APMA 2110: Homework 4

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1. Let μ be the Lebesgue measure in $\mathbb{R}$ and $\mu(E) = 0$. Prove
$$\mu(\{x^2 : x \in E\}) = 0$$

If $E = \emptyset$, certainly $\mu(\{x^2 : x \in E\}) = \mu(\emptyset) = 0$.

Therefore, suppose $E \neq \emptyset$.

$\mu(\{x^2 : x \in E\}) \geq 0$ so it suffices to show that $\mu(\{x^2 : x \in E\}) \leq 0$.

For notational convenience, let $E^2 = \{x^2 : x \in E\}$ and let $\varepsilon > 0$.

Because $E \subseteq \mathbb{R}$,
$$E = (E \cap (-\infty, 0)) \cup (E \cap [0, \infty))$$

First consider $E \cap [0, \infty)$. By monotonicity of x^2 on $[0, \infty)$ and the existence of the covering (a_i, b_i),
$$x \in E \cap [0, \infty) \implies x \in (a_n, b_n) \text{ for some } n \implies x^2 \in (a_n^2, b_n^2)$$

Similarly, for $E \cap (-\infty, 0)$,
$$x \in E \cap (-\infty, 0) \implies -x \in E \cap [0, \infty) \implies (-x)^2 = x^2 \in (a_m^2, b_m^2) \text{ for some } m$$

Then,
$$E^2 \subseteq \bigcup_{i=1}^{\infty} (a_i^2, b_i^2)$$

By monotonicity and subadditivity,
$$\mu(E^2) \leq \sum_{i=1}^{\infty} \mu((a_i^2, b_i^2))$$

By the faithfulness of the measure, $\mu((a_i^2, b_i^2)) = \rho(a_i^2, b_i^2) = |b_i^2 - a_i^2|$.

And
$$\left|b_i^2 - a_i^2\right| = |b_i - a_i| \cdot |b_i + a_i| = \rho(b_i, a_i) \cdot |b_i + a_i|$$

And
$$\mu(E) = 0 \implies \exists (a_i, b_i) \text{ s.t. } \bigcup_{i=1}^{\infty} (a_i, b_i) \supset E \text{ and } \rho(a_i, b_i) = 0 \; \forall i \geq 1$$

So
$$\mu(E^2) \leq \sum i = 1^{\infty} \mu((a_i^2, b_i^2)) = \rho(a_i, b_i) \cdot |b_i + a_i| = 0$$

So $\mu(E^2) = 0$ as well.

2. Define the n-dimensional open intervals

$$I = \{x : a_j < x_j < b_j,\ j = 1, \dots, n\}$$

and their volume

$$\rho(I) = \prod_{j=1}^{n} (b_j - a_j)$$

Construct the Lebesgue measure μ on $\mathbb{R}^n$ by constructing an outer measure μ^* and using the Carathéodory construction.

Show that

1. $\mu(I) = \rho(I)$ and I is measurable
2. μ^* is the same if we choose closed cubes with length less than a fixed $\varepsilon > 0$

We claim that for I an open n-dimensional interval,

$$\mu^*(I) = \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) \,\middle|\, I \subseteq \bigcup_{j=1}^{\infty} E_j \right\}$$

is an outer measure on $\mathbb{R}^n$.

1. $\mu^*(\emptyset) = 0$

Clearly $\rho(\emptyset) = 0$ and $\emptyset \subseteq \bigcup_{j=1}^{\infty} I_j$ for any I_j, so it suffices to take $I_j = \{j\}$ so

$$\mu^*(\emptyset) \leq \sum_{j=1}^{\infty} \rho(I_j) = 0$$

Hence, $\mu^*(\emptyset) = 0$.

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2. Monotonicity

Suppose $A \subseteq B$ and $\{E_j \in \mathcal{I}\}_1^{\infty}$ is a covering of B.

$A \subseteq B$ so $\{E_j\}_1^{\infty}$ is a covering of A as well.

