Everything in a Nutshell

1. **(two-spin particle)** For every measurement direction $n \in \mathbb{R}^3_1$), $\lambda_n \in \{-1, 1\}$, where λ_n is the spin deflection (down or up, respectively) when measured in the n direction.

2. **(state space)** To every direction $n \in \mathbb{R}^3_1$ there is an associated state $|n^+\rangle \in \mathbb{C}^2_1$:

$$\left|n^+\right\rangle \doteq \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \in \mathbb{C}^2_1$$

   where ϕ and θ are the polar and azimuth angles of the polar coordinates of n. Write $|n^-\rangle$ for the state associated with the direction $-n$. Since $-n$ has polar angle $\pi - \phi$ and azimuth angle $\theta + \pi$, $|n^-\rangle$ and $|n^+\rangle$ are orthogonal in $\mathbb{C}^2_1$

3. **(spare degree of freedom)** For any η, $e^{i\eta}\,|n^+\rangle$ and $|n^+\rangle$ refer to the same physical state.

4. **(cardinal directions)**

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad |r\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad |l\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \quad |i\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \quad |o\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix}$$

5. **($\mathbf{n}-$spin particles & $\mathbf{C}^n_1$)** In general, discrete wave functions of single particles are formulated as elements of the vector space $\mathbb{C}^n$ with inner product $\langle\psi|\phi\rangle \in \mathbb{C}$. Complex-valued inner products are assumed to be conjugate linear in the first argument and linear in the second: if $|\Psi\rangle = \sum_{i=1}^{a} \alpha_i\,|\psi_i\rangle$ and $|\Phi\rangle = \sum_{j=1}^{b} \beta_j\,|\phi_j\rangle$ then

$$\langle\Psi|\Phi\rangle = \sum_{i=1}^{a}\sum_{j=1}^{b} \alpha_i^*\beta_j$$

   The dimension n is the number of possible outcomes of any given measurement of the state.

6. **(measurements)** A measurement A returns one of n real numbers, $\lambda_k^{(A)}$, $k = 1 : n$, and leaves the measured particle in one of n corresponding states $\left|e_k^{(A)}\right\rangle$, $k = 1 : n$. The set of possible states that result from a measurement are orthonormal: $\left\langle e_k^{(A)}\middle|e_l^{(A)}\right\rangle = \delta_{k,l}$. (The superscript (A) is rarely needed.)

7. **(definite state)** The state resulting from a measurement is called a *definite state*. But it is just another state, and only "definite" in the sense that repeating the same measurement will give the same answer and will not alter the state.

8. **(Hermitian operators)** Given a measurement A, the set of measured values λ_k and their corresponding definite states $|e_k\rangle$, $k = 1 : n$, can be combined to define a Hermitian matrix, also called A:

$$A \doteq \sum_{k=1}^{n} \lambda_k\,|e_k\rangle\langle e_k|$$

What's more, every $n \times n$ Hermitian matrix corresponds to a measurement (in principle—some might be difficult to implement) through it's *spectral representation*: If A is an $n \times n$ Hermitian matrix, then there exists scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and orthonormal vectors $|e_1\rangle, \ldots, |e_n\rangle \in \mathbb{C}^n$ such that

$$A = \sum_{k=1}^{n} \lambda_k |e_k\rangle\langle e_k|$$

In both cases, $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A with corresponding eigenvectors $|e_1\rangle, \ldots, |e_n\rangle$.

9. **(spin operators in $\mathbb{C}_1^2$)** For any $n = (n_x, n_y, n_z) \in \mathbb{R}_1^3$, the operator that measures spin in the $\pm n$ direction (which is called σ_n) can be written as

$$\sigma_n = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$$

where ϕ and θ are the polar and azimuth angles of n, and σ_x, σ_y, and σ_z are the operators measuring spin in the x, y, and z directions respectively:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

10. **(probabilities & squared overlap)** For any initial state $|\psi\rangle \in \mathbb{C}_1^n$ and measurement A, the probability that the observation results in the definite state $|e_k\rangle$ (and hence yields the value λ_k) is given by the "squared overlap": $|\langle\psi|e_k\rangle|^2$. The inner product $\langle\psi|e_k\rangle$ is sometimes called the probability amplitude.

$$\mathbb{P}\left(|\psi\rangle \xrightarrow{A} |e_k\rangle\right) = |\langle\psi|e_k\rangle|^2$$

In fact, the expression is unambiguous without indicating A on the left-hand side. All that matters is that a measurement was made for which $|e_k\rangle$ is a definite state. Hence we could write

$$\mathbb{P}\left(|\psi\rangle \to |\phi\rangle\right) = |\langle\psi|\phi\rangle|^2$$

for any two states $|\psi\rangle$ and $|\phi\rangle$, and there would be no need to specify a particular measurement.

