APMA 2110: Real Analysis

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Chapter 1

Analysis and Metric Spaces

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Some basic notation:

$$\begin{aligned}
\mathbb{N} &:= \{1, 2, 3, \ldots\} \\
\mathbb{Z} &:= \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
\mathbb{Q} &:= \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \right\} \\
\mathbb{R} &:= \text{the set of real numbers} \\
\mathbb{C} &:= \text{the set of complex numbers}
\end{aligned}$$

Some basic logic:

- $(A \implies B) \iff (\neg B \implies \neg A)$ (contrapositive)
- $E \subset X \implies \forall x \in E,\ x \in X$

Sets

Note that in this course, $\subset$ includes the possibility of equality, while $\subsetneq$ does not.

Power Set: $P(X) = \{E : E \subseteq X\}$

Example: $X = \{1, 2, 3\}$

$$P(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$$

Sets: Let $\mathbb{E}$ be a collection of sets E

- $\bigcup_{E \in \mathcal{E}} = \{x : x \in E, \text{ for some } E \in \mathcal{E}\}$
- $\bigcap_{E \in \mathcal{E}} = \{x : x \in E, \text{ for all } E \in \mathcal{E}\}$
- $\mathcal{E} = \{E_\alpha : \alpha \in A\} = \{E_{\substack{\alpha \\ \alpha \in A}}\}$
- $E_\alpha \cap E_\beta = \emptyset$ for $\alpha \neq \beta \iff E_\alpha$ and E_β are *disjoint*

Limsup and Liminf: For $\{E_n\}_{n=1}^{\infty}$,

$$\limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$
$$\liminf E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$$

Exercise: Prove that

$$\limsup E_n = \{x : x \in E_n \text{ for infinitely many } n\}$$
$$\liminf E_n = \{x : x \in E_n \text{ for all but finitely many } n\}$$

i.e. after first finite n, x is in E_n for all n.

Difference and Symmetric Difference: Let E and F be two sets

$$E \setminus F = \{x : x \in E, x \notin F\}$$
$$E \triangle F = (E \setminus F) \cup (F \setminus E)$$
$$E^c = X \setminus E, \; E \subseteq X$$

De Morgan's Laws:

$$\left(\bigcup_{\alpha \in A} E_\alpha \right)^c = \bigcap_{\alpha \in A} E_\alpha^c$$
$$\left(\bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

Exercise: Prove De Morgan's Laws.

Cartesian Product: If X and Y are sets, then $X \times Y$ is the *ordered* set

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

Relations

Relations: A *relation* R from X to Y is a subset of $X \times Y$ such that

$$xRy \iff (x, y) \in R$$

Equivalence relation: A relation $\sim$ is an equivalence relation in the special case $Y = X$ if it is

- Reflexive: $x \sim x \quad \forall x \in X$
- Symmetric $x \sim y \iff y \sim x$
- Transitive $x \sim y, y \sim z \implies x \sim z$

Functions

Mappings: A mapping/function $f : X \to Y$ is a relation R from X to Y such that $\forall x \in X$, there exists a *unique* $y \in Y$ such that xRy. We write $y = f(x)$.

Composition: If $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$ is a function such that $g \circ f(x) = g(f(x))$

Images: If $D \subseteq X, E \subseteq Y$, the *image* of D (and the *inverse image*/pre-image of E) under $f : X \to Y$ is

$$f(D) = \{f(x) : x \in D\}$$
$$f^{-1}(E) = \{x \in X : f(x) \in E\}$$

For $f : X \to Y$ we further call X the *domain* of f and Y the *codomain* of f. The *range*/*image* of f is $f(X)$.

Inverses: f^{-1} defines an operation on $P(X)$ such that

$$f^{-1}\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} f^{-1}(E_\alpha)$$
$$f^{-1}\left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A} f^{-1}(E_\alpha)$$
$$f^{-1}(E^c) = (f^{-1}(E))^c$$

Exercise: Prove the above properties of inverses. Warning: in general, f also commutes with unions but *not* with intersections. Why?

Bijectivity:

- f is *injective* iff $f(x_1) = f(x_2) \implies x_1 = x_2$
- f is *surjective* iff $\forall y \in Y, \exists x \in X$ s.t. $f(x) = y$
- f is *bijective* iff it is both injective and surjective

In the case of a bijective mapping f, then f^{-1} is a function from Y to X (i.e. f^{-1} has a unique value for bijective f)

Sequences

Sequences: A sequence in a set X is a function $f : \mathbb{N} \to X$. We $\{x_n\}$ for $x_n \in X$

Subsequence: A subsequence $x_{n_k} \subseteq \{x_n\}$ with $n_k \in \{1, \dots, \infty\}$

Ordering

Partial ordering: a partial ordering on a nonempty set X is a relation R on X such that

- If xRy and yRz, then xRz (transitivity)
- If xRy and yRx, then $x = y$ (antisymmetry)
- xRx for all x (reflexivity)

Example: Let E be a set. Consider the relation $\subseteq$. Let $E_1, E_2, E_3 \subseteq E$.

- $E_1 \subseteq E_2$ and $E_2 \subseteq E_3$ implies $E_1 \subseteq E_3$ (transitivity ✓)
- $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ implies $E_1 = E_2$ (antisymmetry ✓)
- $E_1 \subseteq E_1$ (reflexivity ✓)

Therefore, inclusion (with equality) is a partial ordering. (Proof for first two by considering elements, proof for last by equality)

Total ordering: A total ordering/linear ordering is a partial ordering such that for all $x, y \in X$, either xRy or yRx.

Example: Inclusion is not a total ordering on $P(X)$ since (in general) $E_1 \not\subseteq E_2$ and $E_2 \not\subseteq E_1$ for $E_1 \neq E_2$.

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Recall: a *partial ordering* is a relation that satisfies

1. if xRy and yRz, then xRz
2. if xRy and yRx, then $x = y$
3. xRx for all x

Examples:

- In the real numbers, $\leq$ is the typical ordering.
- For a set X and its power set $P(X)$, $\subseteq$ is a partial ordering.

Warning: In this class, we will use $\leq$ to denote an abstract partial ordering.

Total/Linear Ordering: A total ordering is a partial ordering such that for all $x, y \in X$, either $x \leq y$ or $y \leq x$.

Extrema: If X is partially ordered by $\leq$, a *maximal* (resp. *minimal*) element of X is an element $x \in X$ such that $x \leq y \implies y = x$

Bounds: If $E \subseteq X$, an *upper* (resp. *lower*) *bound* for E is an element $x \in X$ such that $y \leq x$ (resp. $x \leq y$) for all $y \in E$.

Zorn's Lemma (transfinite induction): If X is partially ordered by $\leq$, assume every linearly ordered subset of X has an upper bound. Then X has a maximal element.

Proof: We regard this as axiomatic

Well-Ordering: A set X is *well-ordered* if

1. it is linearly ordered by $\leq$
2. every nonempty subset of X has a minimal element.

Well-ordering Principle: Every non-empty set X can be well-ordered

Proof: Consider $\mathcal{W} = \{\text{all well-ordered subsets of } X\}$.

Suppose there exist well-ordered sets $E_1, E_2 \subseteq W$. Then each has a minimal element.

We know $\mathcal{W}$ is non-empty because for all finite subsets of X, we can order them (using the normal linear order on $\mathbb{R}$).

We will proceed by defining a relation R between the linear orderings $\leq_1$ and $\leq_2$ of E_1 and E_2 respectively. We will say $\leq_1 R \leq_2$ if:

1. $\leq_2$ extends $\leq_1$ (i.e. $E_1 \subseteq E_2$ and $\leq_1 = \leq_2$ on E_1)
2. If $x \notin E_1, x \in E_2$, then $y \leq_2 x$ for all $y \in E_1$

Exercise: Prove that R is a partial ordering in $\mathcal{W}$

Assume $\mathcal{S} = \{\leq_\alpha; R\}$ is the set of linear orderings $\leq_\alpha$ of $E_\alpha \subseteq \mathcal{W}$ for $\alpha \in A$. Thus, $\leq_\alpha R \leq_\beta$ for $\alpha, \beta \in A$.