Then by definition of the measure as inf,

$$\mu^*(A) \leq \sum_{j=1}^{\infty} \rho(E_j)$$

and taking the inf of both sides,

$$\mu^*(A) \leq \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) \right\} = \mu^*(B)$$

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3. Subadditivity

Let $\{A_j\}_{j=1}^{\infty} \subseteq P(\mathbb{R}^n)$ and $\varepsilon > 0$. Let $\{E_{jk}\}_{k=1}^{\infty}$ be a cover of A_j such that

$$\sum_{k=1}^{\infty} \rho(E_{jk}) \leq \mu^*(A_j) + \frac{\varepsilon}{2^j}$$

(existence guaranteed by definition of measure as inf)

By construction,

$$\bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j=1}^{\infty} \left(\bigcup_{k=1}^{\infty} E_{jk} \right)$$

so

$$\begin{aligned} \mu^* \left(\bigcup_{j=1}^{\infty} A_j \right) &\leq \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \rho(E_{jk}) \right) \\ &\leq \sum_{j=1}^{\infty} \left(\mu^*(A_j) + \frac{\varepsilon}{2^j} \right) \\ &= \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon \\ &= \sum_{j=1}^{\infty} \mu^*(A_j) \end{aligned}$$

Hence, μ^* is an outer measure on $\mathbb{R}^n$.

Let $\mathcal{M}$ be the μ^*-measurable sets on $\mathbb{R}^n$. By the Carathéodory construction, $\mu = \mu^* \Big|_{\mathcal{M}}$ is a measure.

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1. We claim $\mu(I) = \rho(I)$.

Proof:

Let $\varepsilon > 0$ and $I = \{x : a_j < x_j < b_j,\ j = 1, \ldots, n\}$.

Clearly, $I \subseteq \{x : a_j - \varepsilon < x_j < b_j + \varepsilon,\ j = 1, \ldots, n\}$ so by the definition of μ^*,

$$\begin{aligned} \mu^*(I) &\leq \rho(\{x : a_j - \varepsilon < x_j < b_j + \varepsilon,\ j = 1, \ldots, n\}) \\ &= \prod_{j=1}^{n} (b_j - a_j + 2\varepsilon) \\ &\overset{\varepsilon \to 0}{=} \prod_{j=1}^{n} (b_j - a_j) \\ &= \rho(I) \end{aligned}$$

It remains to show that $\mu^*(I) \geq \rho(I)$.

Define $\lambda \in \mathbb{R}$ such that $\lambda < \rho(I)$. Pick a closed, bounded interval $J \subseteq I$ such that $\lambda < \rho(J) < \rho(I)$.

Let $\{E_j\}_{j=1}^{\infty}$ be an open covering of I (and hence a covering of J). By Heine-Borel, there exists a finite subcover $\{E_j\}_{j=1}^{N}$.

Then

$$\rho(J) \le \sum_{j=1}^{N} \rho(E_j)$$

by finite subadditivity.

Taking the limit,

$$\rho(J) \le \sum_{j=1}^{\infty} \rho(E_j) \implies \lambda < \rho(J) \le \mu^*(I)$$

And since λ was arbitrary with $\alpha < \rho(I)$, we can take $\lambda = \rho(I)$ to show that

$$\rho(I) \le \mu^*(I)$$

Now we can show that I is in fact measurable.

Proof:

It suffices to show

$$\mu^*(E) \ge \mu^*(I \cap E) + \mu^*(I^c \cap E)$$

Notice

$$\mu^*(I \cap E) + \mu^*(I^c \cap E) = \mu^*((\mathbf{a}, \mathbf{b}) \cap E) + \mu^*((-\infty, \mathbf{a}] \cap E) + \mu^*([\mathbf{b}, \infty) \cap E)$$

because $I \subseteq \mathbb{R}^n$.

