APMA 2110: Homework 6

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1. Let $f \geq 0$ be a measurable function $(X, \mathcal{A}, \mu)$. Show that f is integrable if and only if

$$\sum_{n=-\infty}^{\infty} 2^n \mu\{x : f(x) > 2^n\} < \infty$$

Let f be integrable. Then since $f \geq 0$,

$$\int_X f \; d\mu = \int_X |f| \; d\mu < \infty$$

Notice that for the disjoint sets

$$E_n = \{x : 2^n < f(x) \leq 2^{n+1}\}$$

we have that

$$\int_{E_n} f \; d\mu \geq 2^n \mu(E_n)$$

for all n.

Hence,

$$\int_X f \; d\mu \geq \sum_{-\infty}^{\infty} 2^n \mu(E_n)$$

Suppose that $\sum_{n=-\infty}^{\infty} 2^n \mu\{x : f(x) > 2^n\} = \infty$.

There are two possible ways this could happen:

1. There are infinitely many n such that $\mu\{x : f(x) > 2^n\} > 0$.

2. $\exists n$ such that $\mu\{x : f(x) > 2^n\} = \infty$.

CASE 1: There are infinitely many n such that $\mu\{x : f(x) > 2^n\} > 0$.

WLOG, let N be the smallest of all such n. Then $\forall m \geq N$,

$$\{x : f(x) > 2^N\} \subseteq \{x : f(x) > 2^m\} \implies 0 < \mu\{x : f(x) > 2^N\} \leq \mu\{x : f(x) > 2^m\}$$

so $2^n\mu(E_n) \to \infty$ and thus $\int f\, d\mu \to \infty$, which is a contradiction.

CASE 2: $\exists n$ such that $\mu\{x : f(x) > 2^n\} = \infty$.

But for all $m \leq n$,

$$\{x : f(x) > 2^n\} \subseteq \{x : f(x) > 2^m\} \implies \infty = \mu\{x : f(x) > 2^n\} \leq \mu\{x : f(x) > 2^m\}$$

In particular, then infinitely many m have $\mu\{x : f(x) > 2^m\} > 0$ so we are back in Case 1 and have a contradiction.

Hence, $\sum_{n=-\infty}^{\infty} 2^n\mu\{x : f(x) > 2^n\} < \infty$.

Let $\sum_{n=-\infty}^{\infty} 2^n\mu\{x : f(x) > 2^n\} < \infty$.

Lemma: If $f \geq 0$,

$$\int_X f\, d\mu = (R)\int_0^{\infty} \mu\{x : f(x) > t\}\, dt$$

Proof:

Since f is measurable and $f \geq 0$, we can take a sequence of simple functions $\phi_n \to f$ such that $0 \leq \phi_n \leq \phi_{n+1} \leq f$ for $n \geq 1$.

By the Monotone Convergence Theorem,

$$\int f\, d\mu = \int \lim \phi_n\, d\mu = \lim \int \phi_n\, d\mu$$

Each ϕ_n is a simple function, $\phi_n = \sum_{i=1}^{m_n} a_i^{(n)} \mathbb{1}_{A_i^{(n)}}(x)$, so

$$\begin{aligned}
\int \phi_n \, d\mu &= \sum_{i=1}^{m_n} a_i^{(n)} \mu(A_i^{(n)}) \\
&= \sum_{i=1}^{m_n} (a_i^{(n)} - a_i^{(n-1)}) \, \mu(x : \phi_n(x) > a_i^{(n-1)})
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \int \phi_n &= \lim_{n\to\infty} \sum_{i=1}^{m_n} (a_i^{(n)} - a_i^{(n-1)}) \, \mu(x : \phi_n(x) > a_i^{(n-1)}) \\
&= \sum_{i=1}^{\infty} (a_i - a_{i-1}) \, \mu(x : f(x) > a_{i-1}) \\
&= (R) \int_0^\infty \mu(x : f(x) > t) \, dt
\end{aligned}$$

Since $f \geq 0$, it suffices to show that

$$\int_X f \, d\mu < \infty$$

By the Lemma,

$$\begin{aligned}
\int_X f \, d\mu &= (R) \int_0^\infty \mu\{x : f(x) > t\} \, dt \\
&= \sum_{-\infty}^{\infty} \int_{2^n}^{2^{n+1}} \mu\{x : f(x) > t\} \, dt
\end{aligned}$$