11. **(probabilities & overlap in $\mathbb{C}_1^2$)** For any $n, m \in \mathbb{R}_1^3$

$$\mathbb{P}(\lambda_m = 1 \,\big|\, |\psi\rangle = |n^+\rangle) = \frac{1 + m \cdot n}{2} = \mathbb{P}(|n^+\rangle \to |m^+\rangle) = \left|\langle m^+|n^+\rangle\right|^2$$
$$\mathbb{P}(\lambda_m = -1 \,\big|\, |\psi\rangle = |n^+\rangle) = \frac{1 - m \cdot n}{2} = \mathbb{P}(|n^+\rangle \to |m^-\rangle) = \left|\langle m^-|n^+\rangle\right|^2$$

12. **(expectations)** In general, if the measurement A is represented by measurement values λ_k, $k = 1 : n$, and corresponding definite states $|e_k\rangle$, $k = 1 : n$, then given any starting state $|\psi\rangle$ the observable value of the measurement, call it λ^A or just λ, is a random variable with probabilities given by the squared overlap:

$$\mathbb{P}\left(\lambda = \lambda_k \big|\, |\psi\rangle\right) = |\langle\psi|e_k\rangle|^2$$

Its *expected value* is the probability-weighted average measurement:

$$\mathbb{E}\left[\lambda \big|\, |\psi\rangle\right] = \sum_{k=1}^{n} \mathbb{P}\left(\lambda = \lambda_k \big|\, |\psi\rangle\right) \lambda_k = \sum_{k=1}^{n} \lambda_k |\langle\psi|e_k\rangle|^2 \tag{1}$$

This can be rewritten, conveniently, as a simple matrix-vector operation:

$$\mathbb{E}\left[\lambda\middle|\,|\psi\rangle\right] = \langle\psi|A|\psi\rangle \tag{2}$$

13. **(probabilities as expectation)** For any $|\psi\rangle$ and $|\phi\rangle$

$$\mathbb{P}\left(|\psi\rangle \to |\phi\rangle\right) = |\langle\psi|\phi\rangle|^2 = \langle\psi|\phi\rangle\langle\psi|\phi\rangle^* = \langle\psi|\phi\rangle\langle\phi|\psi\rangle = \langle\psi|A|\psi\rangle$$

where A is the particular (rank-one) Hermitian operator $A = |\phi\rangle\langle\phi|$, which has the single non-zero eigenvalue $\lambda = 1$. All the other eigenvalues are zero.

14. **(more expectations)** If H, representing a measurement, is an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $|e_1\rangle, \ldots, |e_n\rangle$, then we can define a new measurement by changing the observables, i.e. the eigenvalues, without changing the eigenvectors: for any function $f : \mathbb{R} \to \mathbb{R}$

$$\tilde{H} \doteq \sum_{k=1}^{n} f(\lambda_k)\,|e_k\rangle\langle e_k|$$

The relationship between H and $\tilde{H}$ is conveniently summarized by writing $\tilde{H} = f(H)$. This extends the reach of expectations to expectations of functions of the observables:

$$\mathbb{E}\left[\lambda\middle|\,|\psi\rangle\right] = \langle\psi|f(A)|\psi\rangle = \sum_{k=1}^{n} f(\lambda_k)|\langle\psi|e_k\rangle|^2$$

(In fact, for well-behaved function f the relationship $\tilde{H} = f(H)$ can be taken literally, e.g. you can check that the matrix HH has the same eigenvectors as H, but the corresponding eigenvalues are squared.)

15. **(Pythagorean principle for spins states in $\mathbb{C}_1^2$)**

$$E\left[\lambda_{\mathbf{x}}\middle|\,|\psi\rangle\right]^2 + E\left[\lambda_{\mathbf{y}}\middle|\,|\psi\rangle\right]^2 + E\left[\lambda_{\mathbf{z}}\middle|\,|\psi\rangle\right]^2 = 1$$

16. **(commutators and the ordering of measurements)** Given any two Hermitian matrices A and B, the measurement A and followed by the measurement B will not typically give the same results and the measurement of B followed by A. In fact, they give the same results for every starting state $|\psi\rangle$ if and only if $[A, B] = 0$, where $[A, B] \doteq AB - BA$ is called the commutator. When $[A, B] = 0$, A and B are said to commute.

