APMA 1690: Quiz 1

Milan Capoor

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Problem 1

Consider the following PDF

$$p(t) = \frac{1}{2\sqrt{t}} \cdot \mathbb{1}_{\{0<t<1\}}.$$

Compute the CDF $F(t)$ associated with the PDF $p(t)$, i.e. $F(t) = \int_{-\infty}^{t} p(x)\,dx$

$$\begin{aligned}
F(t) &= \int_{-\infty}^{t} p(x)\,dx \\
&= \int_{-\infty}^{t} \frac{1}{2\sqrt{x}} \cdot \mathbb{1}_{\{0<x<1\}}\,dx \\
&= \int_{-\infty}^{0} 0\,dx + \int_{0}^{t} \frac{1}{2\sqrt{x}}\,dx \\
&= \int_{0}^{t} \frac{1}{2\sqrt{x}}\,dx \\
&= \frac{1}{2}\left[2\sqrt{x}\right]_0^t \\
&= \sqrt{t}
\end{aligned}$$

So

$$\boxed{F(t) = \begin{cases} 0 & t \le 0 \\ \sqrt{t} & 0 < t < 1 \\ 1 & t \ge 1 \end{cases}}$$

Problem 2

Let $U_1, U_2, \ldots, U_n, \ldots$ be $\mathbb{R}$-valued random variables defined on the probability space $(\Omega, \mathbb{P})$, and $U_1, U_2, \ldots, U_n, \ldots \overset{iid}{\sim} Unif(0,1)$. Define random variables V_n by

$$V_n(\omega) = \frac{1}{n} \sum_{i=1}^{n} (U_{2i-1}(\omega))^2 \cdot \mathbb{1}_{\{U_{2i}(\omega) < 0.5\}}, \qquad n = 1, 2, \ldots$$

What is the value v such that $\mathbb{P}(\omega \in \Omega : \lim_{n\to\infty} V_n(\omega) = v) = 1$? Provide the value of v and your justification.

By the LLN, for large enough n, the sample average of a sequence X_n converges to the expected value of X_1 with probability 1 given that X_n is a sequence of iid random variables and the expected value is defined.

Since V_n is defined as the average of a transformation of the sequence $U_1, \ldots, U_n$ which are iid RVs, $\{V_i\}_{i=1}^n$ is a sequence of iid RVs. Thus, to find v, we just need to determine the expected value of

$$\tilde{V}(\omega) = U_1(\omega)^2 \cdot \mathbb{1}_{\{U_2(\omega) < 0.5\}}$$

First we need to find the PDF of the associated distribution:

$$\begin{aligned} F_{\tilde{V}}(t) &= \mathbb{P}(U_1^2 \cdot \mathbb{1}_{U_2 < 0.5} \leq t) \\ &= \begin{cases} 0 & t < 0 \\ \mathbb{P}(U_1^2 \cdot \mathbb{1}_{U_2 < 0.5} \leq t) & 0 \leq t \leq 1 \\ 1 & t > 1 \end{cases} \end{aligned}$$

Looking at the middle case,

$$\mathbb{P}(U_1^2 \cdot \mathbb{1}_{U_2 < 0.5} \leq t) = \mathbb{P}(U_1 \cdot \mathbb{1}_{U_2 < 0.5} \leq \sqrt{t})$$

because the indicator function takes values only in $\{0, 1\}$ and is invariant under the square root.

