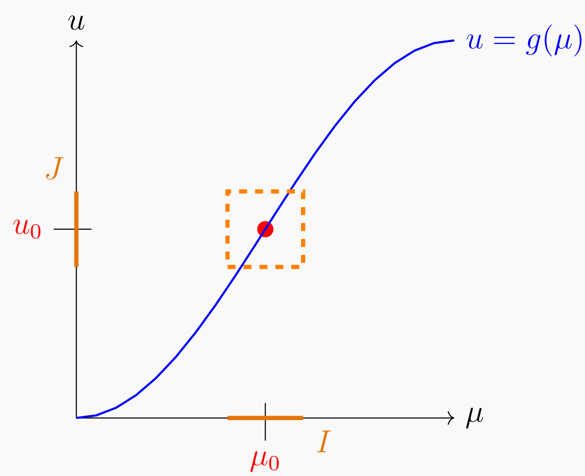
For simplicity, we will consider only functions $f(u,\mu)$ that are infinitely often differentiable and for which all derivatives are continuous in (u,μ), i.e. $f \in C^\infty(\mathbb{R}^2, \mathbb{R}) = C^\infty$

Goal: Assume u_0 is an equilibrium $\dot{u} = f(u,\mu)$ for $\mu = \mu_0$ so that when $f_u(u_0,\mu_0) \neq 0$, there is a function $g(\mu)$ so that $f(u,\mu) = 0$ for (u,μ) near (u_0,μ_0) iff $u = g(\mu)$.



Implicit Function Theorem: Assume $f(u_0, \mu_0) = 0$ and $f_u(u_0, \mu_0) \neq 0$ for $f \in C^\infty$. Then there exists open intervals, I, J with $u_0 \in J, \mu_0 \in I$ and a $g : I \to J$ such that $f(u, \mu) = 0$ for $(u, \mu) \in J \times I$ iff $u = g(\mu)$. Furthermore, $g \in C^\infty$. In particular, if u_0 is an equilibrium of $\dot{u} = f(u, \mu)$ at $\mu = \mu_0$ with $f_u(u_0, \mu_0) \neq 0$, then $\dot{u} = f(u, \mu)$ has an equilibrium in $J \times I$ iff $u = g(\mu)$ and these equilibria share their stability properties with u_0

Example:

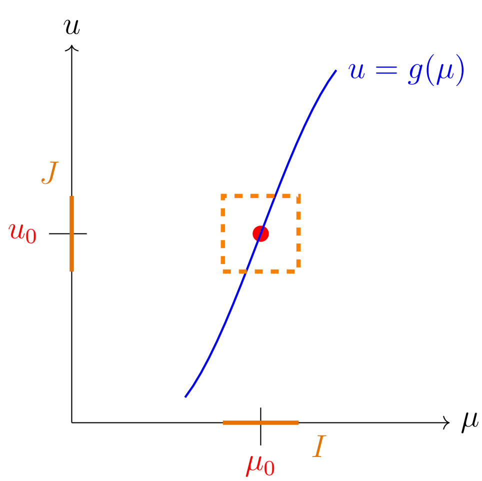


Proof: Omitted

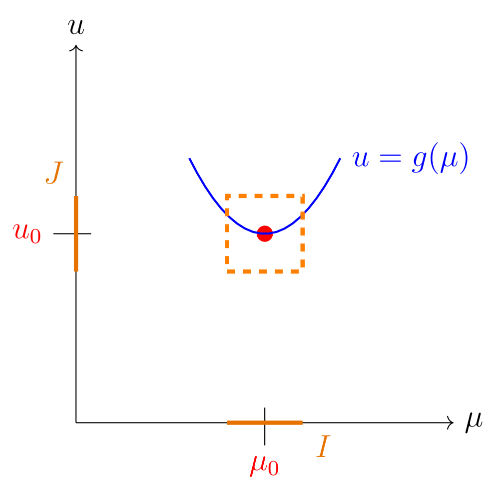
1.4 Jan 29

1.4.1 Implicit Function Theorem

re **Recall:** If we have $f = f(u, \mu) \in C^\infty$ with $f(u_0, \mu_0) = 0$ and $f_u(u_0, \mu_0) \neq 0$, then there exist open intervals I, J with $\mu_0 \in I$, $u_0 \in J$ and a unique $g : I \to J$ with $g(\mu_0) = u_0$ so that $f(u, \mu) = 0$ for $(u, \mu) \in J \times I$ iff $u = g(\mu)$. Furthermore, $g \in C^\infty$.



$f_u(u_0, \mu_0) > 0$ (unstable)

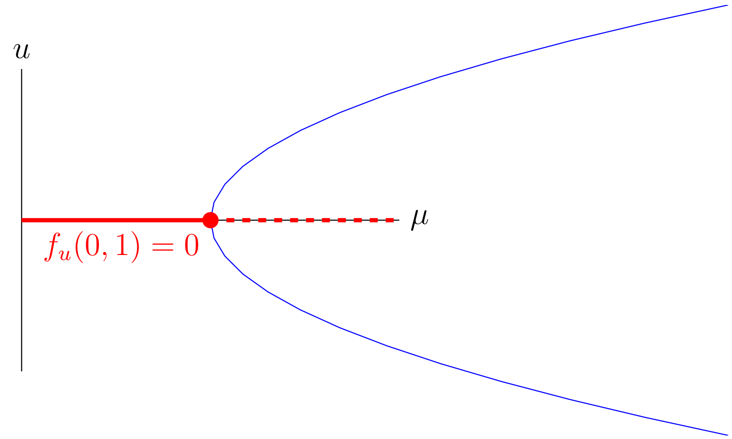


$f_u(u_0, \mu_0) < 0$ (stable)

Definition: we say that u_0 is a **hyperbolic equilibrium** of $\dot{u} = f(u, \mu)$ at $\mu = \mu_0$ if

- $f(u_0, \mu_0) = 0$ (u_0 is an equilibrium)
- $f_u(u_0, \mu_0) \neq 0$ (u_0 is hyperbolic)

Example: if u_0 is *not* hyperbolic, the dynamics can be more complicated when we vary μ near μ_0.



Here the equilibrium on the red line is hyperbolic.

Catalogue of Bifurcations:

- Consider $\dot{u} = f(u, \mu)$ with $u, \mu \in \mathbb{R}$ and $f \in C^\infty$.
- Assume WLOG that $(u, \mu) = (0, 0)$ is an equilibrium with

$$\begin{cases} f(0,0) = 0 \\ f_u(0,0) = 0 \end{cases}$$

  (i.e. $(0,0)$ is not hyperbolic)
- **Goal:** find all equilibria of $\dot{u} = f(u, \mu)$ near $(0,0)$ and determine their stability.

Since we only need to examine the behavior around $(0,0)$, we can use a *Taylor Expansion:*

(where $O : \mathbb{R}^2 \to \mathbb{R}$ goes to 0 at least cubically as $u, \mu \to 0$)

Plugging in our conditions,

$$f(u,\mu) = f_\mu(0,0)\mu + \frac{1}{2}f_{uu}(0,0)u^2 + f_{u\mu}(0,0)u\mu + \frac{1}{2}f_{\mu\mu}(0,0)\mu^2 + O(|u| + |\mu|^3)$$

From here, we will

1. start from terms of lowest order to highest order monomials and assume that coefficients are non-zero.
2. we already assumed $f(0,0) = 0$ and $f_u(0,0) = 0$ so there are no choices left
3. hence, assume the coefficient a of $f_\mu(0,0)$ is non-zero

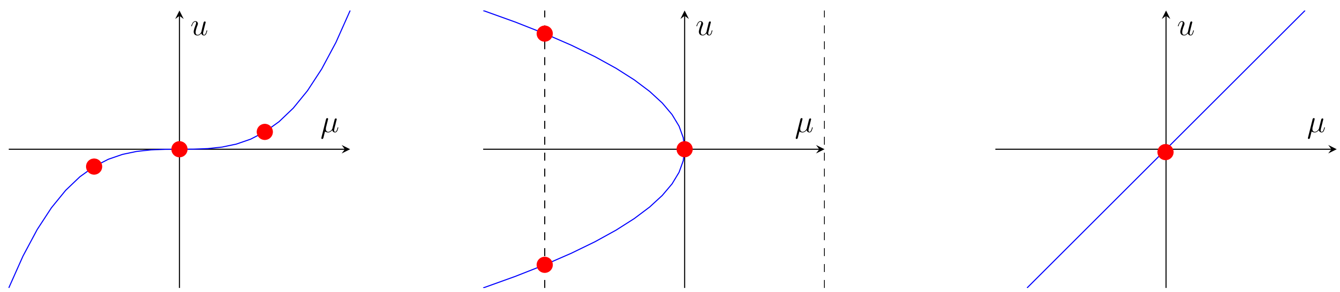
Hence,

$$f(u,\mu) = a\mu + O((|u| + |\mu|)^2) = 0$$

(where we set it to 0 as we are looking for equilibria)

Then, by the Implicit Function Theorem, we have a unique function g in a neighborhood of $(0,0)$ with $g(0) = 0$ and $\mu = g(u)$.

Now we have a few potential cases:



(1): $g'(0) \neq 0, g''(0) = 0$ (2): $g'(0) = 0, g''(0) \neq 0$ (3): $g'(0) \neq 0$

On the left, we gave a unique equilibrium for μ near 0. On the right, as μ increases, two equilibria collide at $\mu = 0$ and disappear. Notice that this is different than the case in the logistic model from HW where only one equilibrium disappeared and from the bead on a loop example where two equilibria merged. In some sense, this is a more complicated bifurcation, but also the most common in applications.

1.5 Jan 31

Setup: $u = 0$ is a non-hyperbolic equilibrium at $\mu = 0$, i.e. $f(0,0) = 0$ and $f_u(0,0) = 0$. We want to find solutions of $f = f(u, \mu)$.

Making the assumption, $f_\mu(0,0) = a \neq 0$, we show that $f(u, \mu) = 0$ for (u, μ) near $(0,0)$ iff $\mu = g(u)$ with $g(0) = 0$ and $g \in C^\infty$.

Formulated differently,we know that $f(u, g(u)) = 0$ for all u. Differentiating in u, we get

$$0 = \frac{d}{du}(f(u, g(u))) = f_u(u, g(u)) + f_\mu(u, g(u))g'(u) \tag{(*)}$$

for all u near 0

Evaluating at $u = 0$,

$$0 = f_u(0,0) + f_\mu(0,0)g'(0) = ag'(0) \implies g'(0) = 0$$

From (*), we know that case (1) above is impossible. Can we determine $g''(0)$?

Differentiating again,

$$\begin{aligned}
0 &= f_u(u, g(u)) + f_\mu(u, g(u))g'(u) \\
0 &= f_{uu}(u, g(u)) + f_{u\mu}(u, g(u))g'(u) + f_{\mu u}(u, g(u))g'(u) + f_{\mu\mu}(u, g(u))g'(u)^2 + f_\mu(u, g(u))g''(u)
\end{aligned}$$

Evaluating at $u = 0$,

$$\begin{aligned}
0 &= f_{uu}(0,0) + 2f_{u\mu}(0,0)g'(0) + f_{\mu\mu}(0,0)g''(0)^2 + f_\mu(0,0)g''(0) \\
&= f_{uu}(0,0) + f_\mu(0,0)g''(0)g''(0) = -\frac{f_{uu}(0,0)}{f_\mu(0,0)}
\end{aligned}$$

We assume $f_{uu}(0,0) \neq 0$ to put us in Case (2) above.

Remark: there is no reason we could not have chosen $f_{uu}(0,0) = 0$ to look at (3). However, in some sense Case (2) is more interesting and also has less tedious calculations. Further, it would be somewhat surprising for there to be neither first nor second derivatives in a Taylor Expansion. In general, though, this choice was arbitrary.

In particular,

$$g(u) = -\frac{1}{2}\frac{f_u(0,0)}{f_\mu(0,0)}u^2 + O(u^3)$$

Conclusion (Existence): Assume $f(0,0) = 0$, $f_u(0,0) = 0$, $f_\mu(0,0) \neq 0$, $f_{uu}(0,0) \neq 0$. Then $f(u,\mu) = 0$ vanishes near $(0,0)$ iff $\mu = g(u)$ with $g = -\frac{1}{2}\frac{f_u(0,0)}{f_\mu(0,0)}u^2 + O(u^3)$.

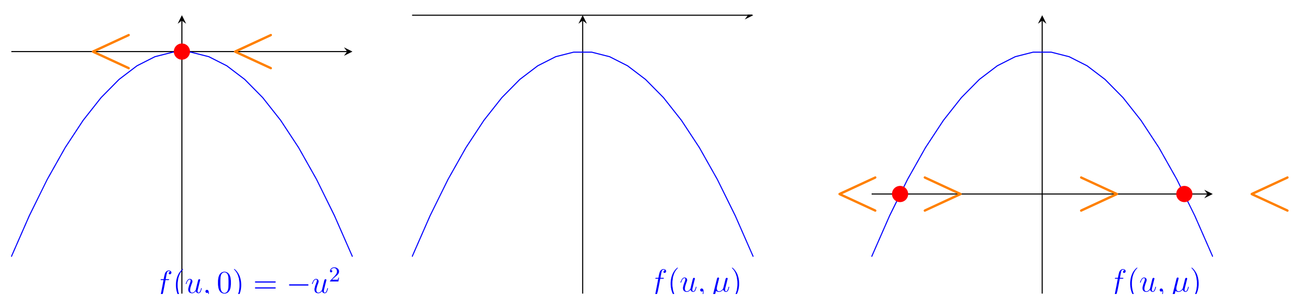
1.5.1 Bifurcation Analysis

Here, $-\frac{f_u(0,0)}{f_\mu(0,0)} < 0$ and $\mu < 0$ corresponds to having precisely two rest states, while $\mu > 0$ has none.

The prototypical equation which satisfies our hypothesis is

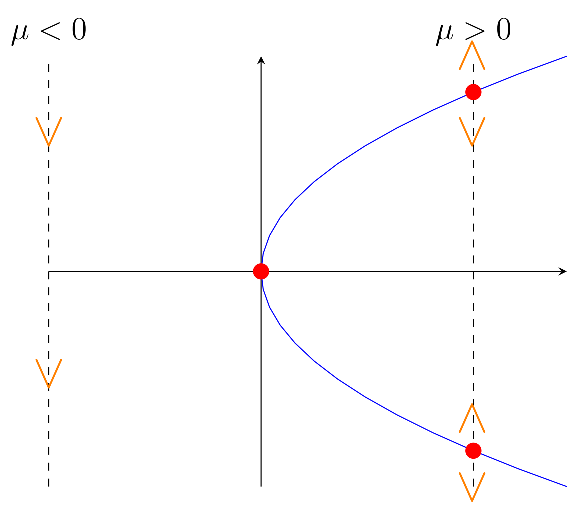
$$f(u,\mu) = \mu - u^2$$

This gives three possible graphs:



$\mu = 0$ $\qquad$ $\mu < 0$ $\qquad$ $\mu > 0$

Which yields the bifurcation diagram:



$$(2): \quad g'(0) = 0, g''(0) \neq 0$$

1.5.2 Stability at the Equilibria

If $u = u_*$ is an equilibrium of $\dot{u} = f(u,\mu)$ at $\mu = \mu_*$, then

$$\begin{cases} f_u(u_*,\mu_*) > 0 & \text{unstable} \\ f_u(u_*,\mu_*) < 0 & \text{stable} \\ f_u(u_*,\mu_*) = 0 & \text{undetermined} \end{cases}$$

We know that our equilibria occur at $(u, \mu) = (u, g(u))$. Hence, we must check the condition $f_u(u, g(u))$. The process is the same as before:

Take the Taylor Expansion:

$$\begin{aligned} f(u,\mu) &= f_\mu(0,0)\mu + \frac{f_{uu}(0,0)}{2}u^2 + O(\mu u + \mu^2 + u^3) \\ f_u(u,\mu) &= f_{uu}(0,0)u + O(\mu + u^2) \end{aligned}$$

Taking $g(\mu) = -\frac{f_{uu}(0,0)}{f_\mu(0,0)}u^2 + O(u^3) = O(u^2)$, notice that evaluating $f_u(u, \mu)$ at $(u, \mu) = (u, g(u))$,

$$O(\mu + u^2) = O(g(u) + u^2) = O(u^2)$$

so

$$f_u(u, g(u)) = f_{uu}(0,0)u + O(u^2)$$

Hence, the equilibrium u at $\mu = g(u)$ is

- stable for $f_{uu}(0,0)u < 0$
- unstable for $f_{uu}(0,0)u > 0$

1.6 Feb 3

Theorem (saddle-node/fold/turning-point bifurcation): Consider $\dot{u} = f(u, \mu)$ with $u, \mu \in \mathbb{R}$ and $f \in C^2$. Assume that u_0 is a non-hyperbolic equilibrium at $\mu = \mu_0$ with $f(u_0, \mu_0) = 0$ and $f_u(u_0, \mu_0) = 0$. Assume further non-degeneracy conditions $f_{uu}(u_0, \mu_0) \neq 0$ and $f_\mu(u_0, \mu_0) \neq 0$.

Then there exist open intervals I, J with $(u_0, \mu_0) \in I \times J$ and a unique $g : I \to J$ with $g(u_0) = \mu_0$ so that $f(u, \mu) = 0$ for $(u, \mu) \in I \times J$ iff $\mu = g(u)$ for some $u \in I$.

Furthermore, $g \in C^2$ with

$$g(u) = -\frac{1}{2}\frac{f_{uu}(u_0,\mu_0)}{f_\mu(u_0,\mu_0)}(u-u_0)^2 + O(|u-u_0|^3)$$

and

$$f_u(u, g(u)) = f_{uu}(u_0,\mu_0)(u-u_0) + O(|u-u_0|^2)$$

so that u is stable if $f_{uu}(u_0, \mu_0)(u - u_0) < 0$ and unstable if $f_{uu}(u_0, \mu_0)(u - u_0) > 0$.

Proof: Follows from example above

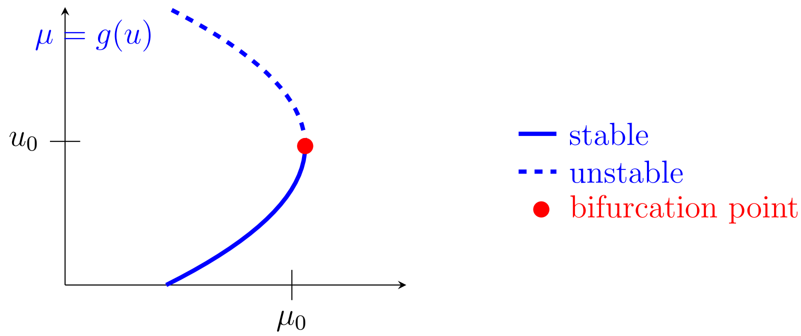
Example: Assume

$$\begin{cases} f_\mu(u_0,\mu_0) > 0 \\ f_{uu}(u_0,\mu_0) < 0 \end{cases}$$

Then

$$g(u) = \underbrace{-\frac{1}{2}\frac{f_{uu}(u_0,\mu_0)}{f_\mu(u_0,\mu_0)}}_{<0}(u-u_0)^2 + O(|u-u_0|^3)$$

hence u is stable if $u < u_0$ and unstable if $u > u_0$.



