APMA 2110: Homework 11

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1. Suppose μ and ν are σ-finite measures on $(X, \mathcal{M})$ with $\nu \ll \mu$, and let $\lambda = \mu + \nu$. If $f = \frac{d\nu}{d\lambda}$, show $0 \leq f < 1$ a.e. and $\frac{d\nu}{d\mu} = \frac{f}{1-f}$.

STEP 1. $0 \leq f$ μ-a.e.

It suffices to show that $E = \{x : f(x) < 0\}$ has measure zero.

Notice though, for $E_n = \{x : f(x) < -\frac{1}{n}\}$,

$$E = \{x : f(x) < 0\} = \bigcup_{n=1}^{\infty} E_n$$

and since $f = \frac{d\nu}{d\lambda}$,

$$\nu(E_n) = \int_{E_n} f \, d\lambda < \int_{E_n} -\frac{1}{n} \, d\lambda = -\frac{1}{n}\lambda(E_n)$$

But $\nu \geq 0$, so

$$-\frac{1}{n}\lambda(E_n) \geq 0 \implies \lambda(E_n) = \mu(E_n) + \nu(E_n) \leq 0 \implies \mu(E_n) \leq 0$$

But again $\mu(E_n) \geq 0$, so $\mu(E_n) = 0$ and hence $\mu(E) = 0$.

STEP 2. $f < 1$ μ-a.e.

Now consider $F = \{x : f(x) \geq 1\}$. By σ-finiteness of ν, $\exists \{F_n\}$ such that $F = \bigcup_{n=1}^{\infty} F_n$ and $\nu(F_n) < \infty$ for all n.

Since $f = \frac{d\nu}{d\lambda}$,

$$\nu(F_n) = \int_{F_n} f \, d\lambda \geq \int_{F_n} 1 \, d\lambda = \lambda(F_n) = \mu(F_n) + \nu(F_n)$$

Since $\nu(F_n) < \infty$, $0 \geq \mu(F_n) \implies \mu(F_n) = 0 \implies \mu(F) = 0$ and $f < 1$ μ-a.e.

$\mu \leq \lambda \implies \mu \ll \lambda$ and $\nu \ll \mu$ by assumption so by the chain rule,

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda} \cdot \frac{d\lambda}{d\mu} = f \cdot \frac{d\lambda}{d\mu}$$

Hence, it suffices to show that $\frac{d\lambda}{d\mu} = \frac{1}{1-f}$.

But in fact, since $\nu \ll \mu$ by assumption,

$$\mu(E) = 0 \implies \nu(E) = 0 \implies \nu(E) + \mu(E) = \lambda(E) = 0$$

so we also have that $\lambda \ll \mu$.

In particular, this means that

$$\frac{d\lambda}{d\mu} \cdot \frac{d\mu}{d\lambda} = 1 \text{ a..e}$$

and it in fact suffices to show that $\frac{d\mu}{d\lambda} = 1 - f$.

Consider

$$\begin{aligned}
\mu(E) + \nu(E) &= \lambda(E) \\
&= \int_E 1 \, d\lambda \\
&= \int_E (1 - f) \, d\lambda + \int_E f \, d\lambda \\
&= \int_E (1 - f) \, d\lambda + \nu(E)
\end{aligned}$$

and since ν is σ-finite,

$$\mu(E) = \int_E (1 - f) \, d\lambda \implies \frac{d\mu}{d\lambda} = 1 - f$$

exactly as desired. ■

2. Let $f \in \mathrm{L}^1(\mathbb{R}^n)$ and recall the average function

$$A_r f(x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)\, dy$$

Prove that

$$\lim_{r \to 0} ||A_r f - f||_{\mathrm{L}^1(\mathbb{R}^n)} \to 0$$

Deduce there is a subsequence $r_n \to 0$ such that $A_{r_n} f \to f$ a.e.

Let $\varepsilon > 0$.

Since $f \in \mathrm{L}^1$, by a theorem from class, we may approximate by a continuous integrable function with compact support g such that

$$\int_{\mathbb{R}^n} |f - g| \ dy < \frac{\varepsilon}{3}$$

By the triangle inequality,

$$\begin{aligned}
\lim_{r \to 0} ||A_r f - f||_{\mathrm{L}^1} &= \lim_{r \to 0} \int_{\mathbb{R}^n} |A_r f - f| \ dm \\
&= \lim_{r \to 0} \int_{\mathbb{R}^n} |A_r f - A_r g + A_r g - g + g - f| \ dm \\
&\leq \lim_{r \to 0} \int_{\mathbb{R}^n} |A_r f - A_r g| \ dm + \lim_{r \to 0} \int_{\mathbb{R}^n} |A_r g - g| \ dm + \lim_{r \to 0} \int_{\mathbb{R}^n} |g - f| \ dm
\end{aligned}$$

For fixed $r > 0$, by continuity of g, $|y - x| < r$ implies $|g(y) - g(x)| < \delta$, so

$$|A_r g - g| = \left| \frac{1}{m(B(r,x))} \int_{B(r,x)} [g(y) - g(x)]\, dy \right| < \delta$$

Now since g has compact support (say on some set $K \subseteq \mathbb{R}^n$),

$$\int_{\mathbb{R}^n} \delta \ dm = \int_K \delta \ dm = \delta m(K) < \infty$$

since the Lebesgue measure is σ-finite so letting $\delta = \frac{\varepsilon}{3m(K)}$, $\lim_{r \to 0} \int_{\mathbb{R}^n} |A_r g - g| < \frac{\varepsilon}{3}$.