For $t \in (2^n, 2^{n+1}]$,

$$\mu\{x : f(x) > t\} \leq \mu\{x : f(x) > 2^n\}$$

so

$$\begin{aligned}\int_{2^n}^{2^{n+1}} \mu\{x : f(x) > t\}\, dt &\le \int_{2^n}^{2^{n+1}} \mu\{x : f(x) > 2^n\}\, dt \\ &\le (2^{n+1} - 2^n)\mu\{x : f(x) > 2^n\} \\ &= 2^n \mu\{x : f(x) > 2^n\}\end{aligned}$$

Hence,

$$\int |f|\ d\mu = \int_X f\ d\mu \le \sum_{-\infty}^{\infty} 2^n \mu\{x : f(x) > 2^n\} < \infty$$

and f is integrable. $\blacksquare$

2. Define $f \in \mathcal{L}^p(X, \mathcal{A}, \mu)$ if $\int |f|^p \ d\mu < \infty$ with $1 \leq p \leq \infty$. If $\mu(X) < \infty$, show that $f \in \mathcal{L}^p(X, \mathcal{A}, \mu)$ for all $1 \leq q \leq p$. What happens when $\mu(X) = \infty$?

If $|f| \geq 1$, then $|f|^q \leq |f|^p$ and by monotonicity,

$$\int |f|^q \ d\mu \leq \int |f|^p \ d\mu < \infty \implies f \in \mathcal{L}^q$$

If $|f| \leq 1$, then $|f|^q \leq 1$ so $|f|^q \leq |f|^p + 1$ and

$$\int |f|^q \ d\mu \leq \int |f|^p + 1 \ d\mu = \int |f|^p + \mu(X)$$

If $\mu(X) < \infty$, then $\int |f|^p + \mu(X) < \infty$ and $f \in \mathcal{L}^q$ for all $1 \leq q \leq p$.

Meanwhile, if $\mu(X) = \infty$, then $\int |f|^p + \mu(X) = \infty$ and $f \notin \mathcal{L}^q$ for $1 \leq q \leq p$ $\blacksquare$.

3. Assume $\mu(X) < \infty$ and $f_n \to f$ a.e.

- Assume $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\sup_n \sup_{E:\mu(E)<\delta} \int_E |f_n| \ d\mu < \varepsilon$$

Show that $\int |f_n - f| \ d\mu \to 0$

Notice

$$\lim \int |f_n - f| \ d\mu = \lim \int_A |f_n - f| \ d\mu + \lim \int_{A^c} |f_n - f| \ d\mu$$

Consider the first term.

First notice that $\forall A \in \{E \subseteq X : \mu(E) < \delta\}$,

$$\int_A |f_n| \ d\mu \leq \sup_n \int_A |f_n| \ d\mu < \sup_n \sup_{E:\mu(E)<\delta} \int_E |f_n| \ d\mu < \varepsilon$$

by assumption.

Then

$$\begin{aligned} \lim \int_A |f_n - f| \ d\mu &\leq \lim \int_A |f_n| \ d\mu + \lim \int_A |f| \ d\mu \\ &< \varepsilon + \lim \int_A |f| \ d\mu \end{aligned}$$

but as $\delta \to 0$, $\int_A |f| \ d\mu \to 0$ (approximation by simple functions), so we are free to choose δ such that $\int_A |f| \ d\mu < \varepsilon$ and

$$\int_A |f_n - f| \ d\mu < 2\varepsilon$$

For the second term, $\mu(A) < \delta$ for all δ, $\mu(X) < \infty$, and $f_n \to f$ a.e. so Egorov's Theorem guarantees $f_n - f \to 0$ uniformly on E^c. Hence, for N sufficiently large,

$$|f_n - f| < \varepsilon \implies \lim \int |f_n - f| \ d\mu < \varepsilon\mu(X)$$

And since $\mu(X) < \infty$, for $\varepsilon \to 0$, $\varepsilon\mu(X) \to 0$.

Hence,

$$\lim \int |f_n - f| \; d\mu = \lim \int_A |f_n - f| \; d\mu + \lim \int_{A^c} |f_n - f| \; d\mu = 0 + 0 = 0 \quad \blacksquare$$

- Assume for some $p > 1$

$$\sup_n \int |f_n|^p \; d\mu < \infty$$

Show that $\int |f_n - f| \; d\mu \to 0$. What if $p = 1$?