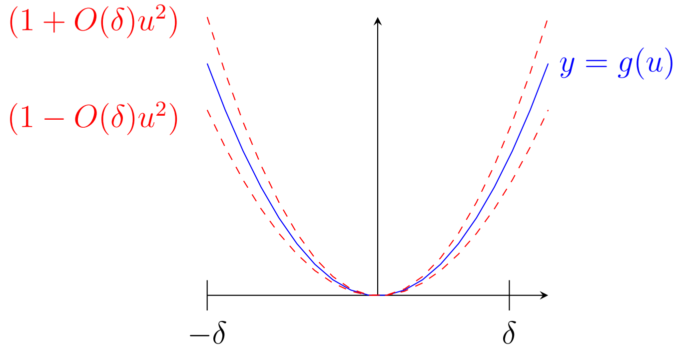
An important question is how we know that the $O(u^3)$ terms do not change the graph of u dramatically. Consider

$$g(u) = u^2 + O(u^3) = (1 + O(u))u^2$$

so

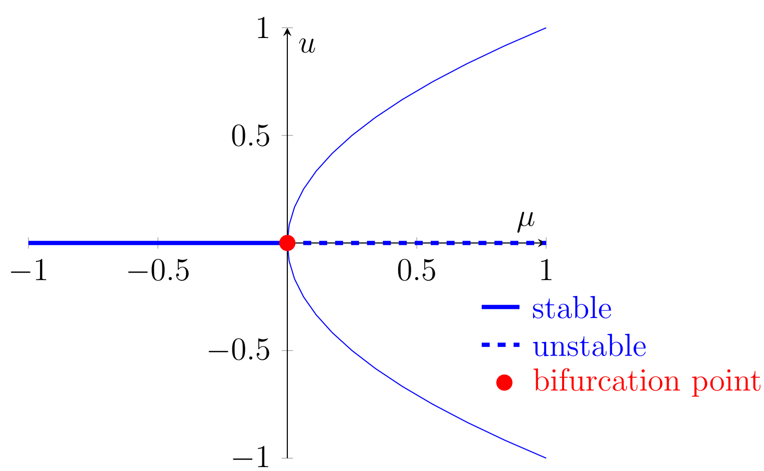
$$\begin{cases} g(0) = 0 \\ g'(0) = 0 \\ g''(0) = 2 \end{cases}$$

Hence:

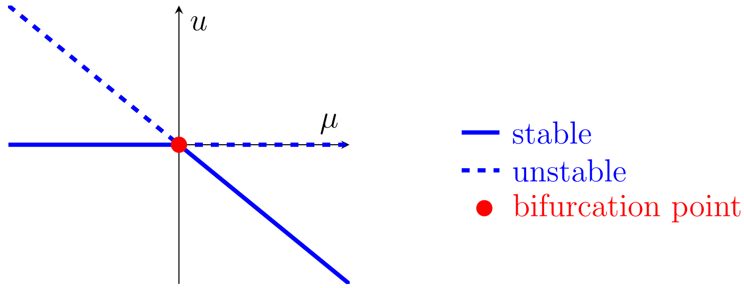


1.6.1 Summary of Bifurcations (so far):

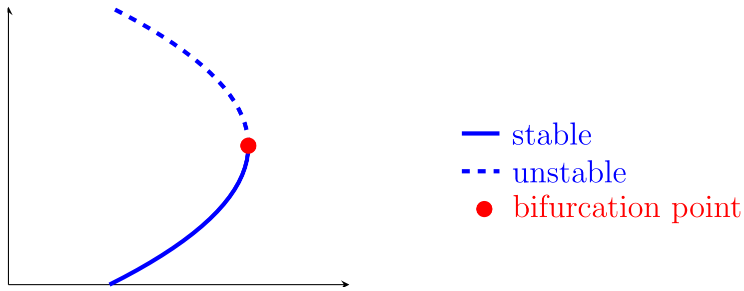
- Pitchfork bifurcation



- Transitional Bifurcation



- Fold/turning-point/saddle-node Bifurcation



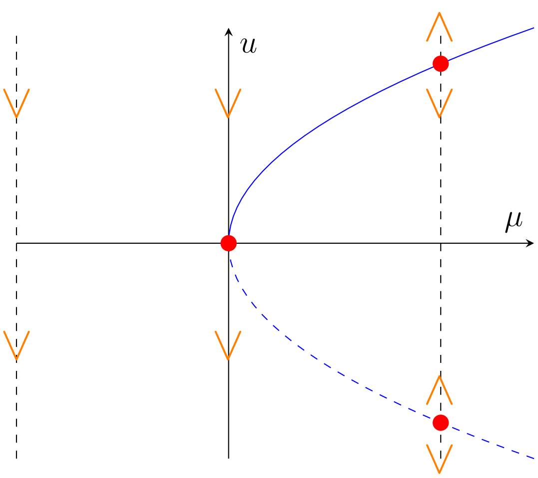
1.7 Feb 5

1.7.1 When do we expect to encounter these bifurcations?

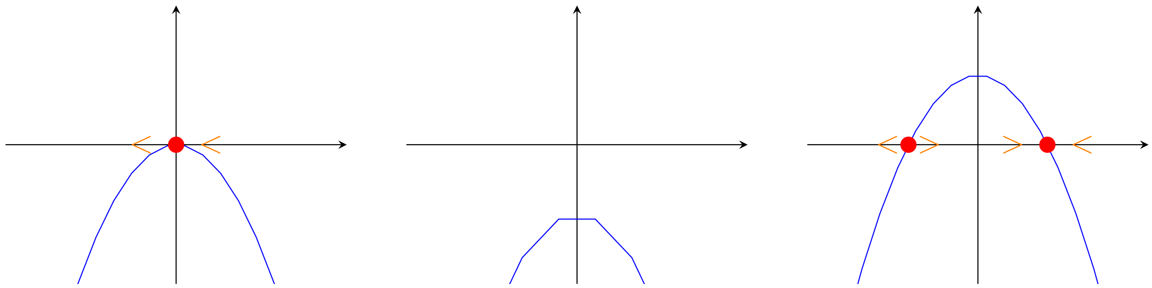
- **Saddle-node Bifurcation:**

$$\begin{cases} f(u_0, \mu_0) = 0 \\ f_u(u_0, \mu_0) = 0 \\ f_{uu}(u_0, \mu_0) \neq 0 \\ f_\mu(u_0, \mu_0) \neq 0 \end{cases}$$

which has prototypical example $\dot{u} = \mu - u^2 = f(u, \mu)$:



and phase diagrams:



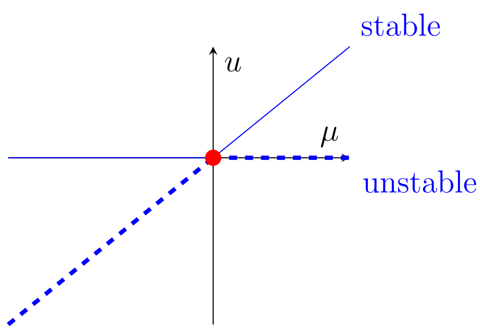
$\mu = 0$ $\qquad$ $\mu < 0$ $\qquad$ $\mu > 0$

In some sense, these are the bifurcations we expect to see most often.

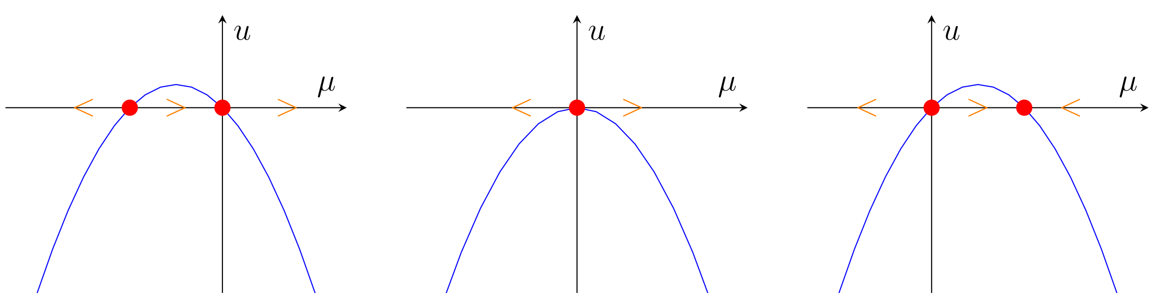
- **Transcritical bifurcation:**

$$\begin{cases} f(0,0) = 0 & \text{existence of equilibrium} \\ f(0,\mu) = 0 \quad \forall \mu & \\ f_u(0,0) = 0 & \text{non-hyperbolic} \\ f_{u\mu}(0,0) \neq 0 & \\ f_{uu}(0,0) \neq 0 & \end{cases}$$

The essential character here is that there always an equilibrium at $u = 0$. Hence, the prototypical example is $\dot{u} = u(u - \mu) = f(u, \mu)$



Which has phase diagrams:



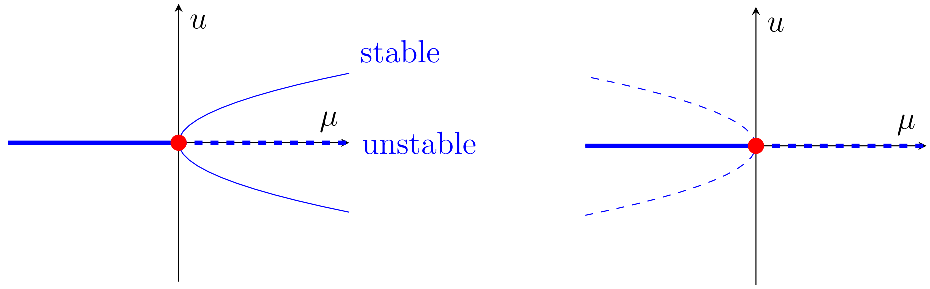
$\mu < 0$ $\qquad$ $\mu = 0$ $\qquad$ $\mu > 0$

- **Pitchfork Bifurcation:**

$$\begin{cases} f(-u,\mu) = -f(u,\mu) \quad \forall (u,\mu) & \text{odd in } u \\ f_u(0,0) = 0 & \text{non-hyperbolic} \\ f_{u\mu}(0,0) \neq 0 & \text{non-degenerate} \\ f_{uuu}(0,0) \neq 0 & \text{non-degenerate} \end{cases}$$

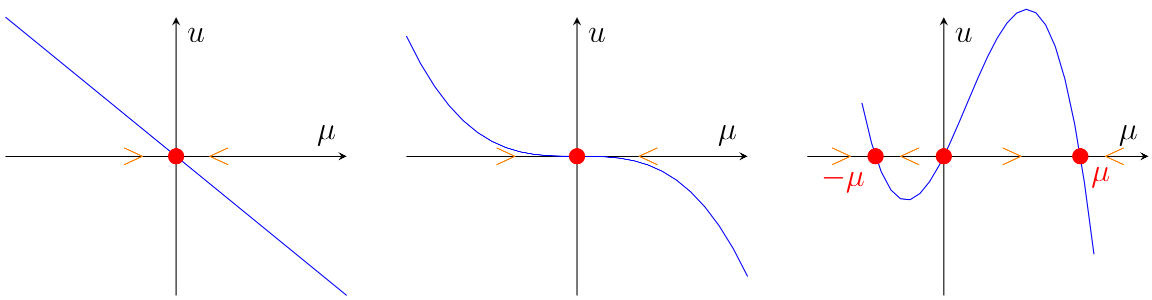
where the first condition comes from the fact that the Taylor Series contains only odd powers of u.

This has two possible bifurcation diagrams:



supercritical   subcritical

The prototypical example is $\dot{u} = u(\mu = u^2) = f(u, \mu)$ which has phase diagrams:



$\mu < 0$ $\mu = 0$ $\mu > 0$

Remark: Transcritical and pitchfork bifurcations only occur for equilibria at 0

What happens if one of the non-degeneracy conditions ($f_\mu(0,0) = 0,\ f_{uu}(0,0) = 0$) is not true? In general, this suggests that the system has another parameter and we might need to consider variations in multiple parameters around the points. In general, this gets very complicated, very fast and we do not yet have a full model.

1.8 Feb 5

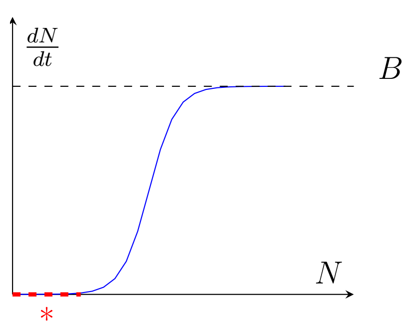
1.8.1 Population model for Budworms

PART 1: Nondimensionalize

Let N be the population density. Hence, as normal,

$$\frac{dN}{dt} = RN(1 - \frac{N}{k}) - \frac{BN^2}{A + N^2}$$

where B is the capacity for predator eating and on (*), the predators search for an alternative food source.



A reasonable first step is to non-dimensionalize the system. We currently have units of

- R: $\frac{1}{\text{time}}$
- K: population
- A: population
- B: $\frac{\text{population}}{\text{time}}$

Hence, let $x = \frac{N}{A}$, so

$$A\dot{x} = ARx(1 - \frac{Ax}{K}) - \frac{Bx^2}{1+x^2}$$

To non-dimensionalize time, we would also like to reduce the parameters. Let $\tau = \frac{B}{A}t$, so

$$\frac{d}{dt} = \frac{d}{d\tau}\frac{d\tau}{dt} = \frac{B}{A}\frac{d}{d\tau}$$

which gives

$$\frac{dx}{d\tau} = \frac{AR}{B}x(1 - \frac{A}{K}x) - \frac{x^2}{1+x^2}$$

Let $a = \frac{AR}{B} > 0$ represent the growth rate and $b = \frac{K}{A} > 0$ represent the carrying capacity. Hence, our final system is

$$\frac{dx}{d\tau} = ax(1 - \frac{x}{b}) - \frac{x^2}{1+x^2}$$

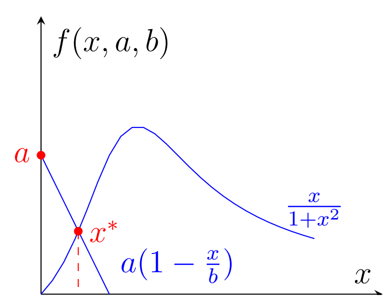
and this looks very familiar.

PART 2: Bifurcation Analysis

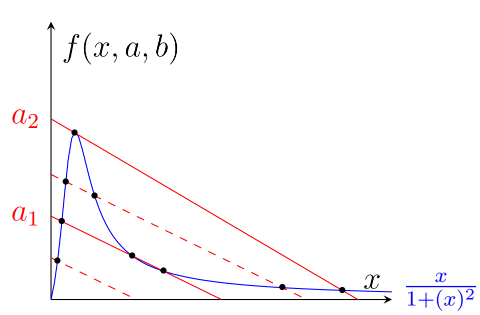
Let $f(x, a, b) = ax(1 - \frac{x}{b}) - \frac{x^2}{1+x^2}$ Clearly, $x = 0$ is always an equilibrium. Also, $f_x(0, a, b) = a > 0$ so $x = 0$ is always unstable.

Now it suffices to consider $f(x, a, b) = a(1 - \frac{x}{b}) - \frac{x}{1+x^2}$

Case 1. $b \ll$, vary a.

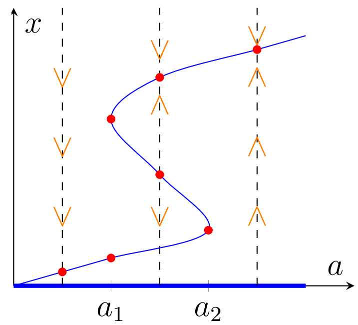


Case 2. $b \gg$, vary a.



a	number of fixed points
$a < a_1$	1
$a = a_1$	2 (saddle)
$a_1 < a < a_2$	3
$a = a_2$	2 (saddle)
$a > a_2$	1

So at last we can draw our bifurcation diagram:



In this diagram, we call the region $[a_1, a_2]$ **bistable** because there are two stable equilibria. Population levels on $[0, a_2)$ represent a "normal" population level, while the node at a_2 represents an insect outbreak when $a > a_2$. At the end of the curve, we say the population is in an "outbreak" population level

Hysteresis: If we start with the budworm population on the lower stable branch and slowly increase the parameter a, the population will remain on the lower branch until $a = a_2$. Beyond this point, the population will jump to the upper branch (at a much higher population level). Then reducing $a < a_2$ will not restore the lower population level since the population will now follow the upper stable branch until it reaches the bifurcation point at a_1.

Chapter 2

Phase Space

2.1 Feb 10

Consider systems of ODEs of the form $\dot{u} = F(u)$ with $u \in \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$ and $F \in C^1/$

Example: $u = (u_1, u_2) \in \mathbb{R}^2$.

$$\begin{cases} \dot{u}_1 = u_1(2 - u_1 - 2u_2) = F_1(u_1, u_2) \\ \dot{u}_2 = u_2(2 - u_1 - u_2) = F_2(u_1, u_2) \end{cases}$$

with $F(u) = F(u_1, u_2) = \begin{pmatrix} F_1(u_1, u_2) \\ F_2(u_1, u_2) \end{pmatrix}$.

$F \in C^1$ since F_1, F_2 are continuously differentiable in (u_1, u_2).

We can calculate the Jacobian

$$F_u(i) = \begin{pmatrix} \frac{\partial F_1}{\partial u_1}(u_1, u_2) & \frac{\partial F_1}{\partial u_2}(u_1, u_2) \\ \frac{\partial F_2}{\partial u_1}(u_1, u_2) & \frac{\partial F_2}{\partial u_2}(u_1, u_2) \end{pmatrix} = \begin{pmatrix} 3 - 2u_1 - 2u_2 & -2u_1 \\ -u_2 & 2 - u_1 - 2u_2 \end{pmatrix}$$

Solution: A function $u : \mathbb{R} \to \mathbb{R}^n$ in C^1 is a *solution* of $\dot{u} = F(u)$ if $\frac{du(t)}{dt} = F(u(t))$ for all $t \in \mathbb{R}$. (Equivalently, we could replace $t \in \mathbb{R}$ by $t \in J$ for some open interval $J \subseteq \mathbb{R}$)

2.1.1 Existence and Uniqueness of Solutions

For

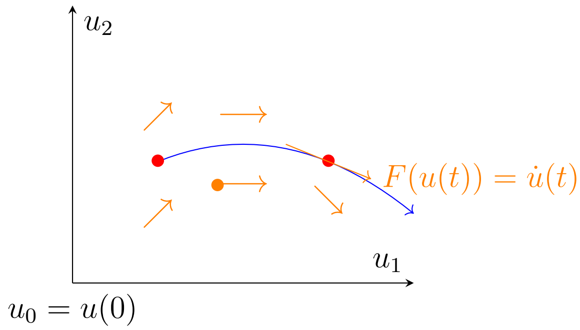
$$\begin{cases} \dot{u} = F(u) \\ u(0) = u_0 \end{cases}$$

let $u \in \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$ in C^1 with initial condition $u_0 \in \mathbb{R}^n$ given.

Theorem (Existence and Uniqueness): Assume $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. For each $u_0 \in \mathbb{R}^n$, there exists a $\delta > 0$ and a unique $u : (-\delta, \delta) \to \mathbb{R}^n$ so that $u \in C^1$ which satisfies the system above for all $t \in (-\delta, \delta)$.

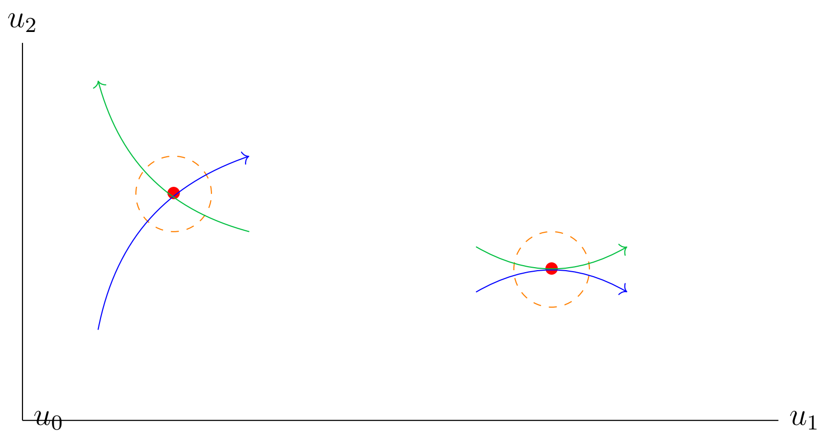
Furthermore, δ can be chosen to depend continuously on u_0 so the map $u_0 \mapsto u(t; u_0)$ is C^1 in u_0 (where $u(t; u_0)$ denotes the unique solution of the system for $t \in (-\delta, \delta)$.)

Consequences: Trajectories $\{u(t) : t \in \mathbb{R}\}$ cannot touch or cross.

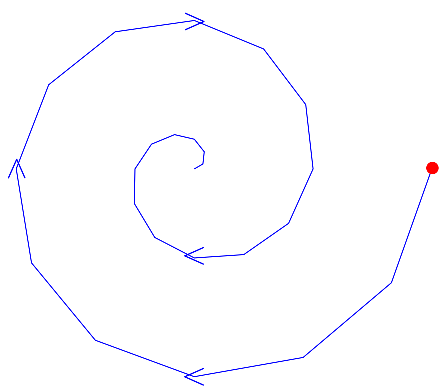


where $\bullet\!\!\longrightarrow$ represents a **vector field** which gives the direction and speed at u.

Example: This is impossible (else the solution through u_0 is not unique)



Planar Systems: uniqueness poses interesting obstacles. For example, how does $u(t)$ evolve as $t \to \infty$?

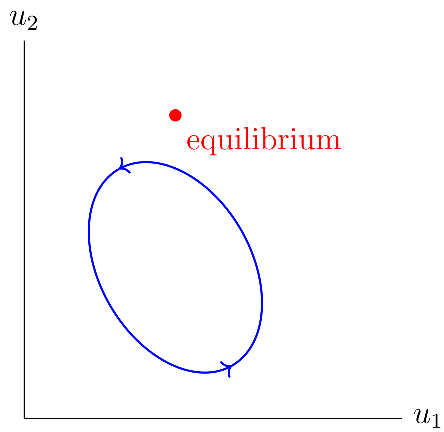


2.1.2 Equilibria, Periodic Orbits, and Heteroclinic Orbits

Let $\dot{u} = F(u)$ with $u \in \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ in C^1.

