APMA 2110 - Homework 9

Milan Capoor

Nov 18, 2024

1. Let $f \in \mathcal{L}^1(\mathbb{R})$ be Lebesgue integrable. Prove that for any $\varepsilon > 0$, $\exists \delta > 0$ such that if $|h| < \delta$, then

$$\int_{\mathbb{R}} |f(x+h) - f(x)|\ dx < \varepsilon$$

By integrability, pick $R_1 > 0$ such that

$$\int_{-\infty}^{-R} |f(x+h)|\ dx < \frac{\varepsilon}{6}, \quad \int_{R}^{\infty} |f(x+h)| < \frac{\varepsilon}{6}$$

Similarly, pick $R_2 > 0$ such that

$$\int_{-\infty}^{-R} |f(x)|\ dx < \frac{\varepsilon}{6}, \quad \int_{R}^{\infty} |f(x)| < \frac{\varepsilon}{6}$$

Let $R = \max\{R_1, R_2\}$. Then,

$$\begin{aligned}\int_{\mathbb{R}} |f(x+h) - f(x)|\ dx &= \int_{-\infty}^{-R} |f(x+h) - f(x)|\ dx + \int_{-R}^{R} |f(x+h) - f(x)|\ dx + \int_{R}^{\infty} |f(x+h) - f(x)|\ dx \\ &\leq \int_{-\infty}^{-R} |f(x+h)| + |f(x)|\ dx + \int_{-R}^{R} |f(x+h) - f(x)|\ dx + \int_{R}^{\infty} |f(x+h)| + |f(x)|\ dx \\ &< \frac{2\varepsilon}{3} + \int_{-R}^{R} |f(x+h) - f(x)|\ dx\end{aligned}$$

It suffices to show that

$$\int_{-R}^{R} |f(x+h) - f(x)|\ dx < \frac{\varepsilon}{3}$$

By reduction to smooth functions, $\exists \phi$ such that

$$\int_{-R}^{R+h} |f(x) - \phi(x)|\ dx < \frac{\varepsilon}{9}$$

so

$$\int_{-R}^{R} |f(x+h) - f(x)|\ dx \leq \int_{-R}^{R} |f(x+h) - \phi(x+h)| + \int_{-R}^{R} |\phi(x+h) - \phi(x)|\ dx + \int_{-R}^{R} |\phi(x) - f(x)|\ dx$$

so $\int_{-R}^{R} |f(x+h) - \phi(x+h)|\ dx < \frac{\varepsilon}{9}$ and $\int_{-R}^{R} |\phi(x) - f(x)|\ dx < \frac{\varepsilon}{9}$.

All that is left is to show that

$$\int_{-R}^{R} |\phi(x+h) - \phi(x)| \ dx < \frac{\varepsilon}{9}$$

But ϕ is smooth, hence continuous: $\exists \delta > 0$ s.t. $|x - y| < \delta \implies |\phi(x) - \phi(y)| < \frac{\varepsilon}{18R}$.

In particular,

$$|x + h - x| = |h| < \delta \implies |\phi(x+h) - \phi(x)| < \frac{\varepsilon}{18R}$$

so by monotonicity,

$$\int_{-R}^{R} |\phi(x+h) - \phi(x)| \ dx \leq \int_{-R}^{R} \frac{\varepsilon}{18R} \, dx = \frac{R\varepsilon}{18R} + \frac{R\varepsilon}{18R} = \frac{\varepsilon}{9}$$

All together, for $|h| < \delta$,

$$\int_{-R}^{R} |f(x+h) - f(x)| \ dx \leq \frac{\varepsilon}{9} + \frac{\varepsilon}{9} + \frac{\varepsilon}{9} = \frac{\varepsilon}{3}$$

and

$$\int_{\mathbb{R}} |f(x+h) - f(x)| \ dx < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \blacksquare$$

2. Let $f \in \mathcal{L}^1(\mathbb{R})$ be Lebesgue integrable. Show that

$$F(t) = \int_{-\infty}^{x} f(t)\, dt$$

is continuous.

Let $\varepsilon > 0$ and suppose $|x - y| < \delta$ for $\delta > 0$. We want to show that

$$|F(x) - F(y)| = \left| \int_{-\infty}^{x} f(t)\, dt - \int_{-\infty}^{y} f(t)\, dt \right| = \left| \int_{y}^{x} f(t)\, dt \right| < \varepsilon$$

Since $f \in \mathcal{L}^1$, from a lemma in class,

$$\left| \int_{y}^{x} f(t)\, dt \right| \leq \int_{y}^{x} |f(t)|\, dt$$

By reduction to smooth functions, $\exists \phi$ such that

$$\int_{y}^{x} |f(t) - \phi(t)|\, dt < \frac{\varepsilon}{2} \implies \int_{y}^{x} |f(t)|\, dt \leq \int_{y}^{x} |f(t) - \phi(t)|\, dt + \int_{y}^{x} |\phi(t)|\, dt$$

Since ϕ is smooth, it is finite on the closed interval $[y, x]$ (otherwise, not continuous). Let $|\phi(t)| \leq M$ for $t \in [y, x]$. Then,

$$\int_{y}^{x} |f(t)|\, dt \leq M\, |x - y|$$

Let $\delta = \frac{\varepsilon}{2M}$. Again by continuity (smoothness),

$$|x - y| < \delta = \frac{\varepsilon}{2M} \implies \int_{y}^{x} |f(t)|\, dt < \frac{\varepsilon}{2}$$

So for $|x - y| < \frac{\varepsilon}{2M}$,

$$|F(x) - F(y)| < \varepsilon$$

and F is continuous. $\blacksquare$

3. Let $X = Y = [0,1]$, $\mathcal{A} = \mathcal{B}[0,1]$ (Borel Sets), μ be the Lebesgue measure, and ν be the counting measure. If $D = \{(x,x) : x \in [0,1]\}$ is the diagonal in $X \times Y$, show

- $\int\int \mathbb{1}_D \, d\mu \, d\nu$
- $\int\int \mathbb{1}_D \, d\nu \, d\mu$
- $\int \mathbb{1}_D \, d(\mu \times \nu)$

are all unequal.