Claim: Let

$$E_\infty = \bigcup_{\alpha \in A} E_\alpha$$

equipped with the partial ordering $\leq_\infty$ such that $\leq_\infty \big|_{E_\alpha} = \leq_\alpha$ for all $\alpha \in A$.

Clearly, $\leq_\alpha R \leq_\infty$ for all $\alpha \in A$. Then for any sequence of well-ordered sets in $\mathcal{W}$, E_∞ is an upper-bound.

Exercise: Verify that $\leq_\alpha R \leq_\infty$ is well defined and that E_∞ is an upper bound for $\mathcal{W}$

By Zorn's Lemma, there exists a maximal element $E_{\max} \in \mathcal{W}$. (Verify it's a well-ordering by extending $\leq_{\max}$ to include any $x_0 \in X \setminus E_{\max}$ such that $x \leq x_0$ for all $x \in E_{\max}$).

Consider $E_{\max} \cup \{x_0\}$. Clearly, $E_{\max} \leq E_{\max} \cup \{x_0\}$, so $E_{\max} \cup \{x_0\}$ and by the extension above, $E_{\max} \cup \{x_0\} \in \mathcal{W}$. This contradicts the maximality of $E_{\max}$, so $E_{\max} = X$.

Definition: Let $\prod_{\alpha \in A} X_\alpha$ be the set of all maps $f : A \to \bigcup_{\alpha \in A} X_\alpha$ such that $f(\alpha) \in X_\alpha$ for all $\alpha \in A$.

Axiom of Choice: If $\{X_\alpha\}_{\alpha \in A}$ is a nonempty collection of nonempty sets, $\prod_{\alpha \in A} X_\alpha$ is nonempty, i.e. there exists at least one choice function f

Proof: Let $X = \bigcup_{\alpha \in A} X_\alpha$. Pick a well-ordering on X and $\alpha \in A$. Let $f(\alpha)$ be the minimal element of X_α. Then $f \in \prod_{\alpha \in A} X_\alpha$

Cardinality

Definition:

- card $X \leq$ card Y if there exists an injective function $f : X \to Y$
- card $X =$ card Y if there exists a bijective function $f : X \to Y$
- card $X \geq$ card Y if card $X \leq$ card Y but card $X \neq$ card Y there exists a surjective function $f : X \to Y$

Property: card $X \leq$ card Y iff card $Y \geq$ card X

Proof: card $X \leq$ card Y implies there exists an injective $f : X \to Y$. Pick $x_0 \in X$ and define $g : Y \to X$ by

$$g(y) = \begin{cases} f^{-1}(y) & y \in f(X) \\ x_0 & \text{otherwise} \end{cases}$$

In the first case, we have injectivity of f so each $f^{-1}(y)$ is unique. In the second case we ensure surjectivity.

Conversely, if $g : Y \to X$ is surjective, consider $g^{-1}(\{x\})$ for $x \in X$. These sets are non-empty and disjoint because f is a map (each x can map to a single y). Then any $f \in \prod_{x \in X} g^{-1}(\{x\})$ is an injection from X to Y.

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Property: For any sets x and Y, either card $X \leq$ card Y or card $Y \leq$ card X

Proof Sketch: Consider the (non-empty) set

$$J = \{\text{all injections } f_E : X \to Y \text{ with respect to } E \subseteq X\}$$

Define a relation R on J such that $f_{E_1} R f_{e_2}$ if $E_1 \subseteq E_2$ and $f_{E_2}\big|_{E_1} = f_{E_1}$, i.e. f_{E_2} is an extension of f_{E_1}.

Repeating the argument of the Well-Ordering Principle, R is a partial ordering.

Then we can find an upper bound for J by considering the union of all $E \in J$ and extending the injections.

By Zorn's Lemma, there exists a maximal element $f_{E_{\max}} \in J$ with respect to the ordering R.

Case 1: Suppose $E_{\max} = X$. Then $f_{E_{\max}}$ is an injection from X to Y so card $X \leq$ card Y

Case 2: Suppose $E_{\max} \subsetneq X$. Then $\exists x_0 \in X \setminus E_{\max}$. Consider the image $f(E_{\max})$. We claim $f(E_{\max}) = Y$ so $f_{E_{\max}}^{-1}$ is defined on all of Y and is injective $Y \to X$ and we are done. Thus, it only remains to show $f(E_{\max}) = Y$.

If the claim is not true, $\exists y_0 \in Y$ but $y_0 \notin f_{E_{\max}}(X)$ but this is a contradiction to maximality (as in the Well-Ordering Principle proof).

Schröder-Bernstein Theorem: If card $X \leq$ card Y and card $Y \leq$ card X, then card $X =$ card Y

Note: This seems trivial but in fact the two functions are not necessarily the same so we must construct our own bijection.

Proof: Denote the cardinality injections $f : X \to Y$ and $g : Y \to X$.

If $f(X) = Y$, then f is a bijection and we are done.

If $f(X) \neq Y$ (i.e. $f(X) \subsetneq Y$), the consider $Y_1 = Y \setminus f(X)$ and $g(Y_1)$. Then $f(Y_1) \subsetneq X$, so call $X_1 = f(Y_1)$. We now have a bijection $X_1 \to Y_1$.

Let's repeat. $f(X \setminus X_1) \subsetneq Y \setminus Y_1$ so define $Y_2 = (Y \setminus Y_1) \setminus f(X \setminus X_1)$.

Now we know $f(X_1) \subseteq Y_2$ and $f^{-1}(Y_1) \subseteq X_1$ so we can define a bijection $X_2 \to Y_2$.

Assume $X_1, \ldots X_n$ and $Y_1, \ldots, Y_n$ are constructed. WLOG assume that this procedure can be repeated infinitely (or else we would already have a bijection).

Define

$$\left(Y \setminus \bigcup_{i=1}^{n} Y_i\right) \setminus f\left(X \setminus \bigcup_{i=1}^{n} X_i\right) = Y_{n+1}$$

since $f(X_i) \subseteq Y_{i+1}$.

Exercise: Verify that

$$g : \bigcup_{i=1}^{\infty} Y_i \to \bigcup_{i=1}^{\infty} X_i$$

is a bijection and further that

$$f : \left(X \setminus \bigcup_{i=1}^{\infty} X_i\right) \to \left(Y \setminus \bigcup_{i=1}^{\infty} Y_n\right)$$

is also a bijection.

Together, these steps show that we have a bijection on the full sets X and Y.

Proposition: For any set X, card $X <$ card $P(X)$

Proof: Clearly, $\forall x \in X$, we have an injection $f : X \hookrightarrow P(X)$ defined by $f(x) = \{x\}$.

We claim there is no surjection $g : X \to P(X)$ and proceed by contradiction.

Let $g : X \twoheadrightarrow P(X)$. Define

$$Y = \{x \in X \text{ s.t. } x \notin g(x)\}$$

We claim $Y \notin g(X)$. If not, assume $x_0 \in X$ such that $g(x_0) = Y$.

Case 1: If $x_0 \in Y$, then $x_0 \notin g(x_0) = Y$ - contradiction

Case 2: If $x_0 \notin Y$, then $x_0 \in g(x_0) = Y$ - contradiction

Therefore, $Y \notin g(X)$ so g is not surjective.

Countable: A set X is *countably infinite* if card $X \leq$ card $\mathbb{N}$.

Proposition:

(a) If X and Y are countable, so is $X \times Y$.

(b) If A is countable and X_α is countable for every $\alpha \in A$, then $\bigcup_{\alpha \in A} X_\alpha$ is countable.

Proof:

(a) card $X =$ card $Y =$ card $\mathbb{N}$ so it suffices to show $\mathbb{N} \times \mathbb{N} =$ card $\mathbb{N}$

$\forall n \in \mathbb{N}$, define $f(n) \hookrightarrow (n, 1) \in \mathbb{N} \times \mathbb{N}$.