Equilibria: Each $u_* \in \mathbb{R}^n$ with $F(u_*) = 0$ gives a time-independent solution $u(t) = u_*$ for all t

Periodic Orbits: A solution $u(t)$ is called a *periodic orbit* if there is a $T > 0$ (the period) so that $u(t+T) = u(t)$ for all t and $u(t)$ is not an equilibrium.



Example from ecology (modeling competing species):

- Suppose we have two species occupying the same spatial region and competing for the same food resources
- We can use a logistic model for each species with species-specific growth rates and carrying capacities
- The competition for resources reduces carrying capacity of other species: we assume that this effect is proportional to population size of competing species

For example,

$$\begin{cases} \dot{x} = x(3-x) - 2xy = x(3-x-2y) = f(x,y) \\ \dot{y} = \underbrace{y(2-4)}_{\text{logistic mode}} - \underbrace{xy}_{\text{competition}} = y(2-x-y) = g(x,y) \end{cases}$$

(e.g. x is rabbit population, y is sheep population)

Then the equilibria are given by $(f(x,y), g(x,y)) = (0,0)$:

$$(x,y) = (0,0),\ (0,2),\ (3,0),\ (1,1)$$

And to find stability, we can take a Taylor Expansion near rest state (x_*, y_*):

$$F(x,y) = \underbrace{F(x_*,y_*)}_{0} + F_u(x_*,y_*)\begin{pmatrix} x-x_* \\ y-y_* \end{pmatrix} + \dots$$

and since we have

$$F_u(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(x,y)=(x_*,y_*)} = \begin{pmatrix} 3-2x_*-2y_* & -2x_* \\ -y_* & 2-x_*-2y_* \end{pmatrix}$$

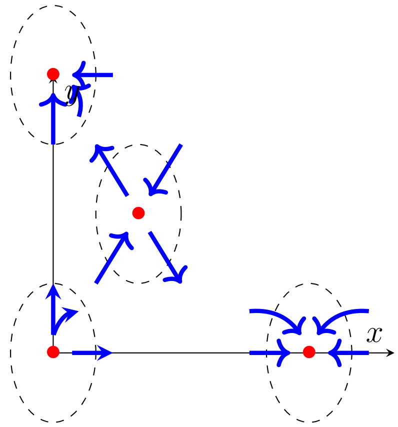
so that

$$F_u(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \qquad \text{eigenvalues } \lambda_{1,2} = 2,3 > 0 \implies \text{unstable}$$

$$F_u(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \qquad \text{eigenvalues } \lambda_{1,2} = -1,-2 < 0 \implies \text{stable}$$

$$F_u(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \qquad \text{eigenvalues } \lambda_{1,2} = -3,-1 < 0 \implies \text{stable}$$

$$F_u(1,1) = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \qquad \text{eigenvalues } \lambda_{1,2} = 1 \pm \sqrt{2} \text{ with } \lambda_1 < 0 < \lambda_2 \implies \text{saddle}$$



Here the x-axis and y-axis are invariant and the behavior around the equilibrium is known from the Jacobian. The behavior everywhere else we can only guess right now.

Can periodic orbits exist? We will see!

2.2 Feb 12

Recall the model

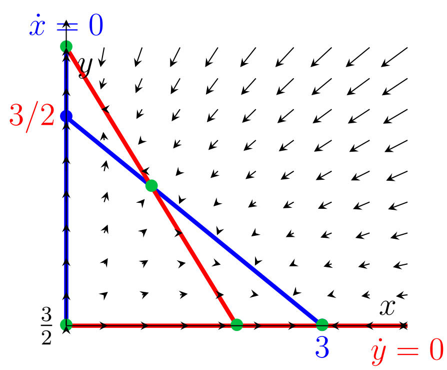
$$\begin{cases} \dot{x} = x(3 - x - 2y) = f(x, y) \\ \dot{y} = y(2 - x - y) = g(x, y) \end{cases}$$

Last time, we just set these equal to 0 and used the Jacobian. As we will see, using nullclines gives us an alternative approach.

Nullclines: The *nullcline* of $f = \dot{x}$ is $\{(x, y) : f(x, y) = 0\} = \{\dot{x} = 0\}$. Similarly, the nullcline of $g = \dot{y}$ is $\{(x, y) : g(x, y) = 0\} = \{\dot{y} = 0\}$.

In the example above, the nullclines are given by

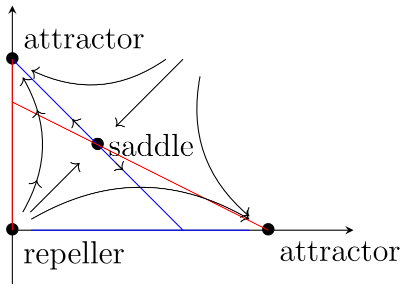
$$\begin{aligned} f : \quad & \{(x, y) : x(3 - x - 27) = 0\} = \{(0, y) : y \in \mathbb{R}\} \cup \{(3 - 2y, y : y \in \mathbb{R})\} \\ g : \quad & \{(x, y) : y(2 - x - y) = 0\} = \{(x, 0) : x \in \mathbb{R}\} \cup \{(x, 2 - x) : x \in \mathbb{R}\} \end{aligned}$$



We notice that these intersect at several points. We are particularly interested in intersections of these curves because they represent $(\dot{x}, \dot{y}) = (0, 0)$ – equilibria!

We can also look at regions created by the curves to consider the signs of $\dot{x}$ and $\dot{y}$, giving us a sense of the direction of the vector field.

This can give us a sense of the full behavior of the system:



Conclusions: in this example, we have two competing species. With our parameter choices, we could see extinction of one species or no stable coexistence

We can do a little more work to analyze the saddle point:

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \implies A_* = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(x,y)=(x_*,y_*)} \implies \lambda_1 < 0 < \lambda_2 \text{ eigenvalues}$$

2.3 Feb 14

2.3.1 Phase Plane Analysis

Goal: Understand the dynamics of $\dot{u} = F(u)$ $(u \in \mathbb{R}^2, F \in C^1)$

Our method is to

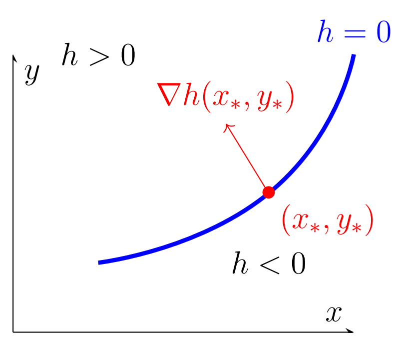
1. Find equilibria (solve $F(u_*) = 0$)
2. Determine their stability using the Jacobian $F_u(u_*)$ and its eigenvalues
3. Compute and plot the nullclines for $F(u) = (F_1(u), F_2(u))$ (find the curves for which $F_i(u) = 0$)
4. Draw phase portrait indicating equilibria, nullclines, and representative solutions

2.3.2 Nullclines (Revisited)

Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function.

We can take the **gradient** $\nabla h(x,y) = \begin{pmatrix} h_x(x,y) \\ h_y(x,y) \end{pmatrix} \in \mathbb{R}^2$.

Suppose we already know the nullcline $\{(x,y) : h(x,y) = 0\}$:



The gradient is perpendicular to nullclines at each point and points in the direction of increasing h.

Now consider

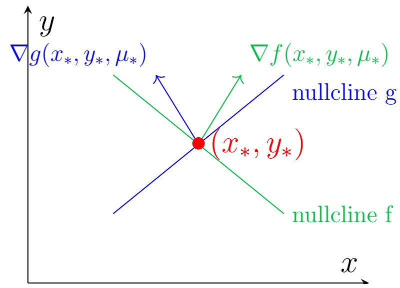
$$\begin{cases} \dot{x} = f(x,y,\mu) \\ \dot{y} = g(x,y,\mu) \end{cases}, \quad f, g \in C^1$$

with equilibrium (x_*, y_*) at $\mu = \mu_*$ so that $f(x_*, y_*, \mu_*) = g(x_*, y_*, \mu_*) = 0$.

Let $\nabla f(x_*, y_*, \mu_*)$ and $\nabla g(x_*, y_*, \mu_*)$ be the gradients of f and g at (x_*, y_*, μ_*).

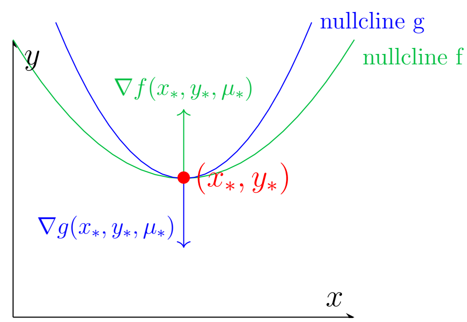
We know

1. ∇f and ∇g are linearly independent



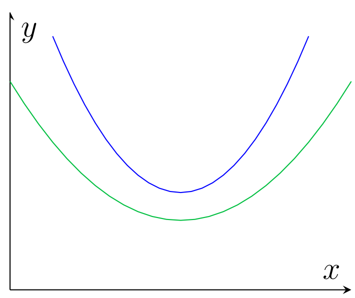
   Since we have continuous differentiability, for all μ near μ_*, we will have a unique equilibrium near (x_*, y_*) with similar nullclines. We say "the equilibrium persists"

2. $\nabla f(x_*, y_*, \mu_*)$ and $\nabla g(x_*, y_*, \mu_*)$ are nonzero and linearly dependent.

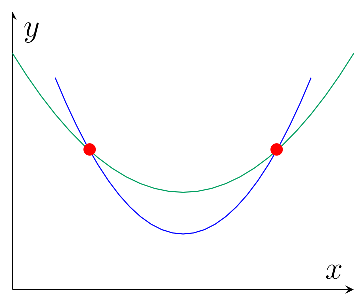


   where the nullclines must be tangent to each other.

   In this case, we have two options for $\mu \neq \mu_*$:



no equilibria



two equilibria

   which tells us that we have a saddle node.

Lemma: Let (x_*, y_*) be an equilibrium of $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x, y, \mu) \\ g(x, y, \mu) \end{pmatrix}$ with Jacobian $A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(x,y)=(x_*,y_*)}$, then A has an eigenvalue at 0 iff $\nabla f(x_*, y_*)$ and $\nabla g(x_*, y_*)$ are linearly dependent.

Proof:

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix}, \quad \nabla g = \begin{pmatrix} g_x \\ g_y \end{pmatrix} \implies A = \begin{pmatrix} \nabla f^T \\ \nabla g^T \end{pmatrix}$$

Hence, it suffices to show A has an eigenvalue at 0 iff $\det A = 0$.

In the plane, the determinant corresponds to the area of the parallelogram spanned by two vectors. Hence, $\det A = 0$ implies that ∇f and ∇g are linearly dependent and vice versa. ■

Remark: For $u \in \mathbb{R}$, we saw the condition for a bifurcation at u_* was $f_u(u_*) = 0$. For $u \in \mathbb{R}^2$, meanwhile, the condition for bifurcation is that the Jacobian has an eigenvalue at 0.

2.3.3 Application (Autocatalytic gene-protein interaction)

Let x be a protein P and y be a gene G where

1. the gene G codes for protein P
2. the protein P upregulates the gene G

$$\begin{cases} \dot{x} = -ax + y = f(x,y) \\ \dot{y} = \frac{x^2}{1+x^2} - \frac{y}{2} = g(x,y) \end{cases}$$

Here, if $y > 0$, the gene is active and produces protein (interaction 1). In the first equation, the protein also naturally degrades at rate a.

In the second equation, the first term models the upregulation of the gene by the protein (interaction 2) and the second models the gene switching off.

Let us now do the phase-plane analysis, focusing on the nullclines.

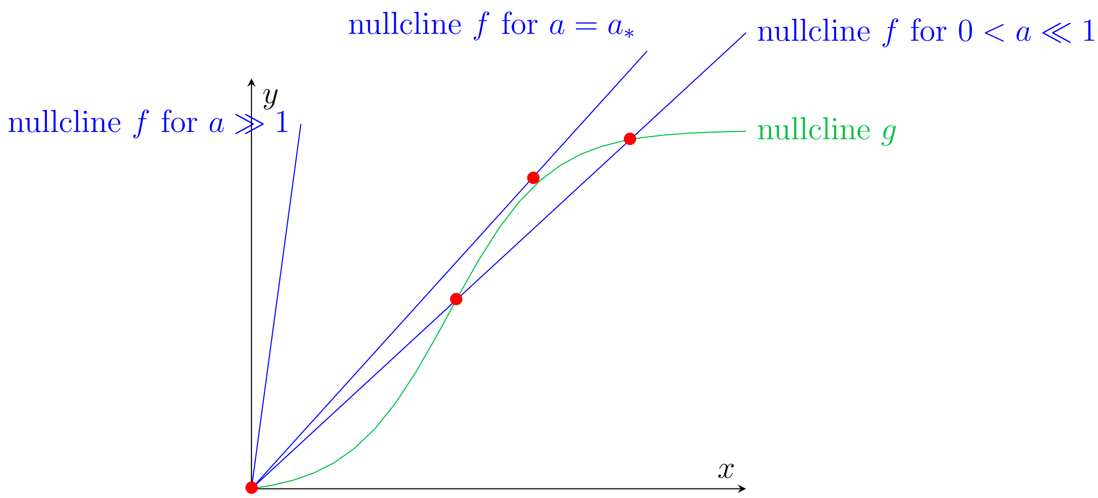
The nullcline of f is given by

$$\{(x,y) : -ax + y = 0\} = \{(x, ax) : x \in \mathbb{R}\}$$

and the nullcline of g is given by

$$\{(x,y) : \frac{x^2}{1+x^2} - \frac{y}{2} = 0\} = \{(x, \frac{2x^2}{1+x^2}) : x \in \mathbb{R}\}$$

We can plot:



Our next step is to fix $a < a_*$ and determine stability.

2.4 Feb 19

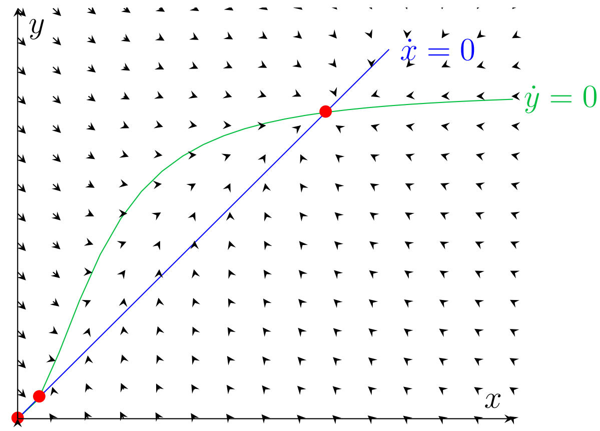
Recall: Last time we began studying the system

$$\begin{cases} \dot{x} = -ax + y \\ \dot{y} = \frac{x^2}{1+x^2} - \frac{y}{2} \end{cases}, \quad a > 0$$

which models the interaction between a gene and a protein.

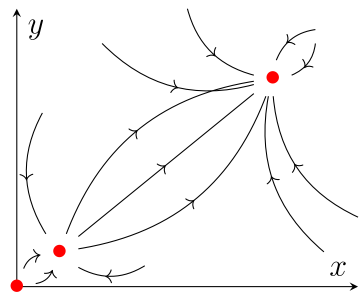
Last time, we found nullclines $y = ax$ and $y = \frac{2x^2}{1+x^2}$. Plotting for different values of a, we found a saddle node bifurcation at $a = a_*$.

Today, we will focus on the case $a < a_*$:

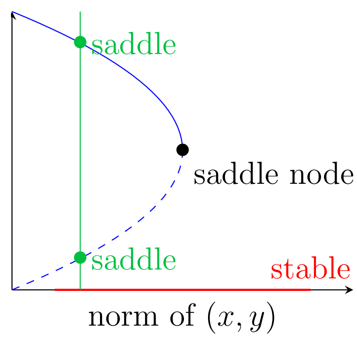


$0 < a < a_*$

which gives the following phase portrait for a fixed:



and bifurcation diagram:



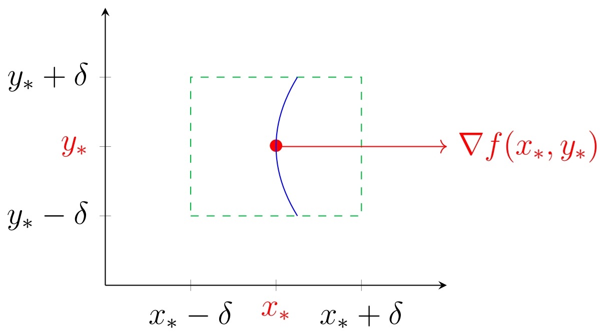
2.4.1 Implicit Function Theorem

Theorem (IFT): Consider $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$ with $f \in C^k$ for some $k \geq 1$. Assume $f(x_*, y_*) = 0$.

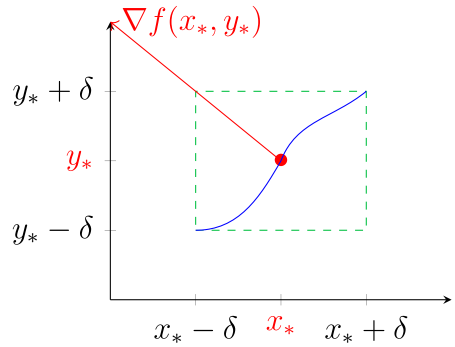
We then have:

1. If $f_x(x_*, y_*) \neq 0$, then there $\exists \delta > 0$ and $\exists! g : (y_* - \delta, y_* + \delta) \to \mathbb{R}$ with $x_* = g(y_*)$ so that $f(x, y) = 0$ for $(x, y) \in (x_* - \delta, x_* + \delta) \times (y_* - \delta, y_* + \delta)$ iff $x = g(y)$.
2. If $f_y(x_*, y_*) \neq 0$, then there $\exists \delta > 0$ and $\exists! h : (x_* - \delta, x_* + \delta) \to \mathbb{R}$ with $y_* = h(x_*)$ so that $f(x, y) = 0$ for $(x, y) \in (x_* - \delta, x_* + \delta) \times (y_* - \delta, y_* + \delta)$ iff $y = h(x)$.
3. g and h are C^k functions.

We can graph these two cases:



$f_x(x_*, y_*) \neq 0$



$f_y(x_*, y_*) \neq 0$

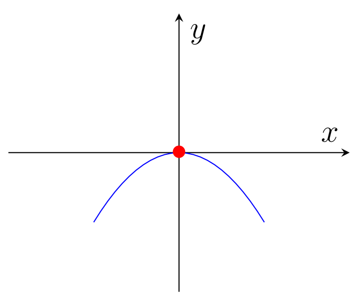
Example: Find all zeros of

$$f(x, y) = y + y^2 e^x + (\sin x)^2 - xy$$

near $(x_*, y_*) = (0, 0)$.

Remark: This is somewhat of a bad example because f is quadratic in y so we can solve for y explicitly. However, we will use the IFT to find the same result.

1. Check conditions
   - f can be differentiated as often as we need ✓
   - $f(0, 0) = 0$ ✓
   - $f_x(0, 0) = (y^2 e^x + 2 \sin x \cos x - y)\Big|_{(0,0)} = 0$ ×
   - $f_y(0, 0) = 1 + 2ye^x - x\Big|_{(0,0)} \neq 0$ ✓
2. Hence we can apply IFT Part 2 and conclude:
   - There exists $\delta > 0$ and $g : (-\delta, \delta) \to \mathbb{R}$ with $g(0) = 0$ so that $f(x, y) = 0$ for $|x|, |y| < \delta$ iff $y = g(x)$ and $g \in C^\infty$
3. We can Taylor Expand g:
   - $g(x) = g(0) + g'(0)x + O(x^2) = g'(0)x + O(x^2) = O(x)$
   - With a little more work, we can show that $g(x) = -x^2 + O(x^3)$



Remark: In the above example, we found $f_x(0, 0) \neq 0$. This does *not* mean a function satisfying $x = g(y)$ does not exist. It just means that the IFT does not apply.

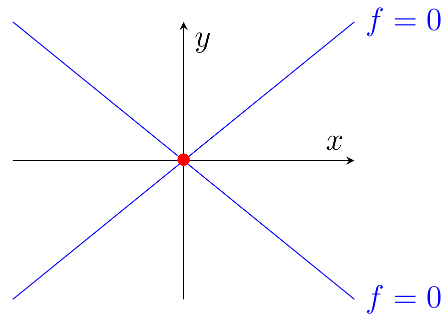
Example: Find all zeros of $f(x, y) = x^2 - y^2$ near $(0, 0)$.

1. Check conditions:
   - $f \in C^\infty$ ✓
   - $f(0, 0) = 0$ ✓

- $f_x(0,0) = 2x\Big|_{(0,0)} = 0$ $\times$

- $f_y(0,0) = -2y\Big|_{(0,0)} = 0$ $\times$

But now we can't apply the IFT. However, we can still solve this problem by hand. We have $x^2 = y^2$ so $|x| = |y|$:



And it makes sense that the IFT does not apply here as any map satisfying this graph could not be a function.