17. **(uncertainty principle)** Given a starting state $|\psi\rangle$ and a measurement A, $\lambda^{(A)}$ is a random variable with some mean μ_A and variance δ_A^2, both of which depend on $|\psi\rangle$:

$$\mu_A = \mathbb{E}\left[\lambda^{(A)}\middle|\,|\psi\rangle\right] \quad \text{and} \quad \delta_A^2 = \mathbb{E}\left[(\lambda^{(A)})^2 - \mu_A^2\middle|\,|\psi\rangle\right]$$

The *uncertainty principle* asserts that for any two measurements A and B and any starting state $|\psi\rangle$, we can not expect to simultaneously accurately predict the outcomes of measuring A and B unless A and B happen to commute:

$$\delta_A\delta_B \geqslant \frac{1}{2}|\langle\psi|[A, B]|\psi\rangle|$$

18. **(Hamiltonians and unitary operators)** The *Hamiltonian*, H, is the Hermitian operator corresponding to the measurement of the total energy. This will usually involve continuous variables like time, position and velocity. Unitary operators (matrices in finite dimensions) are defined by the property $U^\dagger U = UU^\dagger = I$. Unitary operators obey a spectral decomposition that is similar to the decomposition obeyed by Hermitian operators: If U is an $n \times n$ unitary matrix then there exists $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $|e_1\rangle, \ldots, |e_n\rangle \in \mathbb{C}^n$ such that

$$U = \sum_{k=1}^{n} \lambda_k |e_k\rangle\langle e_k|$$

where $|\lambda_k|^2 = 1$ (i.e. $\lambda_k = e^{i\theta_k}$ for some $\theta_k \in \mathbb{R}$) and $\langle e_k|e_l\rangle = \delta_{k,l}$, for all k and l. The importance of unitary operators stems from the fact that they preserve normalization and inner products: If U is unitary, then

$$\langle (U|\phi\rangle)|(U|\psi\rangle)\rangle = \langle\phi| U^\dagger U |\psi\rangle = \langle\phi|\psi\rangle$$

(Think of a unitary operator as a generalization of a rigid rotation.)

19. **(the Schrödinger equation)** The Hamiltonian defines the continuous evolution of a state $|\psi\rangle$, which will typically depend on both time t and position r, $|\psi(t,r)\rangle$. The Hamiltonian determines the evolution via the continuous evolution of a unitary operator:

$$|\psi(t,r)\rangle = U_t |\psi(0,r)\rangle = e^{-\frac{i}{\hbar}tH(r)}$$

where $H(r)$ is the Hamiltonian, assumed here to be time-independent, and the exponential of an operator (or matrix) is defined via the spectral representation used earlier to define $f(A)$ for any Hermitian operator A. Or, written as a differential equation

$$\frac{\partial}{\partial t}|\psi(t,r\rangle = \frac{-i}{\hbar}H(r)|\psi(t,r)\rangle$$

The Schrödinger equation can be used to solve for the time evolution of a state, given the Hamiltonian, or in reverse: given a desired unitary transformation and time t, find the Hamiltonian that will generate the desired transformation in the desired time.

20. **(tensor-product space, $\mathbf{V} \otimes \mathbf{W}$)** For any orthonormal basis $|e_1\rangle, \ldots, |e_n\rangle$ of V and orthonormal basis $|f_1\rangle, \ldots, |f_m\rangle$ of W, the tensor-product space of $V \otimes W$ can be written as

$$V \otimes W = \left\{ \sum_{i=1}^{n}\sum_{j=1}^{m} \gamma_{ij} |e_i f_j\rangle \,\middle|\, \gamma_{ij} \in \mathbb{C},\ \forall\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m \right\} \tag{3}$$

21. **(inner product in $\mathbf{V} \otimes \mathbf{W}$)** For any $|\psi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \gamma_{ij}|e_i f_j\rangle$ and $|\phi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_{ij}|e_i f_j\rangle$

$$\langle\phi|\psi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_{ij}^* \gamma_{ij}$$

and hence

$$\langle\psi|\psi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} |\gamma_{ij}|^2$$

(i) It follows that $\langle e_i j_j | e_k f_l \rangle = \delta_{i,k}\delta_{j,l}$ (take $|\phi\rangle = |e_i f_j\rangle$ and $|\psi\rangle = |e_k, f_l\rangle$)

(ii) The inner product $\langle\phi|\psi\rangle$ is conjugate linear in the first argument and linear in the second.

