APMA 1930X: Homework 6

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Fix an arbitrary time $T > 0$. Let r, θ, σ be given constants with $\sigma > 0$. Consider the controlled process

$$\frac{dX_t}{X_t} = (r + \theta\pi_t)\,dt + \sigma\pi_t\,dBt, \quad X_0 = x_0$$

where $\pi = \pi_t : 0 \leq t \leq T$ is the control.

Our goal is to maximize the expected power utility $\mathbb{E}[\sqrt{X_T}]$ over all control policies π. Your task is to

1. clearly define the value function;
2. write down the Hamilton-Jacobi-Bellman (HJB) equation and give some short explanation as to how you arrive at this equation;
3. solve the HJB equation explicitly;
4. identify the optimal control policy π^* explicitly.

Hint: In order to solve the equation, you can assume that the value function $v(t, x)$ takes the form $f(t)\sqrt{x}$ for some function f. Then you can turn the HJB equation (a partial differential equation) into a very simple ordinary differential equation of f.

Our goal is to maximize the expected power utility $\mathbb{E}[\sqrt{X_T}]$ over all control policies π on $[0, T]$.

Hence,

$$v(t, x) = \sup_{\pi_t} \mathbb{E}\left[\int_t^T \sqrt{X_s}\,ds \mid X_t = x\right]$$

and we seek $v(0, x_0)$.

Let $\varepsilon > 0$ be a small increment in time. As always, applying an optimal control for $[t + \varepsilon, T]$ and an arbitrary control $[t, t + \varepsilon]$,

$$v(t, x) \geq \mathbb{E}\left[\int_t^{t+\varepsilon} \sqrt{X_s}\,ds + v(t + \varepsilon, X_{t+\varepsilon})\right]$$

First consider $\mathbb{E}[v(t + \varepsilon, X_{t+\varepsilon})]$:

$$\begin{aligned} d(v(t, X_t)) &= \partial_t v(t, X_t)\,dt + \partial_x v(t, X_t)\,dX_t + \frac{1}{2}\partial_{xx} v(t, X_t)\,(dX_t)^2 \\ &= \left[\frac{\partial v}{\partial t} + (r + \theta\pi_t)x\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2\pi_t^2 x^2 \frac{\partial^2 v}{\partial x^2}\right] dt + \sigma\pi_t x \frac{\partial v}{\partial x}\,dB_t \end{aligned}$$

with partial derivatives evaluated at (t, X_t).

Now,

$$\begin{aligned} v(t + \varepsilon, X_{t+\varepsilon}) - v(t, X_t) &= \int_t^{t+\varepsilon} dv(s, X_s) \\ &= \int_t^{t+\varepsilon} \left[\frac{\partial v}{\partial t} + (r + \theta\pi_s)x\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2\pi_s^2 x^2 \frac{\partial^2 v}{\partial x^2}\right] ds + \int_t^{t+\varepsilon} \sigma\pi_t x \frac{\partial v}{\partial x}\,dB_s \end{aligned}$$

And because stochastic integrals have zero expectation,

$$\mathbb{E}[v(t+\varepsilon, X_{t,X_t})] - \mathbb{E}[v(t,X_t)] = \mathbb{E}\left[\int_t^{t+\varepsilon}\left[\frac{\partial v}{\partial t} + (r+\theta\pi_s)x\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2\pi_s^2x^2\frac{\partial^2 v}{\partial x^2}\right]\,ds\right] = 0$$

So we have our HJB equation given by

$$\max_{\pi_t}\left[\frac{\partial v}{\partial t} + (r+\theta\pi_t)x\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2\pi_t^2x^2\frac{\partial^2 v}{\partial x^2}\right] = 0$$

For simplicity, we will assume that $v(t,x) = f(t)\sqrt{x}$ for some function f and verify our solution.

With this assumption, the HJB equation becomes

$$\begin{aligned}
f'(t)\sqrt{x} + \frac{(r+\theta\pi_t)xf(t)}{2\sqrt{x}} - \frac{\sigma^2\pi_t^2x^2f(t)}{8x^{3/2}} &= 0\\
f'(t)\sqrt{x} + \frac{\sqrt{x}}{2}(r+\theta\pi_t)f(t) - \frac{1}{8}\sigma^2\pi_t^2\sqrt{x}f(t) = 0\\
f'(t) + \left[\frac{1}{2}(r+\theta\pi_t) - \frac{1}{8}\sigma^2\pi_t^2\right]f(t) = 0
\end{aligned}$$

with terminal condition $v(T,x) = \sqrt{x} \implies f(T) = 1$.

Letting $\beta = \frac{1}{2}(r+\theta\pi_t) - \frac{1}{8}\sigma^2\pi_t^2$, we have the simple ODE

$$f'(t) + \beta f(t) = 0 \implies f(t) = Ce^{-\beta t}$$

Using the terminal condition, $f(T) = 1 \implies C = e^{\beta T}$. Hence,

$$f(t) = e^{\beta(T-t)} \implies \boxed{v(t,x) = e^{\beta(T-t)}\sqrt{x}}$$

Finally, the optimal control is given by

$$\begin{aligned}
\max_{\pi_t}\beta = \max_{\pi_t}\left[\frac{1}{2}(r+\theta\pi_t) - \frac{1}{8}\sigma^2\pi_t^2\right]\\
\frac{d}{d\pi}\left[\frac{1}{2}(r+\theta\pi) - \frac{1}{8}\sigma^2\pi^2\right] = 0\\
\frac{1}{2}\theta - \frac{\sigma^2}{4}\pi = 0\\
\boxed{\pi^* = \frac{2\theta}{\sigma^2}}
\end{aligned}$$

So, at last, the maximized expected power utility is

$$\begin{aligned}
v(0,x_0) &= \sqrt{x_0}\exp\left[\left(\frac{1}{2}(r+\theta\pi^*) - \frac{1}{8}\sigma^2(\pi^*)^2\right)(T)\right]\\
&= \sqrt{x_0}\exp\left[\left(\frac{1}{2}(r+\theta\cdot\frac{2\theta}{\sigma^2}) - \frac{1}{8}\sigma^2\cdot\frac{4\theta^2}{\sigma^4}\right)(T)\right]\\
&= \boxed{\sqrt{x_0}\exp\left[T\left(\frac{r}{2} + \frac{\theta^2}{\sigma^2} - \frac{\theta^2}{2\sigma^2}\right)\right]}
\end{aligned}$$