Consider $g((m, n)) \to 2^m 3^n \in \mathbb{N}$. Is this injective? Consider $g(m_1, n_1) = 2^{m_1} 3^{n_1}$. By the unique prime factorization of integers, $2^{m_1} 3^{n_1} = 2^m 3^n$ iff $(m_1, n_1) = (m, n)$ so g is injective.

Now we can use Schroder-Bernstein and we are done.

(b) As A is countable, $\forall \alpha \in A$, $\exists f_\alpha : \mathbb{N} \to X_\alpha$ So we can define $F : \mathbb{N} \times A \to \bigcup_{\alpha \in A} X_\alpha$ by

$$F(n, \alpha) = f_\alpha(n)$$

which is surjective

Corollary: $\mathbb{Z}$ and $\mathbb{Q}$ are countable

Proof: $\mathbb{Z} = \mathbb{N} \cup \{-\mathbb{N}\} \cup 0$

We can define $f : \mathbb{Z}^2 \to \mathbb{Q}$ by

$$f(m,n) = \begin{cases} \frac{m}{n} & n \neq 0 \\ 0 & n = 0 \end{cases}$$

Convention for this course: We will use $\mathbb{R}$ to denote the standard reals and will define the *extended reals* $\overline{R}$ by $\mathbb{R} \cup \pm\infty$

Under this notation, we can state that for any $E \subseteq \overline{R}$, $\sup \overline{E}$ and $\inf \overline{E}$ are always well-defined, i.e. all sets are bounded above by ∞ and below by $-\infty$.

We define the following rules:

- $X \pm \infty = \pm\infty$
- $\infty + \infty = \infty$
- $-\infty - \infty = -\infty$
- $\infty - \infty$ is undefined
- $x(\pm\infty) = \pm\infty$ for $x > 0$ and $x(\pm\infty) = \mp\infty$ for $x < 0$
- $0 \cdot (\pm\infty) = 0$

Note: this last point does *not* talk about limits, it is just notation

Proposition: Every open set in $\mathbb{R}$ is a countable disjoint union of open intervals

Proof Sketch: For all $x \in U$, there exists an open interval $I_{\alpha,\beta} = (\alpha, \beta) \subseteq U$ with $\alpha < x < \beta$.

Let $\mathcal{J}_x = \{x \in I_{\alpha,\beta} \mid I_{\alpha,\beta} \in U\}$.

Take $\alpha_{\inf} = \inf \alpha$ and $\beta_{\sup} = \sup \beta$.

Exercise: Check that $x \in (\alpha_{\inf}, \beta_{\sup}) \subseteq U$

We call $I_x = (\alpha_{\inf}, \beta_{\sup})$ for all $x \in U$

We claim $\forall x, y \in U$, either $I_x \cap I_y = \emptyset$ or $I_x = I_y$.

Suppose $I_x \cap I_y \neq \emptyset$. Then $I_x \cup I_y$ is an open interval containing x, so $I_x \cup I_y \in \mathcal{J}_x$ but I_x is maximal so this is a contradiction unless $I_x = I_y$

Now we can write

$$U = \bigcup_{x \in U} I_x$$

Why is this countable? We can define an injection $U \to \mathbb{Q}$ by choosing a rational number in each I_x (exist by density of $\mathbb{Q}$).

Metric Spaces

Definition: A *metric space* is a set X together with a *distance function* $\rho : X \times X \to [0, \infty)$ such that

1. $\rho(x, y) = 0 \iff x = y$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

Examples:

- $\mathbb{R}^n$ with $\rho(x, y) = |x - y|$
- Set of continuous functions f over $[0, 1]$ with $\rho_1(f, g) = \int 0^1 |f(x) - g(x)| \, dx$ (or alternatively $\rho_\infty = \sup_{0 \leq x \leq 1} |f(x) - g(x)|$)

Exercise: Check the above are metric spaces

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Closed and Open Sets

Open ball: Let (X, ρ) be a metric space. If $x \in X$, $r > 0$, we define the *open ball* $B(x, r) = \{y \in X \text{ s.t. } \rho(x, y) < r\}$

Open set: a set E is open iff $\forall x \in E, \exists r > 0$ s.t. $B(x, r) \subseteq E$

Closed set: a set E is closed iff E^c is open

Example: $B(x,r)$ is open. Consider $y \in B(x,r)$. Then $\rho(x,y) = s < r$. By the triangle inequality, $B(y, r-s) \subseteq B(x,r)$

Exercise: Prove that $B(x,r)$ is open

Properties:

- $\emptyset$ is open
- If U_x are open sets, $\bigcup_{x\in A} U_x$ is open (as is the finite intersection)
- If F_x are closed sets, $\bigcap_{x\in A} F_x$ is closed (as is the finite union)

Interior: Let $E \subseteq X$. The *interior* of E is

$$\mathring{E} = \bigcup_{O \subseteq E} O$$

(this is the largest open set in E)

Closure: The *closure* of E is

$$\overline{E} = \bigcap_{E \subseteq F} F$$

(this is the smallest closed set containing E)

Proposition: Let (X,ρ) be a metric space. Let $E \subseteq X$ and $x \in X$. Then the following are equivalent:

(a) $x \in \overline{E}$

(b) $B(x,r) \cap E \neq \emptyset$ for all $r > 0$

(c) $\exists (x_n) \subseteq E$ such that $x_n \to x$

Proof: $((a) \to (b))$ Let $x \in \overline{E}$. Suppose $\exists r_0 > 0$ such that $B(x,r) \cap E = \emptyset$. Then $E \subseteq (B(x,r_0))^c$. But $(B(x,r_0))^c$ is closed so $\overline{E} \subseteq (B(x,r_0))^c$ so $x \in B(x,r_0) \subseteq (\overline{E})^c$ but this implies $x \in (\overline{E})^c$ which is a contradiction.

$((b) \to (c))$ Let $r = \frac{1}{n}$. By (b), $B(x,\frac{1}{n}) \cap E \neq \emptyset$. Choose $x_n \in B(x,\frac{1}{n}) \cap E$. Certainly $\rho(x_n,x) < \frac{1}{n}$ so $\lim \rho(x_n,x) = 0$ and $x_n \to x$

$((c) \to (a))$ If $x \notin \overline{E}$, $x \in (\overline{E})^c$ but $(\overline{E})^c$ is open so $\exists r > 0$ s.t. $B(x,r) \subseteq$

$(\overline{E})^c \subseteq E^c$. Then there cannot exist any sequence in E. But this contradicts $x_n \to x$

Density

Dense: E is dense in X if $\overline{E} = X$ (examples $\mathbb{R}^n, \mathbb{Q}^n$)

Nowhere dense: E is nowhere dense if $(\overline{E})^\circ = \emptyset$ (example: emptyset)

Separable: X is separable if there exists a countable dense subset $E \subseteq X$

Limits: In this class, $x_n \to x$ iff $\lim_{n\to\infty} \rho(x_n, x) = 0$

Continuity

Let $\mathcal{C} = \{\text{continuous functions on } [0,1]\}$.

Continuity at a point: If (X_1, ρ_1) and (X_2, ρ_2) are metric spaces, $f : X_1 \to X_2$ is continuous at $x \in X_1$ if $\forall \varepsilon > 0, \exists \delta_x > 0$ such that $\forall y \in X_1$ such that $\rho_1(x, y) < \delta_x$ (i.e. $y \in B_1(x, \delta_x)$),

$$\rho_2(f(x), f(y)) < \varepsilon$$

(i.e. $f(y) \in B_2(f(x), \varepsilon)$)

Continuity on a set: f is continuous in X iff f is continuous at every $x \in X$

Uniform Continuity: f is uniformly continuous if δ is independent of x, i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\rho_1(x, y) < \delta \implies \rho_2(f(x), f(y)) < \varepsilon$$

for all $x \in X$.

Proposition: $f : X_1 \to X_2$ is continuous iff $f^{-1}(U) \subseteq X_1$ is open for all open $U \subseteq X_2$

Proof: Let f be continuous and $U \subseteq X_2$ be open. $f^{-1}(U) = \emptyset$ is open so take $x \in f^{-1}(U)$. Then $f(x) = y \in U$.