First, notice that $D \subseteq \mathcal{B}([0,1]) \otimes \mathcal{B}([0,1])$ so by a Lemma from class,

$$D_x = \{y \in Y : (x,y) \in D\} \in \mathcal{B}([0,1])$$
$$D^y = \{x \in X : (x,y) \in D\} \in \mathcal{B}([0,1])$$

so

$$\begin{aligned}\int\int \mathbb{1}_{D(x,y)} \, d\mu(x) \, d\nu(y) &= \int \left(\int \mathbb{1}_{D^y(x)} \, d\mu \right) \, d\nu \\ &= \int \left(\int_{\{x:x=y\}} 1 \, d\mu \right) \, d\nu \\ &= \int \mu(\{y\}) \, d\nu = 0\end{aligned}$$

Similarly,

$$\begin{aligned}\int \left(\int \mathbb{1}_D \, d\nu(y) \right) \, d\mu(x) &= \int \left(\int_D \mathbb{1}_{D_x} \, d\nu(y) \right) \, d\mu(x) \\ &= \int \nu(\{x\}) \, d\mu \\ &= \int 1 \, d\mu \\ &= \mu([0,1]) = 1\end{aligned}$$

Define

$$(\mu \times \nu)(E) = \inf \left\{ \sum_i \mu(A_i)\nu(B_i) \;\middle|\; E \subseteq \bigcup_i A_i \times B_i \right\}$$

for $A_i \times B_i$ disjoint rectangles.

If $D \subseteq \bigcup_i A_i \times B_i$, then $\bigcup_i (A_i \cap B_i)$ covers $[0,1]$ and we must have that $\mu(A_n \cap B_n) > 0$ for some n. But this implies that $A_n \cap B_n$ is uncountable, so $\nu(A_n \cap B_n) = \infty$. Hence,

$$\sum_i \mu(A_n)\nu(B_n) = \infty$$

for all covers of D by rectangles so $(\mu \times \nu)(D) = \infty$.

Hence,

$$\int \mathbb{1}_D \, d(\mu \times \nu) = \infty$$
$$\int\int \mathbb{1}_D \, d\mu \, d\nu = 0$$
$$\int\int \mathbb{1}_D \, d\nu \, d\mu = 1 \quad \blacksquare$$

4. (Fubini) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and let $(X \times Y, \mathcal{L}, \mu \times \nu)$ be the product measure. If $f \in \mathcal{L}^1(\mu \times \nu)$, show that

1. f_x is $\mathcal{B}$-measurable for a.e. x
2. f^y is $\mathcal{A}$-measurable for a.e. y
3. f_x is integrable for a.e. x
4. f^y is integrable for a.e. y
5. If $x \to \int f_x \, d\nu$ and $y \to \int f^y \, d\mu$ are measurable and integrable, then

$$\int f \, d(\mu \times \nu) = \int \int f(x, y) \, d\mu(x) \, d\nu(y) = \int \int f(x, y) \, d\nu(y) \, d\mu(x)$$

Since $f \in \mathcal{L}^1(\mu \times \nu)$, it is measurable on $\mathcal{M} \otimes \mathbb{N}$. By a lemma in class, we have that f_x and f_y are measurable for a.e. x and y respectively.

In class, we showed the result for the special case $f = \mathbb{1}_E$ for $E \in \mathcal{M} \times \mathbb{N}$. By linearity, the result holds for simple functions. For $f \in \mathcal{L}^1(\mu \times \nu)$, we can sat $f \geq 0$ WLOG and then approximate f by simple functions $\phi_n \nearrow f$.

Let

$$g(x) = \int f_x \, d\nu$$
$$h(y) = \int f^y \, d\mu$$

and g_n, h_n be the corresponding sections of ϕ_n.

By MCT, $g_n \nearrow g$ and $h_n \nearrow h$ so g, h are measurable and

$$\int g \, d\mu = \lim \int g_n \, d\mu = \lim \int \phi_n \, d(\mu \times \nu) = \int f \, d(\mu \times \nu)$$
$$\int h \, d\nu = \lim \int h_n \, d\nu = \lim \int \phi_n \, d(\mu \times \nu) = \int f \, d(\mu \times \nu)$$

In particular, since $f \in \mathcal{L}^1(\mu \times \nu)$, f_x and f^y are integrable for a.e. x and y respectively and the result follows. ■

5. Let ν be a measure on the Borel sets of the positive real line $[0, \infty)$ such that

$$\phi(t) = \nu([0, t))$$

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \geq 0$ measurable. Show that

$$\int_\Omega \phi(f(x))\, d\mu = \int_0^\infty \mu(\{x : f(x) > t\})\, dt$$

$$\begin{aligned}
\int_X \phi(f(x))\, d\mu &= \int_X \nu[0, f(x))\, d\mu \\
&= \int_X \int_0^\infty \mathbb{1}_{[0,f(x))}\, d\nu\, d\mu \\
&= \int_0^\infty \int_X \mathbb{1}_{[0,f(x))}\, d\mu\, d\nu \qquad \text{(Tonelli)} \\
&= \int_0^\infty \mu([0, f(x)))\, d\nu \\
&= \int_0^\infty \mu(\{t : f(x) > t\})\, d\nu \\
&= \int_0^\infty \mu(\{x : f(x) > t\})\, dt
\end{aligned}$$