APMA 2110 - Homework 2

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1. Let A be a countable set. Show that the set of all finite sequences from A

$$X = \{(a_1, a_2, a_3, \ldots, a_n) : a_i \in A, \text{ any fixed } n \geq 1\}$$

is countable.

Define $X_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A\}$.

We claim that $X = \bigcup_{n=1}^{\infty} X_n$.

Proof: Let $x \in X$. Then $x = (a_1, a_2, \ldots, a_n)$ for some n. So $x \in X_n$ for some n. Thus $X \subseteq \bigcup_{n=1}^{\infty} X_n$. Now consider $x \in \bigcup_{n=1}^{\infty} X_n$. Then $x \in X_n$ for some n so $x = (a_1, a_2, \ldots, a_n)$ for some n. Thus $x \in X$. So $\bigcup_{n=1}^{\infty} X_n \subseteq X$.

By a proposition from class, the union of countably many countable sets is countable. So it suffices to show that each X_n is countable.

Let $n = 1$. So $X = \{(a) : a \in A\}$. Trivially, there exists $f : X \hookrightarrow\!\!\!\!\rightarrow A$ by $f(x) = x$. Since A is countable, X is countable.

Let $X_k = \{(a_1, a_2, \ldots, a_k) : a_i \in A\}$. Assume X_k is countable. Then

$$X_{k+1} = \{(a_1, a_2, \ldots, a_k, a_{k+1}) : a_i \in A\}$$

We can define a bijection $f : X_{k+1} \hookrightarrow\!\!\!\!\rightarrow X_k \times A$ by

$$f((a_1, a_2, \ldots, a_k, a_{k+1})) = ((a_1, a_2, \ldots, a_k), a_{k+1})$$

Clearly this is injective by element-wise comparison of the finite sequences. Further, it is surjective because the first k terms of sequences in X_{k+1} span X_k (as it is a subsequence of length k in X) and the last term can be any element of A.

Since the product of two countable sets is countable (by a proposition from class), X_{k+1} is countable. $\blacksquare$

2. Let p be an integer greater than 1 and x a real number $0 < x < 1$. Show that there is a sequence $\{a_n\}$ of integers with $0 \leq a_n < p$ such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

and that this sequence is unique except when $x = q/p^n$, in which case there are two such sequences.

Let $x_1 = x$ and $a_1 = \lfloor px_1 \rfloor$ where $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ is defined by

$$\lfloor x \rfloor = \sup\{n \in \mathbb{Z} : n \leq x\}$$

(well-defined by completeness - x is an upper bound).

Then let $x_2 = px_1 - a_1$. We claim that $0 \leq x_2 < 1$.

Suppose not. Then $x_2 < 0$ or $x_2 \geq 1$.

Case 1: $x_2 < 0$. Then $px_1 < a_1 = \lfloor px_1 \rfloor$ which is a contradiction of the definition of the floor function.

Case 2: $x_2 \geq 1$. Then $px_1 \geq 1 + a_1 = 1 + \lfloor px_1 \rfloor$ so

$$px_1 - \lfloor px_1 \rfloor \geq 1$$

which is again a contradiction of the definition of the floor function.

By similar argument, let $x_3 = px_2 - a_2$ and $a_2 = \lfloor px_2 \rfloor$.

Expanding so far,

$$x_1 = \frac{a_1}{p} + \frac{x_2}{p^2} + \frac{r_3}{p^2}$$

Suppose that for $i = 1, \ldots, k$, we have

$$x_{i+1} = px_i - a_i$$

for $a_i = \lfloor px_i \rfloor \in \{0, 1, \ldots, p-1\}$ and $0 \leq x_i < 1$. Then

$$x = \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} + \frac{x_{k+1}}{p^k}$$

where

$$x_{k+1} = px_k - a_k$$

If $x_{k+1} = 0$, then $a_k = px_k = \lfloor px_k \rfloor$ so

$$x = \sum_{n=1}^{k} \frac{a_n}{p^n}$$

exactly so we can define $a_n = 0$ for $n > k$ and we have successfully constructed a sequence.