Since U is open, $\exists \varepsilon_y > 0$ s.t. $B_2(y, \varepsilon_y) = B_2(f(x), \varepsilon_y) \subseteq U$.

By continuity, $\exists \delta_x > 0$ such that $\forall z \in B_1(x, \delta_2)$,

$$\rho_2(f(x), f(z)) < \varepsilon_y \implies f(z) \in B_2(y, \varepsilon_y) \subseteq U \implies z \in f^{-1}(U)$$

so $B_1(x_1, \delta_x) \subseteq f^{-1}(U)$ and $f^{-1}(U)$ is open.

Conversely, suppose $f^{-1}(U)$ is open for all open $U \subseteq X_2$. Let $\varepsilon > 0$. Consider $y = f(x)X_2$. Then $B_2(y, \varepsilon)$ is open so $f^{-1}(B_2(y, \varepsilon))$ is open by assumption.

Let $x \in f^{-1}(B_2(y, \varepsilon))$. Then $\exists \delta_x$ such that $B_1(x, \delta_x) \subseteq f^{-1}(B_2(y, \varepsilon))$.

Then $f(B_1(x, \delta_x)) \subseteq B_2(y, \varepsilon)$ which is precisely the definition of continuity.

Cauchy Sequences

Cauchy Sequence: A sequence (x_n) in a metric space (X, ρ) is Cauchy if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall m, n \geq N$,

$$\rho(x_m, x_n) < \varepsilon$$

Completeness: A subset $E \subseteq X$ is *complete* if every Cauchy sequence $x_n \in E$ has a limit $x \in E$

Examples:

- In $\mathbb{R}^n$, any bounded closed subset is complete.
- $(\mathcal{C}, \rho_\infty)$ is complete

Exercise: Prove that $(\mathcal{C}, \rho_\infty)$ is complete for

$$\rho_\infty(x, y) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

(though in general this is not true for other metrics)

Proposition: A closed subset (X, ρ) of a complete metric space is complete and complete subsets of a metric space must be closed

Proof:

Exercise

Set Distance:

- Let $x \in X$ and $E \subseteq X$. The *distance* from x to E is

$$\rho(x, E) = \inf\{\rho(x, y) : y \in E\}$$

- For $E, F \subseteq X$,

$$\rho(E, F) = \inf\{\rho(x, y) : x \in E, y \in F\}$$

Diameter: diam $E = \sup\{\rho(x, y) : x, y \in E\}$

Bounded: E is bounded iff diam $E < \infty$

Open cover: Let $\{V_\alpha\}_{\alpha \in A}$ be a family of sets. $\{V_\alpha\}$ *covers* E if

$$E \subseteq \bigcup_{\alpha \in A} V_\alpha$$

Total boundedness: E is *totally bounded* if $\forall \varepsilon > 0$, E can be covered by finitely many balls of radius ε

Example: $\mathbb{R}^n$ is totally bounded. *Proof:* consider a hypercube of side length R. Clearly we can divide this into ε-cubes and then take slightly larger balls to cover the whole space.

Theorem (Characterization of Compactness): The following are equivalent:

1. E is complete and totally bounded
2. Every sequence in E has a convergent subsequence with its limit in E
3. If $\{V_\alpha\}_{\alpha \in A}$ is an open cover of E, then there exists a finite set $F \subseteq A$ such that $\{U_\alpha\}_{\alpha \in F}$ covers E

Proof: HW

1.5 Sept 19

Products of Metric Spaces: Let (X, ρ_1) and (Y, ρ_2) be metric spaces. Define the *product metric* on $X \times Y$ by $(X_1 \times X_2, \rho_1 \times \rho_2)$ where

$$\rho_1 \times \rho_2 = \sqrt{\rho_1^2(x_1, y_1) + \rho^2(x_2, y_2)}$$

(so called *Euclidean Metric*)

Though many other metrics are possible, such as $\max(\rho_1, \rho_2)$ and $\rho_1 + \rho_2$.

In general, we will simply take the Euclidean metric because all these metrics are equivalent in the sense that $\exists C_1, C_2$ such that

$$C_1(\rho_1 \times \rho_2)_1 \leq C_2(\rho_1 \times \rho_2)_2 \leq C_2(\rho_1 \times \rho_2)_3$$

Properties:

- $\rho_1 \times \rho_2 \to 0 \iff \rho_1 \to 0$ and $\rho_2 \to 0$

Chapter 2

Measure Theory

2.1 Sept 19

Measure Theory Motivation

Riemann Integral: Let $f : [a,b] \to \mathbb{R}$. We subdivide $[a,b]$ by

$$a = x_0 < x_1 < \cdots < x_n = b$$

and define subintervals $[x_i, x_{i+1}]$.

Then

$$\int f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot (x_{i+1} - x_i)$$

Convergence: Many times, we are interested in the question:

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx \stackrel{?}{=} \int_0^1 f(x)\, dx$$

for $f_n(x) \to f(x)$.

This is easy when $f_n \to f$ uniformly but in general, we need something else.

In Riemann integration, we divide the domain into intervals and sum the function over these intervals.

In Lebesgue intergration, we instead divide *the range*, i.e. we take a set

$$E_i = \{x : a_n \leq f(x) \leq a_{n+1}\}$$

Measure: We define $\mu(E)$, the *measure* of a subset, by:

1. (Countable Additivity) $\{E_n\}$ such that $E_i \cap E_j = \emptyset$ for $i \neq j$ then $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{i=1}^{n} \mu(E_n)$
2. (Translation invariance) $\mu(E + r) = \mu(\{x + r : x \in E\}) = \mu(E)$
3. $\mu([0,1]) = 1$

Proposition: There is no measure μ satisfying the above properties which is defined for all subsets of $[0,1)$

Proof: Step 1. Let $\mathbb{Q}_1 = \mathbb{Q} \cap [0,1)$. Define an equivalence relation $x \sim y$ iff $x - y \in \mathbb{Q}_1$.

Now consider the equivalence class $\mathcal{E}_x = \{y \in [0,1) : y \sim x\}$. (As it is an equivalence class: $\mathcal{E}_x \cap \mathcal{E}_y \neq \emptyset \implies \mathcal{E}_x = \mathcal{E}_y$)

And clearly,

$$[0,1) = \bigcup_{x \in [0,1)} \mathcal{E}_x$$

By the Axiom of Choice, choose a unique element $e_x \in \mathcal{E}_x$. Define $N = \{e_x\}$. Now $e_x - e_y \notin \mathbb{Q}_1$.

Step 2. $\forall r \in \mathbb{Q}_1$, define

$$N_r = \{e_x + r : e_x \in N \cap [0, 1-r)\} \cup \{e_x + r - 1, e_x \in N \cap [1-r, 1]\}$$

(the first set is the points that don't leave the interval under translation, the second set is the pullback of the points that do)

Step 3. We claim

$$[0,1) = \bigcup N_r,\ N_r \cap N_s = \emptyset \text{ for } r \neq s$$

Proof:

1. (Subset) $\forall y \in [0,1)$, $\exists e_x \in N$ such that $y - e_x \in \mathbb{Q}_1$.

   If $y \geq e_x$, $r = e_x - y + 1$. Otherwise, $r = e_x - y$.

2. (Disjoint Union) Suppose $N_r \cap N_s \neq \emptyset$. Let $r \neq s$. Select $y \in N_r \cap N_s$ so $y - s \in N$ and $y - r \in N$

   Case 1. $y - s \neq y - r$. But then

$$(y - r) - (y - s) = s - r \in \mathbb{Q}_1$$

   which is a contradiction of the construction of N.

   Case 2. $y - s \neq y - r + 1$. Contradiction again by rational difference.

Step 4. By the definition of a measure,

$$\begin{aligned} \mu(N_r) &= \mu(N_r \cap (0, 1 - r)) + \mu(N_r \cap [1 - r, 1]) \\ &= \mu(N) \end{aligned}$$

Exercise: Check that $\mu(N_r) = \mu(N)$

But by countable Additivity,

$$1 = \mu([0, 1)) = \sum_{r \in \mathbb{Q}_1}^{\infty} \mu(N_r) = \begin{cases} 0 \\ \infty \end{cases}$$

which is a contradiction.

