APMA 2110 - Homework 10

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27 Nov 2024

1. Assume the validity of the change of variables formula for the Riemann integral. Let f be Riemann integrable in $\mathbb{R}^n$ and show that f is Lebesgue integrable.

Use this fact to give an alternative proof of the change of variables formula for $f \in \mathcal{L}^1(\mathbb{R}^n)$. In particular, prove the integration in polar coordinates for $f \in \mathcal{L}^1(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} f = \int_0^\infty \int_{S^{n-1}} f(rx')r^{n-1}\, d\sigma(x')\, dr$$

where $\sigma = \sigma_{n-1}$ is the surface measure on the unit sphere S^{n-1}.

We present an alternative proof of the change of variables formula for $f \in \mathcal{L}^1(\mathbb{R}^n)$.

STEP 1. Assume that for $\phi : E \to \mathbb{R}^n$ a C^1 diffeomorphism and $f : \phi(E) \to \mathbb{R}^n$ Riemann integrable,

$$\int_{\phi(E)} f(y)\, dy = \int_E (f \circ \phi)(x)\, |\det D_x\phi|\, dx$$

STEP 2. For smooth functions in $\mathbb{R}^n$, the Riemann integral is equivalent to the Lebesgue integral.

Proof: Suppose f is a smooth function in $\mathbb{R}^n$.

By definition, f is continuous. Trivially, $\{x : f(x) \text{ not continuous}\}$ has measure zero, so by a Theorem from class, f is Riemann integrable.

Note that we showed in class that the Lebesgue and Riemann integrals are equivalent in $\mathbb{R}$. We can extend this to $\mathbb{R}^n$ by considering the product measure $m^n = m \times \cdots \times m$ on $\mathbb{R}^n$ and applying Fubini:

$$\begin{aligned}\int_{\mathbb{R}^n} f\, dm^n &= \int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f\, dm(x_1), dm(x_2) \dots dm(x_n) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f\, dx_1 dx_2 \dots dx_n\end{aligned}$$

STEP 3. By the approximation of $\mathcal{L}^1(m)$ functions (shown in class), there exists a smooth function g such that $||f - g||_{\mathcal{L}^1} < \varepsilon$.

Hence, up to order ε, the change of variables formula holds for $f \in \mathcal{L}^1(\mathbb{R}^n)$.

Denote the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Then for $x \in \mathbb{R}^n \setminus \{0\}$, the polar coordinates of x are given by:

$$\begin{cases} r = |x| \in (0, \infty) \\ x' = \frac{x}{|x|} \in S^{n-1} \end{cases}$$

Consider the map $\Phi : (0, \infty) \times S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ given by $\Phi(r, x') = rx'$.

By the change of variables formula above,

$$
\begin{aligned}
\int_{\mathbb{R}^n} f &= \int_{(0,\infty)\times S^{n-1}} f(\Phi(r,x'))\,|\det D\Phi(r,x')|\;d(\sigma\times r) \\
&= \int_0^\infty \int_{S^{n-1}} f(rx')r^{n-1}\,d\sigma(x')\,dr \qquad \text{(Tonelli)}
\end{aligned}
$$

2. Let ν be a signed measure on $(X, \mathcal{M})$. Prove:

- If E_j is an increasing sequence of sets in $\mathcal{M}$, $\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \nu(E_j)$.
- If E_j is a decreasing sequence of sets in $\mathcal{M}$ and $\nu(E_1) < \infty$, then $\nu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \nu(E_j)$.

We would like to rewrite $\bigcup_j E_j$ as a union of disjoint sets to take advantage of countable additivity.