We now consider all combinations of U_1, U_2 such that the product is less than or equal to $\sqrt{t}$ with $t \in [0, 1]$:

1. $\mathbb{P}(U_1 = 0) = 0$
2. $\mathbb{P}(U_2 \geq 0.5) = \frac{1}{2}$

3. $\mathbb{P}((U_1 \leq \sqrt{t}) \cap (U_2 < 0.5)) = \sqrt{t} \cdot \frac{1}{2} = \frac{\sqrt{t}}{2}$

We can sum these to get the total probability:

$$F_{\tilde{V}} = \begin{cases} 0 & t < 0 \\ \frac{1+\sqrt{t}}{2} & 0 \leq t \leq 1 \\ 1 & t > 1 \end{cases}$$

and differentiate to at last get the PDF:

$$p_{\tilde{V}} = \frac{1}{4\sqrt{t}}$$

Finally, we can confirm the expected value exists:

$$\begin{aligned} \mathbb{E}[\tilde{V}] &= \int_{-\infty}^{\infty} \left| t^2 \cdot \mathbb{1}_{t<0.5} \right| \cdot \frac{1}{4\sqrt{t}} \, dt \\ &= \int_{-\infty}^{\infty} \frac{1}{4} t^{\frac{3}{2}} \cdot \mathbb{1}_{t<0.5} \, dt \\ &= \int_{0}^{0.5} \frac{1}{4} t^{\frac{3}{2}} \, dt \\ &= \frac{1}{10} [t^{5/2}]_0^{0.5} = \frac{\sqrt{32}}{10} = \frac{2}{5}\sqrt{2} < \infty \end{aligned}$$

And in fact, because the expression inside the absolute value is always positive, the expected value is the value of the integral itself. Thus,

$$v = \mathbb{E}[\tilde{V}] = \boxed{\frac{2\sqrt{2}}{5}}$$

Problem 3

Let X be a continuous random variable whose PDF is the following

$$p(x) = cx^2 \cdot \mathbb{1}_{\{0 \le x \le 1\}}$$

where c is some constant.

(a) *What is the value of c?*

By the indicator function, $p(x)$ is only non-zero on the range $0 \le x \le 1$. Thus, the value of the CDF must be 1 at $x = 1$:

$$F(1) = \int_0^1 cx^2 \, dx = \frac{c}{3} = 1 \implies \boxed{c = 3}$$

(b) *Find the CDF of X.*

$$\begin{aligned} F(t) &= \int_{-\infty}^{t} p(x) \, dx \\ &= \int_{-\infty}^{t} 3x^2 \cdot \mathbb{1}_{\{0 \le x \le 1\}} \, dx \\ &= \int_0^t 3x^2 \, dx = [x^3]_0^t = t^3 \end{aligned}$$

So

$$\boxed{F(t) = \begin{cases} 0 & t < 0 \\ t^3 & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}}$$

(c) *Suppose U is a $\mathbb{R}$-valued random variable defined on the probability space $(\Omega, \mathbb{P})$ and $U \sim$ Unif$(0,1)$. Construct a random variable $\tilde{X}$ using U such that the CDF of $\tilde{X}$ is the CDF you got in (b). You have to present your answer in an analytic form, i.e., the infimum sign "inf" is forbidden in your answer.*

By the inverse CDF method,

$$\tilde{X}(\omega) = G(U(\omega))$$

where

$$G(u) = \inf\{t \in \mathbb{R} : F(t) \geq u\}, \qquad u \in (0,1)$$

Here,

$$F(t) = t^3 \qquad (0 \leq t \leq 1)$$

So

$$\begin{aligned} G(u) &= \inf\{t \in \mathbb{R} : F(t) \geq u\} \\ &= \inf\{t \in \mathbb{R} : t^3 \geq u\} \\ &= \inf\{t \in \mathbb{R} : \sqrt[3]{u} \leq t\} \\ &= \sqrt[3]{u} \end{aligned}$$

So

$$\boxed{\tilde{X} = \sqrt[3]{U}}$$

(d) *Justify your answer to (c).*

$$\begin{aligned} F_{\tilde{X}}(t) &= \mathbb{P}(\tilde{X} \leq t) \\ &= \mathbb{P}(\sqrt[3]{U} \leq t) \\ &= \mathbb{P}(U \leq t^3) \\ &= F_U(t^3) \\ &= \begin{cases} 0 & t < 0 \\ t^3 & 0 \leq t \leq 1 \\ 1 & 1 < t \end{cases} \qquad \blacksquare \end{aligned}$$

Problem 4

Let d be any positive integer. Recall/accept the following fact: The d-dimensional simple random walk is a homogeneous Markov chain whose state space is $\mathfrak{X} = \mathbb{Z}^d = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$.