Conclusion: it is not always possible to define a measure so we need to be careful.

Algebras

Algebra: Given a set X, an *algebra* is a collection of subsets $\mathcal{A} \subseteq P(X)$ such that if $E_1, \ldots, E_n \subseteq \mathcal{A}$,

1. $\bigcup_{i=1}^{n} E_i \in \mathcal{A}$
2. $E \in \mathcal{A} \implies E^c \in \mathcal{A}$

Property 2 gives us that $X \in \mathcal{A}$ and $\emptyset \in \mathcal{A}$ $(E \cup E^c = X,\ X^c = \emptyset)$

Sigma Algebra: An algebra $\mathcal{A}$ is a σ-*algebra* if it is closed under countable unions and complements, i.e. for $E_1, E_2, \dots \in \mathcal{A}$,

1. $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$
2. $E \in \mathcal{A} \implies E^c \in \mathcal{A}$

Remark: It suffices to demand closure for disjoint countable unions since

$$\bigcup_{n=1}^{\infty} E_i = \bigcup_{n=1}^{\infty} F_i$$

for $F_k = E_k \setminus \bigcup_{i=1}^{k-1} E_i$ and $F_i \cap F_{i+1} = \emptyset$

Examples:

- $P(X)$
- ϕ, X
- $\mathcal{A} = \{E \subseteq X : E \text{ countable or } E^c \text{ countable}\}$

Proposition: Let $\mathcal{A}_1, \mathcal{A}_2$ be two σ-algebras on X. Then $\mathcal{A}_1 \cap \mathcal{A}_2$ is also a σ-algebra

Exercise: Prove this proposition (easy using definition)

Generated σ-algebra: Given a collection of subsets $\mathcal{E} \subseteq P(X)$, there exists a smallest σ-algebra containing $\mathcal{E}$, denoted

$$M(\mathcal{E}) = \bigcap_{\mathcal{E} \subseteq \mathcal{A}} \mathcal{A}$$

Lemma: $\mathcal{E} \subseteq M(\mathcal{F}) \implies M(\mathcal{E}) \subseteq M(\mathcal{F})$

Proof: Omitted

Metric Spaces

Borel σ-algebra: Let (X, ρ) be a metric space. We call the σ-algebra generated by the open sets of X, the *Borel σ-algebra* B_x on X.

This is a σ-algebra because $X, \emptyset, \bigcup_{i=A} U_i$ are all open since their union is open and we have complements from the generating set.

We define

$$\bigcup_{n=1}^{\infty} F_n = F_\sigma$$

$$\bigcap_{n=1}^{\infty} O_n = G_\delta$$

for F_n closed and O_n open.

Example: The Borel set of $\mathbb{R}$, $B_\mathbb{R}$ can be generated by any of the following:

1. open intervals $\mathcal{E}_1 = \{(a, b) : a < b\}$
2. closed intervals $\mathcal{E}_2 = \{[a, b] : a < b\}$
3. the half-open intervals $\mathcal{E}_3 = \{(a, b] : a < b\}$, $\mathcal{E}_4 = \{[a, b) : a < b\}$
4. open rays $\mathcal{E}_5 = \{(a, \infty) : a \in \mathbb{R}\}$, $\mathcal{E}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$
5. closed rays

Exercise:

1. Prove that $(a, b] = \bigcap_{n=1}^{\infty}(a, b + \frac{1}{n})$ and $(a, b) = \bigcup_{n=1}^{\infty}[a + \frac{1}{n}, b - \frac{1}{n}]$
2. Prove that the above methods all generate $B_\mathbb{R}$

Conclusion: any open set in $\mathbb{R}$ is the countable union of open intervals

2.2 Sept 24

Recall last time, we were trying to characterization the Borel σ-algebra of $\mathbb{R}$, $\mathcal{B}_\mathbb{R}$.

Proposition: We claim that $\mathcal{B}_{\mathbb{R}}$ is generated by:

1. open intervals $\mathcal{E}_1 = \{(a,b) : a < b\}$
2. closed intervals $\mathcal{E}_2 = \{[a,b] : a < b\}$
3. the half-open intervals $\mathcal{E}_3 = \{(a,b] : a < b\}$, $\mathcal{E}_4 = \{[a,b) : a < b\}$
4. open rays $\mathcal{E}_5 = \{(a,\infty) : a \in \mathbb{R}\}$, $\mathcal{E}_6 = \{(-\infty,a) : a \in \mathbb{R}\}$
5. closed rays

Proof:

1. Open intervals.

   Let $\mathbb{E}_1 = \{(a,b) : a < b\}$. Clearly $B_{E_1} \subseteq B_{\mathbb{R}}$ because any open set $O \subseteq \mathcal{B}_{\mathbb{R}}$.

   For the other direction, we also have

   $$O = \bigcup_{i=1}^{\infty}(a_i, b_i)$$

   (a countable union), so $B_{\mathbb{R}} \subseteq B_{\mathcal{E}_1}$

2. Closed intervals.

   We claim

   $$(a,b) = \bigcup_{n=1}^{N}[a + \frac{1}{n}, b - \frac{1}{n}]$$

   for N sufficiently large.

   Proof: HW

   Now $\forall y \in (a,b)$,

   $$y \in \bigcup_{n=1}^{N}[a + \frac{1}{n}, \frac{b}{\frac{1}{n}}] \implies a < y < b$$

   for N sufficiently large.

For the other direction, take $[a, b] \in \mathcal{E}_2$. Then

$$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$$

Proof: $[a, b] \subseteq \bigcap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$ is clear.

For the other direction, let $yin \bigcap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$. Suppose $a \leq y \leq b$ is false. Then $y \notin (a - \frac{1}{N}, b + \frac{1}{N})$ so it cannot be in the intersection

Exercise: Prove the last two versions: half intervals and rays

Recall: For a cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of metric spaces with (X_i, ρ_i), we define the *product metric* by $(X_1 \times X_2 \times \cdots \times X_n, \rho)$ where

$$\rho(\overline{x}, \overline{y}) = \sqrt{\rho_1^2(x_1, y_1) + \cdots + \rho_n^2(x_n, y_n)}$$

where $\overline{x} = (x_1, x_2, \ldots, x_n)$ with $x_i \in X_i$ (and similarly for $\overline{y}$)

Proposition:

$$\lim_{m \to \infty} \rho(\overline{x}, \overline{y}) = 0 \iff \lim_{m \to \infty} \rho_i(x_i^m, y_i^m) = 0$$

Proof: Omitted

In this way, we can consider $\mathbb{R}^n$ as a metric space with this Euclidean metric. What is the Borel set of $\mathbb{R}^n$?

Proposition: $B_{\mathbb{R}^n}$ is

Proof: First take O_i open set in X_i

$$\bigoplus_{i=1}^{n} O_i = O_1 \times O_2 \times \cdots \times O_n$$

We claim that this is an open set in the $X_1 \times X_2 \times \cdots \times X_n$ topology.

Proof: Take $\overline{x} \in \bigoplus_{i=1}^{n} O_i$ with $x_i \in O_i$.

It suffices to show $\exists \varepsilon_0 > 0$ such that $B_{\varepsilon_0}(\overline{x}) \subseteq \bigoplus_{i=1}^{n} O_i$ where

$$B_{\varepsilon_0}(\overline{x}) = \{\overline{y} : \rho(\overline{x}, \overline{y}) < \varepsilon_0\}$$

so $\overline{y} \in B_{\varepsilon}(\overline{x})$ iff $\rho_i(x_i, y_i) < \varepsilon_0$ for all i.