22. **(independence)** The vector

$$|\psi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \gamma_{ij} |e_i f_j\rangle \in V \otimes W$$

represents two *independent particles*, particle 1 in V and particle 2 in W, if and only if there exists $|v\rangle = \sum_{i=1}^{n} \alpha_i |e_i\rangle \in V$ and $|w\rangle = \sum_{j=1}^{m} \beta_j |f_j\rangle \in W$ such that $\gamma_{ij} = \alpha_i\beta_j$ for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. If particles 1 and 2 are not independent, then we say that they are *entangled*.

(i) If follows that $|e_i f_j\rangle$ represents two independent particles (just take $|v\rangle = |e_i\rangle$ and $|w\rangle = |f_j\rangle$).

(ii) We often indicate that the pair of $|v\rangle$ and $|w\rangle$ are independent in $V \otimes W$ by writing their joint state as $|vw\rangle$.

(iii) Another way to write $|vw\rangle$ is to write the pair as a *tensor product*, $|vw\rangle = |v\rangle \otimes |w\rangle$, where the *tensor-product operator*, mapping $V \times W \to V \otimes W$, is bi-linear, and therefore defined by its actions on the basis elements of V and W. The tensor-product notation makes it easy to find the vector in $V \otimes W$ associated with any pair from V and W. For an example, consider $\mathbb{C}^2 \otimes \mathbb{C}^2$ using the up-down basis for both particles:

$$\begin{aligned} |ll\rangle &= \left(\frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle\right) \\ &= \frac{1}{2}|u\rangle \otimes |u\rangle - \frac{1}{2}|u\rangle \otimes |d\rangle - \frac{1}{2}|d\rangle \otimes |u\rangle + \frac{1}{2}|d\rangle \otimes |d\rangle \\ &= \frac{1}{2}|uu\rangle - \frac{1}{2}|ud\rangle - \frac{1}{2}|du\rangle + \frac{1}{2}|dd\rangle \end{aligned}$$

(iv) If $|a\rangle, |c\rangle \in V$ and $|b\rangle, |d\rangle \in W$, then

$$\langle ac|bd\rangle_{V\otimes W} = \langle a|c\rangle_V \langle b|d\rangle_W$$

23. **(measurements in $\mathbf{V \otimes W}$)** If A and B are Hermitian operators (measurements, Hermitian matrices) on V and W then $A \otimes B : V \otimes W \to V \otimes W$ is defined by linear extension from $A \otimes B\,|e_i f_j\rangle = |Ae_i Bf_j\rangle\ \forall i, j$:
for any $|\phi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \gamma_{ij}|e_i f_j\rangle$

$$A \otimes B = \sum_{i=1}^{n}\sum_{j=1}^{m} \gamma_{ij} |Ae_i Bf_j\rangle$$

Which then also applies to any independent pair, $v \in V$ and $w \in W$: $A \otimes B\,|vw\rangle = |AvBw\rangle$.

24. **(characterization of Hermitian operators on $\mathbf{V \otimes W}$)** From (23.), it follows that

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger \tag{4}$$

which implies that $A \otimes B$ is Hermitian if and only if A and B are both Hermitian. And from (4) if follows that every Hermitian operator H on $V \otimes W$ can be written as

$$H = \sum_{k=1}^{N} A_k \otimes B_k$$

for some N, N Hermitian operators $A_1, \ldots A_N$ on V, and N more Hermitian operators $B_1, \ldots B_N$ on W.

25. **(three or more particles)** These definitions and their consequences are recursive, e.g. we can use the basis $\{|e_i f_j\rangle\}_{i=1:n, j=1:m}$ for $V \otimes W$ and $|u_1\rangle, \ldots, |u_l\rangle$ for U to define the tensor-product space

$$(V \otimes W) \otimes U = V \otimes W \otimes U = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} \gamma_{ijk} |e_i f_j u_k\rangle \,\middle|\, \gamma_{ijk} \in \mathbb{C},\ \forall\ 1 \leqslant i \leqslant n\ \ 1 \leqslant j \leqslant m,\ 1 \leqslant k \leqslant l \right\}$$

26. **(probabilities and expectations)** In general, the rules for computing probabilities of outcomes and expectations of observables are the same, which should not be surprising since the tensor-product space is also an inner-product space, but with dimension nm (which id the dimension of $V \otimes W$, and the product of the dimensions of V and W). Thus if H is a Hermitian operator on $V \otimes W$, then H is defined by nm orthonormal eigenvectors, $|\phi_1\rangle, \ldots, |\phi_{nm}\rangle \in V \times W$, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_{nm}$.