Otherwise, $0 < x_{k+1} < 1$ so $a_{k+1} = \lfloor px_{k+1} \rfloor$ and $0 \leq a_{k+1} < p$.

We claim $0 \leq x_{k+2} < 1$ with $x_{k+2} = px_{k+1} - a_{k+1}$, i.e. this process can be continued indefinitely.

Suppose not. Then $x_{k+2} < 0$ or $x_{k+2} \geq 1$.

Case 1: $x_{k+2} < 0$. Then $px_{k+1} < a_{k+1} = \lfloor px_{k+1} \rfloor$ which is a contradiction of the definition of the floor function.

Case 2: $x_{k+2} \geq 1$. Then $px_{k+1} \geq 1 + a_{k+1} = 1 + \lfloor px_{k+1} \rfloor$ so $px_{k+1} - \lfloor px_{k+1} \rfloor \geq 1$ which is again a contradiction of the definition of the floor function.

Then,

$$x_{k+1} = \frac{a_{k+1}}{p} + \frac{x_{k+2}}{p}$$

so

$$x = \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_k}{p^k} + \frac{a_{k+1}}{p^{k+1}} + \frac{x_{k+2}}{p^{k+1}}$$

By induction,

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

such that $0 \leq a_n < p$ for all n.

We claim that this sequence is unique except when $x = q/p^n$ for some $q \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Case 1: $x \neq q/p^n$.

Suppose there are two such sequences $\{a_n\}$ and $\{b_n\}$ such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n} = \sum_{n=1}^{\infty} \frac{b_n}{p^n}$$

Let k be the smallest index such that $a_k \neq b_k$. WLOG, assume $a_k < b_k$. Then

$$\sum_{n=1}^{k-1} \frac{a_n}{p^n} + \frac{a_k}{p^k} + \sum_{n=k+1}^{\infty} \frac{a_n}{p^n} = \sum_{n=1}^{k-1} \frac{b_n}{p^n} + \frac{b_k}{p^k} + \sum_{n=k+1}^{\infty} \frac{b_n}{p^n}$$

and by assumption, the first $k-1$ terms are equal so

$$\frac{b_k - a_k}{p^k} = \sum_{n=k+1}^{\infty} \frac{b_n - a_n}{p^n}$$

but since $0 \leq a_k < b_k < p$, $1 \leq b_n - a_n < p$ for all n so

$$\frac{1}{p^k} < \sum_{n=k+1}^{\infty} \frac{1}{p^{k+1}} = \frac{1}{p^{k+1}(p-1)}$$

but $p \geq 1$ so this is a contradiction.

Case 2: $x = q/p^n$

Lemma: if $x = q/p^n$, then $\exists k$ such that $x_k = 0$ for $k > n$.

Assume to the contrary that $x_k \neq 0$ for all $k > n$.

Consider x_{n+1}:

$$\frac{a_{n+1}}{p^{n+1}} = \frac{\lfloor px_{n+1} \rfloor}{p^{n+1}} \neq 0$$

But $a_{n+1} < p$, so p cannot divide a_{n+1}. Hence,

$$x = \frac{a_1}{p} + \frac{a_2}{p^2} + \dots + \frac{a_n}{p^n} + \frac{a_{n+1}}{p^{n+1}} + \dots$$

cannot be expressed with denominator p^n so $x \neq q/p^n$. Contradiction.

Suppose terms $x_1, \dots, x_k$ have been defined for $k > n$. By assumption, $x_{k+1} \neq 0$ so by the same argument, $\frac{a_{n+1}}{p^{n+1}} \neq 0$ with $p \nmid a_{n+1}$.

Then $x = q/p^n$ cannot be expressed with denominator p^n so $x \neq q/p^n$. Contradiction.

By induction, $x_k = 0$ for $k > n$.

