Set Algebra

$$\limsup E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \tag{1}$$
$$\liminf E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k \tag{2}$$
$$\limsup f_n = -\liminf\{-f_n\} \tag{3}$$
$$\liminf f_n = -\limsup\{-f_n\} \tag{4}$$

Equivalence Relation: $\sim$ is an equivalence relation on X if

1. $x \sim x$
2. $x \sim y \implies y \sim x$
3. $x \sim y \wedge y \sim z \implies x \sim z$

Inverses:

$$f^{-1}\left(\bigcup_{\alpha \in A} E_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(E_\alpha) \tag{5}$$
$$f^{-1}\left(\bigcap_{\alpha \in A} E_\alpha\right) = \bigcap_{\alpha \in A} f^{-1}(E_\alpha) \tag{6}$$
$$f^{-1}(E^c) = (f^{-1}(E))^c \tag{7}$$

Partial Ordering:

1. $xRy \wedge yRx \implies x = y$
2. $xRy \wedge yRz \implies xRz$
3. xRx for all x

Total/Linear Ordering:

1. R is a partial ordering
2. xRy or yRx

Zorn's Lemma: If X is partially ordered and every totally ordered subset of X is bounded, X has a maximal element.

Well-Ordering:

1. X is linearly ordered
2. every nonempty subset of X has a minimal element

Schroder-Bernstein: $\exists f : X \hookrightarrow Y, g : Y \hookrightarrow X \implies \exists h : X \hookrightarrow\!\!\!\!\!\rightarrow Y$

Countability:

- X, Y countable $\implies X \times Y$ countable
- X_n countable $\implies \bigcup_{n=1}^{\infty} X_n$ countable

Topology of $\mathbb{R}$: Every open set in $\mathbb{R}$ is a countable disjoint union of open intervals

Metric: $\rho : X \times X \to [0, \infty)$ s.t.

1. $\rho(x, y) = 0 \iff x = y$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

Point-set topology

- **Open Set:** $\forall x > 0, \exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq U$
- **Closed Set:** E^c open
- **Interior:** $\mathring{E} = \bigcup_{O \subseteq E} O$
- **Closure:** $\overline{E} = \bigcap_{E \subseteq K} K$
- **Dense:** $E \subseteq X$ dense in X if $\overline{E} = X$
- **Nowhere Dense:** $(\overline{E})^\circ = \emptyset$
- **Separable:** $\exists E \subseteq X$ countable, dense
- **Complete Metric Space:** All cauchy sequences have limit in X
- **Set diameter:** diam $E = \sup\{\rho(x, y) : x, y \in E\}$
- **Bounded set:** diam $E < \infty$
- **Totally bounded set:** E may be covered by finitely many balls

Characterization of closed sets: for $E \subseteq X$ and $x \in X$, the following are equivalent:

1. $x \in \overline{E}$
2. $E \cap B(x, \varepsilon) \neq \emptyset$
3. $\exists x_n \in E$ with $x_n \to x \in E$

Continuity:

- **Continuity at a point:** $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\rho(x, y) < \delta_x \implies \rho(f(x), f(y)) < \varepsilon$
- **Uniform Continuity:** $\exists \delta > 0$ $\forall \varepsilon > 0$ s.t. $\rho(x, y) < \delta \implies \rho(f(x), f(y)) < \varepsilon$
- **Cauchy Sequence:** for $mn \geq N$, $\rho(x_n, x_m) < \varepsilon$

Continuity Lemmas:

- $f : X \to Y$ continuous iff $f^{-1}(U) \subseteq X$ open for all $U \subseteq Y$
- A closed subset of a complete metric space is complete
- A complete subset of a metric space is closed

Characterization of Compactness: the following are equivalent:

1. E complete and totally bounded
2. Every sequence in E has a convergent subsequence with limit in E
3. Every open cover of E has a finite subcover