Example (Intuition): Take $ax + by = 0$. If we want to solve for x,

$$x = -\frac{b}{a}y \implies a \neq 0 \implies a = f_x(0,0) \neq 0$$

which can help keep the assumptions versus conclusions straight.

2.5 Feb 21

2.5.1 Implicit Function Theorem (Examples)

Example: Find the zeros of $f(x,y) = e^x - 1 + y\cos x + \sin^2 y$.

As always, we first check the conditions:

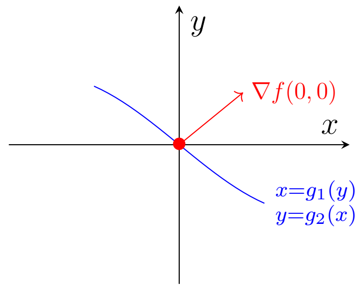
- $f \in C^\infty$ ✓
- By inspection, $f(0,0) = 0$. ✓
- $f_x(0,0) = [e^x - y\sin x]_{(0,0)} = 1 \neq 0$ ✓
- $f_y(0,0) = [\cos x + 2\sin y\cos y]_{(0,0)} = 1 \neq 0$ ✓

so we can apply IFT in both variables.

Specifically, $f(x,y) = 0$ iff $x = g_1(y)$ and $y = g_2(x)$ for some $g_1, g_2 \in C^\infty$.

Now we can make a few observations that will help us plot:

- $f(0,0) = 0$
- $\nabla f(0,0) = (1,1) \perp \{f = 0\}$ which gives us a tangent line at $(0,0)$. And in fact, this gives the curve of zeros for both x and y.



$$\{(x,y) : f(x,y) = 0\}$$

Example: Find all zeros of $f(x,y) = x^3 - y^3$ near $(0,0)$.

We check the conditions:

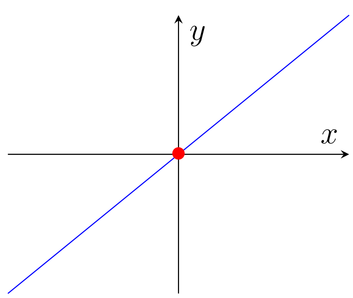
- $f \in C^\infty$
- $f(0,0) = 0$
- $f_x(0,0) = 0$
- $f_y(0,0) = 0$

so the IFT does not apply.

But we can still solve this by hand!

$$f(x,y) = x^3 - y^3 \iff x^3 = y^3 \overset{*}{\iff} x = y$$

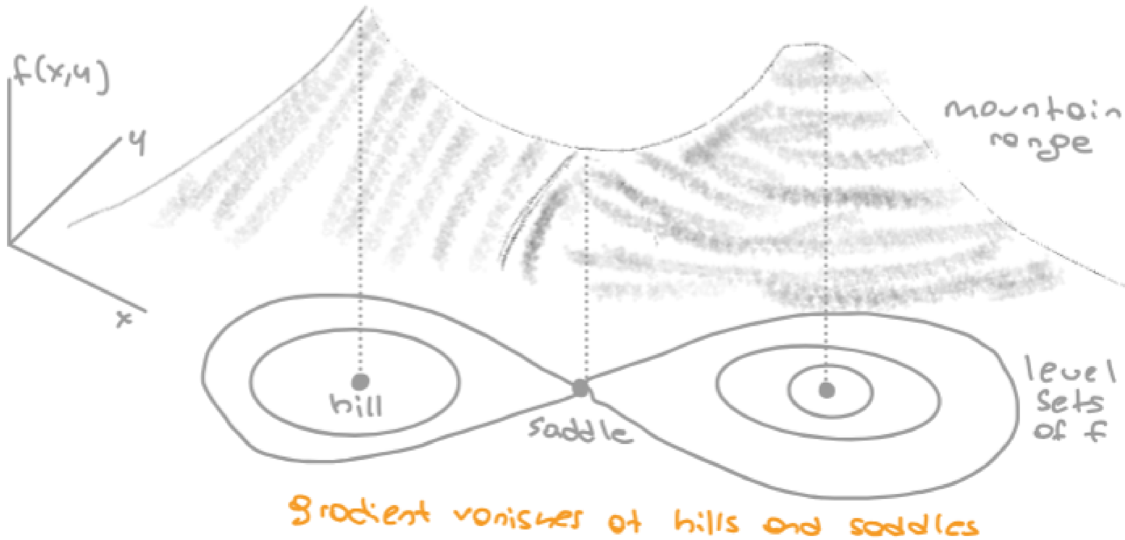
(note that the $(*)$ equality works because there is no ambiguity in signs).



$$\{(x,y) : x = y\} = \{(x,y) : f(x,y) = 0\}$$

2.5.2 Applications

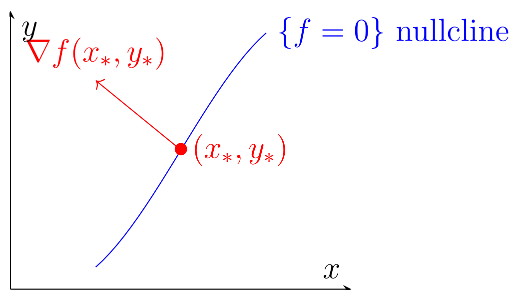
Topography and Intuition: Imagine a mountain range with two peaks. Let $f(x,y)$ represent a point on the mountain range. In between the two peaks, we have a saddle point. We can project the level sets of the height function down to the xy-plane to get a contour map.



In this case, the IFT applies everywhere except valleys, hills, and saddles – $\nabla f(x,y) = 0$.

Nullclines: Consider a nullcline $\{(x,y) : f(x,y) = 0\}$

If $f(x_*, y_*) = 0$ and $\nabla f(x_*, y_*) \neq 0$, then as we have seen, the nullcline is (locally) a curve.



and the IFT makes this rigorous:

$$\nabla f(x_*, y_*) = \begin{pmatrix} f_x(x_*, y_*) \\ f_y(x_*, y_*) \end{pmatrix} \neq 0$$

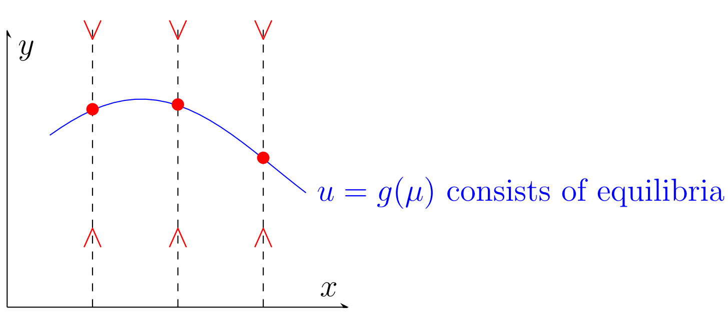
so we really have the curve as the IFT applies.

Hyperbolic Equilibria: Consider a system $\dot{u} = f(u, \mu)$ with $u, \mu \in \mathbb{R}$ and

- $f(u_*, \mu_*) = 0$, i.e. u_* is an equilibrium for $\mu = \mu_*$
- $f_u(u_*, \mu_*) \neq 0$, i.e. u_* is hyperbolic

Here, the IFT guarantees that $\exists g \in C^k$ with $g(\mu_*) = u_*$ so that $f(u,\mu) = 0$ for (u, μ) near (u_*, μ_*) iff $u = g(\mu)$.

In this case, we say "u_* persists for μ near μ_*"



2.5.3 Multidimensional IFT

Theorem: Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ in C^k for some $u \geq 1$. Assume $f(x_*, y_*) = 0$.

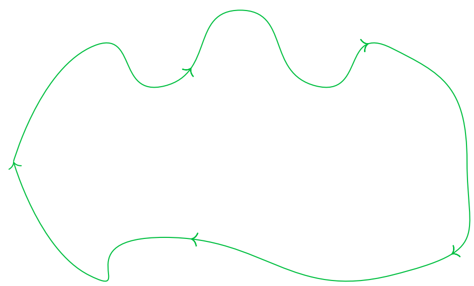
If the Jacobian

$$f_x(x_*, y_*) = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1:n} \in \mathbb{R}^{n\times n}$$

is invertible (so has nonzero determinant), then $\exists g : \mathbb{R}^m \to \mathbb{R}^n$ in C^k with $g(y_*) = x_*$ so that $f(x, y) = 0$ for (x, y) near (x_*, y_*) iff $x = g(y)$.

2.5.4 Periodic Orbits

Definition: Let $\dot{u} = f(u)$. A solution $u(t)$ is periodic if there is a $T > 0$ so that $u(t + T) = u(t)$ for all t (and u is not an equilibrium)

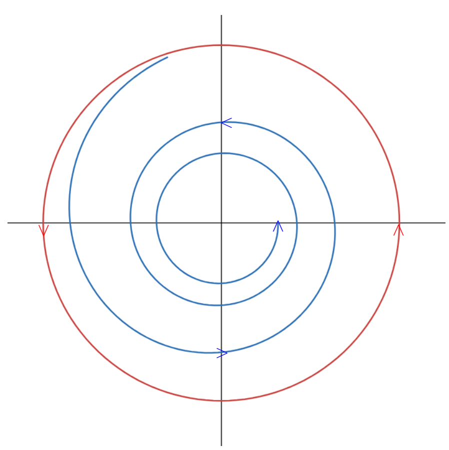


In the following days, we will do the following:

1. Index theory to exclude periodic orbits
2. Poincare-Bendixson Theorem and trapping regions to find large amplitude periodic orbits
3. Hopf bifurcation theorem to find small amplitude periodic orbits

Remark: 1-2 work only in $\mathbb{R}^2$ but the Hopf bifurcation theorem works in $\mathbb{R}^n$.

Index Theory:



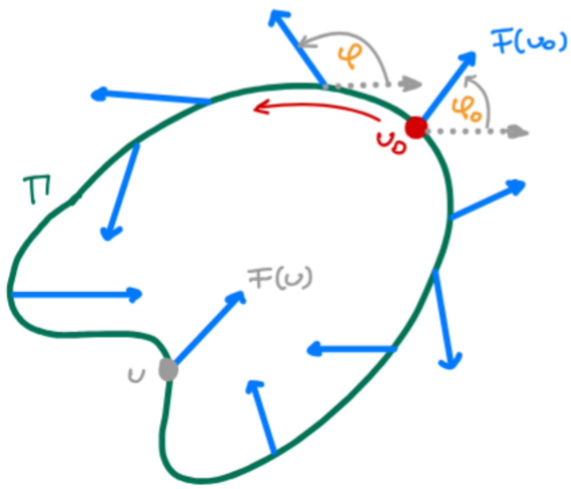
We conjecture that the interior of the periodic orbit contains an equilibrium (most likely an attractor)

2.6 Feb 24

2.6.1 Index Theory

Definition: Pick a continuous closed curve without self-intersections that does not pass through any equilibria. We call this curve a *simple closed curve* or *simple loop*

Example: Let Γ be a simple closed loop.



Starting at some vector, say $(1,0)$, let ϕ_0 represent the angle to the next vector along the curve. The number of full rotations we make around the curve is the *index* of the curve.

Specifically,

1. Pick any $u_0 \in \Gamma$ and compute the angle ϕ_0 of $F(u_0)$ with $(1,0)$
2. Traverse the curve Γ counterclockwise and trace the angle ϕ of $F(u)$ with $(1,0)$ at each $u \in \Gamma$.
3. ϕ will vary continuously as u traverses Γ since Γ and F are continuous.
4. We traverse Γ exactly once and record the final angle ϕ_1 upon returning to u_0.
5. *Key Observation:* $\phi_1 = \phi_0 + 2\pi n$ for $n \in \mathbb{Z}$. This n is the index.

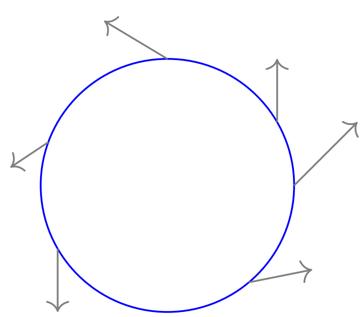
Definition: The *index* of the simple closed curve Γ (for the vector field F) is given by

$$I_\Gamma = \frac{\phi_1 - \phi_0}{2\pi} \in \mathbb{Z}$$

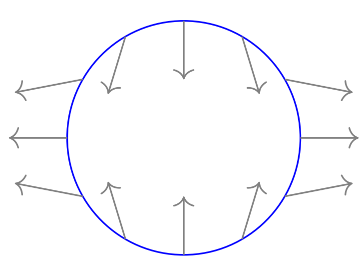
which represents the number of net counterclockwise rotations of $F(u)$ as u traverses Γ once in the counterclockwise direction.

Examples:

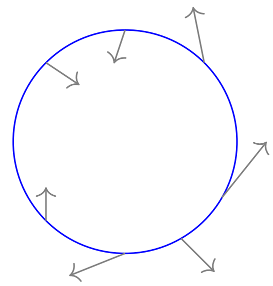
1. $I_\Gamma = 1$



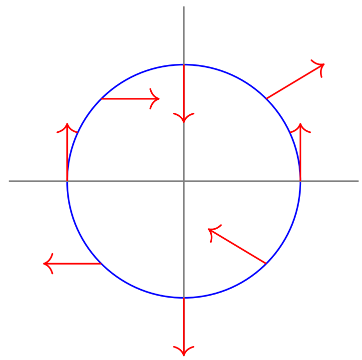
2. $I_\Gamma = -1$ (since the rotation is clockwise)



3. $I_\Gamma = 2$

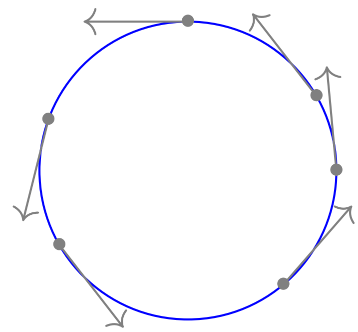


4. $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x^2 y \\ x^2 - y^2 \end{pmatrix}$ with Γ the unit circle.

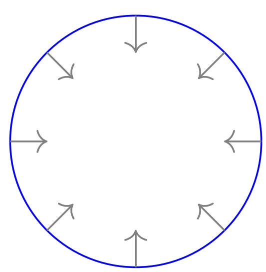


leads to $I_\Gamma = 0$

5. In the following, Γ is a periodic orbit of $\dot{u} = F(u)$ so $F(u)$ is tangent to Γ at all points.



6. $I_\Gamma = 1$



Remark: It is not too difficult to convince ourselves that every periodic orbit has $I_\Gamma = 1$.

2.7 Feb 26

Recall: For $\dot{u} = F(u),\ u \in \mathbb{R}^2$, Γ is a *simple closed loop* if it is

- continuous
- has no self-intersections

- does not pass through any equilibria

Last time, we calculated the *index*, $I_\Gamma \in \mathbb{Z}$, of various simple closed loops (the net number of counterclockwise rotations of $F(u)$ as u traverses Γ once in the counterclockwise direction).

Properties of the Index:

(i) If we continuously deform a simple closed loop Γ to another simple closed loop $\tilde{\Gamma}$ without introducing self-intersections and without passing through equilibria during the deformation, then $I_\Gamma = I_{\tilde{\Gamma}}$.

(ii) If we deform F continuously without creating any equilibria on Γ during the deformation, then $I_\Gamma(F) = I_\Gamma(\tilde{F})$.

(iii) If Γ does not contain any equilibria in its interior, then $I_\Gamma = 0$.

(iv) If Γ is a periodic orbit of $\dot{u} = F(u)$, then $I_\gamma = 1$

(v) If we replace $F(u)$ by $-F(u)$ (e.g. reverse time), then the index stays the same.

Proof:

(i) and (ii). Contradiction via continuity of ϕ_0 and $\frac{\phi_1 - \phi_0}{2\pi} \in \mathbb{Z}$

(iii). Can shrink Γ continuously to a point in its interior

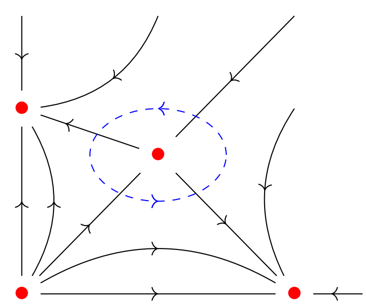
(iv). Deform to a circle and then use the fact that $F(u)$ is tangent to Γ at all points.

(v). The transformation changes each ϕ to $\phi + \pi$ so in the end, all π's cancel

Theorem: If Γ is a periodic orbit of $\dot{u} = F(u)$ with $u \in \mathbb{R}^2$, then there exists at least one equilibrium in the interior of Γ.

Proof: Follows trivially from (iii) and (iv) above

2.7.1 Competing Species Revisited



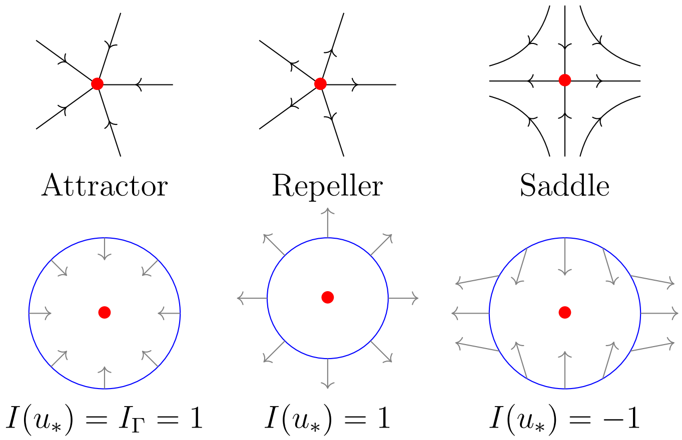
We know from the above that the blue dashed region is the only candidate for a periodic orbit. We will now use index theory to show that there also cannot be a periodic orbit here.

2.7.2 Isolated Equilibria

Isolated Equilibrium: For $\dot{u} = F(u)$, let u_* be an equilibrium. We say it is *isolated* if $F(u) \neq 0$ for all u near u_* (with $u \neq u_*$)

Definition: Assume u_* is an isolated equilibrium. Choose a simple closed loop Γ so that its interior contains only one equilibrium, u_*. We define $I(u_*) = I_\Gamma$. (And this is well-defined by property (i))

We now categorize the indices of various equilibria:



Consequence: If Γ is a periodic orbit that contains a single equilibrium u_* in its interior, then u_* cannot be a saddle ($I(u_*) = -1$ but $I_\Gamma = 1$)

2.8 Feb 28

Recall: Last time, we were considering the case $\dot{u} = F(u)$ with $u \in \mathbb{R}^2$ and $f \in C^1$.

We proved two key results:

Theorem: Inside each periodic orbit, there is at least one equilibrium.

Theorem: Assume u_* is an isolated equilibrium (no other equilibria near u_*). Define

$$\begin{aligned} I(u_*) &= \text{index of} u_* \\ &= \text{index } I_\Gamma \text{ of a simple closed loop containing only } u_* \end{aligned}$$

Then, necessarily, $I_\Gamma = I(u_*) = 1$.

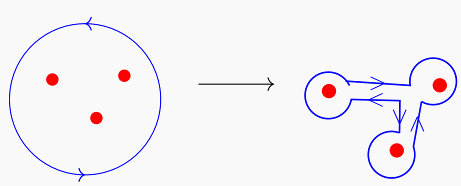
Examples: Assume u_* is a hyperbolic equilibrium of $F(u)$ with Jacobian $A = F_u(u_*)$ and eigenvalues λ_1, λ_2.

- If $\lambda_1, \lambda_2 < 0$ (attractor), then $I(u_*) = 1$
- If $\lambda_1, \lambda_2 > 0$ (repeller), then $I(u_*) = 1$
- If $\lambda_1 > 0 > \lambda_2$ (saddle), then $I(u_*) = -1$

Theorem: Assume Γ is a simple closed loop that encircles exactly n equilibria $u_1, \ldots, u_n$. Then

$$I(\Gamma) = \sum_{j=1}^{n} I(u_j)$$

Proof:



But any contribution from one path of one of the arms is precisely cancelled out by the other path of the arm. So the net contribution is 0.

By a theorem earlier,

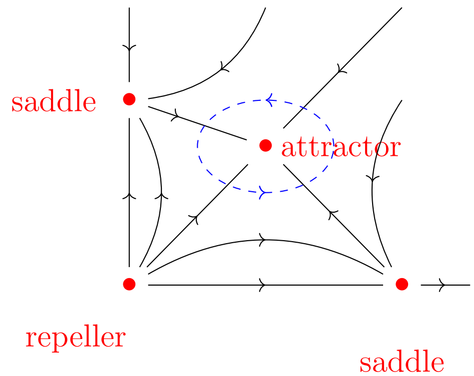
$$I_\Gamma = I_{\tilde{\Gamma}} = \sum_{j=1}^{n} I(u_j)$$

Consequence: If a periodic orbit encloses exactly n equilibria, then their indices must sum to 1.