Thus,

$$x = \frac{q}{p^n} = \sum_{n=1}^{k} \frac{a_n}{p^n}$$

for finite k.

It remains to show that there is one more sequence that satisfies this condition.

Consider

$$b_n = \begin{cases} a_n & n \leq k-1 \\ a_n - 1 & n = k \\ p-1 & n > k \end{cases}$$

Then

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n} - \sum_{n=1}^{\infty} \frac{b_n}{p^n} = \left(\sum_{n=1}^{k-1} \frac{a_n}{p^n} + \frac{a_k}{p^k} + \sum_{n=k+1}^{\infty} \frac{a_n}{p^n} \right) - \left(\sum_{n=1}^{k-1} \frac{a_n}{p^n} + \frac{a_k - 1}{p^k} + \sum_{n=k+1}^{\infty} \frac{p-1}{p^k} \right)$$

But $a_n = 0$ for $n > k$ so

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_n}{p^n} - \sum_{n=1}^{\infty} \frac{b_n}{p^n} &= \frac{1}{p^k} - \sum_{k+1}^{\infty} \frac{p-1}{p^k} \\ &= \frac{1}{p^k} - \frac{1}{p^k} = 0 \end{aligned}$$

which implies

$$x = \sum_{n=1}^{\infty} \frac{b_n}{p^n}$$

and a_n and b_n are valid sequences if $x = q/p^n$.

Now suppose that c_n is another such sequence

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n} = \sum_{n=1}^{\infty} \frac{c_n}{p^n}$$

but $\exists k$ such that $a_k \neq c_k$.

WLOG, let $a_k > c_k$. Then

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n} - \sum_{n=1}^{\infty} \frac{c_n}{p^n} = \frac{a_k - c_k}{p^k} + \sum_{n=k+1}^{\infty} \frac{a_n - c_n}{p^n}$$

Certainly, $a_n - c_n \geq 1$. But then

$$\begin{aligned}\frac{a_k - c_k}{p^k} + \sum_{n=k+1}^{\infty} \frac{a_n - c_n}{p^n} &\geq \frac{1}{p^k} + \sum_{n=k+1}^{\infty} \frac{1}{p^n} \\ &= \frac{1}{p^k} + \frac{1}{p^k(p-1)} \\ &= \frac{p}{p^k(p-1)} > 0\end{aligned}$$

but this is a contradiction of the equality of the series.

Thus, a_n as constructed above is unique except when $x = q/p^n$ for some $q \in \mathbb{Z}$ and $n \in \mathbb{N}$, in which case there are two such sequences. ∎

3. Show that the set of infinite sequences from two numbers $\{0, 1\}$

$$E = \{(a_1, a_2, a_3, \dots) : a_i \in \{0, 1\}\}$$

is not countable. Furthermore, show that $(0, 1)$ is uncountable.

(Cantor's Diagonalization) Suppose that E is countable. Then we can define a sequence S_n of elements in E indexed by the natural numbers.

Let the notation $S_m[k]$ represent the k-th element of the m-th sequence in (S_n).

Define a new sequence (X) by

$$X_n = 1 - S_n[n]$$

That is, the elements of X are the complements of the diagonal elements of (S_n).

(X) is a sequence of 0's and 1's, so $(X) \in E$. However, for every n, $X \neq S_n$ since X differs from S_n in the n-th element. But this suggests there are sequences in E that are not in S_n, a contradiction. ■

Assume $(0, 1)$ is countable. Then $(0, 1) = \{x_1, x_2, x_3, \dots\}$.

Let $I_1 \subseteq (0, 1)$ be a closed interval which does not contain x_1. Let $I_2 \subseteq I_1$ be a closed interval which does not contain x_2. Continue this process to get a nested sequence of closed intervals $I_1 \supset I_2 \supset I_3 \supset \dots$ such that $I_{n+1} \subseteq I_n$ and $x_n \notin I_n$.

By construction,

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

but every finite intersection of I_n's is nonempty.