    (i) The probability, given the current state $|\psi\rangle \in V \otimes W$, that the measurement results in the definite state $|\phi_k\rangle$, for some $k = 1, 2, \ldots nm$:

    $$\mathbb{P}\left(|\psi\rangle \xrightarrow{H} |\phi_k\rangle\right) = |\langle\psi|\phi_k\rangle|^2$$

    Or, as noted in item (10.), just drop the H and write $\mathbb{P}(|\psi\rangle \to |\phi\rangle) = |\langle\psi|\phi\rangle|^2$ for any pair $|\psi\rangle, |\phi\rangle \in V \otimes W$.

    (ii) Consider $\mathbb{C}_1^2 \otimes \mathbb{C}_1^2$ and the singlet state $|sing\rangle = \frac{1}{\sqrt{2}} |ud\rangle - \frac{1}{\sqrt{2}} |du\rangle$, for example. Conclude that

    $$\mathbb{P}\left(|sing\rangle \to |ud\rangle\right) = |\langle sing|ud\rangle|^2 = \frac{1}{2} \quad \text{and} \quad \mathbb{P}\left(|sing\rangle \to |du\rangle\right) = |\langle sing|du\rangle|^2 = \frac{1}{2}$$

    (iii) Item (i) leads to the expected value of $\lambda^{(H)}$ given $|\psi\rangle$:

    $$\mathbb{E}\left[\lambda^{(H)} \middle| |\psi\rangle\right] = \sum_{k=1}^{nm} \mathbb{P}\left(|\psi\rangle \xrightarrow{H} |\phi_k\rangle\right) \lambda_k = \sum_{k=1}^{nm} \lambda_k |\langle\psi|\phi_k\rangle|^2 = \langle\psi|H|\psi\rangle$$

    which is identical to the expression derived for single spin-state particles.

    (iv) Following the same reasoning used in item (13.), for any $|\psi\rangle$ and $|\phi\rangle$ we can write

    $$\mathbb{P}\left(|\psi\rangle \to |\phi\rangle\right) = \langle\psi|H|\psi\rangle$$

    where $H = |\phi\rangle\langle\phi|$.

27. **(special case: expectations when $\mathbf{H} = A \otimes B$)** If A and B are the Hermitian operators that correspond, respectively, to the measurements of $\lambda^{(A)}$ and $\lambda^{(B)}$, then, for example:

    (a)
    $$\mathbb{E}[\lambda_A \lambda_B \big| \, |\psi\rangle] = \langle\psi| \, A \otimes B \, |\psi\rangle$$

    (b)
    $$\mathbb{E}[\lambda_A \big| \, |\psi\rangle] = \langle\psi| \, A \otimes I \, |\psi\rangle$$

    (c)
    $$\mathbb{E}[\lambda_A + \lambda_B \big| \, |\psi\rangle] = \langle\psi| \, A \otimes I \, |\psi\rangle + \langle\psi| \, I \otimes B \, |\psi\rangle$$

28. **(flattening)** Tensor-product spaces are inner-product spaces, i.e. vector spaces equipped with an inner product. As such, it is sometimes convenient to translate ("flatten") them into the familiar vector/matrix representation. This can be done by the following rules: Let

    $$V \otimes W = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij} \, |e_i f_j\rangle \, \Big| \, \gamma_{ij} \in \mathbb{C}, \; \forall \, 1 \leqslant i \leqslant n, \; 1 \leqslant j \leqslant m \right\}$$

    and assume that A is represented in the basis $|e_1\rangle, \ldots, |e_n\rangle$ as an $n \times n$ matrix with components a_{ij}, and B is represented in the basis $|f_1\rangle, \ldots, |f_m\rangle$ as an $m \times m$ matrix with components b_{ij}. Then $A \otimes B$ can be represented by an $nm \times nm$ matrix

    $$A \otimes B \to \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ & & \vdots & \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}$$

    where $a_{ij}B$ is the $m \times m$ matrix whose k, l component is $a_{ij}b_{kl}$. The corresponding "flattening" of $|\psi\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij} \, |e_i f_j\rangle$ is then

    $$|\psi\rangle \to \begin{pmatrix} \gamma_{11} \\ \gamma_{12} \\ \vdots \\ \gamma_{1m} \\ \vdots \\ \vdots \\ \gamma_{n1} \\ \gamma_{n2} \\ \vdots \\ \gamma_{nm} \end{pmatrix}$$