Example: (Competing Species)

If there is a periodic orbit around the saddle point, then to make the indices match, we must have another equilibrium inside the periodic orbit. But in this case, the periodic orbit must cross the x or y axes, which is not possible by existence and uniqueness.

The only option we could not rule out on the basis of index theory was a situation like



2.9 March 3

2.9.1 Poincare-Bendixson Theorem and Trapping Regions

Remark: The following discussion works only in $\mathbb{R}^2$.

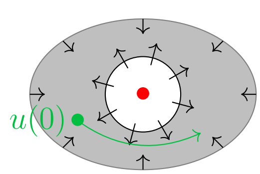
Poincare-Bendixson Theorem: Consider $\dot{u} = f(u)$ with $u \in \mathbb{R}^2$ and $f \in C^1$. Assume R is a closed, bounded subset of $\mathbb{R}^2$ so that

- R does not contain any equilibria
- There exists $u(0) \in R$ so that $u(t) \in R$ for all $t \geq 0$.

Then R contains a periodic orbit of $\dot{u} = F(u)$.

Proof: Not so easy

Typical Application: construct a trapping region R shaped like an annulus



The strategy is to ensure R is forward invariant so all solutions with $u(0) \in R$ remain in R for all $t \geq 0$.

Application (Selkov 1968): Model glycolysis in yeast cell.

$$\begin{cases} \dot{x} = -x + ay + x^2 y = f(x,y) & \text{concentration of ADP} \\ \dot{y} = b - ay - x^2 y = g(x,y) & \text{concentration of IGP (fructose)} \end{cases}$$

where $a, b > 0$ and a is an enzyme concentration and b is the fructose intake.

First, we find equilibria:

$$\begin{array}{rl} 0 & = -x + ay + x^2 y \\ 0 & = b - ay - x^2 y \\ \hline 0 & = -x + b \end{array}$$

Hence $x = b$ and we can substitute into any of these equations to get $y = \frac{b}{a+b^2}$, implying there is a unique equilibrium at $(x_*, y_*) = (b, \frac{b}{a+b^2})$.

Now, we evaluate stability:

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -a - x^2 \end{pmatrix}$$

which yields

$$J\left(b, \frac{b}{a+b^2}\right) = \begin{pmatrix} -1 + \frac{2b^2}{a+b^2} & a + b^2 \\ -\frac{2b^2}{a+b^2} & -(a+b^2) \end{pmatrix}$$

We could just calculate eigenvalues here, but there is an easier way!

Recall: If λ_1, λ_2 are eigenvalues of A then

$$\begin{aligned} \det A &= \lambda_1 \lambda_2 \\ \operatorname{tr} A &= \lambda_1 + \lambda_2 \end{aligned}$$

Hence, we can calculate

$$\begin{aligned} \det J(x_*, y_*) &= \left(-1 + \frac{2b^2}{a+b^2}\right)\left(-(a+b^2)\right) + \frac{2b^2}{a+b^2}\left(a+b^2\right) \\ &= a + b^2 - 2b^2 + 2b^2 \\ &= a + b^2 \\ \operatorname{tr} A &= -1 + \frac{2b^2}{a+b^2} - a - b^2 \end{aligned}$$

Since $\det J > 0$, we conclude $\operatorname{Re}\lambda_1$ and $\operatorname{Re}\lambda_2$ have the same sign (or $\lambda = \pm ic$ for $c \neq 0$) so we cannot have a saddle.

Now, we consider the trace:

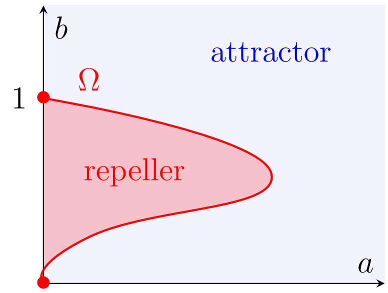
$$\begin{cases} \text{repeller} & \operatorname{tr} J > 0 \\ \lambda = \pm ic & \operatorname{tr} J = 0 \\ \text{attractor} & \operatorname{tr} J < 0 \end{cases}$$

Let's plot the region where

$$\begin{aligned} \operatorname{tr} A &= -1 - a - b^2 + \frac{2b^2}{a+b^2} = 0 \\ &= -(a+b^2) - (a+b^2)^2 + 2b^2 = 0 \\ &= -a - b^2 - a^2 - 2ab^2 - b^4 + 2b^2 = 0 \\ &= a^2 + a(1+2b^2) + b^2(b^2-1) = 0 \end{aligned}$$

which is nicely a quadratic in a.

Notice, for $a = 0$ we need $b = \{0, 1\}$ (assuming $b > 0$):

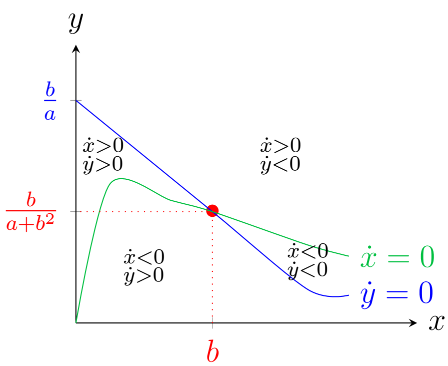


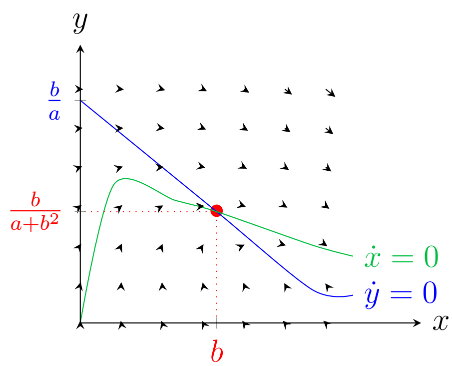
so the region $\Omega = \{(a,b) : \text{tr } > 0\}$ is a repeller where the rest of the plane is an attractor.

We can now plot nullclines:

$$f(x,y) = 0 \implies -x + ay + x^2 y = 0 \implies y = \frac{x}{a+x^2}$$

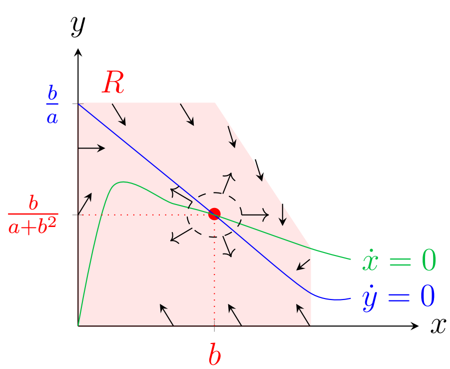
$$g(x,y) = 0 \implies b - ay - x^2 y = 0 \implies y = \frac{b}{a+x^2}$$





and now we can construct our trapping region.

We choose a slope so that arrows cross from upper-right to lower-left.



We want $\dot{x} + \dot{y} = b - x$ so let us try slope -1: it suffices to show

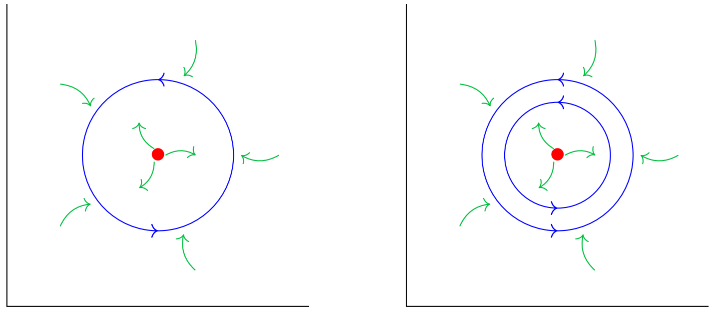
$$\frac{\dot{y}}{\dot{x}} < 1 \implies \dot{y} < -\dot{x} \implies \dot{x} + \dot{y} < 0 \implies b - x < 0 \implies x > b$$

The region R satsifes the condition of the Poicare-Bendixson theorem:

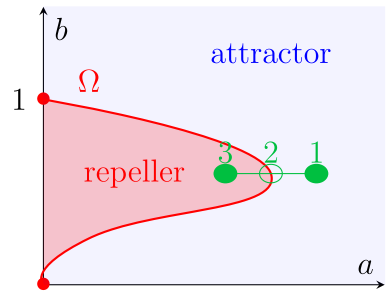
- R closed, bounded ✓
- R does not contain any equilibria ✓
- For all $u(0) \in R$ we know that $u(t) \in R$ for $t \geq 0$ ✓

We conclude that our model has a periodic orbit in R for each $(a,b) \in \Omega$

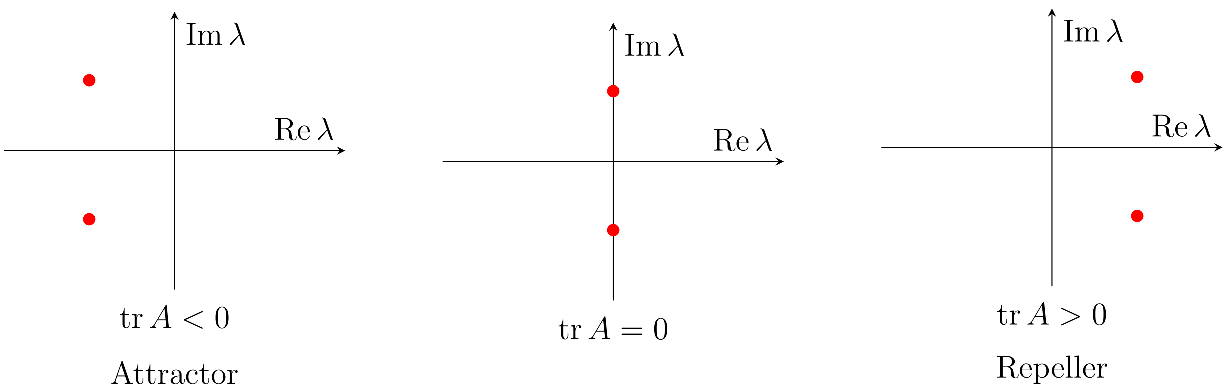
Remark: This theorem makes no statement about the number of periodic orbits or thier stability:



Let us return to the region in parameter space



where the eigenvalues of the Jacobian can satisfy



$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

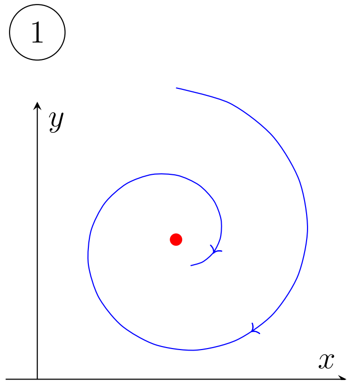
We have eigenvalues $\lambda = \pm i$ and we can check

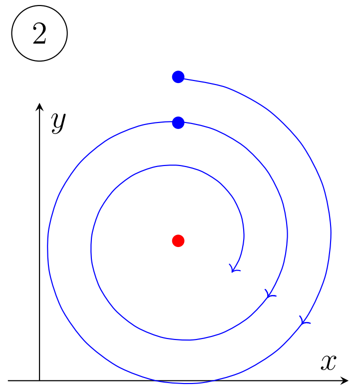
$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases} \implies \ddot{x} = \dot{y} = -x \implies \ddot{x} = -x \implies x(t) = \cos t$$

and similarly,

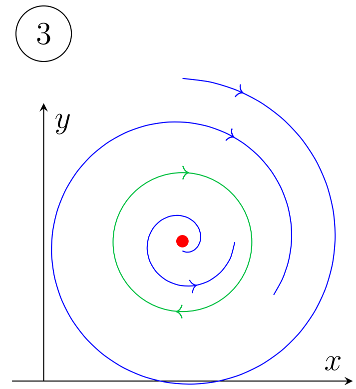
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

So





Assumption: Nonlinear terms stabilize equilibrium

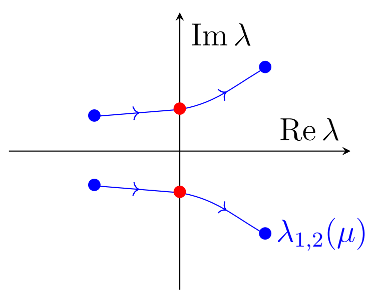


Here we have a small amplitude periodic orbit

2.10 March 10

2.10.1 Hopf Bifurcations of Equilibria

1. Consider $\dot{u} = F(u, \mu)$ with $u \in \mathbb{R}^2$, $\mu \in \mathbb{R}$ for $F \in C^3$.
2. Assume that u_* is an equilibrium for all μ near $\mu = \mu_*$ so that $F(u_*, \mu) = 0$ for μ near μ_*
3. Denote the eigenvalues of $F_u(u_*, \mu)$ by $\lambda_{1,2}(\mu)$ then we assume $\lambda_{1,2}(\mu) = \alpha(\mu) \pm i\beta(\mu)$ where $\alpha(\mu_*) = 0$, $\frac{d\alpha}{d\mu}(\mu_*) \neq 0$ and $\beta(\mu_*) \neq 0$.

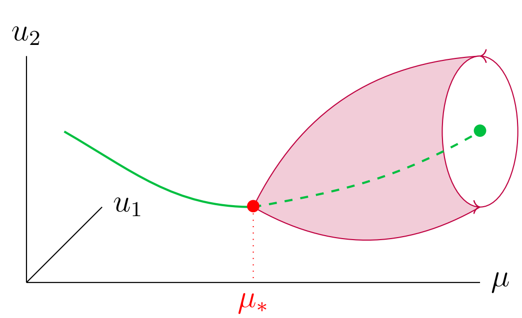


(Assumes $\alpha'(\mu_*) > 0$)

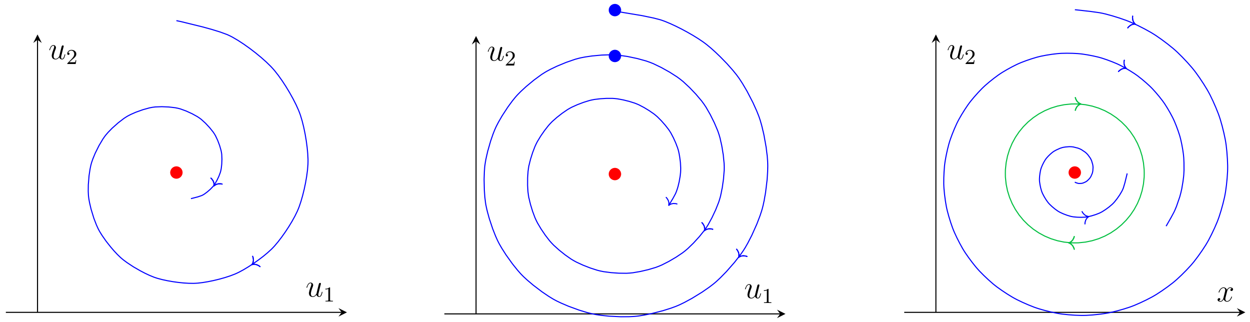
Remark: the assumption that u_* does not depend on μ can be relaxed as we will see.

If (1)-(3) hold, then typically, one of the two scenarios will occur:

(a) **Supercritical Hopf Bifurcation:** We have a stable periodic orbit with period $\approx \frac{2\pi}{\beta(\mu_*)}$ and amplitude $\approx \sqrt{\mu - \mu_*}$.



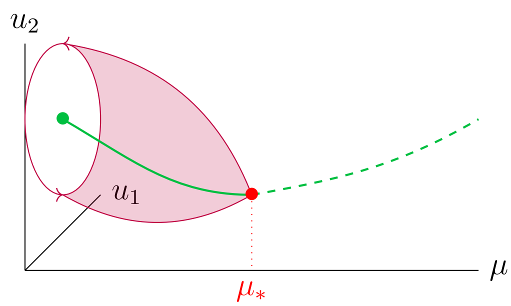
Now these graphs we saw last week have context:



Attractor | Attractor | Repeller + Stable Periodic

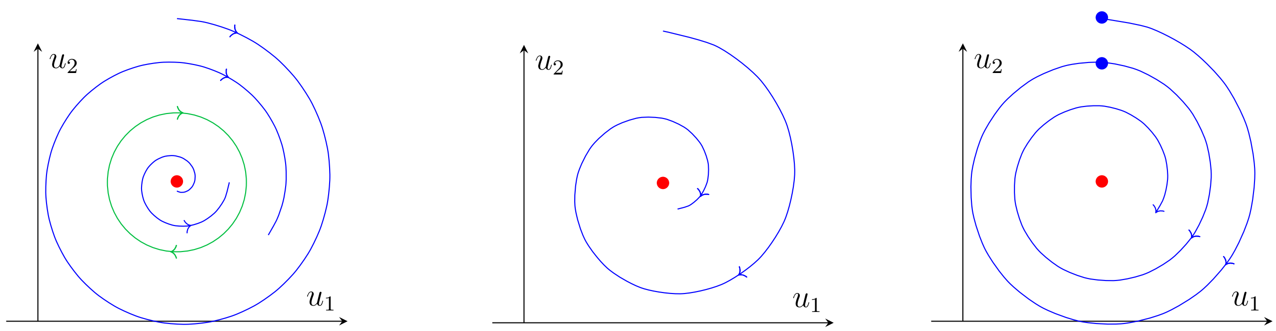
$\mu < \mu_*$ | $\mu = \mu_*$ | $\mu > \mu_*$

(b) **Subcritical Hopf Bifurcation:**



(Assumes $\alpha'(\mu_*) > 0$)

So now we have an unstable periodic orbit with period $\approx \frac{2\pi}{\beta(\mu_*)}$ and amplitude $\approx \sqrt{\mu - \mu_*}$.



Attractor + Unstable Periodic | Repeller | Repeller

$\mu < \mu_*$ | $\mu = \mu_*$ | $\mu > \mu_*$

Remark: Conditions (1)-(3) guarantee the emergence of periodic orbits at $\mu = \mu_*$

Paradimatic Equation:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \alpha(x^2 + y^2) \begin{pmatrix} x \\ y \end{pmatrix}, \qquad a, \omega \neq 0$$

Note that

1. the eigenvalues are $\lambda = \mu \pm i\omega$
2. $(x, y) = 0$ is an equilibrium for all μ
3. $\alpha(0) = 0$, $\alpha'(0) = 1$, and $\beta(0) = \omega \neq 0$

so the three conditions are satisfied.

We see that $(x, y) = 0$ is an attractor for $\mu < 0$ and a repeller for $\mu > 0$.

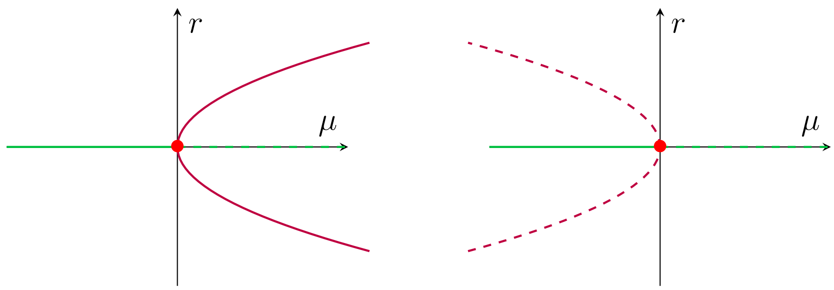
To find periodic orbits, we can use polar coordinates:

$$r^2 = x^2 + y^2 \implies \begin{cases} \tan\phi = \frac{y}{x} \\ x = r\cos\phi \\ y = r\sin\phi \end{cases} \implies \begin{cases} r\dot{r} = x\dot{x} + y\dot{y} \\ \dot{\phi} = \frac{x\dot{y} - y\dot{x}}{r^2} \end{cases}$$

so that

$$\begin{cases} \dot{r} = \mu r - ar^3 = r(\mu - ar^2) & \text{(Pitchfork in radial direction)} \\ \dot{\phi} = \omega & (\phi(t) = \phi_0 + \omega t) \end{cases}$$

And we also know that for $a > 0$ we will be supercritical and $a < 0$ we will be subcritical:



$a > 0$ $a < 0$

Remark: It is generally quite difficult to determine analytically whether a Hopf bifurcation is supercritical or subcritical.

Chapter 3

Equilibria in Higher Dimensions

Let $\dot{u} = F(u, \mu)$ for $u \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ with $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ in C^3.

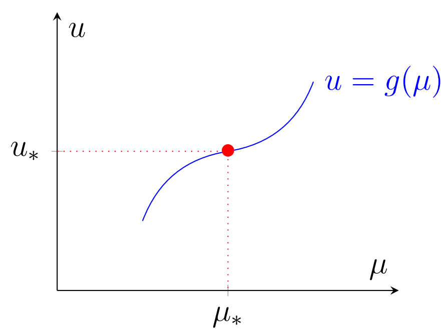
Assume that $F(u_*, \mu_*) = 0$ with Jacobian $A = F_u(u_*, \mu_*)$ and eigenvalues $\lambda_1, \dots, \lambda_n$ and associated eigenvectors $v_1, \dots, v_n$.