> *Proof:* Let $I_n = [a, b]$. Then $\bigcap_{k=1}^{n} I_k \subseteq [a, b]$. But exists $x_0 \in \mathbb{Q} \cap [a, b]$ by the density of the rationals in the reals. So $\exists x_0 \in \bigcap_{k=1}^{n} I_k$ for all n.

so by HW 1.1.e, the bounded intersection of closed sets with the Finite Intersection Property is nonempty. This is a contradiction, so $(0, 1)$ is uncountable. ■

4. Take $p = 3$ in (1). The Cantor ternary set C consists of those real numbers in $[0, 1]$ for which $a_n \neq 1$ for all n in (1). (In case there are two ternary expansions, we put x in the Cantor set if *one* of the expansions has no term $a_n = 1$. Prove

- C is closed, and C is obtained by first removing the middle third $(\frac{1}{3}, \frac{2}{3})$ from $[0, 1]$, then removing the middle $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ of the remaining intervals and so on.

  Let $I_1 = (\frac{1}{3}, \frac{2}{3})$ and

$$C_1 = [0, 1] \setminus I_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

  Let $I_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$ and

$$C_2 = [0, 1] \setminus (I_1 \cup I_2) = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$$

  Inductively, we can define a sequence of closed intervals I_n such that I_{n+1} is the union of the middle thirds of the intervals in I_n and

$$C_n = [0, 1] \setminus \bigcup_{i=1}^{n} I_i$$

  We claim that

$$C = \bigcap_{n=1}^{\infty} C_n$$

  for

$$C = \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n} \middle| a_n \in \{0, 2\} \right\}$$

  Part I: $C \subseteq \bigcap_{n=1}^{\infty} C_n$

  Let $x \in C$. Then

$$x = \frac{a_1}{3} + \sum_{n=2}^{\infty} \frac{a_n}{3^n}$$

  And

$$\sum_{n=2}^{\infty} \frac{a_n}{3^n} \leq \sum_{n=2}^{\infty} \frac{2}{3^n} = -\frac{2}{3} + \sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{1}{3}$$

  Therefore, if $a_1 = 0$, $x \in [0, \frac{1}{3}]$. If $a_1 = 2$, then $x \in [\frac{2}{3}, 1]$.

  Regardless, $x \in C_1$.

  Now suppose $x \in C_n$. We can write $C_n = \bigcup_{i=1}^{2^n} F_i$ where F_i is a closed interval, so $x \in F_i$ for some i. Let $F_i = [a, b]$.

  We want to show that $x \in C_{n+1}$, for which it suffices to show that

$$x \in [a, a + \frac{b-a}{3}] \cup [b - \frac{b-a}{3}, b]$$

  But each F_i is 1/3 of the length of the interval that contains it, so $b - a = \frac{1}{3^n}$. Thus, we want to show that

$$x \in [a, a + \frac{1}{3^{n+1}}] \cup [a + \frac{2}{3^{n+1}}, b]$$

By construction,

$$\begin{aligned} x &= \sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{a_{n+1}}{3^{n+1}} + \sum_{i=n+2}^{\infty} \frac{a_i}{3^i} \\ &= a + \frac{a_{n+1}}{3^{n+1}} + \sum_{i=n+2}^{\infty} \frac{a_i}{3^i} \\ &\leq a + \frac{a_{n+1}}{3^{n+1}} + \sum_{i=n+2}^{\infty} \frac{2}{3^i} \\ &= a + \frac{a_{n+1}}{3^{n+1}} + \frac{1}{3^{n+1}} \end{aligned}$$

If $a_{n+1} = 0$, then

$$x \leq a + \frac{1}{3^{n+1}} \implies x \in [a, a + \frac{1}{3^{n+1}}]$$

If $a_{n+1} = 2$, then

$$x \leq a + \frac{2}{3^{n+1}} + \frac{1}{3^{n+1}} = a + \frac{1}{3^n} = b \implies x \in [a + \frac{2}{3^{n+1}}, b]$$

Therefore, $x \in C_{n+1}$. By induction, $x \in \bigcap_{n=1}^{\infty} C_n$ so

$$C \subseteq \bigcap_{n=1}^{\infty} C_n$$

Part II: $\bigcap_{n=1}^{\infty} C_n \subseteq C$

Let $x \in \bigcap_{n=1}^{\infty} C_n$.