WLOG define $E_0 = \emptyset$. Then, because $E_n \subseteq E_{n+1}$,

$$\begin{aligned}
E_1 &= E_1 \setminus \emptyset \\
E_2 &= E_1 \cup (E_2 \setminus E_1) \\
E_3 &= E_2 \cup (E_3 \setminus E_2) \\
&= (E_1 \setminus \emptyset) \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2)
\end{aligned}$$

Hence, inductively define

$$E_n = \bigcup_{k=0}^{n-1} E_{k+1} \setminus E_k$$

so

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=0}^{\infty} (E_n \setminus E_{n-1})$$

Then, by countable additivity,

$$\begin{aligned}
\nu\left(\bigcup_{n=1}^{\infty} E_n\right) &= \nu\left(\bigcup_{n=0}^{\infty} (E_n \setminus E_{n-1})\right) \\
&= \sum_{n=0}^{\infty} \nu(E_n \setminus E_{n-1}) \\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \nu(E_n \setminus E_{n-1}) \\
&= \lim_{n\to\infty} \nu(E_n)
\end{aligned}$$

Let $E_1 \supset E_2 \supset \dots$ and $\nu(E_1) < \infty$. Define $F_j = E_1 \setminus E_j$. Clearly, $F_n \subseteq F_{n+1}$.

By the previous part,

$$\nu\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \nu(F_n)$$

$$F_j = E_1 \setminus E_j \implies \bigcup_{j=1}^{n} F_j = E_1 \setminus \bigcup_{j=1}^{n} E_j$$

$$\nu\left(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \nu(E_1 \setminus \bigcap_{j=1}^{n} E_j)$$

In a signed measure, we do not have monotonicity but we do have that for $E \subseteq F$,

$$\nu(F) = \nu(E \cup F \setminus E) = \nu(E) + \nu(F \setminus E)$$

Proof: Consider $F \setminus E = F \cap E^c$. But $E \cap (F \cap E^c) = \emptyset$ so

$$\mu(F) = \mu(E \cup F \setminus E) = \mu(E) + \mu(F \setminus E)$$

Because $\nu(E_1) < \infty$,

$$\begin{aligned} \nu(E_1) - \nu\left(\bigcap_{n=1}^{\infty} E_n\right) &= \lim_{n\to\infty}\left[\nu(E_1) - \nu\left(\bigcap_{j=1}^{n} E_j\right)\right] \\ &= \lim_{n\to\infty}\left[\nu(E_1) - \nu(E_n)\right] \\ &= \nu(E_1) - \lim_{n\to\infty} \nu(E_n) \\ \nu\left(\bigcap_{n=1}^{\infty} E_n\right) &= \lim_{n\to\infty} \nu(E_n) \quad \blacksquare \end{aligned}$$

3. If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, show $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

ν is a signed measure, so by Hahn decomposition, $\exists P, N$ unique with P positive and N negative such that

$$P \cup N = X, \ P \cap N = \emptyset$$

Define

$$\nu^+(A) = \nu(A \cap P), \ \nu^-(A) = -\nu(A \cap N)$$

Then, $\nu = \nu^+ - \nu^-$.

First consider λ. Because λ, μ are positive measures, we have $\lambda \geq \lambda - \mu = \nu$.

In particular, $\forall A \in \mathcal{M}$,

$$\lambda(A) \overset{*}{\geq} \lambda(A \cap P) \geq \nu(A \cap P) = \nu^+(A)$$

where $\overset{*}{\geq}$ follows from the fact that λ is a positive measure (and hence monotonic).

Now consider μ. We have $-\nu = \mu - \lambda$ so in particular, $\mu \geq -\nu$. Hence, for any $A \in \mathcal{M}$,

$$\mu(A) \geq \mu(A \cap N) \geq -\nu(A \cap N) = \nu^-(A)$$

again by monotonicity of μ. $\blacksquare$

4. Suppose that ν is a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$. Prove

1.
$$\nu^+(E) = \sup\{\nu(F) : F \subseteq E, F \in \mathcal{M}\}$$
$$\nu^-(E) = -\inf\{\nu(F) : F \subseteq E, F \in \mathcal{M}\}$$

2.
$$|\nu|(E) = \sup\left\{\sum_{j=1}^{n} |\nu(E_j)| \;\middle|\; n \in \mathbb{N},\ E_1 \ldots E_n \text{ disjoint}, \bigcup_{1}^{n} E_j = E\right\}$$

Let $X = P \cup N$ be the Hahn decomposition of X and

$$\nu^+(E) = \nu(E \cap P)$$
$$\nu^-(E) = -\nu(E \cap N)$$

Define $\overline{F} = \arg\sup\{\nu(F) : F \subseteq E,\ F \in \mathcal{M}\}$.