For notational convenience, let

$$\begin{aligned} E_1 &= (\mathbf{a}, \mathbf{b}) \cap E \\ E_2 &= (-\infty, \mathbf{a}] \cap E \\ E_3 &= [\mathbf{b}, \infty) \cap E \end{aligned}$$

Let $\varepsilon > 0$. By the sharpness of the outer measure, $\exists \bigcup_{n=1}^{\infty} I_n$ such that

$$E \subseteq \bigcup_{n=1}^{\infty} I_n$$

and

$$\sum_{n=1}^{\infty} \rho(I_n) < \mu^*(E) + \varepsilon$$

So

$$\begin{aligned} E_1 &\subseteq \bigcup_{n=1}^{\infty} I_n \cap (\mathbf{a}, \mathbf{b}) \\ E_2 &\subseteq \bigcup_{n=1}^{\infty} I_n \cap (-\infty, \mathbf{a}] \\ E_3 &\subseteq \bigcup_{n=1}^{\infty} I_n \cap [\mathbf{b}, \infty) \end{aligned}$$

Then by subadditivity,

$$\mu^*(E_1) \leq \sum_{n=1}^{\infty} \mu^*(I_n \cap (\mathbf{a}, \mathbf{b}))$$
$$\mu^*(E_2) \leq \sum_{n=1}^{\infty} \mu^*(I_n \cap (-\infty, \mathbf{a}])$$
$$\mu^*(E_3) \leq \sum_{n=1}^{\infty} \mu^*(I_n \cap [\mathbf{b}, \infty))$$

And by the faithfulness of the measure (as shown above),

$$\mu(I_n) = \mu^*(I_n \cap (\mathbf{a}, \mathbf{b})) + \mu^*(I_n \cap (-\infty, \mathbf{a}]) + \mu^*(I_n \cap [\mathbf{b}, \infty))$$

so

$$\mu^*(E_1) + \mu^*(E_2) + \mu^*(E_3) \leq \sum_{n=1}^{\infty} \mu(I_n) \leq \mu^*(E)$$

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2. Let $\varepsilon > 0$ and define

$$F = \{x : a_j \leq x_j \leq b_j,\ j = 1, \ldots, n,\ b_j - a_j < \varepsilon\}$$

with

$$\rho(F) = \prod_{j-1}^{n} (b_j - a_j)$$

We claim that

$$\mu^*(F) = \inf \left\{ \sum_i E_i : F \subseteq \bigcup_i E_i, \right\}$$

is an outer measure and $\mu^*(F) = \rho(F)$

First, we show that $\mu^*(F)$ is an outer measure.

1. $\mu^*(\emptyset) = 0$. Let $\{E_i\}$ be any collection with $\rho(E_i) = 0$ for all i. Then $\emptyset \subseteq \bigcup_i E_i$ so $\mu^*(\emptyset) = 0$.

2. Monotonicity. Let $A \subseteq B$. By sharpness of the measure, there exists a covering of B (and thus a covering of A) such that

$$\mu^*(A) \leq \sum_i \rho(E_i) \implies \mu^*(A) \leq \inf\{\sum_i \rho(E_i)\} = \mu^*(B)$$

3. Subadditivity. Let $\{A_j\}$ be a collection of sets and $\varepsilon > 0$. Let $\{E_{jk}\}$ be a covering of A_j such that $\sum_k \rho(E_{jk}) \leq \mu^*(A_j) + \frac{\varepsilon}{2^j}$.

Then $\bigcup_j A_j \subseteq \bigcup_{jk} E_{jk}$ so

$$\mu^* \left(\bigcup_j A_j \right) \leq \sum_{j,k} \rho(E_{jk}) \leq \sum_j \mu^*(A_j) + \varepsilon \overset{\varepsilon \to 0}{=} \sum_j \mu^*(A_j)$$

Hence, μ^* is an outer measure.