Measure Theory

Algebra: $\mathcal{A} \subseteq P(X)$ such that

1. $E \in \mathcal{A} \implies E^c \in \mathcal{A}$
2. $\{E_k\}_1^N \in \mathcal{A} \implies \bigcup_1^N \mathbb{E}_k \in \mathcal{A}$

Sigma Algebra: $\mathcal{M} \subseteq P(X)$ such that

1. $E \in \mathcal{A} \implies E^c \in \mathcal{A}$
2. $\{E_k\}_1^\infty \in \mathcal{A} \implies \bigcup_1^\infty E_k \in \mathcal{A}$

Generated Sigma Algebra: $M(E) = \bigcap_{E \subseteq \mathcal{A}} \mathcal{A}$ for algebras $\mathcal{A}$

Borel Sigma Algebra: σ-algebra generated by open sets

Lemma: An algebra $\mathcal{A}$ is a σ-algebra if $\exists E_n \nearrow \in \mathcal{A}$ with $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$

Measure: $\mu : X \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$
2. $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} E_n$ for E_n disjoint

Outer measure: $\mu^* : X \to [0, \infty]$ with

1. $\mu^*(\emptyset) = 0$
2. $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} E_n$

Properties of Measure:

1. $E \subseteq F$, $\mu(E) \leq \mu(F)$
2. $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$
3. $E_j \nearrow$,
$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j \to \infty} \mu(E_j)$$
4. $E_j \searrow$ and $\mu(E_1) < \infty$,
$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j \to \infty} \mu(E_j)$$

μ^*-measurability: A measurable if $\forall E \subseteq P(X)$,

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c)$$

Lebesgue Measure:

$$m(E) = \inf \left\{ \sum \rho(E_j) \mid E \subseteq \bigcup E_j \right\}$$

Properties of the Lebesgue Measure:

- $m(I) = m(I)$ for intervals I on $\mathbb{R}$
- $m(E + a) = m(E)$
- $m(rE) = |r|\, m(E)$
- $\exists O$ open such that $m(O) \geq m(E) \geq m(O) - \varepsilon$
- $\exists K$ closed such that $m(K) \leq m(E) \leq m(K) + \varepsilon$

Integration

Measurable Function: $f : X \to \mathbb{R}$ measurable if $\{x : f(x) > \alpha\} \in \mathcal{M}$ or $\{f \geq \alpha\} \in \mathcal{M}$.

Composition of Measurable Functions:

- f, g measurable and $c \in \mathbb{R} \implies f + c$, $f + g$, fg, cf measurable
- $\{f_n\}$ measurable $\implies \max_i f_i$, $\min_i f_i$, $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, $\liminf_n f_n$ measurable

Simple Functions:

- $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$
- $\int \phi \, d\mu = \sum_{i=1}^{n} a_i \mu(E_i)$

Approximation by Simple Functions:

- $f \geq 0$ measurable implies $\exists \phi_n \nearrow f$, $\phi_n \to f$ pointwise on X, $\phi_n \to f$ uniformly on sets with f bounded.
- f measurable implies $\exists 0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq |f|$ with $\phi_n \to f$ pointwise on X and uniformly on sets with f bounded.

General Integral: for $f \geq 0$

$$\int f \, d\mu = \sup_{\phi \leq f} \left\{ \int \phi \, d\mu \right\}$$

General Integral Properties: for $f \geq 0$,

- $c \geq 0 \implies \int cf \, d\mu = c \int f \, d\mu$
- $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$
- $f \leq g \implies \int f \, d\mu \leq \int g \, d\mu$

Monotone Convergence Theorem: if $f_n \geq 0$ and $f_n \leq f_{n+1}$,

$$\lim_{n \to \infty} \int f_n \, d\mu = \int \lim_{n \to \infty} f \, d\mu$$

Fatou's Lemma: If $f_n \geq 0$,

$$\int \liminf f_n \, d\mu \leq \liminf \int f_n \, d\mu$$

Lebesgue Dominated Convergence: If $0 \leq f_n \leq g$ for $g \in \mathcal{L}^1$,

$$\lim_{n \to \infty} \int f_n \, d\mu = \int \lim_{n \to \infty} f_n \, d\mu$$

Almost Everywhere Properties:

1. $\mu(E) = 0 \implies \int_E f = 0$ for $f \geq 0$
2. $\int_X f = 0$ for $f \geq 0 \implies f = 0$ a.e.
3. $\int_X f < \infty \implies f < \infty$ a.e.

