APMA 2110 - Homework 8

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1. Let $f_n \geq 0$ for all n, $\lim_{n\to\infty} f_n = f$ a.e., and $\lim_{n\to\infty} \int_X f_n = \int_X f$. Prove that for all $E \subseteq X$,

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

STEP 1. Since $f_n \to f$ a.e. on X, $f_n \to f$ a.e. on E for all $E \subseteq X$.

> *Proof:* Let A be the set of measure zero of points where $f_n \to f$ does not hold. Let B be the set of points in E where $f_n \to f$ does not hold. Since $E \subseteq X$, $B \subseteq A$. By monotonicity of measure, $\mu(B) \leq \mu(A) = 0$. Hence $f_n \to f$ a.e. on E.

STEP 2. By Fatou's Lemma,

$$\int_E \liminf f_n = \int_E f \leq \liminf \int_E f_n$$

STEP 3. We claim $\int_X f < \infty$

> *Proof:* Suppose not. We have, by assumption, $\lim \int_X f_n = \int_X f$.
>
> But then, for $n > N$ sufficiently large such that $\left|\int_X f_n - \int_X f\right| < \varepsilon$, we have $\infty - \infty < \varepsilon$ which is not defined.

STEP 4. WTS $\lim \int_E f_n \leq \int_E f$.

$$\int_{X\setminus E} f \leq \liminf \int_{X\setminus E} f_n = \liminf \int_X f_n - \limsup \int_E f_n = \int_X f - \limsup \int_E f_n$$

where the first inequality follows from Fatou again.

Then by finiteness of the integral,

$$\limsup \int_E f_n \leq \int_X f - \int_{X\setminus E} f \implies \limsup \int_E f_n \leq \int_E f$$

All together,

$$\limsup \int_E f_n \leq \int_E f \leq \liminf \int_E f_n$$

so

$$\lim \int_E f = \int_E f \quad \blacksquare$$

2. (Riemann-Lebesgue Theorem) Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{n\to\infty} \int f(x)\cos(nx)\ dx = 0$$

First notice that since $f \in \mathcal{L}^1$, we may choose $R > 0$ such that

$$\int_R^\infty |f(x)|\ dx < \varepsilon, \qquad \int_{-\infty}^{-R} |f(x)|\ dx < \varepsilon$$

for all ε.

Now by the reduction to smooth functions, we can take a C^∞ function with compact support ϕ such that

$$\int_{-R}^R |f - \phi| < \varepsilon$$

Further, since $|\cos nx| \leq 1$,

$$\left|\int_{-R}^R (f(x) - \phi(x))\cos(nx)\ dx\right| \leq \int_{-R}^R |f(x) - \phi(x)|\ dx < \varepsilon$$

(where we have the switching of the absolute value by a lemma from class).

But by integration by parts,

$$\int_{-R}^R \phi(x)\cos(nx)\ dx = \frac{\phi(R)\sin(nR)}{n} + \frac{\phi(-R)\sin(nR)}{n} - \frac{1}{n}\int_{-R}^R \phi'(x)\sin(nx)\ dx$$

so for n sufficiently large, $\left|\int_{-R}^R \phi(x)\cos(nx)\ dx\right| < \varepsilon/2$.

Hence,

$$\begin{aligned}\left|\int_{-R}^R (f(x) - \phi(x))\cos(nx)\ dx\right| \leq \left|\int_{-R}^R f(x)\cos(nx)\ dx\right| + \left|\int_{-R}^R \phi(x)\cos(nx)\ dx\right| < \varepsilon \\ \implies \left|\int_{-R}^R f(x)\cos(nx)\ dx\right| < \varepsilon/2\end{aligned}$$

All together,

$$\begin{aligned}\lim_{n\to\infty}\int f(x)\cos(nx)\ dx &= \lim_{n\to\infty}\left[\int_{-\infty}^{-R} f(x)\cos(nx)\ dx + \int_{-R}^R f(x)\cos(nx)\ dx + \int_R^\infty f(x)\cos(nx)\ dx\right] \\ &< 2\varepsilon + \lim_{n\to\infty}\int_{-R}^R f(x)\cos(nx)\ dx \\ &< 2\varepsilon + \frac{\varepsilon}{2} = \frac{5\varepsilon}{2} \to 0 \quad \blacksquare\end{aligned}$$

3. Let $f \in L^1(\mathbb{R}^d)$ and $f \neq 0$. Prove that there exists C (dependent on f) and $R > 0$ such that

$$\sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)| \, dy \geq C \, |x|^{-d} \qquad |x| > R$$

Pick $R > 0$ and x such that $|x| > R$.

Since $f \in \mathcal{L}^1$, $\exists C$ such that

$$\int_{B(R,x)} |f| = C \leq \int |f| < \infty$$

Further, by the faithfulness of the Lebesgue measure, $m(B(r,x)) \propto r^d$.