Now we show that $\mu^*(F) = \lambda(F)$, where

- μ^* is the outer measure constructed above
- λ is the Lebesgue measure on $\mathbb{R}^n$ (defined in part 1)
- F is the closed cube defined above

Certainly $\mu^*(F)$ is well defined by the construction above. Similarly,

$$F \subseteq I = \{x : a_j - \delta < x_j < b_j + \delta,\ b_j - a_j < \varepsilon\}$$

for any $\delta > 0$ (an open n-dimensional interval) and

$$\rho(F) = \prod_{i=1}^{n}(b_i - a_i) \overset{\delta \to 0}{=} \prod_{i=1}^{n}(b_i - a_i + 2\delta) = \rho(I)$$

Fix $\delta > 0$.

By the sharpness of the outer measure, $\exists\{E_i\}$ such that $F \subseteq \bigcup_i E_i$ and

$$\sum_i \rho(E_i) < \mu^*(F) + \delta$$

But by definition of λ, $\lambda(F) \leq \sum_i \rho(E_i)$ for any covering $\{E_i\}$ of F so

$$\lambda(F) \leq \sum_i \rho(E_i) < \mu^*(F) + \delta \implies \lambda(F) \leq \mu^*(F)$$

But we can argue identically that for any cover $\{J_i\}$ of F, $\sum_i \rho(J_i) < \lambda(F) + \delta$ and

$$\mu^*(F) \leq \sum_i \rho(J_i) < \lambda(F) + \delta \implies \mu^*(F) \leq \lambda(F)$$

and we are done.

3. (Hausdorff Measure) Let (X, ρ) be a metric space. Show that

$$\mathcal{H}_\alpha^\varepsilon(A) = \inf_{\substack{A \subseteq \bigcup_{k=1}^\infty A_k \\ \operatorname{diam} A_k \leq \varepsilon}} \sum_k \operatorname{diam}(A_k)^\alpha$$

is an outer measure, where $\operatorname{diam} A = \sup_{x,y \in A} \rho(x, y)$.

Prove that $\mu^* = \lim_{\varepsilon \to 0} \mathcal{H}_\alpha^\varepsilon$ is again an outer measure.

The resulting measure $\mathcal{H}_\alpha$ via Carathéodory construction is called the Hausdorff measure. Show that if $\mathcal{H}_\alpha(A) < \infty$, then $\mathcal{H}_\beta(A) = 0$ for all $\beta > \alpha$.

We claim that $\mathcal{H}_\alpha^\varepsilon$ is an outer measure on X.

1. $\mathcal{H}_\alpha^\varepsilon(\emptyset) = 0$ because $\emptyset \subseteq \bigcup_{k=1}^\infty A_k$ for any A_k so we can simply choose $A_k = \{k\} \implies \operatorname{diam}(A_k)^\alpha = 0 \forall k$ and then

$$\mathcal{H}_\alpha^\varepsilon(\emptyset) \leq \sum_k \operatorname{diam}(A_k)^\alpha = 0 \implies \mathcal{H}_\alpha^\varepsilon(\emptyset) = 0$$

2. (Monotonicity)

Let $A \subseteq B$ and $\{E_j\}_1^\infty$ be a covering of B (and hence also a covering of A)

By definition of $\mathcal{H}_\alpha^\varepsilon$,

$$\mathcal{H}_\alpha^\varepsilon(A) \leq \sum_{j=1}^\infty \operatorname{diam}(E_j)^\alpha$$
$$\implies \mathcal{H}_\alpha^\varepsilon(A) \leq \inf\left\{\sum_{j=1}^\infty \operatorname{diam}(E_j)^\alpha\right\} = \mathcal{H}_\alpha^\varepsilon(B)$$

3. (Subadditivity)

Let $\{A_j\}_1^\infty \subseteq P(X)$ and $\delta > 0$. Let $\{E_{jk}\}_1^\infty$ be a cover of A_j such that

$$\sum_{k=1}^\infty \operatorname{diam}(E_{jk})^\alpha \leq \mathcal{H}_\alpha^\varepsilon(A_j) + \frac{\delta}{2^j}$$