$\mathcal{L}^1$ (integrable): $\int |f| \, d\mu < \infty$

Properties of $\mathcal{L}^1$ functions: $f, g \in \mathcal{L}^1$ and $a, b \in \mathbb{R}$:

1. $af + bg \in \mathcal{L}^1$
2. $\int af + bg \, d\mu = a \int f + b \int g$
3. $\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$

Approximation by simple functions: $f \in \mathcal{L}^1 \implies \exists \phi$ simple s.t. $\int |f - \phi| \, d\mu < \varepsilon$

Reduction to smooth functions: For μ the Lebesgue-Stieltjes on $\mathbb{R}$, $\exists g \in C^{\infty}$ with compact support such that

$$\int |f - g| \, d\mu < \varepsilon$$

Integrals with Parameter: For $F(t) = \int f(x, t) \, d\mu$ integrable in t,

- If $|f(x, t)| \leq g(x) \in \mathcal{L}^1$ and $f(x, t)$ is continuous in a.e. t, $F(t)$ is continuous

- If $\exists g \in \mathcal{L}^1$ with $\left|\frac{\partial f}{\partial t}\Big|_{(x,t)}\right| \leq g(x)$,

$$F'(t) = \int \frac{\partial f}{\partial t}\Big|_{(x,t)} \, d\mu$$

Lebesgue and Riemann: f bounded real,

- f Riemann integrable $\implies f$ Lebesgue integrable with same value
- f Riemann integrable $\iff f$ has countably many discontinuities

Modes of Convergence:

- **Convergence in $\mathcal{L}^1$:** $\int |f_n - f| \, d\mu \to 0$
- **Convergence in measure:** $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \to 0$
- **Chebysheb inequality:**

$$\int |f_n - f| \, d\mu \geq \varepsilon\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\})$$

- **Cauchy in measure:** $\forall \varepsilon, \delta > 0$, $\exists N$ s.t. $n, m \geq N \implies \mu(\{x : |f_n(x) - f_m(x)| \geq \varepsilon\}) < \delta$

Convergence theorems:

- $f_n \to f \in \mathcal{L}^1 \implies f_n \xrightarrow{\mu} f$
- f_n Cauchy in measure $\implies f_{n_j} \to f$ a.e. and $f_n \xrightarrow{\mu} f$
- $f_n \xrightarrow{\mu} f \implies f_{n_j} \to f$ a.e.
- $f_n \to f \in \mathcal{L}^1 \implies f_{n_j} \to f$ a.e.

Egorov: $f_n \to f$ a.e. pointwise $\implies \exists E$ s.t. $\mu(E) = 0$ and $f_n \to f$ uniformly on E^c

Fubini's Theorem:

Sections:

- E_x
- E_y
- $E \in \mathcal{M} \otimes \mathbb{N} \implies E_x \in \mathbb{N} \wedge E_y \in \mathcal{M}$
- $f(x, y)$ measurable on $\mathcal{M} \otimes \mathbb{N}$ implies $f_x(x, y)$ and $f^y(x, y)$ measurable

Fubini-Tonelli: $(X, \mathcal{M}, \mu)$ and $(Y, \mathbb{N}, \nu)$ σ-finite and $f \in \mathcal{L}^+$ implies

$$\int f \, d(\mu \times \nu) = \int \int f(x, y) \, d\nu(y) \, d\mu(x) = \int \int f(x, y) \, d\mu(x) \, d\nu(y)$$

