APMA 2110 - Homework 3

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1. An algebra $\mathcal{A}$ is a σ-algebra iff $\{E_j\}_1^\infty \in \mathcal{A}$ and $E_1 \subseteq E_2 \subseteq \dots$, then $\bigcup_{j=1}^\infty E_j \in \mathcal{A}$.

Suppose $\mathcal{A}$ is a σ-algebra. Then by definition, $\mathcal{A}$ is closed under countable unions. Trivially, $\bigcup_{j=1}^\infty E_j \in \mathcal{A}$ since $E_j \in \mathcal{A}$ for countably many j.

Conversely, suppose $\{E_j\}_1^\infty \in \mathcal{A}$ and $E_1 \subseteq E_2 \subseteq \dots$, then $\bigcup_{j=1}^\infty E_j \in \mathcal{A}$. We want to show that $\mathcal{A}$ is a σ-algebra. Clearly, $\mathcal{A}$ is closed under countable unions. So it suffices to show that $\mathcal{A}$ is closed under complements.

Take $E_1 \in \mathcal{A}$. Then $E_1^c = (E_1^c \cap E_2) \cup E_2^c$. Certainly $E_1^c \cap E_2 \in \mathcal{A}$. Further, $E_2^c \in \mathcal{A}$ since $\mathcal{A}$ is an algebra and closed under complements for finitely many elements. Since $\mathcal{A}$ is closed under finite disjoint unions, $E_1^c \in \mathcal{A}$.

Suppose $E_1^c, \dots, E_n^c \in \S$. Let $E_n \in \mathcal{A}$. We want to show that $E_n^c \in \mathcal{A}$. Notice that

$$\begin{aligned} E_n^c &= E_{n-1}^c \setminus (E_n \cap E_{n-1}^c) \\ &= E_{n-1}^c \cap (E_n \cap E_{n-1}^c)^c \\ &= (E_{n-1}^c \cap E_n^c) \cup (E_{n-1}^c \cap E_{n-1}^c) \\ &= (E_{n-1}^c \cap E_n^c) \cup E_{n-1}^c \\ &\subseteq E_{n-1}^c \cup E_{n-1}^c \\ &= E_{n-1}^c \end{aligned}$$

but by assumption, $E_{n-1}^c \in \mathcal{A}$ so $E_n^c \in \mathcal{A}$.

2. Prove the Borel set of $\mathbb{R}$, $\mathcal{B}_\mathbb{R}$ is generated by each of the following:

- the half-open intervals $\{(a,b] : a < b\}$ or $\{[a,b) : a < b\}$.

Lemma: $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F}) \implies \mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$

Proof: By definition,

$$\mathcal{M}(\mathcal{E}) = \bigcap_{\mathcal{E} \in \mathcal{A}} \mathcal{A}$$

where $\mathcal{A}$ is a σ-algebra containing $\mathcal{E}$.

By assumption, $\mathcal{M}(\mathcal{F})$ is a σ-algebra containing $\mathcal{E}$. Hence, $\mathcal{M}(\mathcal{F}) = \mathcal{A}$ for some $\mathcal{A}$ and $\mathcal{M}(\mathcal{E})$ is the intersection of all $\mathcal{A}$, so $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$.

Let $\mathcal{E} = \{(a,b] : a < b\}$. We want to show that $\mathcal{B}_\mathbb{R}$ is generated by $\mathcal{E}$, i.e.

$$\mathcal{B}_\mathbb{R} = \mathcal{M}(\mathcal{E}) = \bigcap_{\mathcal{E} \subseteq \mathcal{A}} \mathcal{A}$$

Certainly $\mathcal{B}_\mathbb{R} \subseteq \mathcal{M}(\mathcal{E})$ by the Lemma above because for any open set $O \subseteq \mathcal{B}_\mathbb{R}$,

$$O = \bigcup_{i=1}^{\infty} (a_i, b_i) \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]$$

which is a countable union so $\mathcal{B}_\mathbb{R} \subseteq \mathcal{M}(\mathcal{E})$.

It remains to show that $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{B}_\mathbb{R}$.

We claim

$$(a,b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n})$$

Proof: Let $(a,b] = (a,b) \cup \{b\}$. Certainly $(a,b) \in \bigcap_{n=1}^{\infty}(a, b+\frac{1}{n})$. Further,

$$\lim_{n\to\infty} b + \frac{1}{n} = b \implies b \in (a, b + \frac{1}{n})$$

for sufficiently large n. Hence $b \in \bigcap_{n=1}^{\infty}(a, b+\frac{1}{n})$ and $(a,b] \subseteq \bigcap_{n=1}^{\infty}(a, b+\frac{1}{n})$.