Hence $y_i \in B_{\varepsilon_0}(x_i) \subseteq O_i$

Let $\bigotimes_{i=1}^{n} \mathcal{B}_{x_i}$ be the Borel set generated by $\bigoplus_{i=1}^{n} O_i$

Clearly, $\bigoplus_{i=1}^{n} \mathcal{B}_{x_i} \subseteq \mathcal{B}_{x_1 \times x_2 \times \cdots \times x_n}$

Lemma: If x_i is separable then

$$\bigotimes_{i=1}^{n} B_{x_i} = \mathcal{B}_{x_1 \times x_2 \times \cdots \times x_n}$$

In particular:

$$\bigotimes_{i=1}^{n} \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^n}$$

Proof: It suffices to show that $\forall \overline{x}, \varepsilon$,

$$\mathcal{B}_\varepsilon(\overline{x}, \varepsilon) \subseteq \bigotimes_{i=1}^{n} B_{x_i}$$

Let $\mathcal{C}_i$ be a countable subset of X_i such that $\overline{\mathcal{C}_i} = X_i$ for all $1 \leq i \leq n$

We claim

$$B_\varepsilon(\overline{x}) \subseteq \bigcup_{\substack{c_i \in \mathcal{C}_i \\ r_i \in \mathbb{Q}}} \bigotimes_{i=1}^{n} B_{r_i}(c_i)$$

for $\sqrt{r_1^2 + \cdots + r_n^2} < \varepsilon$

(And this has cardinality $\mathbb{N}^{2n}$ so countable)

Pick

$$\overline{y} \in B_\varepsilon(\overline{x}) \subseteq \bigcup_{\substack{c_i \in \mathcal{C}_i \\ r_i \in \mathbb{Q}}} \bigotimes_{i=1}^{n} B_{r_i}(c_i) \subseteq \bigotimes_{i=1}^{n} \mathcal{B}_{x_i}$$

Then

$$\sigma(\overline{x}, \overline{y}) = \sqrt{\rho_1^2(y_1, x_2), \ldots, \rho_n^2(y_n, x_n)} < \varepsilon$$

but each $\rho_i^2(y_i, x_i)$ is fixed so we may choose $c_i \in \mathcal{C}, r_i \in \mathbb{Q}$ such that

$$\rho_i(y_i, c_i) < r_i = \rho_i(y_i, x_i) - [\rho(y_i, x_i) - \rho(y_i, c_i)]$$

by density (from separability)

Since $\mathbb{Q}^n \subseteq \mathbb{R}^n$ which is countable and dense, $\mathbb{R}^n$ is separable and we are done.

Measure Spaces

Recall that we could not always define a measure except on a σ-algebra. Therefore, we limit our attention.

Measure space: $(X, \mathcal{M})$ where X is a set and $\mathcal{M}$, a σ-algebra, is the "measureable sets"

Measure: For a measure space $(X, \mathcal{M})$, we define a *measure* $\mu : \mathcal{M} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$
2. (Countable additivity) if $\{E_j\}_1^\infty$ is a sequence of pairwise disjoint sets in $\mathcal{M}$, then
$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

Intuitively, this countable additivity property lets us pull out the limits:

$$\mu\left(\lim_{n\to\infty} \bigcup_{1}^{n} E_j\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_j)$$

σ-finite: If $\mu(X) = \infty$ but
$$X = \bigcup_{i=1}^{\infty} X_i$$
where $\mu(X_i) < \infty$ for all i, then we call X *σ-finite*

Example: Let $(X, P(X))$ be a measure space. Let $f : X \to [0, \infty]$. For each $E \in P(X)$, we define

$$\mu(E) = \sum_{x\in E} f(x) = \sup\{\sum_{x\in F} f(x) : F \subseteq E \wedge F \text{ finite}\}$$

Exercise: Prove that μ is a measure on $P(X)$

In particular:

- $f(x) = 1$ for all x, then $\mu(E)$ is the *counting measure*
- Take $x_0 \in X$ and define
$$f(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$
is the *Dirac-Delta Mass* at x_0

Example: Let X be an uncountable set. Let $\mathcal{M} = \{E \text{ is finite or } E^c \text{ is finite}\}$

Define

$$\mu(E) = \begin{cases} 0 & E \text{ is countable} \\ 1 & E^c \text{ is countable} \end{cases}$$

Exercise: Check that $\mathcal{M}$ is a σ-algebra and that μ is a measure

2.3 Sept 26

Theorem (Properties of Measures): Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:

1. (Monotonicity) with $E \subseteq F$ with $E, F \in \mathcal{M}$, then
$$\mu(E) \leq \mu(F)$$

2. (Subadditivity) If $\{E_j\}_1^\infty \in \mathcal{M}$, then
$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$$

3. (Continuity from Below) If $\{E_j\}_1^\infty \subseteq \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \ldots$, then
$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$$

4. (Continuity from Above) If $\{E_j\}_1^\infty \subseteq \mathcal{M}$ and $E_1 \supset E_2 \supset \ldots$ and $\mu(E_1) < \infty$, then
$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$$

Proof: (1) Take $E \subseteq F \in \mathcal{M}$. We want to use finite additivity. Consider $F \setminus E = F \cap E^c$. Certainly, $E \cap (F \cap E^c) = \emptyset$ so

$$\mu(F) = \mu(E \cup F \setminus E) = \mu(E) + \mu(F \setminus E)$$

but the measure is nonnegative so $\mu(E) \leq \mu(F)$.

And in fact, if $\mu(F) < \infty$, then $\mu(F) - \mu(E) = \mu(F \setminus E)$.

---

(2) Once again, we would like to take advantage of finite additivity by expressing $\bigcup_{j=1}^{\infty} E_j$ as a countable disjoint union.

Let

$$F_k = E_k \setminus \bigcup_{i=1}^{k-1} E_i = \bigcup_{j=1}^{\infty} F_k$$

so

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \sum_{j=1}^{\infty} \mu(F_j) \leq \sum_{j=1}^{\infty} \mu(E_j)$$

---

(3) Let $E_1 \subseteq E_2 \subseteq \ldots \subseteq E_n \subseteq \ldots$. Denote $E_0 = \emptyset$. Then

$$\begin{aligned}
E_1 &= E_1 \setminus \emptyset \\
E_2 &= E_1 \cup (E_2 \setminus E_1) \\
E_3 &= E_2 \cup (E_3 \setminus E_2) \\
&= E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2) \\
&= (E_1 \setminus \emptyset) \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2)
\end{aligned}$$

and each of these sets are disjoint.

Inductively define

$$E_n = \bigcup_{k=0}^{n-1} E_{k+1} \setminus E_k$$

We claim

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=0}^{\infty} (E_n \setminus E_{n-1})$$

By additivity,

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} E_n\right) &= \sum_{n=0}^{\infty} \mu(E_{n+1} \setminus E_n) & &= \lim_{N\to\infty} \sum_{n=0}^{N} \mu(E_{n+1} \setminus E_n) \\
&= \lim_{n\to\infty} \mu(E_n)
\end{aligned}$$

---

(4) Let $E_1 \supset E_2 \supset \ldots$ and $\mu(E_1) < \infty$. Define $F_j = E_1 \setminus E_j$.

Clearly, $F_n \subseteq F_{n+1}$. By part 3,

$$\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \mu(F_n)$$

and

$$F_j = E_1 \setminus E_j \implies \bigcup_{n=1}^{n} F_j = E_1 \setminus \bigcup_{n=1}^{n} E_j$$

(by $E_1 \supset E_2 \supset \dots$)

So

$$\mu\left(E_1 \setminus \bigcap_{j=1}^{\infty}\right) = \lim_{n\to\infty} \mu(E_1 \setminus \bigcap_{j=1}^{n} E_j)$$

By Part 1,

$$\begin{aligned}
\lim_{n\to\infty} \mu(E_1 \setminus \bigcap_{j=1}^{n} E_j) &= \lim_{n\to\infty} \left[\mu(E_1) - \mu(\bigcap_{j=1}^{n} E_j)\right] \\
&= \lim_{n\to\infty} [\mu(E_1) - \mu(E_n)] \\
&= \lim_{n\to\infty} \mu(E_n)
\end{aligned}$$

Constructing Measures

We have showed that finding measures is hard in general. Let's construct them instead.