Monotone CLass: $\mathcal{C} \subseteq P(X)$ if either

- $E_n \in \mathcal{C}$ and $E_n \nearrow \implies \bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$
- OR $E_n \in \mathbb{C}$ and $E_n \searrow \implies \bigcap_{n=1}^{\infty} E_n \in \mathcal{C}$

Sigma Algebras and Monotone Classes: for $\mathcal{A} \subseteq P(X)$ an algebra, the sigma algebra $\mathcal{M}(A) = \mathcal{C}(A)$ the monotone class generated by $\mathcal{A}$

The Lebesgue Measure on $\mathbb{R}^n$

If E is a Lebesgue measurable set,

- $m(E) = \inf\{m(U) : E \subseteq U\} = \sup\{m(K) : L \subseteq E\}$
- $E = A_1 \cup N_1 = A_2 \setminus N_2$ where A_1, A_2 Borel and $m(N_1) = m(N_2) = 0$
- $m(E) < \infty \implies m\left(E \triangle \bigcup_{j=1}^{N} R_j\right) < \varepsilon$ for disjoint rectangles R_j

Approximation of $\mathcal{L}^1$ functions: if $f \in \mathcal{L}^1_m(\mathbb{R}^n)$, $\exists \phi$ simple with

$$\int |f - \phi| < \varepsilon$$

and f_c continuous with compact support such that

$$\int |f_c - \phi| < \varepsilon$$

Dyadic Cubes:

Let Q_k be the lattice of cubes of side length 2^{-k}. Define

$$\underline{A}(E, k) = \bigcup\{Q \in Q_k : Q \subseteq E\}$$
$$\overline{A}(E, k) = \bigcup\{Q \in Q_k : Q \cap E \neq \emptyset\}$$

Then define

- Inner content: $\underline{\kappa}(E) = \lim_{k \to \infty} m(\underline{A}(E, k))$
- Outer content: $\overline{\kappa}(E) = \lim_{k \to \infty} m(\overline{A}(E, k))$
- Jordan Content: $m(E) = \kappa(E) = \underline{\kappa}(E) = \overline{\kappa}(E)$

Letting

$$\underline{A}(E) = \bigcup_{k=1}^{\infty} \underline{A}(E, k)$$
$$\overline{A}(E) = \bigcap_{k=1}^{\infty} \overline{A}(E, k)$$

then

$$\underline{A}(E) \subset E \subset \underline{A}(E)$$

and $\underline{A}(E), \underline{A}(E)$ are Borel.

Lemma: $U = \underline{A}(U)$ and U is a countable union of disjoint cubes

Change of Variables:

- For $\tau_a(x) = x + a \in \mathbb{R}^n$, $m(\tau_a(E)) = m(E)$
- If $f : \mathbb{R}^n \to \mathbb{C}$ is in $\mathcal{L}^1$ or ≥ 0, $\int f(x + a) \, dm = \int f \, dm$

If If $G : X \to \mathbb{R}^n$ is C^1-differentiable and invertible, and $G(E)$ is lebesgue measurable,

- $m(G(E)) = \int_E |\det D_x G| \, dx$
- If $f \circ G \in L^1$ and $f \geq 0$,

$$\int_{G(X)} f(y) \, dy = \int_X f \circ G(x) \, |\det D_x G| \, dx$$

Signed Measures

Signed Measure: $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ with

1. $\nu(\emptyset) = 0$
2. ν assumes at most one of $\infty, -\infty$
3. $\nu\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \nu(E_j)$ for E_j disjoint
4. If $\nu\left(\bigcup_{j=1}^\infty E_j\right) < \infty$, $\sum_{j=1}^\infty \nu(E_j) < \infty$

Properties of signed measures:

- If $E^j \nearrow$, $\nu\left(\bigcup_{j=1}^\infty E_j\right) = \lim_{j\to\infty} \nu(E_j)$
- If $E_j \searrow$, $\nu\left(\bigcap_{j=1}^\infty E_j\right) = \lim_{j\to\infty} \nu(E_j)$