CASE 1: $\lambda_j \neq 0$ for all j

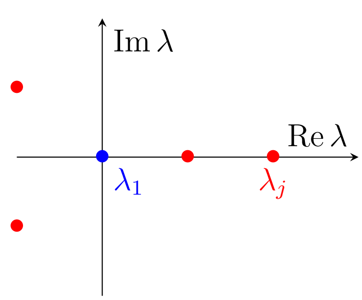
Hence, A invertible so by the IFT, for all $\mu \approx \mu_*$, $\dot{u} = F(u, \mu)$ has a unique equilibrium $u_*(\mu)$ near u_*.

Implicit Function Theorem: Let $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be in C^k for some $k \geq 1$. Assume $F(u_*, \mu_*) = 0$ and that $F_u(u_*, \mu_*) \in \mathbb{R}^{n \times n}$ is invertible. Then $\exists \delta > 0$ and a $g : \mathbb{R} \to \mathbb{R}^n$ in C^k such that $F(u, \mu) = 0$ for $u \in B_\delta(u_*)$ and $\mu \in B_\delta(\mu_*)$ if and only if $u = u_*(\mu)$ for $\mu \in B_\delta(\mu_*)$.

Here, the equilibrium *persists*



CASE 2: $\lambda_1 = 0$ and $\mathrm{Re}\,\lambda_j \neq 0$ for $j = 1$



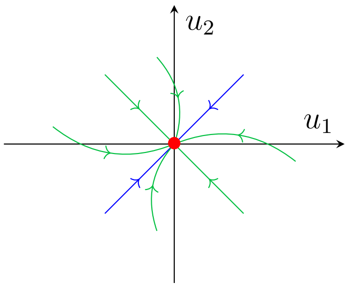
eigenvalues of A

which yields essentially the same bifurcations as for $u \in \mathbb{R}$:

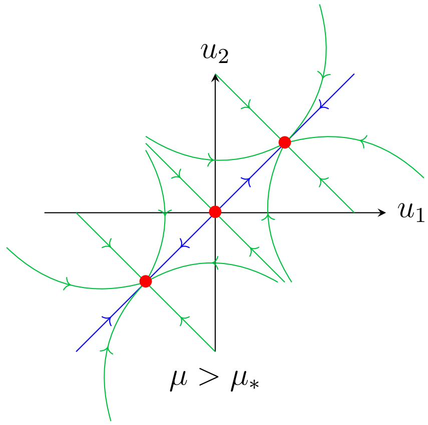
1. **Pitchfork**

If $F(-u,\mu) = -F(u,\mu)$ for all (u,μ) thn we expect a pitchfork.

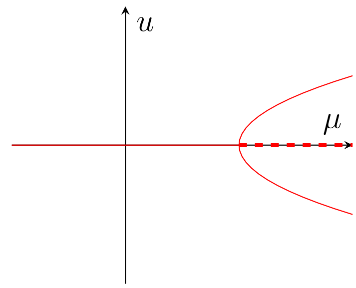
Supercritical Pitchfork Phase Diagrams:



$\mu < \mu_*$



$\mu > \mu_*$

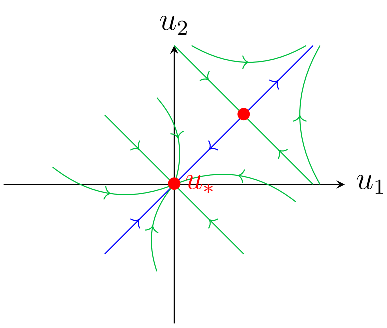


Bifurcation Diagram

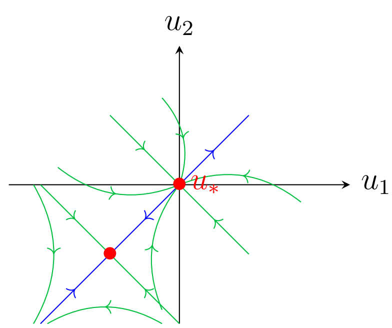
2. **Transcritical**

   If $F(u_*,\mu) = 0$ for all μ, then we expect a transcritical bifurcation.

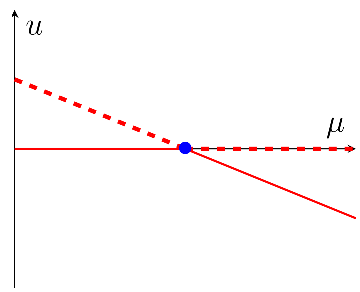
   Phase Diagrams ($\lambda_1 < 0 < \lambda_2$)



$\mu < \mu_*$



$\mu > \mu_*$

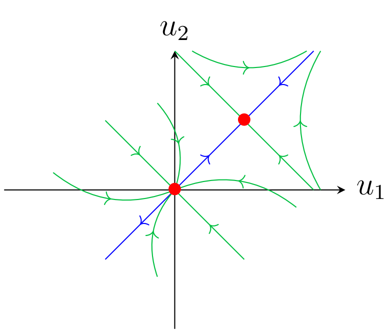


Bifurcation Diagram

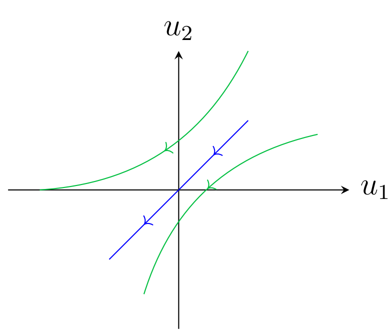
3. **Saddle Node**

   Without these additional structures, we expect to see a saddle-node bifurcation.

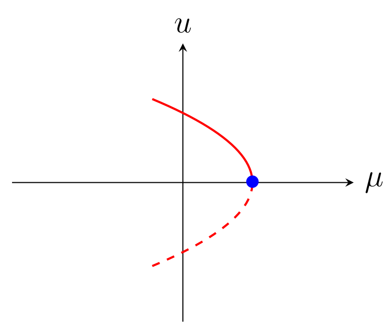
   Phase Diagrams ($\lambda_2 < \lambda_1 < 0$):



$\mu < \mu_*$

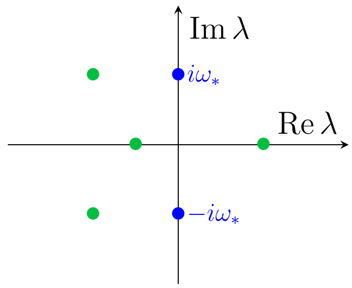


$\mu > \mu_*$

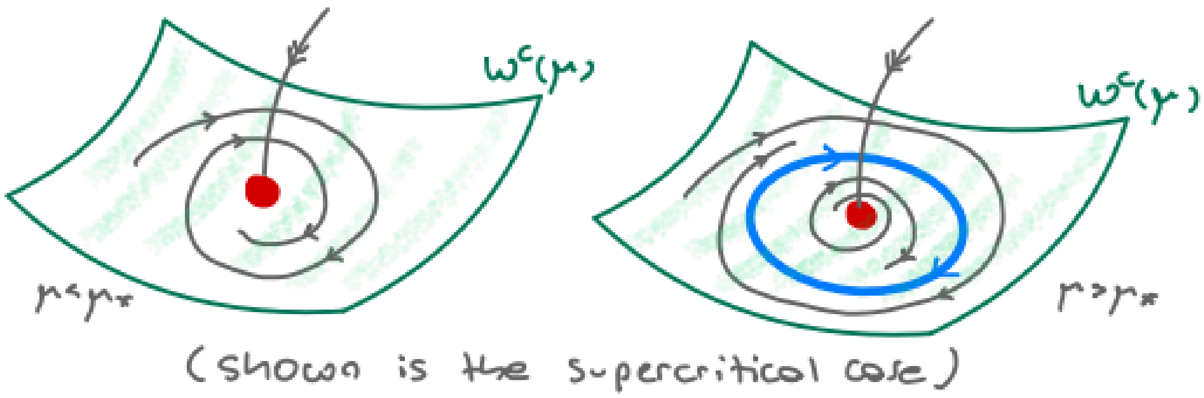


Bifurcation Diagram

CASE 3: $\lambda_{1,2} = \pm i\omega_*$ for $\omega_* \neq 0$ and $\operatorname{Re}\lambda_j \neq 0$ for $j \neq 1,2$



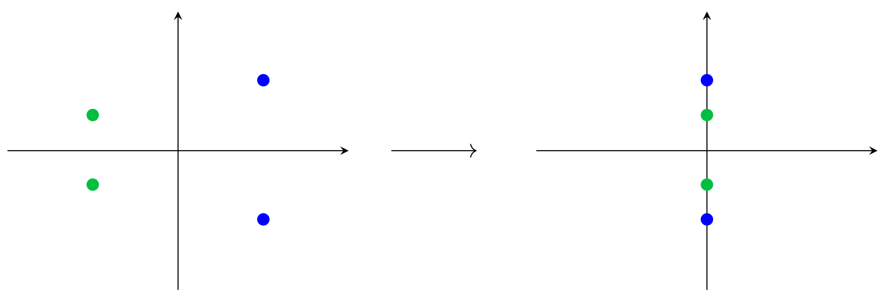
Hence we expect to encounter a Hopf bifurcation.



where the manifold is approximately the space spanned by v_1, v_2 and the period of the orbits is approximately $\frac{2\pi}{\omega_*}$.

What other cases could we see?

One suggestion is that all the eigenvalues are on the imaginary axis. But this would require a transformation



But this would require at least two parameters to be able to transform the green and the blue eigenvalues to the same point. In applications, this could happen with exceeding rarity with one parameter but would just be a coincidence.

And in fact, every other case will require more than one parameter.

Chapter 4

Dissipitative and Conservative Systems

4.1 March 14

Goal: Develop a framework that caputres systems that dissipate energy (e.g. mechanical systems with friction)

Consider $\dot{u} = F(u)$ with $u \in \mathbb{R}^n$ and $F \in C^1$. Let $E : \mathbb{R}^n \to \mathbb{R}$ be C^2.

Lemma 1: If $\langle \nabla E(u), F(u) \rangle < 0$ for all $u \in \mathbb{R}^n$ to which $F(u) \neq 0$, then $E(u(t))$ decreases strictly in t for each solution $u(t)$ that is not an equilibrium.

Proof:

$$\frac{d}{dt}(E(u(t))) = E'(u(t)) \cdot \frac{du}{dt}(t) = \langle \nabla E(u(t)), F(u(t)) \rangle < 0$$

by assumption (provided $u(t)$ is not an equilibrium)

Lyapunov Function: $\mathbb{E} : \mathbb{R}^n \to \mathbb{R}$ in C^1 is a *Lyapunov Function* for $\dot{u} = F(u)$ if $\langle \nabla E(u), F(u) \rangle < 0$ for all $u \in \mathbb{R}^n$ to which $F(u) \neq 0$.

Lemma 2: Assume that E is a Lyapunov functional for $\dot{u} = F(u)$. Then:

(i) $\dot{u} = F(u)$ cannot have any nontrivial periodic orbits

(ii) If $E(u) \to \infty$, then for each solution $u(t)$ of $\dot{u} = F(u)$, there exists a unique equilibrium u_* so that $u(t) \to u_*$ as $t \to \infty$

Proof:

(i) Assume that $u(t)$ is a nontrivial periodic orbit so that there is a $T > 0$ with $u(t + T) = u(t)$ for all t but $u(t)$ not constant. Lemma 1 implies that

$$E(u(0)) \overset{u(T)=u(0)}{=} E(u(T)) \overset{\text{Lemma 1}}{<} E(u(0)) \qquad \text{Contradiction}$$

(ii) The ω-limit of a solution $u(t)$ is defined

$$\omega(u(0)) = \left\{ v \in \mathbb{R}^n : \exists t_k \nearrow \infty \text{ s.t. } \lim_{k \to \infty} u(t_k) = v \right\}$$

We want to show that $\omega(u(0))$ consists of a single equilibrium. Fix a solution $u(t) = u_0$ of $\dot{u} = F(u)$.

1. There is a ball B of finite radius so that $u(t) \in B$ for all $t \geq 0$: Indeed, $E(u(t)) < E(u(0))$ and $E(u) \nearrow \infty$ as $|u| \to \infty$ so $u(t)$ must be bounded as t increases

2. *(E bounded below:)* Since B is bounded, there is a $c \in \mathbb{R}$ so that $E(u) \geq c$ for all $u \in B$.
3. *($E \to E^*$:)* We conclude that there is an $E_0 \in \mathbb{R}$ with $E(u(t)) \searrow E_0$ as $t \to \infty$. In particular, $E(v) = E_0$ for all $v \in \omega(u_0)$.
4. *($v(t) \in \omega(u_0)$ for all t:)* If $v(0) \in \omega(u_0)$, then $v(t) \in \omega(u_0)$ for all $t \geq 0$. Indeed, if $u(t_k) \to v(0)$ as $k \to \infty$, then for each $t \in \mathbb{R}$, wr have $u(t + t_k) \to v(t)$, hence $v(t) \in \omega(u_0)$
5. Combining (3) and (4) with Lemma 1, we conlclude $\omega(u_0)$ can contain only equilibria
6. We claim that $\omega(u_0)$ is not empty: indeed, $u(t) \in B$ for all t and B bounded, hence the sequence $u(k)$ has a limit point by Bolzano-Weierstrass (and Monotone Convergence).
7. Assume $\omega(u_0)$ contains distinct equilibria v_1, v_2. Since we assumed that equilibria are isolated, we can find a ball centered at v_1 containing only v_1. Let S be the boundary of this ball. Then there is a sequence $S_k \nearrow \infty$ so that $u(S_k) \in S$ for all u (if not, $u(t)$ stays in the ball for all sufficiently large t so $v_2 \notin \omega(u_0)$). But by (6), there is a point $w \in S \cap \omega(u_0)$ which contradicts that $\omega(u_0)$ consists of equilibria while S does not contain any equilibria

4.2 March 17

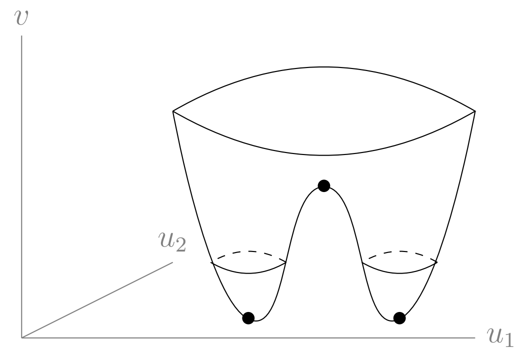
Recall: Last time, we worked on the proving that if a Lyapunov function $E \to \infty$, then each solution u has a unique equilibrium u_* so that $u(t) \to u_*$ as $t \to \infty$.

Proof Sketch: Fix $u(0) = u_0$. Then:

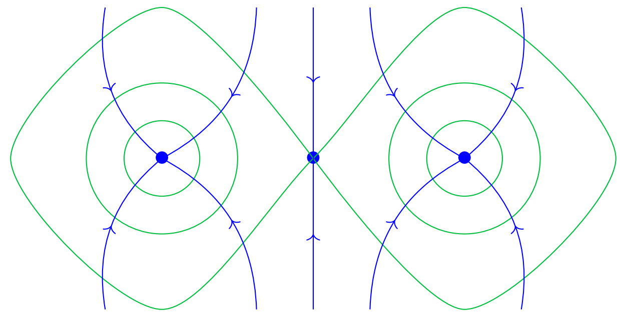
1. $u(t)$ is bounded in some (closed) ball
2. E is bounded below
3. $E \to E*$
4. $v(0) \in \omega(u_0)$ implies $v(t) \in \omega(u_0)$
5. $\omega(u_0)$ contains only equilibria
6. Bolzano Weierstrass implies $\exists t_k$ with $u(t_k) \to u^*$. Thus, $\omega(u_0)$ is nonempty
7. For two isolated equilibria $v_1, v_2 \in \omega(u_0)$, we can find a ball around v_1 that contains no other equilibria. Choose a sequence in ∂B. But then by BW, $\exists v^* \in \partial B(v_1) \cap \omega(u_0)$ which contradicts that B has only one equilibrium

Corollary: The ω-limit set $\omega(u(0)) = \{v \in \mathbb{R}^n : \exists t_k \nearrow \infty \text{ s.t. } u(t_k) \to v\}$ is invariant: if $v(0) \in \omega(u_0)$, then $v(t) \in \omega(u_0)$ for all $t \in \mathbb{R}$

Example: Let $v \in C^2$ and consider $\dot{u} = -\nabla v$. Then v is a Lyapunov function of this system (*pf:* $\langle \nabla v, -\nabla v \rangle = -||\nabla v||^2 < 0$). We call these systems *gradient systems.*



But our solutions are perpendicular to level sets of v so we can look at the contour plot:

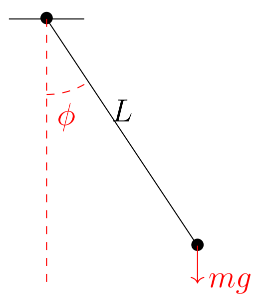


Remark: Every gradient system has a Lyapunov function. But not all systems are gradient systmes. What other kinds of systems can we consider?

4.3 March 19

Example (Pendulum with friction): Consider the model

$$mL\ddot{\phi} = -mg\sin\phi - aL\dot{\phi}$$



Which we can rewrite as a 2D system:

$$\begin{cases} \dot{\phi} = v \\ \dot{v} = -bv - \frac{g}{L}\sin\phi = F(\phi, v) \end{cases}$$

where $b = \frac{a}{m} > 0$.

Can we find a Lyapunov function?

Consider the energy function

$$E(\phi, v) = \frac{v^2}{2} + \frac{g}{L}\cos\phi$$

Check:

$$\langle \nabla E(\phi, v), F(\phi, v)\rangle = \left\langle \begin{pmatrix} \frac{g}{L}\sin\phi \\ v \end{pmatrix}, \begin{pmatrix} v \\ -bv - \frac{g}{L}\sin\phi \end{pmatrix} \right\rangle = -bv^2 \leq 0$$

But to be a Lyapunov functional, we need this to be strictly less than 0 except at equilibria. To be true, we would need $(\phi, 0)$ to be an equilibrium for all ϕ but we can check:

$$-\frac{g}{L}\sin\phi = 0$$

is certainly not true for all ϕ.

Hence, E is not a Lyapunov function for this system. Recall that we used the Lyapunov function to apply the monotone convergence theorem. Let's try to prove properties directly:

Claim: $E(\phi(t), v(t))$ decreases strictly in t except at equilibrium solutions

Proof: Pick a time t_0 and a duraction $\tau > 0$:

$$\begin{aligned} E(\phi(t_0+\tau), v(t_0+\tau)) - E(\phi(t_0), v(t_0)) &= \int_0^T \frac{d}{dx} E(\phi(t_0+s), v(t_0+s))\ ds \\ &= \int_0^\tau \langle \nabla E(\phi(t_0+s), v(t_0+s)), F(\phi(t_0+s), v(t_0+s))\rangle\ ds \\ &= \int_0^\tau -bv(t_0+s)^2\ ds \end{aligned}$$

and now it suffices to find conditions such that $-b\int_0^\tau v(t_0+s)^2\ dx < 0$ except at equilibria.

But note that if $E(\phi(t_0+\tau), v(t_0+\tau)) - E(\phi(t_0), v(t_0)) = 0$ for some $t_0 \in \mathbb{R}$ and $\tau > 0$, then

$$-b\int_0^\tau v(t_0+s)^2\ ds = 0 \implies v(t_0+s) = 0$$

for all $s \in [0, \tau]$.

Hence, in particular, $v'(t) = 0$ for all $t \in (t_0, t_0+\tau)$ and also $\phi'(t) = v(t) = 0$ for all such t.

By uniqueness, $(\phi(t), v(t))$ does not depend on t for $t \in (t_0, t_0+\tau)$ and must therefore be an equilibrium.

Conclusion: All results from Lyapunov functions are applicable to the pendulum with friction system.

Notice that since

$$\frac{d}{dt}E(\phi(t), v(t)) = -bv(t)^2$$

where b is the friction coefficient, if $b = 0$ (no friction), then $\frac{d}{dt}E = 0$ for all t so E is constant: without friction, the energy is conserved.

4.3.1 Conservative Systems (Hamiltonian Systems)

Goal: Develop a framework for processes that conserve a quantity in time (e.g. mechanical energy or population size)

Consider $\dot{u} = F(u)$ with $u \in \mathbb{R}^n$ and $F \in C^1$. Let $H : \mathbb{R}^n \to \mathbb{R}$ be C^2.

Definition: We say that H is a *conserved quantity* for $\dot{u} = F(u)$ if $\langle \nabla H(u), F(u)\rangle = 0$ for all $u \in \mathbb{R}^n$.

Lemma: Assume H is a conserved quantity for $\dot{u} = F(u)$. Then $H(u(t))$ is independent of t for each solution $u(t)$ of $\dot{u} = F(u)$ and therefore

$$H^{-1}(c) = \{u \in \mathbb{R}^n : H(u) = c\}$$

is invariant under solutions for each fixed $c \in \mathbb{R}$.