Outer Measure: Let X be a set and $\mu^* : P(X) \to [0, \infty]$ be an outer measure if

1. $\mu^*(\emptyset) = 0$
2. (Monotonicity) $E \subseteq F \implies \mu^*(E) \leq \mu^*(F)$
3. (Subadditivity) $\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu^*(E_j)$

(note: this is *almost* a measure and would be if we allowed additivity rather than subadditivity)

Proposition: Let $\mathcal{E} \subseteq P(X)$ such that $X, \emptyset \in \mathcal{E}$. Define $\rho : \mathcal{E} \to [0, \infty]$ with $\rho(\emptyset) = 0$. $\forall A \subseteq P(X)$, let

$$\mu^*(A) = \inf\left\{\sum_{i=0}^{\infty} \rho(E_j) \middle| E_j \in \mathcal{E},\ A \subseteq \bigcup_{j=1}^{\infty} E_j\right\}$$

(i.e. take the inf of the sum of all coverings of A). Then μ^* is an outer measure.

Proof:

First note that μ^* is well-defined: certainly $A \subseteq X$ so the set will not be empty and the inf is well defined.

Clearly, $\mu^*(\emptyset) = 0$ because $\rho(\emptyset) = 0$.

(Monotonicity) Let $A \subseteq B$ and $\{E_j \in \mathcal{E}\}_1^{\infty}$ be any covering of B. Since $A \subseteq B$,

$$\mu^*(A) \leq \sum_{n=0}^{\infty} \rho(E_j)$$

Taking the inf,

$$\mu^*(A) \leq \inf\left\{\sum_{i=1}^{\infty} \rho(E_j)\right\} \subseteq \mu^*(B)$$

(Subadditivty) Take $\bigcup_{j=1}^{\infty} A_j$ for all $A_j \in P(X)$.

By definition of inf, $\forall \varepsilon > 0$ there exists $E_{jk} \subseteq \mathcal{E}$ such that

$$\sum_{i=1}^{\infty} \rho(E_{jk}) \leq \mu^*(A_j) + \frac{\varepsilon}{2^j}$$

so

$$\bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{jk}^{\infty} E_{jk}$$

for $E_{jk} \in \mathcal{E}$.

Then

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j,k}^{\infty} \rho(E_{jk})$$
$$\leq \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon$$

Then certainly,

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$$

μ^*-measurable (Carathéodory Criterion): a collection $\mathcal{M}$ of subsets of X is μ^*-measurable if, given $A \in \mathcal{M}$, for all $E \subseteq P(X)$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

And in fact, it suffices to show

$$\mu^*(E) \geq \mu^*(A \cup E) + \mu^*(E \cap A^c)$$

2.4 Oct 01

Carathéodory Procedure: If μ^* is an outer measure and $\mathcal{M}$ are μ^*-measurable sets, then $\mathcal{M}$ is a σ-algebra and $\mu^*\big|_{\mathcal{M}}$ is a measure on $\mathcal{M}$

Proof:

STEP 1. $A \in \mathcal{M}$, $A^c \in \mathcal{M}$ by definition

---

STEP 2. Let $A, B \in \mathcal{M}$ and $A \cup B \in \mathcal{M}$. Then

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B),\ A \cap B = \emptyset$$

Since $\mu^* < \infty$ by definition, it suffices to show

$$\begin{aligned}\mu^*(E) &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)\end{aligned}$$

Since $E \in \mathcal{M}$ satisfies the Carathéodory Criterion (by assumption),

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c)\end{aligned}$$

Now consider $A \cup B$. Using set algebra,

$$A \cup B = (A \cap B) \cup (A \cap B^c) \cup (B \cap A^c)$$

(and this matches the first three terms above very nicely)

Then

$$E \cap (A \cup B) = (E \cap A \cap B) \cup (E \cap A \cap B^c) \cup (E \cap B \cap A^c)$$

By subadditivity of the outer measure,

$$\mu^*(E \cap (A \cup B)) \leq \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap B \cap A^c)$$

Further,

$$\begin{aligned}&\mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &\qquad \geq mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap B \cap A^c)\end{aligned}$$

which is just

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$$

Now take $\mu^*(A \cup B)$. Using the above,

$$\mu^*(A \cup B) = \mu^*(A \cup B \cap A) + \mu^*(A \cup B \cap A^c) = \mu^*(A) + \mu^*(B)$$

which is what we wanted to show.

Now we can inductively extend this pairwise additivity to a finite union.

Let $A_i \in \mathcal{M}$ and

$$\bigcup_{i=1}^{N} A_i \in \mathcal{M}$$

with $A_i \cap A_j = \emptyset$ for $i \neq j$

By induction,

$$\mu^* \left(\bigcup_{i=1}^{N} A_i \right) = \sum_{i=1}^{N} \mu^*(A_i)$$

STEP 3 (Countable Additivity): Let $A_i \in \mathcal{M}$ and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$.

Define

$$B_n = \bigcup_{j=1}^{n} A_j$$

Take a test set E with $\mu^*(E) < \infty$.

By induction on the Carathéodory Criterion,

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap B_j) + \mu^*(E \cap B_j^c) \\
&= \mu^*\left(E \cap \bigcup_{j=1}^{n} A_j\right) + \mu^*\left(E \cap \bigcap_{j=1}^{n} A_j^c\right) \\
&= \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu^*(E \cap \bigcap_{j=1}^{n} A_j^c) \qquad \text{Step 2} \\
&\geq \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu^*(E \cap \bigcap_{j=1}^{\infty} A_j^c) \\
&\geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(E \cap \bigcap_{j=1}^{n} A_j^c) \\
&\geq \mu^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) + \mu^*\left(E \cap \bigcap_{i=1}^{\infty} A_j^c\right) \qquad \text{Subadditivity}
\end{aligned}$$

STEP 4 (Completeness) Let $\mu^*(A) = 0$, then $A \in \mathcal{M}$.

Take any $E \subseteq P(X)$. We want to show

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

But by monotonicity,

$$\mu^*(E \cap A) \leq \mu^*(A) = 0 \implies \mu^*(E \cap A) = 0$$

Using monotonicity again, we get the inequality. Hence, every set with outer-measure 0 is in $\mathcal{M}$.

Now take $A_1 \subseteq A$. By monotonicity, $\mu^*(A_1) = 0 \in \mathcal{M}$

Completeness: A measure space $(X, \mathcal{M}, \mu)$ is complete if $\forall A \in \mathcal{M}$ with $\mu(A) = 0$, then $B \in \mathcal{M}$ for all $B \subseteq A$.

Exercise: Use the Carathéodory Procedure to produce the Hausdorff Measure (HW 4)

Lebesgue Measure

On the real numbers, it would be very nice to have a measure μ such that $\mu((a,b)) = \rho(a,b) = b-a$.

Lemma: If $A \subseteq \bigcup_{i=1}^{\infty}(a_i,b_i)$,

$$\mu^*(A) = \inf \sum_{n=1}^{\infty} \rho(a_i,b_i)$$

and $\mu^*((a,b)) = b-a$.

Then using the Carathéodory process, we get the Lebesgue Measure on $(\mathbb{R}, \mathcal{M}, \mu)$

Proposition (Faithfulness of the Lebesgue Measure): Let I be any interval (closed, open, half-open, etc.) on $\mathbb{R}$. Then $\mu(I) = \rho(I)$

Proof:

STEP 1. Suppose $I = [a,b]$ is closed and finite. $\forall \varepsilon > 0$, consider $(a-\varepsilon, b+\varepsilon) \supset [a,b]$.

By definition of inf,

$$\mu^*([a,b]) \leq \rho((a-\varepsilon, b+\varepsilon)) = b-a+2\varepsilon$$

But by arbitrariness of ε, $\mu^*([a,b]) \leq b-a$

On the other hand, take $\bigcup_{i=1}^{\infty}(a_i,b_i) \supseteq [a,b]$. By Heine-Borel, there exists a finite cover for $[a,b]$ so we can take

$$[a,b] \subseteq \bigcup_{i=1}^{N}(a_i,b_i)$$

We want to show that

$$\sum_{i=1}^{N}(b_i-a_i) \geq b-a \implies \mu^*([a,b]) \geq b-a$$

a must be in some open interval, so call it $a \in (a_1, b_1)$. WLOG, suppose $b_1 \leq b$.