From (2), we know that x has a unique ternary expansion. It remains to show that x has no 1's in its expansion.

Suppose $a_1 = 1$. Then

$$\begin{aligned} x &= \frac{1}{3} + \sum_{n=2}^{\infty} \frac{a_n}{3^n} \\ &\leq \frac{1}{3} + \sum_{n=2}^{\infty} \frac{2}{3^n} \\ &= \frac{1}{3} + \frac{1}{3} \\ &\implies x \in (\frac{1}{3}, \frac{2}{3}) \end{aligned}$$

But then $x \notin C_1$, a contradiction. So $a_1 \neq 1$.

Now suppose $a_k \neq 1$ for all $k \leq n$ for some n. We want to show that $a_{n+1} \neq 1$.

Let the closed interval in C_n which contains x be $[a, b]$. Since $x \in C_{n+1}$ too, we further know that

$$x \in [a, a + \frac{1}{3^{n+1}}] \quad \text{or} \quad x \in [a + \frac{2}{3^{n+1}}, b]$$

Suppose $a_{n+1} = 1$. Then

$$\begin{aligned}
x &= \sum_{n=1}^{n} \frac{a_n}{3^n} + \frac{1}{3^{n+1}} + \sum_{n=n+2}^{\infty} \frac{a_n}{3^n} \\
&= a + \frac{1}{3^{n+1}} + \sum_{n=n+2}^{\infty} \frac{a_n}{3^n} \\
&\leq a + \frac{1}{3^{n+1}} + \sum_{n=n+2}^{\infty} \frac{2}{3^n} \\
&= a + \frac{1}{3^{n+1}} + \frac{1}{3^{n+1}} \\
&= a + \frac{2}{3^{n+1}} \\
&\implies x \in [a + \frac{1}{3^{n+1}}, a + \frac{2}{3^{n+1}}]
\end{aligned}$$

So

$$x \in \left([a, a + \frac{1}{3^{n+1}}] \cup [a + \frac{2}{3^{n+1}}, b]\right) \bigcap \left([a + \frac{1}{3^{n+1}}, a + \frac{2}{3^{n+1}}]\right) \implies x \in \left\{a + \frac{1}{3^{n+1}}, a + \frac{2}{3^{n+1}}\right\}$$

Case 1: $x = a + \frac{1}{3^{n+1}}$. Then

$$\{a_n\} = \{a_1, \dots, a_n, 1, 0, 0, \dots\} = \{a_1, \dots, a_n, 0, 2, 2, \dots\}$$

because (letting the partial sums of the first series be S_n and the second be T_n),

$$S_n - T_n = \frac{1}{3^{n+1}} - \sum_{k=n+2}^{\infty} \frac{2}{3^k} = \frac{1}{3^{n+1}} - \frac{1}{3^{n+1}} = 0 \implies S_n = T_n$$

Case 2: $x = a + \frac{2}{3^{n+1}}$. Then

$$\{a_n\} = \{a_1, \dots, a_n, 2, 0, 0, \dots\}$$

which contradicts that $a_{n+1} = 1$.

Thus, x can always be written in a form such that $a_{n+1} \neq 1$. Hence,

$$\bigcap_{n=1}^{\infty} C_n \subseteq C$$

In our construction, we defined C as the complement in $[0, 1]$ of the union of countably many open intervals. Hence, C is closed.

- The set of accumulation points of C is C itself.

  Let $\varepsilon > 0$ and choose $x \in C$. Consider the open interval $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$.