Proof: $B(r,x)$ is a ball of radius r centered at x so necessarily a subset of the hypercube

$$C(r+1,x) = \{h_j : x_j - \frac{r+1}{2} \leq h_j \leq x_j + \frac{r+1}{2},\ j = 1, \ldots, d\}$$

which has measure $m(C(r,x)) = r^d$.

But further, $C(r-1,x) \subseteq B(r,x)$ so

$$r^{d-1} < m(B(r,x)) < r^{d+1}$$

Now, we have $|x| > R \implies |x|^d > R^d \geq m(B(r,x))$ so

$$\frac{1}{m(B(R,x))} \geq |x|^{-d}$$

and

$$\frac{1}{m(B(R,x))} \int_{B(R,x)} |f(y)| \, dy \geq C \, |x|^{-d}$$

The RHS is independent of r so we can taking the supremum,

$$\sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)| \geq C \, |x|^{-d} \quad \blacksquare$$

4. Suppose $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$. Prove that

(a) $f_n + g_n \xrightarrow{\mu} f + g$

Since $f_n \xrightarrow{\mu} f$, $\mu\{x : |f_n(x) - f(x)| \geq \varepsilon\} \to 0$.

Let $\varepsilon > 0$. Consider the set

$$\{x : |f_n + g_n - f - g| \geq \varepsilon\}$$

,

By the triangle inequality,

$$|f_n + g_n - f - g| \leq |f_n - f| + |g_n - g|$$

So by monotonicity of measure,

$$\mu\{x : |f_n + g_n - f - g| \geq \varepsilon\} \leq \mu\{x : |f_n - f| + |g_n - g| \geq \varepsilon\}$$

For notational convenience, define

$$A_n = \{x : |f_n(x) - f(x)| \geq \varepsilon/2\}$$
$$B_n = \{x : |g_n(x) - g(x)| \geq \varepsilon/2\}$$

By assumption, $\mu(A_n) \to 0$ and $\mu(B_n) \to 0$.

But also,

$$\{x : |f_n - f| + |g_n - g| \geq \varepsilon\} \subseteq A_n \cup B_n$$

so again by monotonicity of measure,

$$\mu\{x : |f_n - f| + |g_n - g| \geq \varepsilon\} \leq \mu(A_n \cup B_n)$$

And by a property from HW 3,

$$\mu(A_n \cup B_n) \leq \mu(A_n) + \mu(B_n) \to 0$$

Hence,

$$\mu\{x : |f_n + g_n - f - g| \geq \varepsilon\} \to 0$$

and so $f_n + g_n \xrightarrow{\mu} f + g$.

(b) $f_n g_n \xrightarrow{\mu} fg$ if $\mu(X) < \infty$

Let n_k be any subsequence in $\mathbb{N}$. Since $f_n \xrightarrow{\mu} f$, we also have $f_{n_k} \xrightarrow{\mu} f$.

By a Theorem from class, $\exists n_{k_j}$ such that $f_{n_{k_j}} \to f$ a.e.

But now we can repeat the same argument for g. Take the sequence n_{k_j}. Since $g_n \xrightarrow{\mu} g$, $\exists n_{k_{j_l}}$ such that $g_{n_{k_{j_l}}} \to g$ a.e.

Lemma: If $f_{n_k} \to f$ a.e. and $g_{n_k} \to g$ a.e., then $f_{n_k} g_{n_k} \to fg$ a.e.

Proof: Since $f_{n_k} \to f$ a.e., $\exists N_1$ such that for almost all $n >\geq N_1$, $|f_{n_k} - f| < \delta$. Similarly, $\exists N_2$ such that for almost all $n \geq N_2$, $|g_{n_k} - g| < \delta$.

Pick $N = \max\{N_1, N_2\}$. Then for almost all $n \geq N$,

$$|f_{n_k} g_{n_k} - fg| < |(f+\delta)(g+\delta) - fg| = \left| f\delta + g\delta - \delta^2 \right| \to 0$$

Hence, $f_{n_{k_{j_l}}} g_{n_{k_{j_l}}} \to fg$ a.e.

Now suppose $f_n g_n$ does not converge to fg in measure. Then $\forall \delta > 0$, $\exists \varepsilon > 0$ such that

$$\mu\{|f_n g_n - fg| \geq \delta\} > \varepsilon$$

for infinitely many n.

But this means that there is a sequence $\{n_k\} \in \mathbb{N}$ such that $\mu\{|f_{n_k} g_{n_k} - fg| \geq \delta\} > \varepsilon$. But then clearly, $f_{n_k} g_{n_k} \not\to fg$ a.e., which contradicts our work above.