But then $b_1 \in (a_2, b_2)$. If $b_2 > b$, then $(a_1, b_1) \cup (a_2, b_2)$ would cover $[a, b]$ and

$$b_2 - a_2 + b_1 - a_1 \geq b - a$$

Therefore, assume $b_2 \leq b$. Inductively define $b_n \in (a_{n+1}, b_{n+1})$. But because we have a finite cover, this process is not infinite, i.e. for some N, $b_N > b$.

Then it suffices to show

$$b_N - a_N + b_{N-1} - a_{N-1} + \cdots + b_1 - a_1 \geq b - a$$

and by construction, each $-a_i + b_{i-1} > 0$

STEP 2. Now take any interval I. Consider

$$[a + \varepsilon, b - \varepsilon] \subseteq I \subseteq (a - \varepsilon, b + \varepsilon)$$

But this is an open cover so

$$\mu^*(I) \leq b - a + 2\varepsilon$$

But in part 1, we showed that

$$b - a - 2\varepsilon \leq \mu^*(I)$$

So by arbitrariness of ε, $\mu^*(I) = b - a$

2.5 Oct 03

Corollary: If A is a countable subset of $\mathbb{R}$, $\mu^*(A) = 0$. Furthermore, $[0,1]$ is not countable.

Proof: $\forall x \in \mathbb{R}$, $\{x\} = (x-\varepsilon, x+\varepsilon)$, so

$$\mu^*(\{x\}) \leq 2\varepsilon \implies \mu^*(\{x\}) = 0$$

Now suppose $A = \bigcup_{n=1}^{\infty} a_n$ with $a_n \in \mathbb{R}$.

By subadditivity,

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(\{a_n\}) = 0$$

Since $\mu^*([0,1]) = 1 \neq 0$, $[0,1]$ is not countable.

Proposition: $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}$

Proof: By the characterization of the Borel set on $\mathbb{R}$, it suffices to show that $(a,\infty) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

$\forall E \in \mathcal{M}$ with $\mu^*(E) < \infty$, we want to show

$$\begin{aligned} \mu(E) &\geq \mu^*(E \cap (a,\infty)) + \mu^*(E \cap (a,\infty)^c) \\ &= \mu^*(E \cap (a,\infty)) + \mu^*(E \cap (-\infty, a]) \end{aligned}$$

For notational convenience, let

$$\begin{aligned} E_1 &= E \cap (a,\infty) \\ E_2 &= E \cap (-\infty, a] \end{aligned}$$

Let $\varepsilon > 0$. By the sharpness of the outer measure, $\exists \bigcup_{n=1}^{\infty} I_n$ such that

$$E \subseteq \bigcup_{n=1}^{\infty} I_n$$

and

$$\sum_{n=1}^{\infty} |I_n| < \mu^*(E) + \varepsilon$$

Then

$$E_1 \subseteq \bigcup_{n=1}^{\infty} I_n \cap (a, \infty)$$
$$E_2 \subseteq \bigcup_{n=1}^{\infty} I_n \cap (-\infty, a]$$

so

$$\mu^*(E_1) \leq \sum_{n=1}^{\infty} \mu^*(I_n \cap (a, \infty))$$
$$\mu^*(E_2) \leq \sum_{n=1}^{\infty} \mu^*(I_n \cap (-\infty, a])$$

Now by the faithfulness of the Lebesgue Measure,

$$\mu(I_n) = \mu^*(I_n \cap (a, \infty)) + \mu^*(I_n \cap (-\infty, a])$$

so

$$\mu^*(E_1) + \mu^*(E_2) \leq \sum_{n=1}^{\infty} \mu(I_n) \leq \mu^*(E)$$

Transformations

Definitions: Given $E \subseteq \mathbb{R}$, we define

- (Translation) $E + a := \{x + a : x \in E\}$
- (Dilation) $rE := \{rx : x \in E\}$

Lemma: For the Lebesgue outer measure and $E \in \mathcal{M}$,

1. $\mu^*(E + a) = \mu^*(E)$
2. $\mu^*(rE) = |r|\,\mu^*(E)$

Proof Sketch: Let $E \subseteq \bigcup_{n=1}^{\infty} I_n$.

Certainly,

$$E + a \subseteq \bigcup_{n=1}^{\infty} \{I_n + a\}$$

$$rE \subseteq \bigcup_{n=1}^{\infty} \{|r|\,I_n\}$$

Then

$$\sum_{n=1}^{\infty} \rho(I_n) = \sum_{n=1}^{\infty} \rho(I_n + a) \geq \mu^*(E + a)$$

And taking the infimum,

$$\mu(E) \geq \mu^*(E + a)$$

For dilation, notice $|I_n| = \frac{1}{|r|}\,|rI_n|$. The result follows similarly.

The other direction is exactly the same.

Approximation of Measurable Sets

Lemma:

1. (Approximation from Above) $\forall E \subseteq P(X)$ and $\forall \varepsilon > 0$, then exists an open set O such that $E \subseteq O$ and
$$\mu(O) \geq \mu^*(E) \geq \mu(O) - \varepsilon$$
2. (Approximation from Below) $\forall E \subseteq \mathcal{M}$ and $\forall \varepsilon > 0$, $\exists K$ closed such that
$$\mu(K) \leq \mu(E) \leq \mu(K) + \varepsilon$$

Proof:

1. By definition of $\mu^*(E)$, $\exists O = \bigcup_{n=1}^{\infty} I_n \supset E$ such that
$$\mu^*(E) \geq \sum_{n=1}^{\infty} \rho(I_n) - \varepsilon$$
and by subadditivity, $\mu^*(E) \geq \mu^*(O) - \varepsilon$.

2. Assume $E \subseteq [a, b]$. Consider $E^c \cap [a, b]$. By part 1, $\exists O \supset E^c \cap [a, b]$ such that
$$\mu^*(E^c \cap [a, b]) \geq \mu^*(O) - \varepsilon$$
and with some algebra,
$$|b - a| - \mu^*(E^c) \leq |b - a| - \mu^*(O) + \varepsilon$$

By measurability,
$$\begin{aligned} \mu(E) &\leq |b - a| - \mu^*(O \cap [a, b]) - \mu^*(O \cap [a, b]^c) + \varepsilon \\ &\leq |b - a| - \mu^*(O \cap [a, b]) \\ &= \mu^*([a, b] \cap O^c) + \varepsilon \end{aligned}$$

Exercise: Complete the case $E \not\subseteq [a, b]$

Chapter 3

Measurable functions

3.1 Oct 03

Measurable function: Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f : X \to \mathbb{R}$. f is *measurable* iff $\forall \alpha \in \mathbb{R}$,

$$\{x \in X, f(x) > \alpha\} \in \mathcal{M}$$

Equivalently, f is measurable iff $\forall \alpha \in \mathbb{R}$, $\{f \geq \alpha\}$ is measurable.

Proposition: If f, g are measurable, so is

(a) $f + c$

(b) $f + g$

(c) cf

(d) fg

for $c \in \mathbb{R} \setminus \{0\}$

Proof: 1a. For all $\alpha \in \mathbb{R}$,

$$\{x : f(x) + c > \alpha\} = \{x : f(x) > \alpha - c\} \in \mathcal{M}$$

‘

1b. Consider $\{x : f(x) + g(x) > \alpha\}$. We claim

$$\{x : f(x) + g(x) > \alpha\} = \bigcup_{r \in \mathbb{Q}} \{f(x) > r\} \cap \{g(x) > x - r\}$$

Proof: Certainly,

$$\{x : f(x) + c > \alpha\} = \{x : g(x) > \alpha - f(x)\}$$

and for fixed x, we can invoke the density and countability of the rational numbers...

Proposition: If $\{f_n\}$ are measurable, so are

(a) $\sup_n f_n$

(b) $\inf_n f_n$

(c) $\limsup_n f_n$

(d) $\liminf_n f_n$

Proof: ...