Now

$$\bigcup_j A_j \subseteq \bigcup_{jk} E_{jk}$$

so

$$\begin{aligned}
\mathcal{H}_\alpha^\varepsilon\left(\bigcup_j A_j\right) &\leq \sum_{j,k} \operatorname{diam}(E_{jk})^\alpha \\
&\leq \sum_j \left(\mathcal{H}_\alpha^\varepsilon(A_j) + \frac{\delta}{2^j}\right) \\
&= \sum_j \mathcal{H}_\alpha^\varepsilon(A_j) + \delta \\
&= \sum_j \mathcal{H}_\alpha^\varepsilon(A_j)
\end{aligned}$$

Now, let $\mu^* = \lim_{\varepsilon\to 0} \mathcal{H}^\varepsilon_\alpha$. We claim that μ^* is an outer measure on X.

1. Choose $A_k = \{k\}$ so $\operatorname{diam}(A_k) = 0 \leq \varepsilon$ for all $\varepsilon \geq 0$ and $\emptyset \subseteq \bigcup_{k=1}^\infty A_k$ so

$$\mu^*(\emptyset) \leq \sum_k \operatorname{diam}(A_k)^\alpha = 0 \implies \mu^*(\emptyset) = 0$$

2. (Monotonicity)

Let $A \subseteq B$ and $\bigcup_{k=1}^\infty A_k \supset B$. Then

$$\begin{aligned}\mu^*(A) &= \lim_{\varepsilon\to 0} \mathcal{H}^\varepsilon_\alpha(A)\\ &\leq \lim_{\varepsilon\to 0} \sum_k \operatorname{diam}(A_k)^\alpha\\ &\leq \lim_{\varepsilon\to 0} \sum_k \varepsilon^\alpha\\ &= 0\end{aligned}$$

and since $\mu^* : P(X) \to [0,\infty]$, $\mu^*(A) = 0 \leq \mu^*(B)$.

3. (Subadditivity)

Let $\bigcup_j A_j \subseteq P(X)$ and $\varepsilon > 0$.

Define

$$O_\varepsilon = \left\{ E_{jk} : \bigcup_k E_{jk} \supset A_j,\ \operatorname{diam}(E_{jk}) \leq \varepsilon,\ \sum_k \operatorname{diam}(E_{jk})^\alpha \leq \mu^*(E_{jk}) + \frac{\varepsilon}{2^j} \right\}$$

Now

$$\bigcup_j A_j \subseteq \bigcup_{jk} E_{jk}$$

for $E_{jk} \in O_\varepsilon$. and

$$\mu^*\left(\bigcup_j A_j\right) = \lim_{\varepsilon\to 0} \inf \left\{ \sum_{j,k} \operatorname{diam}(E_{jk})^\alpha \,\middle|\, E_{jk} \in O_\varepsilon \right\}$$

But as $\varepsilon \downarrow 0$, $O_\varepsilon \downarrow \emptyset$.

Suppose $\exists \varepsilon_1 > 0$ such that $O_{\varepsilon_1} = \emptyset$. Then by (1), $\forall \varepsilon \leq \varepsilon_1$,

$$\mu^*(\bigcup_j A_j) = \lim_{\varepsilon\to 0} \inf\{0\} = 0 \leq \sum_j \mu^*(A_j)$$

Therefore, assume WLOG that $O_\varepsilon \neq \emptyset$ for all $\varepsilon > 0$.