  Let $\{F_n\}$ be the set of 2^n closed intervals that define the partitioning construction of C. Let $\{I_n\} \subseteq \{F_n\}$ be the closed intervals in C which contain x.

Define a sequence $x_n \in C$ by picking $x_n \in I_n$ for all n.

For n large enough, the length of $I_n = \frac{1}{3^n} < \varepsilon$ so $I_n \subseteq B_\varepsilon(x)$. Hence, x is an accumulation point of C, i.e. every point in C is an accumulation point of C.

Now suppose $x \notin C$ but x is an accumulation point of C. Then there exists a sequence $(x_n) \subseteq C$ such that $x_n \to x$. Therefore, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $x_n \in B_\varepsilon(x)$. But C is closed, so C^c is open which implies that $B_\varepsilon(x) \subseteq C^c$. But then $\exists x_0 \in \{x_n\} \cap C^c$ which is a contradiction.

Hence, the set of accumulation points of C is C itself. ■

5. Let X be the set of all continuous functions f over $[0, 1]$. Define

$$\rho_1(f, g) = \int_0^1 |f(x) - g(x)| \ dx$$

Show

- (X, ρ_1) is a metric space.

  It suffices to show that

  1. $\rho_1(f, g) = 0$ iff $f = g$
  2. $\rho_1(f, g) = \rho_1(g, f)$
  3. $\rho_1(f, g) \leq \rho_1(f, h) + \rho_1(h, g)$

  For (1), suppose $f = g$. Then

  $$\rho_1(f, g) = \rho_(f, f) = \int_0^1 |f(x) - f(x)| \ dx = 0$$

  Conversely, if $\rho_1(f, g) = 0$, then

  $$\int_0^1 |f(x) - g(x)| \ dx = 0$$

  but $|f(x) - g(x)| \geq 0$ so $|f(x) - g(x)| = 0$ for all $x \in [0, 1]$.

  Proof: a Riemann integral is by definition a limit of Riemann sums. So for non-negative integrand, the integral is monotonically increasing. Therefore, if the integral is 0, then the integrand must be 0 almost everywhere.

  For (2),

  $$\rho_1(f, g) = \int_0^1 |f(x) - g(x)| \ dx = \int_0^1 |-[g(x) - f(x)]| = \int_0^1 |g(x) - f(x)| \ dx = \rho_1(g, f)$$

  For (3),

  $$\begin{aligned} \rho_1(f, g) &= \int_0^1 |f(x) - g(x)| \ dx \\ &= \int_0^1 |f(x) - h(x) + h(x) - g(x)| \ dx \\ &\leq \int_0^1 |f(x) - h(x)| + |h(x) - g(x)| \ dx \\ &= \int_0^1 |f(x) - h(x)| \ dx + \int_0^1 |h(x) - g(x)| \ dx \\ &= \rho_1(f, h) + \rho_1(h, g) \end{aligned}$$

- (X, ρ_1) is not complete.

  It suffices to find a Cauchy sequence in X that does not converge in X. Consider the sequence of functions $f_n(x) = x^n$

  We claim this is is continuous and further, that it is Cauchy.

  Let $\varepsilon > 0$. Fix $n \in \mathbb{N}$. Then let $\delta = \varepsilon/n$. Then if $|x - y| < \delta$,

$$\begin{aligned}
|f_n(x) - f_n(y)| &= |x^n - y^n| \\
&= |x - y| \left| x^{n-1} + x^{n-2}y + \cdots xy^{n-2} + y^{n-1} \right| \\
&\leq |x - y| \; |1 + 1 + \cdots + 1 + 1| \\
&= n\,|x - y| < n\delta = \varepsilon
\end{aligned}$$

  so f_n is continuous.

