APMA 1690: Quiz 2

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Problem 1

Consider an irreducible and aperiodic Markov chain $\{X_n\}_{n=0}^{\infty}$ with state space $\{1,2,3,4\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

(a) *(2 points) Write down the system of equations (including inequalities) for the stationary distribution $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$. NO NEED TO SOLVE IT. [Do not write the equations in a matrix form. I would like to see each equation explicitly.]*

Since the MC is irreducible and has a finite state space, it has a unique stationary distribution π given by

$$\pi^T = P^T \vec{\pi}$$

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} &\pi_1 = \frac{1}{3}\pi_2 + \frac{1}{2}\pi_4 \\ &\pi_2 = \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3 \\ &\pi_3 = \frac{1}{3}\pi_2 + \frac{1}{2}\pi_4 \\ &\pi_4 = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_3 \end{aligned}}$$

(b) *(1 point) Determine the limit (express it in terms of π_i's)*

$$\lim_{n\to\infty} \mathbb{P}(X_n = 1)$$

$\{X_n\}_{n=0}^{\infty}$ is an irreducible, aperiodic, Markov chain with a finite state space. By the Ergodic Theorem,

$$\lim_{n\to\infty} \mathbb{P}(X_n = x_j) = \pi(x_j)$$

so

$$\lim_{n\to\infty} \mathbb{P}(X_n = 1) = \pi(1) = \boxed{\frac{1}{3}\pi_2 + \frac{1}{2}\pi_4}$$

(c) *(1 point) Determine the value of v in the following equation (express them in terms of π_i's)*

$$\mathbb{P}\left\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i(\omega)=1 \text{ or } 2\}} = v\right\} = 1$$

where $(\Omega, \mathbb{P})$ is the underlying probability space.

The MC is irreducible and has a finite state space so we can use the 2nd Ergodic Theorem. Let $f : \mathfrak{X} \to \mathbb{R}$ such that $f(X_i) = \mathbb{1}_{\{X_i=1 \text{ or } 2\}}$. Then,

$$\sum_{x\in\mathfrak{X}} \pi(x) \cdot \big|f(x)\big| = \sum_{x\in\mathfrak{X}} \pi(x) \cdot \mathbb{1}_{\{x=1 \text{ or } 2\}} = \pi(x_1) + \pi(x_2)$$

Since the MC is also aperiodic, we can use the 1st Ergodic Theorem to get

$$\pi(x_1) + \pi(x_2) = \lim_{n\to\infty} \mathbb{P}(X_n = x_1) + \mathbb{P}(X_n = x_n) \leq 2 < \infty$$

so

$$\sum_{x\in\mathfrak{X}} \pi(x) \cdot \big|f(x)\big| < \infty$$

so

$$\mathbb{P}\left\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i(\omega)=1 \text{ or } 2\}} = \sum_{x\in\mathfrak{X}} \pi(x)\mathbb{1}_{\{x=1 \text{ or } 2\}}\right\} = 1$$

Thus,

$$v = \sum_{x\in\mathfrak{X}} \pi(x)\mathbb{1}_{\{x=1 \text{ or } 2\}} = \boxed{\pi(x_1) + \pi(x_2)}$$

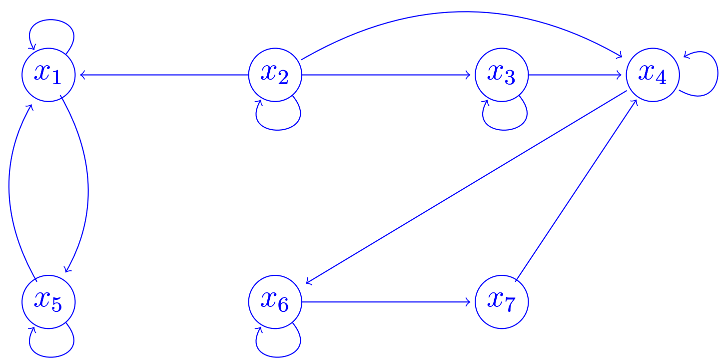
Problem 2

(2 point) Suppose $\{X_n\}_{n=0}^{\infty}$ is a homogeneous Markov chain taking values in $\mathfrak{X} = \{x_1, x_2, \dots,\ x_7\}$, and its transition matrix is the following

$$P = \begin{pmatrix} 0.3 & 0 & 0 & 0 & 0.7 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0.6 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Is this Markov chain irreducible? Justify your answer.

The transition matrix can be represented by the (unlabelled but weighted) directed graph:



We proved in HW 6 that a Markov Chain is irreducible if and only if $\forall x_i, x_j \in \mathfrak{X}$, there is a directed path from $x_i \to x_j$ and a directed path from $x_j \to x_i$.

The diagram helps make it clear that this Markov chain is $\boxed{\text{not}}$ irreducible. Notice that there is no directed path from $x_1 \to x_2$ – just from $x_2 \to x_1$.

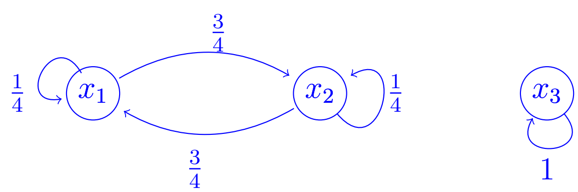
Problem 3

When the irreducibility assumption fails, a homogeneous Markov chain may have infinitely many stationary distributions. For example, suppose $\{X_n\}_{n=0}^{\infty}$ is a homogeneous Markov chain taking values in $\mathfrak{X} = x_1, x_2, x_3$, and its transition probability matrix is the following

$$P = \begin{pmatrix} 1/4 & 3/4 & 0 \\ 3/4 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) *(2 points) Prove that the Markov chain $\{X_n\}_{n=0}^{\infty}$ is reducible (i.e., not irreducible).*

This transition matrix can be represented by the graph



Clearly, there is no path between x_1 and x_3 nor between x_2 and x_3 so the MC is reducible. ■

(b) *(2 points) Find all the stationary distributions of this Markov chain.*

The stationary distributions are given by $\pi^T = \pi^T P$.

$$\begin{pmatrix} \pi(x_1) & \pi(x_2) & \pi(x_3) \end{pmatrix} = \begin{pmatrix} \pi(x_1) & \pi(x_2) & \pi(x_3) \end{pmatrix} \begin{pmatrix} 1/4 & 3/4 & 0 \\ 3/4 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} \pi(x_1) = \frac{1}{4}\pi(x_1) + \frac{3}{4}\pi(x_2) \\ \pi(x_2) = \frac{3}{4}\pi(x_1) + \frac{1}{4}\pi(x_2) \\ \pi(x_3) = \pi(x_3) \end{cases} \implies \begin{cases} \pi(x_1) = \pi(x_2) \\ \pi(x_3) = \pi(x_3) \end{cases}$$

Since the Markov Chain is reducible, there is not a unique stationary distribution.

Adding in the condition that $\pi(x_1) + \pi(x_2) + \pi(x_3) = 1$, we get

$$\boxed{\begin{cases} \pi(x_1) = \pi(x_2) = \frac{1-\pi(x_3)}{2} \\ \pi(x_3) \ \text{ free} \end{cases}}$$