APMA 2110: HW 5

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1. Show that the characteristic function $\chi_E = \mathbb{1}_E$ is a measurable function iff E is a measurable set.

$\implies$. Assume $\mathbb{1}_E$ is a measurable function.

By definition, $\forall \alpha \in \mathbb{R}$, $\{x \in X : \mathbb{1}_E(x) > \alpha\}$ is measurable.

WLOG, let $\alpha = 0$. Then

$$\{x \in X : \mathbb{1}_E(x) > \alpha\} = \{x \in X : \mathbb{1}_E(x) > 0\}$$

But since $\mathbb{1}_E : X \to \{0, 1\}$,

$$\{x \in X : \mathbb{1}_E(x) > 0\} = \{x \in X : \mathbb{1}_E(x) = 1\} = \{x \in X : x \in E\} = E$$

Hence, E is measurable.

$\impliedby$. Now assume E is measurable.

We want to show that $A = \{x \in X : \mathbb{1}_E(x) > \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$.

CASE 1. If $\alpha > 1$, then $A = \emptyset \in \mathcal{M}$.

CASE 2. If $0 \leq \alpha < 1$, then

$$\begin{aligned} A &= \{x \in X : \mathbb{1}_E(x) > \alpha\} \\ &= \{x \in X : \mathbb{1}_E(x) = 1\} \\ &= \{x \in X : x \in E\} \\ &= E \end{aligned}$$

so A is measurable.

CASE 3. If $\alpha < 0$, then $A = X \in \mathcal{M}$. $\blacksquare$

2. Let $\{f_n\}$ be a sequence of measurable functions on X then $\{x : \exists \lim f_n(x)\}$ is a measurable set.

Lemma: $\exists \lim f_n(x) \iff \limsup f_n(x) = \liminf f_n(x) = \lim f_n(x)$.

Proof: ($\implies$) Let $\varepsilon > 0$. If $\lim f_n(x) = f(x)$, then $\exists N \in \mathbb{N}$ such that $\rho(f_n(x), f(x)) < \varepsilon$ for all $n \geq N$.

Then for $n \geq N$, $\{f_k(x) : k \geq n\} \in B(f_n(x), \varepsilon)$ so

$$\limsup f_n = \lim_{n\to\infty} (\sup\{f_k(x) : k \geq n\}) \in B(f_n(x), \varepsilon)$$

and

$$\liminf f_n = \lim_{n\to\infty} (\inf\{f_k(x) : k \geq n\}) \in B(f_n(x), \varepsilon)$$

Since ε arbitrary, $\limsup f_n(x) = \liminf f_n(x) = f(x)$.

($\impliedby$) Let $\varepsilon > 0$ and denote $f(x) = \lim f_n(x)$. Since $\limsup f_n(x) = \liminf f_n(x) = f(x)$, $\exists N \in \mathbb{N}$ such that for $n \geq N$,

$$\rho(\limsup f_n(x), f(x)) = \rho(\sup_{k\geq n} f_k(x), f(x)) < \varepsilon \implies \sup_{k\geq n} f_k(x) \in B(f(x), \varepsilon)$$
$$\rho(\liminf f_n(x), f(x)) = \rho(\inf_{k\geq n} f_k(x), f(x)) < \varepsilon \implies \inf_{k\geq n} f_k(x) \in B(f(x), \varepsilon)$$

But

$$\inf_{k\geq n} f_k(x) \leq f_n(x) \leq \sup_{k\geq n} f_k(x)$$

by the definitions of inf and sup so

$$\rho(f_n(x), f(x)) = \max\left(\rho(\inf_{k\geq n} f_k(x), f(x)), \rho(\sup_{k\geq n} f_k(x), f(x))\right) = \max(\varepsilon, \varepsilon) = \varepsilon$$

so for n sufficiently large, $\rho(f_n(x), f(x)) < \varepsilon$. Hence, $\lim f_n(x) = f(x)$.

Call $f(x) = \limsup f_n(x)$ and $g(x) = \liminf f_n(x)$. By propositions from class, f, g, and $f - g$ are measurable functions because $\{f_n\}$ are measurable.

Therefore, by the Lemma,

$$\begin{aligned}\{x : \exists \lim f_n(x)\} &= \{x : f(x) = g(x)\} \\ &= \{x : f(x) - g(x) = 0\}\end{aligned}$$

Since $f - g$ is measurable, $\{x : f(x) - g(x) > \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$.

Let $\varepsilon > 0$ so $\{x : f(x) - g(x) > \varepsilon\} \in \mathcal{M}$ and $\{x : f(x) - g(x) > -\varepsilon\} \in \mathcal{M}$.

Then since $\mathcal{M}$ is closed under complements and countable intersections,

$$\{x : f(x) - g(x) > \varepsilon\}^c = \{x : f(x) - g(x) \leq \varepsilon\} \in \mathcal{M}$$

and

$$\{x : f(x) - g(x) = 0\} = \{x : f(x) - g(x) \leq \varepsilon\} \cap \{x : f(x) - g(x) > -\varepsilon\} \in M \quad \blacksquare$$

3. Let E be a Lebesgue measurable set in $\mathbb{R}$ and $\mu(E) > 0$. Show that for any $\alpha < 1$, there exists an open interval I_α such that $\mu(E \cap I_\alpha) > \alpha\mu(I_\alpha)$.

Suppose not. Then for all open intervals I, $\mu(E \cap I) \leq \alpha\mu(I)$.

Let $\varepsilon > 0$. By approximation from above, $\exists O$ open such that $E \subseteq O$ and

$$\mu(O) - \varepsilon \leq \mu(E) \leq \mu(O)$$

But since $O \subseteq \mathbb{R}$, by a proposition from class, we can write O as a countable union of disjoint open intervals, $O = \bigcup_n I_n$.

Then,

$$\begin{aligned}
\mu(E) &= \mu(\bigcup_n E \cap I_n) \qquad (E \subseteq O) \\
&= \sum_n \mu(E \cap I_n) \qquad (I_n \text{ disjoint}) \\
&\leq \sum_n \alpha\mu(I_n) \qquad (\text{by assumption}) \\
&= \alpha \sum_n \mu(I_n) \\
&= \alpha\mu(O) \qquad (\text{disjoint union})
\end{aligned}$$

So

$$\mu(O) - \varepsilon \leq \mu(E) = \alpha\mu(O) \leq \mu(O)$$

Taking $\varepsilon \to 0$, we have $\mu(O) = \alpha\mu(O)$ but $\alpha \neq 1$ and $\mu(E) > 0 \implies \mu(O) > 0$ by monotonicity, so we have a contradiction.