STEP 1. $\overline{F}$ is a positive set and $E \setminus \overline{F}$ is a negative set.

Proof: Let $\overline{F} = A \cup B$ where A is a positive set and B is a negative set such that $A \cap B = \emptyset$ (guaranteed by Hahn).

Suppose, for contradiction, that $\overline{F}$ is not positive, i.e. $B \neq \emptyset$

We know $\nu(\overline{F}) = \nu(\overline{F} \setminus B) + \nu(B)$ (countable additivity), so in particular, $\nu(\overline{F} \setminus B) \geq \nu(\overline{F})$ (as $\nu(B) < 0$). But this contradicts the maximality of $\overline{F}$.

Hence, $\overline{F}$ is a positive set.

Similarly, suppose $E \setminus \overline{F}$ is not negative, i.e. it contains some positive set C. But then, C and $\overline{F}$ are disjoint so by countable additivity,
$$\nu(\overline{F} \cup C) = \nu(\overline{F}) + \nu(C) \geq \nu(\overline{F})$$
which again contradicts the maximality of $\overline{F}$.

STEP 2. $\nu(\overline{F}) = \nu(E \cap P) = \nu^+(E)$.

Proof: In fact, we have the strictly stronger claim that $\overline{F} = E \cap P$: clearly, $\overline{F} \subseteq E$ and further, $\overline{F}$ is positive (by Step 1), so $\overline{F} \subseteq E \cap P$.

Then, suppose $\exists D \in (E \cap P) \setminus \overline{F}$. But then D is a positive set in E which is disjoint from $\overline{F}$ so $\nu(D \cup \overline{F}) = \nu(D) + \nu(\overline{F}) > \nu(\overline{F})$, contradicting the maximality of $\overline{F}$. Hence, $E \cap P \subseteq \overline{F}$

Certainly, $\overline{F} = E \cap P \implies \nu(\overline{F}) = \nu(E \cap P) = \nu^+(E)$, by definition.

ν^- follows by similar argument on $\underline{\mathrm{F}} = \arg\inf\{\nu(F) : F \subseteq E,\ F \in \mathcal{M}\}$. ■

Once again, let $X = P \cup N$ be the Hahn decomposition of X and let $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$.

Notice that $E = (E \cap P) \cup (E \cap N)$ and $E \cap P, E \cap N$ are disjoint, so, by definition of the supremum,

$$\begin{aligned}
\sup \left\{ \sum_{j=1}^{n} |\nu(E_j)| \;\middle|\; n \in \mathbb{N},\ E_1 \ldots E_n \text{ disjoint}, \bigcup_{1}^{n} E_j = E \right\} &\geq |\nu(E \cap P)| + |\nu(E \cap N)| \\
&= \nu^+(E) + \left|-\nu^-(E)\right| \\
&= \nu^+(E) + \nu^-(E) \\
&= |\nu|(E)
\end{aligned}$$

Conversely, let $E = \bigcup_{j=1}^{n} E_j$ for E_j disjoint. Then

$$\begin{aligned}
|\nu|(E) &= |\nu| \left(\bigcup_{j=1}^{n} E_j \right) \\
&= \sum_{j=1}^{n} |\nu|(E_j) \qquad \text{(by countable additivity)} \\
&= \sum_{j=1}^{n} \nu^+(E_j) + \nu^-(E_j) \\
&\geq \sum_{j=1}^{n} \left| \nu^+(E_j) - \nu^-(E_j) \right| \\
&= \sum_{j=1}^{n} |\nu(E_j)|
\end{aligned}$$

Taking the supremum over all such decompositions, we have the desired result. ■