Proof: We need to show that $H(u(t))$ is a constant, that is

$$\frac{d}{dt}(H(u(t))) = 0$$

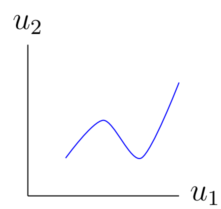
but this happens iff

$$\begin{aligned}\frac{d}{dt}(H(u(t))) &= \langle \nabla H(u(t)), \dot{u}(t)\rangle \\ &= \langle \nabla H(u(t)), F(u(t))\rangle = 0\end{aligned}$$

by assumption

Example ($n = 2$): $H : \mathbb{R}^2 \to \mathbb{R}$ gives $\{u \in \mathbb{R}^2 : H(u) = c\}$ for c fixed.

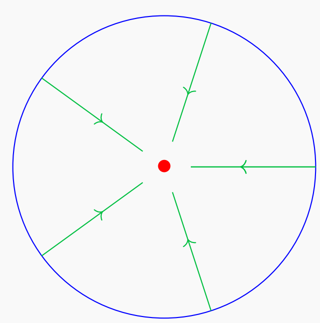
But this reminds us of the setup of the IFT, so we expect $H^{-1}(c)$ to be curves or lines in $\mathbb{R}^2$:



Conclusion: This Lemma lets us reduce the dimensionn of the system.

Corollary: Assume H is a conserved quantity for $\dot{u} = F(u)$ and suppose that $\nabla H(u) = 0$ only at isolated points, then $\dot{u} = F(u)$ cannot have any attractors or repellers.

Proof: WLOG assume that u_* is an attractor (for the repeller case, just reverse time).



u_* is an attractor, hence there is a $\delta > 0$ so that for each u_0 with $|u_0 - u_*| < \delta$ we have $u(t) \to u_*$ as $t \to \infty$. We also know that $H(u(t)) = H(u(0)) = H(u_0)$ for all t by the preceding lemma. Furthermore, since H is continuous, we have

$$H(u_0) = \lim_{t\to\infty} H(u(t)) = H(\lim_{t\to\infty}) = H(u_*)$$

But then $H(u) = H(u_*)$ for all u with $|u - u_*| < \delta$ which contradicts that $\nabla H(u)$ cannot vanish in open sets.

Example (Hamiltonian Systems): Consider $\ddot{x} = f(x)$ for $x \in \mathbb{R}$ and $f \in C^1$.

We can rewrite this as a first-order system:

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ f(x) \end{pmatrix}$$

Pick the mechanical energy:

$$H(x, v) = \frac{v^2}{2} - \int_0^x f(y)\, dy$$

Claim: $H(x, v)$ is a conserved quantity

Proof: Suffices to check

$$\langle \nabla H(x, v), F(x, v) \rangle = 0$$

for all (x, v):

$$\nabla H(x, v) = \begin{pmatrix} H_x(x, v) \\ H_v(x, v) \end{pmatrix} = \begin{pmatrix} -f(x) \\ v \end{pmatrix}$$

$$\langle \nabla H(x, v), F(x, v) \rangle = \begin{pmatrix} -f(x) \\ v \end{pmatrix} \cdot \begin{pmatrix} v \\ f(x) \end{pmatrix} = -f(x)v + vf(x) = 0 \quad \blacksquare$$

Note: $\nabla H(x, v) = \begin{pmatrix} -f(x) \\ v \end{pmatrix}$ vanishes only at equilibria! Hence, no attractors and no repellers.

4.4 March 21

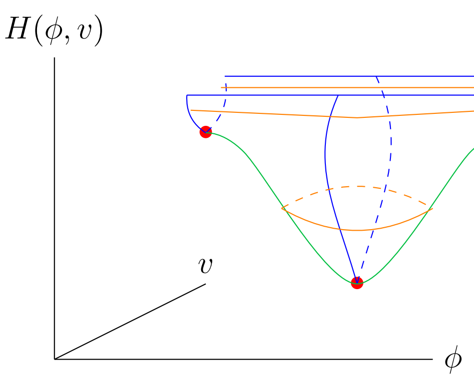
Example (Pendulum): Consider the pendulum system

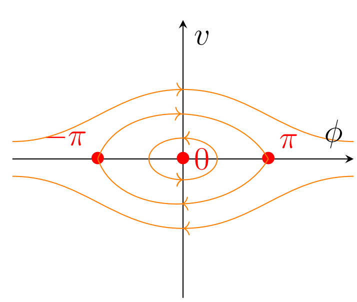
$$\ddot{\phi} = -\frac{g}{L} \sin \phi \iff \begin{pmatrix} \dot{\phi} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ -\frac{g}{L} \sin \phi \end{pmatrix}$$

which has Hamiltonian

$$H(\phi, v) = \frac{v^2}{2} + \int^{\phi} \frac{g}{L} \sin y\, dy = \frac{v^2}{2} - \frac{g}{L} \cos \phi$$

which is conserved along solutions.





Note: the level sets (orange) are circles near the trough but not closed near the top. When they are circles, the pendulum swings back and forth. When they are not closed, the pendulum overturns.

Chapter 5

Chaotic Dynamics

5.1 March 31

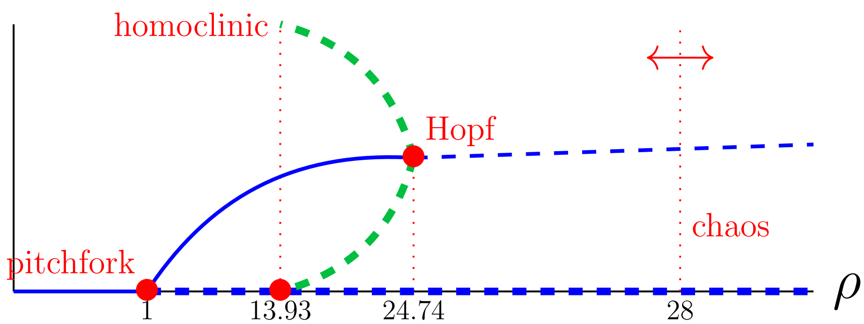
Goals:

- explore chaotic dynamics through numercial solutions of two differential equations in $\mathbb{R}^3$
- define what we mean by "chaotic dynamics"
- characterize, analyze, and classify chaotic dynamics

5.1.1 Lorenz Equations:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = xy - \beta z \end{cases}$$

given by a 3-mode truncation of the Fourier-sereis solution of equations describing Rayleigh-Bernard Convection. The classical parameter values are $\sigma = 10$, $\rho = 28$, and $\beta = \frac{8}{3}$.



5.1.2 Rossler System

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = b - cz + xz \end{cases}$$

was proposed as a simpler model for chaos and exhibits periodic-doubling bifurcations. The classical parameter values are $a = b = 0.1, c \geq 9$.

5.2 April 2

5.2.1 Attractors

Attractors: Consider $\dot{u} = f(u)$ with $u \in \mathbb{R}^n$ and denote the solution $u(t)$ with initial condition $u(0) = u_0$ by $u(t; u_0) = \phi_t(u_0)$, where $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ maps $u_0 \mapsto u(t; u_0)$. We assume that there is a ball $B \subseteq \mathbb{R}^n$ such that $\phi_t(B) \subseteq B$ for all $t \geq 0$ so that B is forward invariant (and is therefore a trapping region).

The attractor A of $\dot{u} = f(u)$ in the ball is then given by

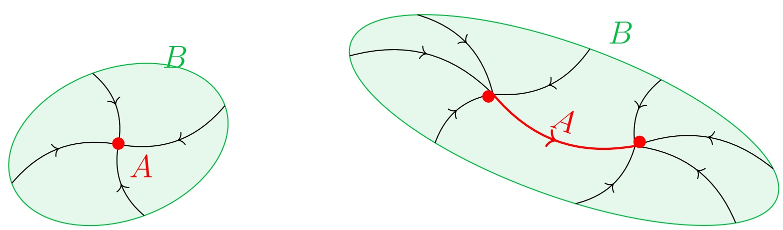
$$A = \bigcap_{t \geq 0} \phi_t(B) = \bigcap_{t \geq 0} \{\phi_t(u_0) : u_0 \in B\}$$

Lemma: Let A be the attractor in B. Then

- A is non-empty
- A is (backwards and forwards) invariant: if $u_0 \in A$, then $\phi_t(u_0) \in A \quad \forall t \in \mathbb{R}$
- A is the largest invariant set in B: if $u_0 \in B$ so that $\phi_t(u_0) \in B$ for all $t \in \mathbb{R}$, then $u_0 \in A$.
- For each $u_0 \in B$, we have $\operatorname{dist}(\phi_t(u_0), A) = \min_{u \in A} |\phi_t(u_0) - u| \to 0$ as $t \to \infty$.

Remark: we want to find equilibria for Lorenz and Rossler and eventually study the dynamics. This lemma tells us that it suffices to focus on the attractor A.

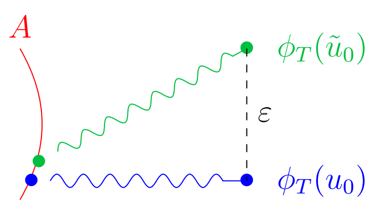
Examples:



Note: In the second example, we know the attractor because all other orbits would leave B in backwards time. The path between the equilibria, however, would just move back and forth between the two equilibria.

5.2.2 Sensitive Dependence on Initial Conditions

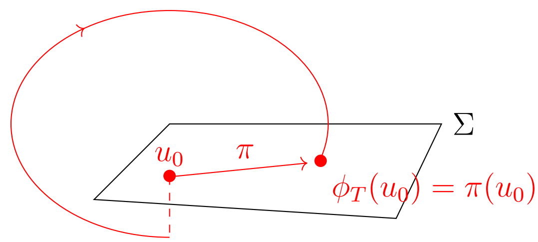
Definition: We say that an attractor A has *sensitive dependence on initial conditions* if $\forall u_0 \in A$, $\exists \varepsilon > 0$ so that for each $\delta > 0$, there is a $\tilde{u}_0 \in A$ with $|u_0 - \tilde{u}_0| < \delta$ and a $T > 0$ so that $|\phi_T(u_0) - \phi_T(\tilde{u}_0)| \geq \varepsilon$.



Intuition: Every solution acts like a saddle

5.2.3 Poincare Maps

Goal: try to reduce the the system $\dot{u} = f(u)$, $u \in \mathbb{R}^3$ to a map on a 2D surface (leads to planar maps).



Here T is chosen as the first time $t > 0$ for which $\phi_t(u_0) \in \Sigma$. Let $Y = T(u_0)$ denote the first return time to Σ: $\pi(u_0) = \phi_{T(u_0)}(u_0)$.

Then we can iterate this process to have a collection of maps between points in Σ.

5.3 April 4

5.3.1 Recall:

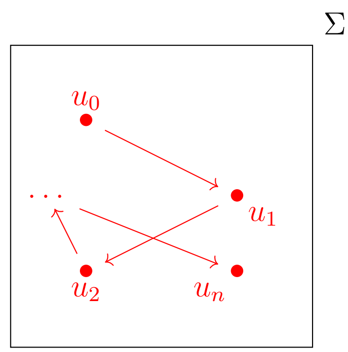
We are interested in $\dot{u} = f(u)$ for $u \in \mathbb{R}^3$. We want to remove the time direction and reduce the ODE to a system of planar maps. Assume we cna find a bounded two-dimensional rectangle Σ ("the section") so that each solution starting in Σ will immediately leave Σ and eventually return to it.

Let $T(u_0) > 0$ denote the first time that $\phi_t(u_0)$ intersects Σ for each $u_0 \in \Sigma$.

We can then define the *Poincare map* $\pi : \Sigma \to \Sigma$ by $u_0 \mapsto \phi_{T(u_0)}(u_0)$.

Idea: If we iterate π we get a discrete dynamical system on Σ:

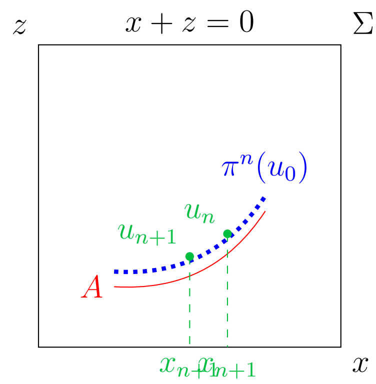
$$\begin{aligned}
&u_0 \in \Sigma \\
&u_1 = \pi(u_0) \in \Sigma \\
&u_2 = \pi(u_1) = (\pi \circ \pi)(u_0) = \pi^2(u_0) \\
&\quad \vdots \\
&u_n = \pi^n(u_0)
\end{aligned}$$



5.3.2 Poincare Maps for the Rossler System

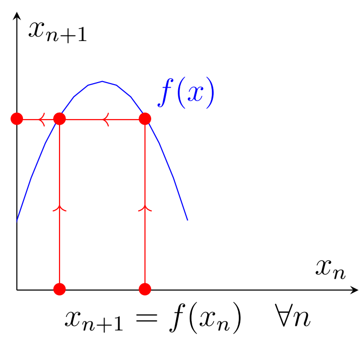
The numerical approach is to

1. Define $\Sigma = \{(x, y, z) : y + z = 0\}$
2. Pick $u_0 = (x_0, y_0, z_0) \in \Sigma$ and use Matlab's ODE solver to compute the intersections of the solution with Σ.
3. To visualize the attractor A we iterate a large number of times $n \geq N \gg 1$ to be close to the attractor (since $|u(t) - A| \to 0$). Only then do we start to plot the iterations



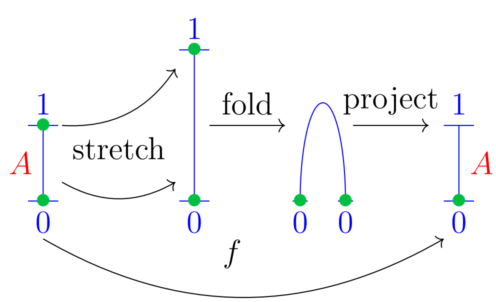
We suspect the attractor A to follow these points. In this case, A appears to be one dimensional and parameterized by x.

We can then plot x_n vs x_{n+1} to see the structure of the attractor.



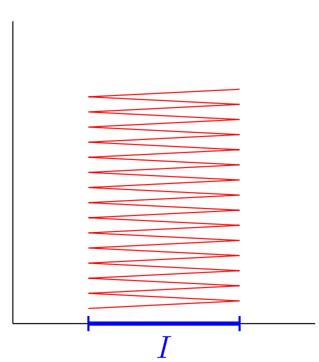
What is the behavior of $f(x)$?

- Sensitive dependence on initial conditions: f needs to stretch out A.



Conclusion: contrary to our expectation, A cannot be a one-dimensional curve or else $\pi\big|_A$ would violate the existence and uniqueness of solutions since f is not one-to-one

We update our conjecture



which is a collection of copies of a one-dimensional interval I all glued together.

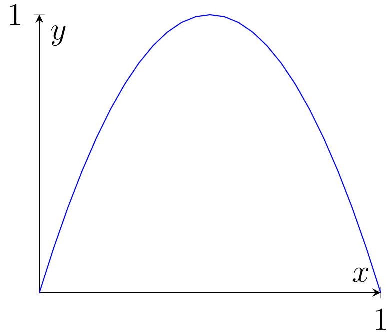
How many copies do we need? It turns out this graph is a Cantor set with fractional dimensional less than 1 (hence why we were fooled earlier). We conclude $A \approx C \times I/$"glue"

5.4 April 7

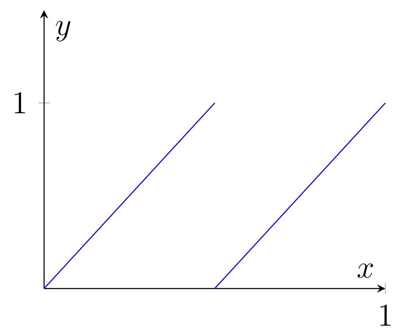
5.4.1 Interval Maps

Consider $f : [0,1] \to [0,1]$ defining a discrete dynamical system.

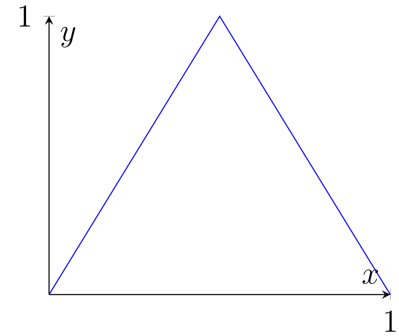
Examples:



quadratic map



$f(x) = 2x \bmod 1$

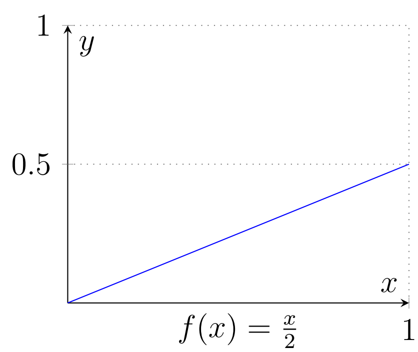


tent map

Example: Let

$$x_n = \frac{1}{2}x_{n-1} = \frac{1}{2}\left(\frac{1}{2}x_{n-2}\right) = \frac{1}{4}x_{n-2} = \cdots = \frac{1}{2^n}x_0$$

Hence $f^n(x) = \frac{1}{2^n}x$



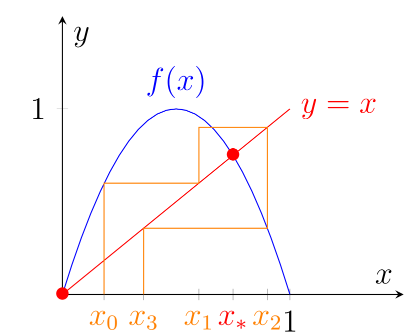
$f(x) = \frac{x}{2}$

Orbit: $\gamma(x_0) = \{x_n : x_n = f(x_{n-1})\ n \geq 1\}$ is the set of all forward iterates of x_0 under f

Fixed point: x_* is a fixed point of f if $f(x_*) = x_*$. Equivalently, $\gamma(x_*) = \{x_*\}$.

Periodic Orbit: x_0 is a periodic orbit of period $k \geq 2$ is $x_k = x_0$ but $x_n \neq x_0$ for $0 < n < k$

Cobweb:



Here, x_* is a fixed point (intersection of $y = x$ and $y = f(x)$). We can determine $\gamma(x_0)$ graphically using the cobweb: given x_{n-1}, find $x_n = f(x_{n-1})$ by

- $x_n = f(x_{n-1})$ on y-axis (height)
- x_n on x-axis by using diagonal $y = x$
- iterate this process

5.5 April 9

5.5.1 Stability of Fixed Points

Assume $f \in C^1$. Assume x_* is a fixed point of f. Then

$$\begin{cases} |f'(x)| < 1 & x_* \text{ stable} \\ |f'(x)| > 1 & x_* \text{ unstable} \end{cases}$$

so we expect bifurcations when $f'(x_*) = \pm 1$.

Proof: Take $x = x_* + d$. Then

$$\begin{aligned} f(x) &= f(x_* + d) \\ &= f(x_*) + f'(x_*)d + O(d^2) \\ &= x_* + [\underbrace{f'(x_*)}_{<1} + O(d)]d \end{aligned}$$

We know that there is a $C > 0$ such that $|O(d)| \leq C\,|d|$. In particular, $|O(d)| \leq C\delta$ for all d with $|d| < \delta$.