  Now we show it is Cauchy. WLOG suppose $m \geq n$ so $x^m \leq x^n$. Then

$$\begin{aligned}
\rho_1(f_n, f_m) &= \int_0^1 |x^n - x^m| \; dx \\
&= \int_0^1 x^n - x^m \; dx \\
&= \frac{1}{n+1} - \frac{1}{m+1}
\end{aligned}$$

  Let $\varepsilon > 0$. Then set $N = 1/2\varepsilon$ so that for $n, m \geq N$,

$$\rho_1(f_n, f_m) = \frac{1}{n+1} - \frac{1}{m+1} < \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} = \varepsilon$$

  so (f_n) is Cauchy and in X.

  But

$$\lim f_n(x) = \lim x^n = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$$

  which is not continuous on $[0, 1]$. Therefore, $\lim f_n \notin X$. So (X, ρ_1) is not complete. ■

6. If E is a subset of the metric space (X, ρ), then prove the following are equivalent:

(a) E is complete and totally bounded

(b) Every sequence in E has a subsequence that converges to a point in E

(c) If $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ is an open covering of E, there is a finite set $\mathcal{F} \subseteq \mathcal{A}$ such that $\{V_\alpha\}_{\alpha \in \mathcal{F}}$ covers E

We will first show $a \iff b$ and then $b \iff c$.

$(a \to b)$ Let (x_n) be a sequence in E. As E is totally bounded, it can be covered by finitely many balls – say of radius $1/2$. Let

$$N_1 = \{n \in \mathbb{N} : x_n \in B_1\}$$

where B_1 is one of the balls that contains infinitely many points in x_n (guaranteed to exist because E is covered by finitely many balls, so at least one must contain infinitely many points).

But now $E \cap B_1 \subseteq E$ so it can also be covered by finitely many balls of radius $1/4$. Again, at least one of these balls – call it B_2 – contains infinitely many x_n. Let $N_2 = \{n \in \mathbb{N} : x_n \in B_2\}$.

Inductively define a sequence of balls B_k of radius $1/2^k$ and a sequence of subsets N_k of $\mathbb{N}$ such that $x_n \in B_k$ for $n \in N_k$.

Using the Axiom of Choice, pick $n_1 \in N_1, n_2 \in N_2, \ldots$ such that $n_1 < n_2 < \cdots$.

By construction, $\{x_{n_k}\}$ is a Cauchy sequence because $\rho(x_{n_j}, x_{n_k}) < \frac{1}{2^{1-k}}$ for $j > k$. Since E is complete, it has a limit in E

$(b \to a)$ First suppose that E is not complete, i.e. $\exists (x_n) \in E$ with no limit in E. But then x_n can have no subsequence which converges in E (or the whole sequence would converge to the same limit).

Now suppose E is not totally bounded and choose $\varepsilon > 0$ so E cannot be covered by finitely many balls of radius ε.

We will inductively construct a sequence that can have no convergent subsequence. First pick any $x_1 \in E$. Then (assuming $x_1, \ldots, x_n$), pick

$$x_{n+1} = E \setminus \bigcup_{i=1}^{n} B_\varepsilon(x_i)$$

so $\rho(x_n, x_m) > \varepsilon$ for any n, m. Thus, no convergent subsequence is possible.

$(b \to c)$ Suppose every sequence in E has a subsequence that converges to a point in E. By our above work, we know E is complete and totally bounded. Thus we just need to show that for any open cover $\{V_\alpha\}_{\alpha \in A}$, there exists an $\varepsilon > 0$ such that for all $x_n \in E$, $B_\varepsilon(x_n) \subseteq V_\alpha$ for some $\alpha \in A$.

Suppose to the contrary that for all $n \in \mathbb{N}$, there is a ball B_n of radius $1/2^n$ such that $B_n \cap E \neq \emptyset$ but $B_n \not\subseteq V_\alpha$ for any α.

Since it is not empty by assumption, pick $x_n \in B_n \cap E$. These x_n then form a sequence which (by b), has a subsequence that converges to some $x \in E$.

As $\{V_\alpha\}_{\alpha \in A}$ is an open cover of E, $x \in V_\alpha$ for some α and (since it is open), $B_\varepsilon(x) \subseteq V_\alpha$ for all $\varepsilon > 0$.