Then

$$\mu^*\left(\bigcup_j A_j\right) = \lim_{\varepsilon\to 0}\inf\left\{\sum_{j,k}\text{diam}\,(E_{jk})^\alpha \;\middle|\; E_{jk}\in O_\varepsilon\right\}$$
$$\leq \lim_{\varepsilon\to 0}\left[\sum_{j,k}\text{diam}\,(E_{jk})^\alpha\right]$$
$$\leq \lim_{\varepsilon\to 0}\left[\sum_{j}\mu^*(E_{jk}) + \frac{\varepsilon}{2^j}\right]$$
$$= \lim_{\varepsilon\to 0}\left[\sum_{j}\mu^*(A_j) + \varepsilon\right]$$
$$= \sum_j \mu^*(A_j)$$

Hence μ^* is an outer measure on X. Henceforth, call the measure associated with μ^* as $\mathcal{H}_\alpha$.

Finally, we seek to show that if $\mathcal{H}_\alpha(A) < \infty$, then $\mathcal{H}_\beta(A) = 0$ for all $\beta > \alpha$.

By definition of the measure as inf, choose $\{E_i\}$ as the covering of A with diam $E_i < \varepsilon$ such that

$$\sum_i \text{diam}\,(E_i)^\alpha \leq \sum_i \text{diam}\,(F_i)^\alpha$$

for all other coverings of A with diam $F_i < \varepsilon$.

Then

$$H_\beta(A) \leq \sum_i (\text{diam}\,E_i)^\beta$$
$$= \sum_i (\text{diam}\,E_i)^{\beta-\alpha}(\text{diam}\,E_i)^\alpha$$
$$\leq \varepsilon^{\beta-\alpha}\sum_i (\text{diam}\,E_i)^\alpha$$
$$= \varepsilon^{\beta-\alpha}\mathcal{H}_\alpha(A) < \infty$$

Letting $\varepsilon \to 0$, $H_\beta(A) \leq 0 \implies H_\beta(A) = 0$.

4. (Lebesgue-Stieltjes Measure) Let f be a monotone increasing function ($f(x) \leq f(y)$ for $x \leq y$). Define $\rho((a,b]) = f(b) - f(a)$ and

$$\mu^*(A) = \inf\left\{\sum \rho((a,b]) : A \subseteq \bigcup_{k=1}^{\infty}(a_k, b_k]\right\}$$

Prove that μ^* is an outer measure in $\mathbb{R}$ and the corresponding measure μ from the Carathéodory construction is called the Lebesgue-Stieltjes measure.

Is it true $\mu^*((a,b]) = \rho((a,b])$? If not, give a sufficient condition on f such that this is true, then show $(a,b]$ is measurable in this case.

1. Choose $a_k = b_k$ so $\rho((a_k,b_k]) = 0$ and $\emptyset \subseteq \bigcup_{k=1}^{\infty}(a_k,b_k]$.

Then

$$\mu^*(\emptyset) \leq \sum_k \rho((a_k,b_k]) = 0 \implies \mu^*(\emptyset) = 0$$

2. (Monotonicity)

Let $A \subseteq B$ and choose a covering $\{(a_j,b_j]\}_{j=1}^{\infty}$ of B (which is hence a covering of A).

Then

$$\begin{aligned}\mu^*(A) &\leq \sum_j \rho((a_j,b_j]) \\ &\implies \inf \mu^*(A) \leq \inf\{\sum_j \rho((a_j,b_j])\} \\ &\implies \mu^*(A) \leq \mu^*(B)\end{aligned}$$

3. (Subadditivity)

Let $\{A_j\}_1^{\infty} \subseteq P(\mathbb{R})$ and $\varepsilon > 0$. Let $\{(a_{jk},b_{jk}]\}_1^{\infty}$ be a cover of A_j such that

$$\sum_k \rho((a_{jk},b_{jk}]) \leq \mu^*(A_j) + \frac{\varepsilon}{2^j}$$

Then

$$\bigcup_j A_j \subseteq \bigcup_{jk}(a_{jk},b_{jk}]$$

and

$$\begin{aligned}\mu^*\left(\bigcup_j A_j\right) &\leq \sum_{j,k} \rho((a_{jk},b_{jk}]) \\ &\leq \sum_j \left(\mu^*(A_j) + \frac{\varepsilon}{2^j}\right) \\ &= \sum_j \mu^*(A_j) + \varepsilon \\ &= \sum_j \mu^*(A_j)\end{aligned}$$

Hence, μ^* is an outer measure on $\mathbb{R}$ inducing the Lebesgue-Stieltjes measure μ.