Conversely, $b \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$ so $(a, b+\frac{1}{n}) \subseteq (a,b]$ for all $n \in \mathbb{N}$. Hence $\bigcap_{n=1}^{\infty}(a, b+\frac{1}{n}) \subseteq (a,b]$.

Then, any $X \in \mathcal{M}(\mathcal{E})$ is a countable intersection of open sets in $\mathbb{R}$, so $X \in \mathcal{B}_\mathbb{R} \implies \mathcal{M}(\mathcal{E}) \subseteq \mathcal{B}_\mathbb{R}$.

The argument for $\{[a,b) : a < b\}$ is similar with

$$[a,b) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b)$$

- the closed rays $\{[a,\infty) : a \in \mathbb{R}\}$ or $\{(-\infty, a] : a \in \mathbb{R}\}$.

Let $\mathcal{E} = \{[a,\infty) : a \in \mathbb{R}\}$.

Because $\mathcal{E}$ is generated by closed sets in $\mathbb{R}$, $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{B}_{\mathbb{R}}$.

For the reverse inclusion, we want to show that $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{E})$.

We claim

$$(a,b) = \bigcap_{n=1}^{\infty} [a - \frac{1}{n}, b)$$

Proof: If $a < x < b$, certainly $a - \frac{1}{n} \leq x < b$ for all $n \in \mathbb{N}$. Hence $(a,b) \subseteq \bigcap_{n=1}^{\infty}[a - \frac{1}{n}, b)$.

Conversely, if $x \in \bigcap_{n=1}^{\infty}[a - \frac{1}{n}, b)$, then $a - \frac{1}{n} \leq x < b$ for all $n \in \mathbb{N}$. But $a - \frac{1}{n} \to a$ as $n \to \infty$ so $a \leq x < b \implies x \in (a,b)$.

Therefore, $\bigcap_{n=1}^{\infty}[a - \frac{1}{n}, b) = (a,b)$.

But we can write any interval $[a,b)$ by

$$[a,b) = [a,\infty) \cup [b,\infty)^c$$

So any open set in $\mathbb{R}$ is a countable union of sets in $\mathcal{E}$ (and their complements).

Hence, $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{E})$.

The argument for $(-\infty, a]$ is similar with

$$(a,b) = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n}]$$

3. If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$$

First notice that

$$\begin{aligned} E &= (E \setminus F) \cup (E \cap F) \\ F &= (F \setminus E) \cup (E \cap F) \end{aligned}$$

which are each disjoint unions.

So

$$\begin{aligned} \mu(E) + \mu(F) &= \mu((E \setminus F) \cup (E \cap F)) + \mu((F \setminus E) \cup (E \cap F)) \\ &= \mu(E \setminus F) + \mu(E \cap F) + \mu(F \setminus E) + \mu(E \cap F) \\ &= \mu(E \cap F) + \mu((E \setminus F) \cup (F \setminus E) \cup (E \cap F)) \end{aligned}$$

But

$$\begin{aligned} (E \setminus F) \cup (F \setminus E) &= (E \cap F^c) \cup (F \cap E^c) \\ &= [(E \cup F) \cap (E \cup E^c)] \cap [(E \cup E^c) \cap (F^c \cup E^c)] \\ &= (E \cup F) \cap (F^c \cup E^c) \\ &= (E \cup F) \cap (E \cap F)^c \end{aligned}$$

So

$$\begin{aligned} \mu((E \setminus F) \cup (F \setminus E) \cup (E \cap F)) &= \mu((E \cup F) \cap (E \cap F)^c \cup (E \cap F)) \\ &= \mu((E \cup F) \cap X) \\ &= \mu(E \cup F) \end{aligned}$$

Therefore,

$$\mu(E) + \mu(F) = \mu(E \cap F) + \mu(E \cup F) \quad \blacksquare$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$, then

$$\mu(\liminf E_j) \leq \liminf \mu(E_j)$$

Also, if $\mu(\bigcup_{j=1}^\infty E_j) < \infty$, then

$$\mu(\limsup E_j) \geq \limsup \mu(E_j)$$

Consider $\mu(\liminf E_j)$. By definition,

$$\mu(\liminf E_j) = \mu\left(\bigcup_{k=1}^\infty \bigcap_{j=k}^\infty E_j\right)$$

Let

$$F_k = \bigcap_{j=k}^\infty E_j$$

so $F_1 \subseteq F_2 \subseteq \dots$.