Signed measure sets: A set is

- **Positive** if $\forall F \subseteq E$, $\nu(F) \geq 0$
- **Negative** if $\forall F \subseteq E$, $\nu(F) \leq 0$
- **Null** if $\forall F \subseteq E$¡ $\nu(F) = 0$

Lemma:

- Any measurable subset of a positive set is positive
- The union of countably many positive sets is positive

Hahn Decomposition: for ν signed measure, there exists a unique P positive and N negative with $X = P \cup N$ where $P \cap N = \emptyset$

Jordan Decomposition: for ν signed measure, $\exists \nu^+, \nu^-$ unique such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$

Total Variation: $|\nu| = \nu^+ + \nu^-$

Mutual singularity: $\mu \perp \nu \iff X = E \cup F \wedge \mu(E) = \nu(F) = 0$

- $\nu \perp \mu \iff |\nu| \perp \mu$

Absolute continuity: $\nu \ll \mu$ if $\mu(E) = 0 \implies \nu(E) = 0$

- $\nu \ll \mu \iff |\nu| \ll \mu \iff \nu^+ \ll \mu \wedge \nu^- \ll \mu$
- $\nu \ll \mu$ iff $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\mu(E) < \delta \implies |\nu(E)| < \varepsilon$
- $f\ d\mu \ll \mu$ for $f \in \mathcal{L}^1$

Lemma: If $f \in \mathcal{L}^1_\mu$, then $\mu(E) < \delta_f \implies f\ d\mu(E) < \varepsilon$

Dichotomy of $\perp$ and $\ll$: If μ, ν positive, finite measures, either $\mu \perp \nu$ or exists a positive set E WRT $\nu - \varepsilon\mu$

Derivatives

Lebesgue-Radon-Nikodym Theorem:ν sigma-finite and $\implies \exists f \in \mathcal{L}^1_\mu$ unique a.e. such that

$$\nu = \lambda + f\ d\mu \quad \lambda \perp \mu$$

Radon-Nikodym Derivative: $\nu \ll \mu \implies \exists f = \frac{d\nu}{d\mu} \implies \nu(E) = \int_E f\ d\mu$

Chain Rule: ν sigma finite, λ, μ positive, $\nu \ll \mu \ll \lambda$,

1. $g \in \mathcal{L}^1 \implies g\frac{d\nu}{d\mu} \in \mathcal{L}^1_\mu$ and
$$\int g\ d\nu = \int g \frac{d\nu}{d\mu}\ d\mu$$
2. $\nu \ll \lambda$ and
$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}$$

Vitali Covering: For $c < m\left(\bigcup_{B\in\mathcal{C}} B\right)$ open balls, $\exists \{B_j\}_{j=1}^k \in \mathcal{C}$ with $\sum_{j=1}^k m(B_j) > 3^{-n}c$

Average Value: For $f \in \mathcal{L}^1_{\text{loc}}$ (integrable on bounded measurable subsets),

$$A_r f(x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f\ d\mu$$

is continuous in $r > 0$ and $x \in \mathbb{R}^n$ and

- (Strong Derivative) $\lim_{r\to 0} A_r f(x) = f(x)$ a.e.
- (Weak Derivative) $\lim_{r\to 0} ||A_r f(x) - f(x)||_{\mathcal{L}^1} \to 0$ and $\exists r_n \to 0$ where $A_{r_n} f(x) \to f(x)$ a.e.

Hardy-Littlewood Maximal Function: for $f \in \mathcal{L}^1_{\text{loc}}$,

$$Hf(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)|\ dy$$

Maximal Theorem: $\exists C$ such that for $f \in \mathcal{L}^1$,

$$m(\{x : Hf > \alpha\}) \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(y)|\ dy$$

$\mathcal{L}^1$-weak functions: $g \in \mathcal{L}^1_W$ if $m\{x : |g| > \alpha\} \leq \frac{C}{\alpha} \int |g|\ d\mu$

Homework Results

Liminf/Limsup Fatou: For $\{E_j\} \in \mathcal{M}$, $\mu(\liminf E_j) \leq \liminf \mu(E_j)$.