Consider

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Then $\rho((-1,0]) = 1$ but $(-1,0] \subseteq \bigcup_n (-1, -\frac{1}{n})$ so

$$\mu((-1,0]) \leq \sum_n \rho((-1,-\frac{1}{n})) = \sum_n (0-0) = 0$$

Hence $\mu^*((a,b]) \neq \rho((a,b])$ in general.

We claim that if f is right continuous, then $\mu^*((a,b]) = \rho((a,b])$.

We have that $(a,b]$ covers itself so certainly $\mu^*((a,b]) \leq \rho((a,b])$.

Now suppose $(a,b] \subseteq \bigcup_{k=1}^{\infty}(a_i, b_i]$. Let $\varepsilon > 0$ and pick $x \in (a,b)$ such that $f(x) - f(a) < \frac{\varepsilon}{2}$ (possible by monotonicity and right continuity of f).

Choose $b_i' > b_i$ such that $f(b_i') - f(b_i) < \frac{\varepsilon}{2^{i+1}}$ and let $E_i = (a_i, b_i')$.

Now $[x,b]$ is compact and $\{E_i\}$ is an open cover. By Heine-Borel, there exists a finite subcover of $[x,b]$ by $\{E_i\}_{i=1}^N$.

Lemma: For $\{E_i\}_{i=1}^n$, a finite subcover of $[a,b]$ with $E_k = (a_k, b_k)$ and f a non-decreasing, right continuous function,

$$\sum_{k=1}^{n}[f(b_k) - f(a_k)] \geq f(b) - f(a)$$

Proof: $\exists E_{k_1}$ s.t. $a \in E_{k_1}$. If $[a,b] \subseteq E_{k_1}$, then the result is immediate. Otherwise, $E_{k_1} = (a_{k_1}, b_{k_1})$ and $b_{k_1} \leq b$ so $\exists E_{k_2}$ s.t. $b_{k_1} \in E_{k_2}$.

If $[a,b] \subseteq E_{k_1} \cup E_{k_2}$, we stop. Otherwise, $b_{k_1} < b_{k_2} \leq b$ and we continue.

Inductively choose E_{k_j} such that $b_{k_{j-1}} \in E_{k_j}$.

Because $\{E_i\}$ is a finite subcover, this process must terminate for some $m \leq n$.

Then, by construction,

$$a_{k_1} < a < b_{k_1}$$
$$a_{k_m} < b < b_{k_m}$$

and $a_{k_j} < b_{k_{j-1}} < b_{k_j}$ so

$$\begin{aligned} f(b) - f(a) &\leq f(b_{k_m}) - f(a_{k_1}) \\ &= [f(b_{k_1}) - f(a_{k_1})] + \sum_{j=2}^{m}[f(b_{k_j}) - f(a_{k_{j-1}})] \\ &\leq \sum_{j=1}^{m}[f(b_{k_j}) - f(a_{k_j})] \end{aligned}$$

By the Lemma,

$$\begin{aligned}
\rho((a,b]) &\leq f(b) - f(x) + \frac{\varepsilon}{2} \\
&\leq \sum_{i=1}^{n} (f(b_i') - f(a_i)) + \frac{\varepsilon}{2} \\
&\leq \sum_{i=1}^{n} \rho((a_i, b_i]) + \varepsilon \\
&\overset{\varepsilon \to 0}{=} \sum_{i=1}^{n} \rho((a_i, b_i])
\end{aligned}$$

Taking the inf over all such covers, $\rho(I) \leq \mu^*(I)$ and hence $\mu^*(I) = \rho(I)$.