We can pick $\delta > 0$ small enough that

$$|f'(x_*) + O(d)| \leq |f'(x_*) \pm C\delta| \leq a < 1$$

Now we iterate. Pick $x_0 = x_* + d_0$ with $|d_0| < \delta$ and write $x_n = x_* + d_n$ for $n \geq 1$. We need $d_n \to 0$, and in fact,

$$\begin{aligned} x_n &= f(x_{n-1}) \\ x_* + d_n &= f(x_* + d_{n-1}) \\ &= x_* + [f'(x_*) + O(d_{n-1})]d_{n-1} \\ d_n &= [f'(x_*) + O(d_{n-1})]d_{n-1} \\ |d_n| &\leq |f'(x_*) + O(d_{n-1})|\,|d_{n-1}| \end{aligned}$$

As long as $|d_{n-1}| \leq \delta$, we have

$$|f'(x_*) + O(d_{n-1})| \leq a < 1$$

In particular, this holds for $n = 1$:

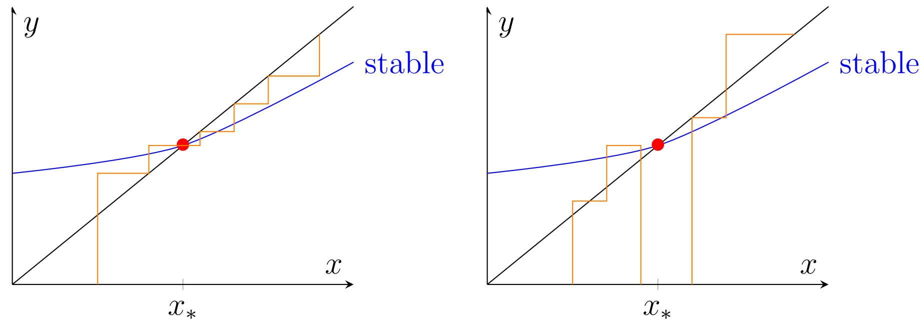
$$\begin{aligned} |d_n| &\leq |f'(x_*) + O(d_{n-1})|\,|d_{n-1}| \\ &\leq a \underbrace{|d_{n-1}|}_{\leq a|d_{n-2}| \leq \dots} < a\delta < \delta \end{aligned}$$

so

$$|d_n| \leq a^n \, |d_0|$$

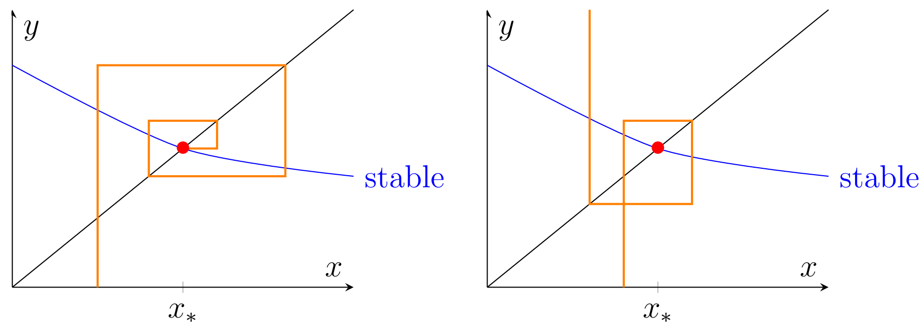
Since $a < 1$, we have $d_n \to 0$ as $n \to \infty$, hence $f(x_n) \to x_*$

We could determine this graphically using cobweb:



$0 < f'(x_*) < 1$ $\quad$ $1 < f'(x_*)$

Monotone Orbits



$-1 < f'(x_*) < 0$ $\quad$ $f'(x_*) < -1$

Oscillatory Orbits

Example: $f(x) = 2x(1-x)$

- Fixed points: $x = f(x) = 2x(1-x) \implies x = 0, 1/2$
- Stability:

$$f'(x) = 2 - 4x \implies \begin{cases} |f'(0)| = 2 > 1 & \text{unstable} \\ |f'(1/2)| = 0 < 1 & \text{stable} \end{cases}$$

5.5.2 Periodic Orbits

Example: $f(x) = -x^3$ with $f : \mathbb{R} \to \mathbb{R}$ and $x_n = f(x_{n-1})$.

- Fixed points: $x = f(x) = -x^3 \implies x = 0$
- Stability: $f'(0) = -3(0)^2 = 0 < 1$ so $x = 0$ is stable
- Period two orbits: we want x_0 with $f^2(x_0) = f(f(x_0)) = x_0$:

$$f^2(x) = f(f(x)) = f(-x^3) = -(-x^3)^3 = x^9 = x \implies x = 0, \pm 1$$

  Since x_* is the unique fixed point of f, we conclude that there is a unique period-2 orbit given by ± 1 or $x_0 = 1$ and $x_1 = -1 = f(x_0)$.

- Stability of period-2 orbit $\{x_0 = 1, x_1 = -1\}$: our criterion is for fixed orbits so we can apply the criterion to $x_{0,1} = \pm 1$ as fixed points of $f^2(x) = x^9$.

$$\frac{d}{dx} f^2(x) = 9x^8 \implies |f'(\pm 1)| = 9 > 1$$

so the period-2 orbit is unstable.

5.6 April 14

5.6.1 Sensitivity to Initial Conditions

How can we detect and measure sensitivity with respect to initial conditions?

Pick y_0 close to x_0 and compute the orbit $y_n = f(y_{n-1})$. Let $y_n = x_n + d_n$ so that d_n is the offset from x_n. Then

$$\begin{aligned} d_n &= y_n - x_n \\ &= f(y_{n-1}) - f(x_{n-1}) \\ &= f(x_{n-1} + d_{n-1}) - f(x_{n-1}) \\ &= f'(x_{n-1})d_{n-1} + O(|d_{n-1}|^2) \\ &\approx f'(x_{n-1})d_{n-1} \end{aligned}$$

Hence,

$$d_n \approx f'(x_{n-1})d_{n-1} \approx f'(x_{n-1})f'(x_{n-2})d_{n-2} \approx \cdots \approx \left(\prod_{j=1}^{n-1} f'(x_j)\right) d_0$$

where the product measures the separation of nearby orbits.

We expect $|d_n| \approx p^n |d_0|$ for some p. Therefore, define the **Lyapunov Multiplier**

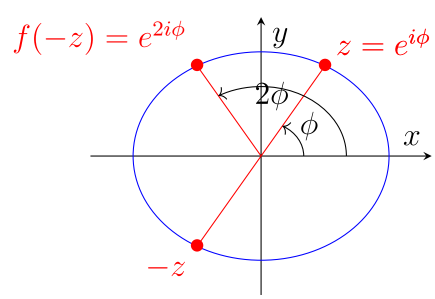
$$L(x_0) = \lim_{n\to\infty} \left(\prod_{j=0}^{n-1} |f'(x_{j-1})|\right)^{1/n}$$

Then if

- $L(x_0) > 1$, we have an *exponential separation* of nearby trajectories
- $L(x_0) < 1$, we have an *exponential attraction* of nearby trajectories

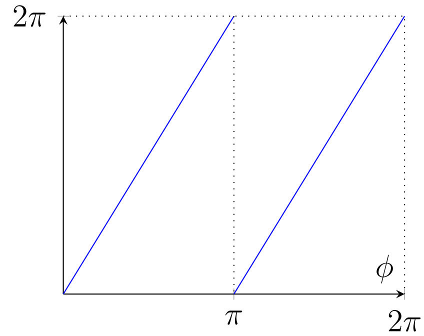
5.6.2 Expanding Circle Map

Let $f : S^1 \to S^1$ by $e^{i\phi} \mapsto e^{2i\phi}$ (equivalently $\phi \mapsto 2\phi \bmod 2\pi$)



or equivalently since

$$\begin{aligned} f^k(z) = (f \circ \cdots \circ f)(z) = z^{2k} \\ f^k(e^{i\phi}) = e^{i2k\phi} \end{aligned}$$



$$f(\phi) = 2\phi \bmod 2\pi$$

1. *Sensitive dependence on initial conditions*

   Since $|f'(z)| = 2$ for all z, we have $L(z_0) = 2 > 1$ for all $z_0 \in S^1$. Hence, we have sensitive dependence on initial data: $\exists \varepsilon > 0$ s.t. $\forall z_0 \in S^1, \delta > 0$, $\exists \tilde{z}_0 \in B_\delta(z_0), n \geq 0$ so that $|f^n(z_0) - f^n(\tilde{z}_0)| \geq \varepsilon$.

2. *The set of periodic orbits of f is dense in S^1*

   Dense Subsets: Let $A \subseteq B \subseteq \mathbb{R}$. We say that A is dense in B if the following is true for each $b \in B$: for each $\varepsilon > 0$, $\exists a \in A$ with $|a - b| < \varepsilon$. In particular, this means that each $b \in B$ can be approximated arbitrarily well by points in A.

   Let $P_k = \{z \in S^1 : f^k(z) = x\}$ be the set of points with not necessarily smallest period k and $P = \bigcup_{k \geq 1} P_k$ be the sset of all periodic orbits.

   We have $z = e^{i\phi} \in P_k$ iff $f^k(z) = z$, i.e. $e^{i2^k\phi} = e^{i\phi}$ or, equivalently, $2^k\phi = \phi + 2\pi n$ for some $n \in [0, 2^k - 1]$.

   Therefore, $z \in P_k$ iff z is a $(2^k - 1)$-th root of unity. Hence,

   $$P = \left\{ z = e^{i\phi} : \exists k \geq 1, 0 \leq n < 2^{k-1} \text{ s.t. } \phi = \frac{2\pi n}{2^{k-1}} \right\}$$

   is dense in S^1

3. *The attractor S^1 is "topologically transitive" or "indecomposable"*

   Topologically transitive: the attractor S^1 of f is *topologically transitive* if for any two open non-empty arcs $U, V \subseteq S^1$, there is an $n > 0$ with $f^n(U) \cap V \neq \emptyset$.

   The attractor of the circle map $f(z) = z^2$ is topologically transitive: Let U be any small arc of length $\delta > 0$ in S^1. Then the length of its n-th iterate under f is $2^n\delta$, so $f^n(u)$ will eventually cover S^1.

4. *The attractor contains a dense orbit: $\exists z_0 \in S^1$ s.t. $\gamma(z_0) \subseteq S^1$ is dense*

   We consider the representation of $f(z) = f(e^{i\phi}) = e^{2i\phi}$ via the doubling map $\phi \mapsto 2\phi \bmod 2\pi$. It is easier to consider the scaled version $\phi = 2\pi x$ with $x \in [0, 1]$ for which we obtain the equivalent map $f(x) = 2x \bmod 1$.

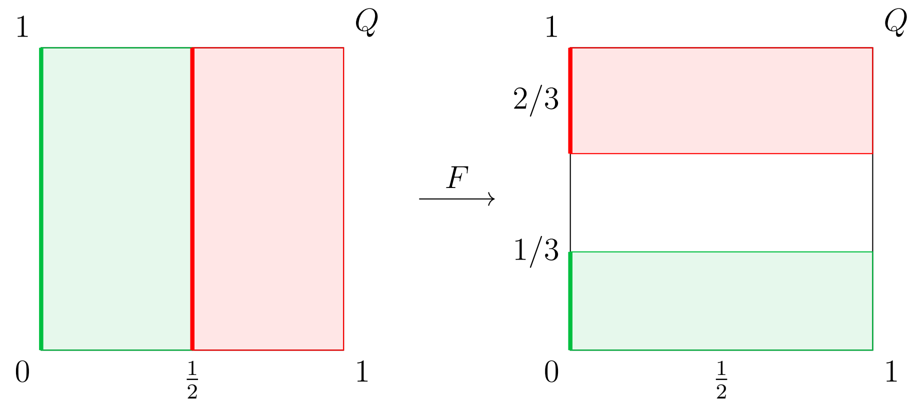
Chaotic: we say $f : I \to I$ is chaotic if

- It has sensitive dependence on initial conditions
- the set of periodic orbits of f is dense in I
- it is topologically transitive
- it has a dense orbit

We conclude the expanding circle map $f(z) = z^2$ $(f(e^{i\phi}) = e^{2i\phi})$ is chaotic (though it is not volume-contracting!)

5.7 April 21

5.7.1 Baker's Map



i.e. a cut, shrink, and rotation.

Formally, $F : Q \to Q$ is defined by

$$F(x,y) = \begin{cases} (2x, y/3) & 0 \le x < 1/2 \\ (2x-1, y/3 + 2/3) & 1/2 \le x \le 1 \end{cases}$$

In the x-direction,

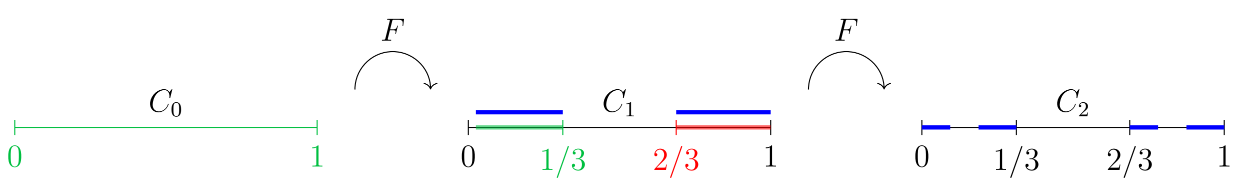
$$x \mapsto \begin{cases} 2x & 0 \le x < 1/2 \\ 2x - 1 & 1/2 \le x < 1 \end{cases} = 2x \bmod 1$$

hence it is chaotic.

Note that $F(Q) \subseteq Q$ so we can iterate F to obtain an attractor of F in Q

$$A = \bigcap_{n \ge 0} F^n(Q) \subseteq Q$$

Since the x-direction will always cover $[0,1]$, we are interested mostly in the y-direction.



At each iteration, we remove the middle third of each interval and continue recursively. At the n-th step, we find that C_n is the disjoint union of 2^n intervals each of length 3^{-n}. Hence, C_n has length $(2/3)^n$ and these sets are nested: $C_{n+1} \subseteq C_n \ \forall n$.

Hence, $F^n(Q) = [0,1] \times C_n$ and

$$A = \bigcap_{n \ge 0} F^n(Q) = [0,1] \times \bigcap_{n \ge 0} C_n$$

Define $C = \bigcap_{n \ge 0} C_n$, the **Cantor Set.**

Theorem: Let C be a Cantor set. Then

- C can be written as a ternary expansion

$$C = \left\{ y = \sum_{n=1}^{\infty} \frac{a_n}{3^n} : a_n \in \{0, 2\} \right\}$$

- C is uncountable
- C is *totally disconnected* (i.e. does not contain any non-empty open intervals)
- C is *perfect* (i.e. every point in C is a limit point of C)
- C is closed and bounded

a ternary expansion

Proof:

- The ternary representation gives the representation of C by construction

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- *Uncountable*:

  Assume C is countable, then $C = \{y_j : j \in \mathbb{N}\}$ where $y_j = \sum_{n=1}^{\infty} \frac{a_n^{(j)}}{3^n}$ with $a_n^{(j)} \in \{0, 2\}$ for all j, n.

  Define $y = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$ where $b_n = 2 - a_n^{(n)}$ so $b_n \neq a_n$ for all n.

  We claim $y \neq y_j$ for all j.

  Assume $y = y_k$ for some k. Then $b_n = a_n^{(k)}$ for all n. In particular, $b_k = a_n^{(k)}$ but this contradicts our choice of b_k.

- *Totally disconnected:*

  We have $C \subseteq C_n$ for all n and $|C_n| \to 0$ so C cannot contain any nonempty open intervals

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- *Perfect:*

  By part (1) if $y \in C$, then

$$y = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$$

  with $b_n \in \{0, 2\}$.

  We want to find $y_j \in C$ so that $y_j \neq y$ for all j yet $y_j \to y$.

  Let

$$b_n^j = \begin{cases} b_n & n \neq j \\ 2 - b_n & n = j \end{cases}$$

  Then $y_j \neq y$ as $b_j^j \neq b_j$ but $y_j \to y$ since the ternary representations agree for more indices.

Conclusion: Baker's map $F : Q \to Q$ has an attractor $A = [0, 1] \times C$ where C is a Cantor Set. $F|_A$ has chaotic dynamics in the x-component since it agrees with the expanding circle map.

5.7.2 Dimensions of Attractors

Box dimension: Tile the plane with an grid of boxes of length ε. Let S be a set. For each $\varepsilon > 0$, let $N(\varepsilon)$ denote the number of boxes of size ε that S intersects.

Intuitively,

- A line has $N(\varepsilon) \propto 1/\varepsilon$
- A surface has $N(\varepsilon) \propto 1/\varepsilon^2$

So the box dimension of S is defined by

$$\dim_{\text{box}}(S) = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}$$

(if it exists)

Example: Let C be the Cantor set. For each C_n, choose $\varepsilon = 3^{-n}$ so $N(\varepsilon) = 2^n$. Hence,

$$\dim_{\text{box}}(C) = \frac{\ln 2}{\ln 3} \approx 0.63$$

Example: $S = \{0\} \cup \{1/n : n \geq 1\}$. We intuit that this should have dimension 0 since it is countable. In fact,

$$\dim_{\text{box}}(S) = 1/2$$

as an artifact of requiring all boxes to be the same size.

Hausdorff dimension: Pick a set $S \subseteq \mathbb{R}$ and $\delta > 0$. Cover S by finitely many intervals I_j of length less than $\varepsilon > 0$. Define

$$\mu_\delta(S) = \lim_{\varepsilon \to 0} \inf_{\text{covers}} \sum_j |I_j|^\delta$$

Then there exists a $d \geq 0$ so that

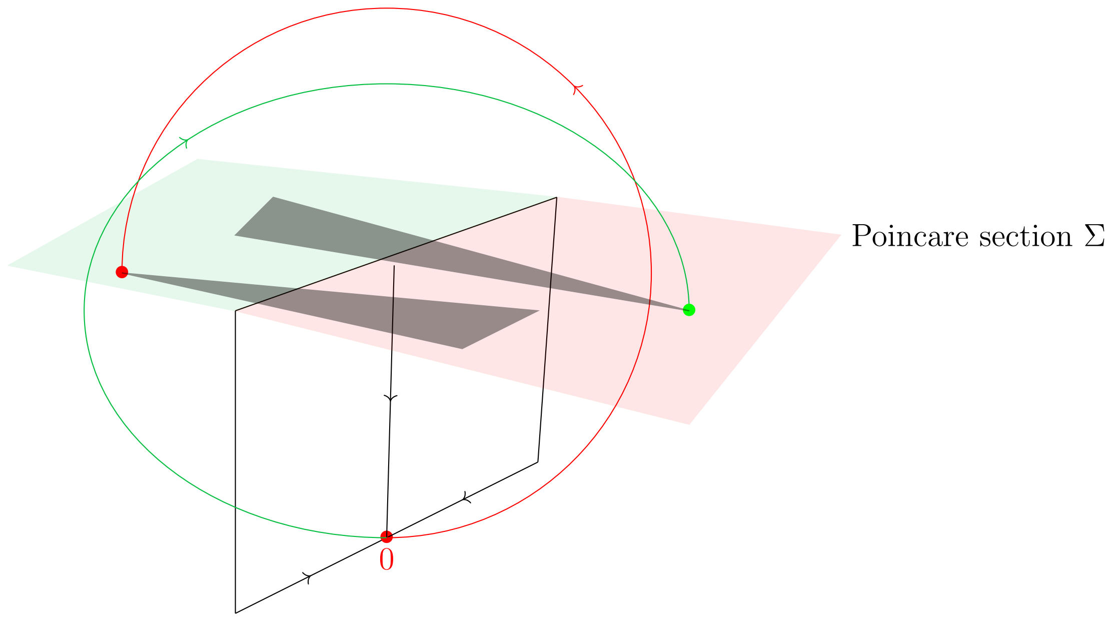
$$\mu_{\delta(S)} = \begin{cases} 0 & \delta > d \\ \infty & \delta < d \end{cases}$$

we call d the **Hausdorff dimension** of S.

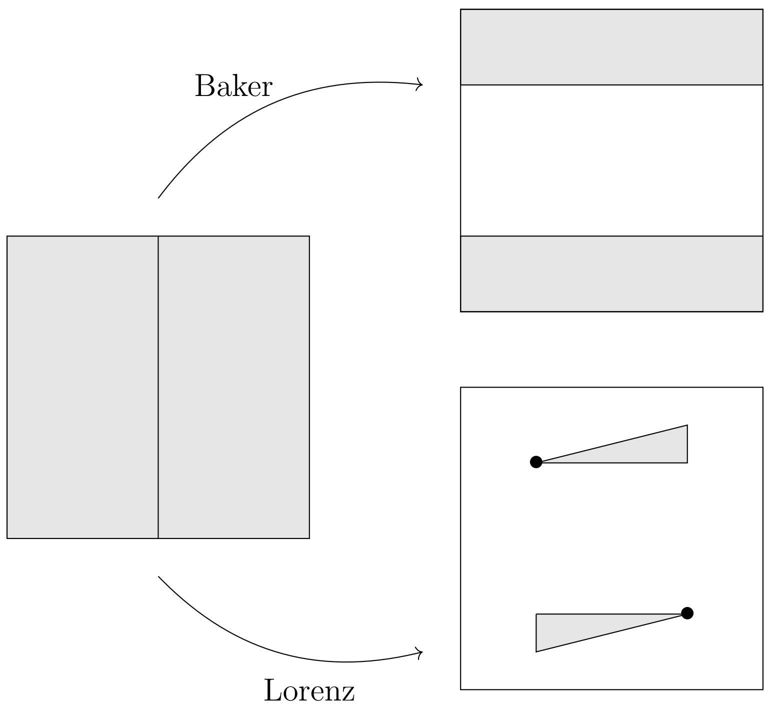
Clearly, this is exceptionaly difficult to compute. But it does have some nice properties!

- $\dim_{\text{Haus}}(C) = \frac{\ln 2}{\ln 3}$
- $\dim_{\text{Haus}}(S) = 0$ for any countable set $S \subseteq \mathbb{R}$

5.7.3 The Lorenz Attractor



Notice that the Lorenz map operates quite like the Baker's map:



The attractor A of the Lorenz equation satisfies:

- sensitive dependence
- infinitely many periodic orbits (though density is not known!)
- topologically transitive