But as x is the limit of a sequence, choose n large enough so that $\rho(x, x_n) < \varepsilon$ and $\frac{1}{2^n} < \varepsilon$. Thus,

$$B_n \subseteq B_\varepsilon(x) \subseteq V_\alpha$$

which is a contradiction.

$(c \to b)$ Suppose every open cover of E has a finite subcover. Let (x_n) be a sequence in E with no convergent subsequence.

Let $\varepsilon > 0$. For each $x \in E$, there must exist an open ball $B_\varepsilon(x)$ which contains only finitely many elements in x_n (or (x_n) would have a convergent subsequence).

But then $\{B_\varepsilon(x)\}_{x \in E}$ is an open cover of E which can have no finite subcover. This is a contradiction, so (x_n) must have a convergent subsequence. ■

7. (Ascoli-Arzela) Assume that (X, ρ) is bounded and separable. Consider all continuous functions $f : (X, \rho) \to (Y, \sigma)$. A family $\mathcal{F}$ of f is *equicontinuous* if for all $x \in X$, for any $\varepsilon > 0$, there is an open set $O_x > 0$ such that

$$\sigma(f(x), f(y)) < \varepsilon$$

for all $y \in O_x$ and *uniformly* for all $f \in \mathcal{F}$.

Take any $f_n \in \mathcal{F}$ such that for each $x \in X$, the closure of $\{f_n(x) : 0 \le n < \infty\}$ is compact. Prove that there exists a subsequence f_{n_k} that converges pointwise to a continuous function f.

Let f_n be a sequence in $\mathcal{F}$.

Since X is separable, it contains D, a countable dense subset. Enumerate D by $\{x_n\}_{n=1}^{\infty}$.

Since $f_n(x_1) \in \overline{\{f_n(x_1) : n \in \mathbb{N}\}}$, which is compact, $f_n(x_1)$ is bounded. Then by Bolzano-Weierstrass, there exists a convergent subsequence $f_{n,1}(x_1)$.

But similarly, the sequence $f_{n,1}(x_2)$ is bounded for all n so it contains a convergent subsequence of its own, $f_{n,2}(x_2)$, which converges to some f at x_1 and x_2.

Suppose we have constructed $f_{n,1}, f_{n,2}, \ldots, f_{n,k}$ such that $f_{n,k}$ converges to f at $x_1, \ldots, x_k$ for all n.

Then $f_{n,k}(x_{k+1}) \in \overline{\{f_n(x_{k+1}) : n \in \mathbb{N}\}}$ so it is bounded. Therefore, it contains a convergent subsequence $f_{n,k+1}(x_{k+1})$ which converges to some f at $\{x_1, x_2, \ldots, x_{k+1}\}$.

By induction, we can take the subsequence $f_{n,n}$ of f_n which converges pointwise to f for all $x \in D$.

Now it remains to show that f is continuous. Let $\varepsilon > 0$.

Pick $x \in D$ and choose O_x such that $B_\varepsilon(x) \subseteq O_x$.

Since D is dense, $\exists y \in B_\varepsilon(x) \cap D$ for all x. Then $y \in O_x$ and by equicontinuity,

$$\sigma(f_{n,n}(x), f_{n,n}(y)) < \frac{\varepsilon}{3}$$

By pointwise convergence, $\exists N \in \mathbb{N}$ such that $\forall n \ge N$,

$$\sigma(f_{n,n}(x), f(x)) < \frac{\varepsilon}{3}$$
$$\sigma(f_{n,n}(y), f(y)) < \frac{\varepsilon}{3}$$

for all $x, y \in D$.

Together, for $n \ge N$ and $x, y \in D$ as picked above,

$$\sigma(f(x), f(y)) \le \sigma(f(x), f_{n,n}(x)) + \sigma(f_{n,n}(x), f_{n,n}(y)) + \sigma(f_{n,n}(y), f(y)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

hence f is continuous. $\blacksquare$.