By continuity from below,

$$\mu(\liminf E_j) = \mu\left(\bigcup_{k=1}^\infty F_k\right) = \lim_{k\to\infty} \mu(F_k) = \lim_{k\to\infty} \mu\left(\bigcap_{j=k}^\infty E_j\right)$$

But for any $n \geq k$, $\bigcap_{j=n}^\infty E_j \subseteq E_n$ so by monotonicity,

$$\mu\left(\bigcap_{j=n}^\infty E_n\right) \leq \mu(E_n)$$

And indeed it suffices to choose the smallest:

$$\mu\left(\bigcap_{j=k}^\infty E_j\right) \leq \inf_{j\geq k} \mu(E_j)$$

Therefore,

$$\mu(\liminf E_j) \leq \lim_{k\to\infty} \inf_{j\geq k} \mu(E_k) = \liminf \mu(E_j)$$

Now suppose $\mu(\bigcup_{j=1}^\infty E_j) < \infty$. As before,

$$\begin{aligned}
\mu(\limsup E_j) &= \mu\left(\bigcap_{k=1}^\infty \bigcup_{j=k}^\infty E_j\right) \\
&= \mu\left(\bigcap_{k=1}^\infty F_k\right) \\
&= \lim_{k\to\infty} \mu(F_k) \qquad \text{(Continuity from above since } \mu(F_1) < \infty) \\
&= \lim_{k\to\infty} \mu\left(\bigcup_{j=k}^\infty E_j\right) \\
&\geq \lim_{k\to\infty} \sup_{j\geq k} \mu(E_j) \qquad \text{(Monotonicity)} \\
&= \limsup \mu(E_j) \quad \blacksquare
\end{aligned}$$

5. Let μ^* be an outer measure. Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of sets such that

$$\sum_{k=1}^{\infty} \mu^*(E_k) < \infty$$

show that $\mu^*(\limsup E_k) = 0$

Certainly $\mu^*(\limsup E_k) \geq 0$. We will seek to further show that $\mu^*(\limsup E_k) \leq 0$.

On the contrary, suppose $\mu^*(\limsup E_k) = m > 0$.

By definition of $\limsup$,

$$\mu^*(\limsup E_k) = \mu^*\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right)$$

For notational convenience, let $F_n = \bigcup_{k=n}^{\infty} E_k$. Then $F_1 \supseteq F_2 \supseteq \ldots$.

Then

$$\mu^*(\limsup E_k) = \mu^*\left(\bigcap_{n=1}^{\infty} F_n\right) = m$$

Note however, that any element $x \in \bigcap_{n=1}^{\infty} F_n$ is in $\bigcup_{k=n}^{\infty} E_k$ for infinitely many n by definition of F_n. Hence,

$$\bigcap_{n=1}^{\infty} F_n \subseteq \bigcup_{k=n}^{\infty} E_k$$

so by monotonicity,

$$\mu^*\left(\bigcap_{n=1}^{\infty} F_n\right) = m \leq \mu^*\left(\bigcup_{k=n}^{\infty} E_k\right)$$

Lemma: If μ^* is an outer measure and $\{E_k\}_1^{\infty}$ a sequence of sets,

$$\mu^*(\bigcup_{k=n}^{\infty} E_k) \leq \sum_{k=n+1}^{\infty} \mu^*(E_k)$$

Proof: Define the sequence of sets $\{F_k\}_1^{\infty}$ by $F_k = E_{n+k}$. This is still a countably infinite sequence of sets in $\mathcal{M}$ so by subadditivity,

$$\mu^*\left(\bigcup_{k=n+1}^{\infty} E_k\right) = \mu^*\left(\bigcup_{k=1}^{\infty} F_k\right) \leq \sum_{k=1}^{\infty} \mu^*(F_k) = \sum_{k=n+1}^{\infty} \mu^*(E_k)$$

By assumption, $\sum_{k=1}^{\infty} \mu^*(E_k) < \infty$ so it must converge to a finite value, say S. Let S_n be its sequence of partial sums. By definition of series convergence, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N$ implies

$$|S - S_n| < \varepsilon$$

Choose $\varepsilon = \frac{m}{2}$. Then $\exists N \in \mathbb{N}$ such that $n \geq N$ implies

$$|S - S_n| = \sum_{k=1}^{\infty} \mu^*(E_k) - \sum_{k=1}^{n} \mu^*(E_k) = \sum_{k=n+1}^{\infty} \mu^*(E_k) < \frac{m}{2}$$

But by the Lemma,

$$\mu^*(\limsup E_k) = m \leq \mu^*\left(\bigcup_{k=n+1}^{\infty} E_k\right) \leq \sum_{k=n+1}^{\infty} \mu^*(E_k) < \frac{m}{2}$$

And $0 < m < \frac{m}{2}$ is a contradiction, so $\mu^*(\limsup E_k) = 0$,