Since $f_n \to f$ a.e., on $X \setminus E$ for some $E \subseteq X$ with $\mu(E) = 0$,

$$\lim |f_n - f| = 0$$

Hence,

$$\begin{aligned}\lim \int |f_n - f| \; d\mu &= \lim \int_{X\setminus E} |f_n - f| \; d\mu + \lim \int_E |f_n - f| \; d\mu \\ &= \lim \int_{X\setminus E} |f_n - f| \; d\mu\end{aligned}$$

by a lemma from class ($\mu(E) = 0$ and $|f_n - f| \geq 0$).,

We claim that $\exists g : X \to \mathbb{R}$ such that $\int g \; d\mu < \infty$ and $|f_n - f| \leq g$ for all n.

Proof: Suppose not. Then for all n,

$$\int |f_n - f| \; d\mu = \infty$$

(or else $g = |f_n - f| + 1$ would suffice as $\mu(X) < \infty$.)

But,

$$\int |f_n - f| \; d\mu \leq \int |f_n| \; d\mu + \int |f| \; d\mu$$

By assumption,

$$\int |f_n|^p \; d\mu \leq \sup_n \int |f_n|^p \; d\mu < \infty$$

and $\mu(X) < \infty$, so $f_n \in \mathcal{L}^p$ for all n. In particular, by Problem 2, $f_n \in \mathcal{L}^1$ for all n so

$$\int |f_n| \ d\mu < \infty$$

Further, by Fatou's lemma,

$$\begin{aligned}\int |f| \ d\mu &= \int \liminf |f_n| \ d\mu \\ &\leq \liminf \int |f_n| \ d\mu \\ &\leq \limsup \int |f_n| \ d\mu < \infty\end{aligned}$$

so $f \in \mathcal{L}^1$ also.

Hence,

$$\int |f_n - f| \ d\mu \leq \int |f_n| \ d\mu + \int |f| \ d\mu < \infty$$

and we have a contradiction.

Now by LDC,

$$\lim \int_{X \setminus E} |f_n - f| \ d\mu = \int_{X \setminus E} \lim |f_n - f| \ d\mu = \int_{X \setminus E} 0 \ d\mu = 0$$

Now consider the case $p = 1$.

Consider $f_n(x) = n \cdot \mathbb{1}_{[0, \frac{1}{n}]}$. Then $f_n \to 0$ a.e. and

$$\int |f_n| \ d\mu = \int n \cdot \mathbb{1}_{[0, \frac{1}{n}]} \ d\mu = n \int_0^{1/n} 1 \ dx = \frac{n}{n} = 1 < \infty$$

for all n.

But

$$\int |f_n - f| \ d\mu = \int |f_n| \ d\mu = 1 \not\to 0$$

4. Assume $f_n \to f$ a.e. Prove that if $\lim_{n\to\infty} \int |f_n| \; d\mu = \int |f| \; d\mu$,

$$\lim_{n\to\infty} \int |f_n - f| \; d\mu = 0$$

First notice that by the triangle inequality,

$$|f_n - f| \leq |f_n| + |f| \implies |f_n| + |f| - |f_n - f| \geq 0$$

Therefore, by Fatou's Lemma,

$$\int \liminf |f_n| + |f| - |f_n - f| \; d\mu \leq \liminf \int |f_n| + |f| - |f_n - f| \; d\mu$$

Since $f_n \to f$ a.e., we have that

$$\liminf \left(|f_n| + |f| - |f_n - f|\right) = 2\,|f| = |f| + |f| - 0 = 2\,|f|$$

almost everywhere.

Hence,

$$\begin{aligned}
2 \int |f| \; d\mu &\leq \liminf \int |f_n| + |f| - |f_n - f| \; d\mu \\
&= \liminf \int |f_n| \; d\mu + \liminf \int |f| \; d\mu + \liminf \int -|f_n - f| \; d\mu \\
&= \liminf \int |f_n| \; d\mu + \liminf \int |f| \; d\mu - \limsup \int |f_n - f| \; d\mu \qquad \text{(by parity of } \limsup) \\
&\leq \lim \int |f_n| \; d\mu + \lim \int |f| \; d\mu - \limsup \int |f_n - f| \; d\mu \\
&= 2 \int |f| \; d\mu - \limsup \int |f_n - f| \; d\mu \qquad \text{(by assumption)}
\end{aligned}$$

so

$$\limsup \int |f - f_n| \; d\mu \leq 0 \implies \lim \int |f - f_n| = 0 \quad \blacksquare$$