If $\mu\left(\bigcup_{j=1}^\infty E_j\right) < \infty$, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$.

Outer measure: For μ^* outer measure,

$$\sum_{k=1}^\infty \mu^*(E_k) < \infty \implies \mu^*(\limsup E_k) = 0$$

Verifying Outer Measure:

1. $\mu^*(\emptyset) = 0$. Trivial (take singletons)
2. Monotonicity. Trivial (choose cover of larger set, invoke inf)
3. Subadditivity. Notice $\exists\{E_{jk}\}$ with
$$\sum \rho(E_{jk}) \leq \mu^*(A_j) + \frac{\varepsilon}{2^j} \implies \bigcup_j A_j \subseteq \bigcup_j \bigcup_k E_{jk}$$

Hausdoff Outer Measure:

$$\mathcal{H}^\varepsilon_\alpha(A) = \inf\left\{\sum_k \text{diam } (A_k)^\alpha \,\middle|\, A \subseteq \bigcup A_k, \quad \text{diam } A_k \leq \varepsilon\right\}$$

Hausdorff Measure:

$$\mathcal{H}_\alpha = \lim_{\varepsilon\to 0} \mathcal{H}^\varepsilon_\alpha$$

satisfies $H_\alpha(A) < \infty \implies H_\beta(A) = 0$ for $\beta > \alpha$

$\mathcal{L}^p$-spaces:

- *Definition:* $f \in \mathcal{L}^p$ if $\int |f|^p \; d\mu < \infty$ for $1 \leq p \leq \infty$
- *Property:* For $\mu(X) < \infty$, $f \in \mathcal{L}^p \implies f \in \mathcal{L}^q$ for $1 \leq q \leq p$.

Layercake Representation: For $f \geq 0$ measurable,

$$\int |f| \; d\mu = \int_0^\infty \mu\{x : |f(x)| > \lambda\} \; d\lambda$$

Convergence results:

- If $f_n \to f$ a.e. and $\lim_{n\to\infty} \int |f_n| \; d\mu = \int |f| \; d\mu$,

$$\lim_{n\to\infty} \int |f_n - f| \; d\mu = 0$$

- If $f_n \xrightarrow{\mu} f$,

$$\liminf_n \int |f_n| \; d\mu \geq \int |f| \; d\mu$$

- If $f_n \xrightarrow{\mu} f$ and $|f_n| \leq g \in \mathcal{L}^1$, $f_n \to f \in \mathcal{L}^1$
- $f_n \geq 0$, $f_n \to f$ a.e., $\int_X f_n \to \int_X f$, then for $E \subseteq X$,

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

- $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$,

$$f_n + g_n \xrightarrow{\mu} f + g$$

- $f_n \xrightarrow{\mu} f$, $g_n \xrightarrow{\mu} g$, and $\mu(X) < \infty$,

$$f_n g_n \xrightarrow{\mu} fg$$

Riemann Lebesgue Theorem: $f \in \mathcal{L}^1(\mathbb{R})$,

$$\lim_{n\to\infty} \int f(x) \cos(nx) \; dx = 0$$

N-Balls: for $f \in \mathcal{L}^1(\mathbb{R}^d)$,

$$Hf(x) \geq \frac{C}{|x|^d} \int |f| \; d\mu$$

Continuity of Lebesgue Integral: $f \in \mathcal{L}^1_m(\mathbb{R})$.

- If $|h| < \delta$,

$$\int_{\mathbb{R}} |f(x+h) - f(x)| \; dx < \varepsilon$$

- The following is continuous:

$$F(t) = \int_{-\infty}^{x} f(t) \; dt$$

Signed Measures: If $\nu = \lambda - \mu$, $\lambda \geq \nu^+$ and $\mu \geq \nu^-$