Further, by definition,

$$\begin{aligned}
\int_{\mathbb{R}^n} |A_r f - A_r g| &= \int_{\mathbb{R}^n} \left| \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y) - g(y)\, dy \right| \, dx \\
&\leq \int_{\mathbb{R}^n} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - g(y)| \ dy\, dx \\
&\leq \frac{1}{m(B(r,x))} \int_{\mathbb{R}^n} \int_{B(r,x)} |f(y) - g(y)| \ dy\, dx \qquad \text{(since } m(B(r,x)) \text{ is constant WRT } x\text{)} \\
&= \frac{1}{m(B(r,x))} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y) - g(y)|\, \mathbb{1}_{B(r,x)}\, dy\, dx \\
&= \frac{1}{m(B(r,x))} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y) - g(y)|\, \mathbb{1}_{B(r,x)}\, dx\, dy \qquad \text{(Fubini)} \\
&= \frac{1}{m(B(r,x))} \int_{\mathbb{R}^n} |f(y) - g(y)| \int_{\mathbb{R}^n} \mathbb{1}_{B(r,x)}\, dx\, dy \\
&= \frac{1}{m(B(r,x))} \int_{\mathbb{R}^n} |f(y) - g(y)|\, m(B(r,x))\, dy \\
&= \int_{\mathbb{R}^n} |f(y) - g(y)| \ dy < \frac{\varepsilon}{3}
\end{aligned}$$

Hence,

$$\lim_{r \to 0} ||A_r f - f||_{\mathrm{L}^1} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Therefore, $A_r f \to f$ in L^1 (take $r = \frac{1}{n}$ for the limit $n \to \infty$), and by a theorem from class, there is a subsequence $r_n \to 0$ such that $A_{r_n} f \to f$ a.e. $\blacksquare$

3. Define a variant of the maximal function $H(f)$ in $\mathbb{R}^n$ as

$$H^*f(x) = \sup\left\{\frac{1}{m(B)}\int_B |f(y)|\ dy \;\middle|\; B \text{ is a ball and } x \in B\right\}$$

Show $Hf \leq H^*f \leq 2^n Hf$.

Recall that

$$Hf(x) = \sup_{r>0} \frac{1}{m(B(r,x))}\int_{B(r,x)} |f(y)|\ dy$$

where $B(r,x)$ is the n-ball of radius r centered at x.

Pick any $r > 0$. Let $B(r,x)$ be a ball of radius r centered at x. Certainly, $x \in B(r,x)$ so by definition of $H^*f(x)$,

$$\frac{1}{m(B(r,x))}\int_{B(r,x)} |f(y)|\ dy \leq H^*f(x)$$

Taking the sup over $r > 0$, we have that $Hf(x) \leq H^*f(x)$.

Conversely, for any ball B_r of radius r with $x \in B_r$, $B_r \subseteq B(2r,x)$ (the ball of radius $2r$ centered at x). Hence,

$$\frac{1}{m(B_r)}\int_{B_r} |f(y)|\ dy \leq \frac{1}{m(B_r)}\int_{B(2r,x)} |f(y)|\ dy = \frac{2^n}{m(B(2r,x))}\int_{B(2r,x)} |f(y)|\ dy \leq 2^n Hf$$

Again taking the sup on both sides, $H^*f(x) \leq 2^n Hf(x)$. ■

4. Assume μ is a positive measure and $f_n \to f$ in $\mathrm{L}^1(\mu)$. Prove that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\left|\int_E f_n \, d\mu\right| < \varepsilon$ for all n and any $\mu(E) < \delta$.

Let $\varepsilon > 0$.

By a lemma from class,

$$\left|\int_E f_n \, d\mu\right| \leq \int_E |f_n| \, d\mu$$

for any set $E \subseteq X$.

By the triangle inequality,

$$\int_E |f_n| \, d\mu = \int_E |f_n - f + f| \, d\mu \leq \int_E |f_n - f| \, d\mu + \int_E |f| \, d\mu$$

Since $f \in \mathrm{L}^1$, $\int |f| \, d\mu = M < \infty$. Hence, for $\delta_1 = \frac{\varepsilon}{2M}$ and $\mu(E) < \delta_1$,

$$\int_E |f| \, d\mu < \frac{\varepsilon}{2}$$

Further, since $f_n \to f$ in L^1, $\exists N$ such that $\forall n \geq N$ and any measurable set E,

$$\int_E |f_n - f| \, d\mu < \frac{\varepsilon}{2}$$

It remains to show that $\int_E |f_n - f| \, d\mu < \frac{\varepsilon}{2}$ for $n < N$.

Consider the finite set $\{f_n\}_{n=1}^{N-1} \subseteq \mathrm{L}^1$. By a corollary from class, $\forall \varepsilon > 0$, $\exists \delta_{f_n} > 0$ such that $\mu(E) < \delta$ implies $\int_E |f_n| \, d\mu < \frac{\varepsilon}{2}$.
Let $\delta_2 = \max\{\delta_{f_1}, \ldots, \delta_{f_{N-1}}\}$. Then, $\mu(E) < \delta_2$ implies

$$\int_E |f_n| \, d\mu < \frac{\varepsilon}{2}$$

for all $n < N$.

Taking $\delta = \max\{\delta_1, \delta_2\}$, we have that $\mu(E) < \delta$ implies

$$\int_E |f_n| \, d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \blacksquare$$