Prove the following: The d-dimensional simple random walk is irreducible. (Hint: You may consider applying the following result that you have proved: if $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.)

A homogeneous Markov chain is irreducible if for all $x, y \in \mathfrak{X}$,

$$\rho_{xy} = \mathbb{P}(T_y < \infty \mid X_0 = x) > 0$$

The condition $T_y < \infty$ is equivalent to the condition that $X_n = y$ for some finite n. Thus,

$$\rho_{xy} = \mathbb{P}(X_n = y \mid X_0 = x)$$

But we notice that the 1-dimensional random walk is a subset of the d-dimensional walk: Consider states $x_n, x_{n+1} \in \mathbb{Z}^d$:

$$x_n = \begin{pmatrix} x_n^{(1)} \\ x_n^{(2)} \\ \vdots \\ x_n^{(d)} \end{pmatrix}, \qquad x_{n+1} = \begin{pmatrix} x_{n+1}^{(1)} \\ x_{n+1}^{(2)} \\ \vdots \\ x_{n+1}^{(d)} \end{pmatrix}$$

Then $x_{n+1} = x_n + \xi_n$ and element wise, $x_{n+1}^{(1)} = x_n^{(1)} + \xi_n^{(1)}$ where $x_{n+1}^{(1)}, x_n^{(1)} \in \mathbb{Z}$ and $\xi_n^{(1)} \in \{1, -1\}$.

Thus, we can say that

$$\mathbb{P}(X_{n+1}^{(1)} = x_{n+1}^{(1)} \mid X_n^{(1)} = x_n^{(1)}) \leq \mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

because for any events $A \subset B \implies \mathbb{P}(A) \leq \mathbb{P}(B)$.

In HW 5 we proved the 1-d SRW is homogeneous, so $\exists p$ which is not a function of n such that:

$$p(x_n^{(1)}, x_{n+1}^{(1)}) \leq \mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

So calculating for the 1-d random walk,

$$\begin{aligned}
p(y^{(1)}, x^{(1)}) &= \mathbb{P}(X_n^{(1)} = y^{(1)} \mid X_0^{(1)} = x^{(1)}) \\
&= \frac{\mathbb{P}(X_n^{(1)} = y^{(1)}, X_0^{(1)} = x^{(1)})}{\mathbb{P}(X_0^{(1)} = x^{(1)})} \\
&= \frac{\mathbb{P}(X_n^{(1)} = y^{(1)}, X_{n-1}^{(1)} = x_{n-1}^{(1)}, \ldots,\ X_0^{(1)} = x^{(1)})}{\mathbb{P}(X_0^{(1)} = x^{(1)})} \\
&= \frac{\mu(x^{(1)}) \prod_{i=1}^{n-1} p(x_i^{(1)}, x_{i+1}^{(1)})}{\mu(x^{(1)})} \\
&= \prod_{i=1}^{n-1} p(x_i^{(1)}, x_{i+1}^{(1)})
\end{aligned}$$

Since, $p(x_i^{(1)}, x_{i+1}^{(1)})$ is a probability, $0 \le p(x_i^{(1)}, x_{i+1}^{(1)}) \le 1$. But by definition of the 1-dim SRW,

$$p(x, y) = \mathbb{P}(X_n = y \mid X_{n-1} = x) = \mathbb{P}(\xi_n = y - x) = \frac{1}{2} > 0$$

So all the terms in the product are greater than 0, and

$$0 < p(y^{(1)}, x^{(1)}) < \mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

so the MC is irreducible. $\blacksquare$