APMA 1930X: Homework 3

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1. (Kelly Criterion) Suppose you are playing a betting game with the initial stake $S_0 = x_0 >_0$. Let S_n be your wealth at the end of the n-th round. Let Y_{n+1} be the amount you bet at the $(n+1)$-th round. Your winning per-unit-stake on the $(n+1)$-th round of betting can be represented by X_{n+1}, where $\{X_1, X_2, \dots\}$ are iid random variables with

$$P(X_{n+1} = a) = p, P(X_{n+1} = -b) = q = 1 - p$$

In other words, you will win the amount aY_{n+1} with probability p, and lose the amount bY_{n+1} with probability q. The dynamics of the wealth process is given by

$$S_{n+1} = S_n + X_{n+1}Y_{n+1}, \quad n \geq 0$$

Here we should assume that $a, b > 0$ and the game is in favor of you, that is,

$$\mathbb{E}[X_{n+1}] = ap - bq > 0$$

and

$$0 \leq Y_{n+1} \leq \frac{S_n}{b}$$

The upper bound on Y_{n+1} is to ensure that the loss will not exceed S_n.

Suppose the game ends at the N-th round. We wish to find an optimal betting strategy $\{Y_1^*, Y_2^*, \dots, Y_N^*\}$ so as the maximize the expected utility $\mathbb{E}[\log S_N]$

Hint: Mimic Example 2.9. Verify the the value function takes the form

$$v_n(x) = \log x + (N - n)c$$

for some constant c.

Let $V_n(x) = \sup \mathbb{E}[\log S_N \mid S_n = x]$, i.e. the maximum expected utility at time n given that the wealth is x.

Clearly,

$$V_N(x) = \sup \mathbb{E}[\log S_n \mid S_N = x] = \sup \mathbb{E}[\log x] = \log x$$

Now, per usual, suppose that (with $0 \leq n \leq N-1$), $S_n = x$ and we bet $Y_{n+1} = y$ at the $n+1$-th round but then bet optimally for $n+2, \dots, N$.

If we win (with probability p), then $S_{n+1} = x + ay$ and if we lose (with probability q), then $S_{n+1} = x - by$. In the case we win, our maximum utility is just $V_{n+1}(x + ay)$ and in the case we lose, our maximum utility is $V_{n+1}(x - by)$.

Hence,

$$\mathbb{E}[V_{n+1}(S_{n+1})] = pV_{n+1}(x+ay) + qV_{n+1}(x-by)$$

So to find the optimal betting strategy, we introduce the DPE

$$\begin{cases} V_n(x) = \sup_{0\le y\le x}[pV_{n+1}(x+ay) + qV_{n+1}(x-by)] \\ V_N(x) = \log x \end{cases}$$

Let's calculate a few cases:

$$\begin{aligned} V_{N-1}(x) &= \sup_{0\le y\le x}[pV_N(x+ay) + qV_N(x-by)] \\ &= \sup_{0\le y\le x}[p\log(x+ay) + q\log(x-by)] \end{aligned}$$

And

$$\begin{aligned} 0 &= \frac{d}{dy}[p\log(x+ay) + q\log(x-by)] \\ &= \frac{ap}{x+ay} - \frac{bq}{x-by} \\ &\implies y = \frac{xap - xbq}{ab(p+q)} \end{aligned}$$

So

$$y^* = \frac{x(ap-bq)}{ab}$$

Let $c^* = \frac{ap-bq}{ab}$. Then

$$\begin{aligned} V_{N-1}(x) &= p\log(x+ac^*x) + q\log(x-bc^*x) \\ &= p\log(x(1+ac^*)) + q\log(x(1-bc^*)) \\ &= p\log x + p\log(1+ac^*) + q\log x + q\log(1-bc^*) \\ &= p\log x + p\log(1+ac^*) + (1-p)\log x + q\log(1-bc^*) \\ &= \log x + p\log(1+ac^*) + q\log(1-bc^*) \end{aligned}$$

But the last two terms are constants so $V_{n-1}(x) = \log x + c$ for $c = p\log(1+ac^*) + q\log(1-bc^*)$.

Similarly,

$$\begin{aligned} V_{N-2}(x) &= pV_{N-1}(x+ay^*) + qV_{N-1}(x-by^*) \\ &= p(\log(x+axc^*) + c) + q(\log(x-bxc^*) + c) \\ &= p\log x + p\log(1+ac^*) + pc + q\log x + q\log(1-bc^*) + qc \\ &= \log x + (p+q)c + p\log(1+ac^*) + q\log(1-bc^*) \\ &= \log x + c + c \\ &= \log x + 2c \end{aligned}$$

Suppose that $V_{n+1}(x) = \log x + (N-n-1)c$. Then

$$\begin{aligned} V_n(x) &= \sup_{0\le y\le x} [pV_{n+1}(x+ay) + qV_{n+1}(x-by)] \\ &= \sup_{0\le y\le x} [p\log(x+ay) + p(N-n-1)c + q\log(x-by) + q(N-n-1)c] \end{aligned}$$

Taking the derivative with respect to y, we get

$$\begin{aligned} 0 &= \frac{d}{dy}[p\log(x+ay) + p(N-n-1)c + q\log(x-by) + q(N-n-1)c] \\ &= \frac{ap}{x+ay} - \frac{bq}{x-by} \\ &\implies y = x\frac{ap-bq}{ab(p+q)} = xc^* \end{aligned}$$

so

$$\begin{aligned} V_n(x) &= [p\log(x+ay^*) + p(N-n-1)c + q\log(x-by^*) + q(N-n-1)c] \\ &= p\log(x+axc^*) + p(N-n-1)c + q\log(x-bxc^*) + q(N-n-1)c \\ &= (p+q)\log x + p\log(1+ac^*) + q\log x\log(1-bc^*) + (p+q)(N-n-1)c \\ &= \log x + c + (N-n-1)c \\ &= \log x + (N-n)c \end{aligned}$$

Hence, by induction, $V_n(x) = \log x + (N-n)c$ for

$$c = p\log(1+\frac{ap-bq}{b}) + q\log(1-\frac{ap-bq}{a})$$

So the optimal betting strategy is $Y_n^* = \frac{S_n(ap-bq)}{ab}$.