APMA 2110: Homework 1

Milan Capoor

1. Prove $(a) \implies (b) \implies (c) \implies (d) \implies (e) \implies (a)$:

   (a) (Least Upper Bound) Any nonempty subsets E of $\mathbb{R}$ with an upper bound has a least upper bound.

   (b) (Monotone Convergence Theorem) Any bounded monotone sequence is convergent.

   (c) (Bolzano-Weierstrass) Every bounded sequence in R has a convergent subsequence.

   (d) (Heine-Borel) Let F be a closed and bounded set of real numbers. Then each open covering of F has a finite subcovering.

   (e) (Finite Intersection Property) Let $\mathcal{C}$ be the collection of closed sets F (of real numbers) with the property that every finite subcollection of $\mathcal{C}$ has nonempty intersection, and suppose that one of the sets is bounded. Then $\bigcap_{F\in\mathcal{C}} F \neq \emptyset$

$(a \to b)$ WLOG let $a_n \in \mathbb{R}$ be a bounded monotone increasing sequence. Let $A = \{a_n : n \in \mathbb{N}\}$. By (a), $\sup A$ exists. We claim $\lim a_n = \sup A$. Let $\varepsilon > 0$. By the definition of $\sup A$, $\sup A - \varepsilon$ is not an upper bound of A. Thus, there exists $N \in \mathbb{N}$ such that $a_N > \sup A - \varepsilon$. Since a_n is monotone increasing, for all $n \geq N$, $a_n \geq a_N > \sup A - \varepsilon$. Further, $a_n \leq \sup A + \varepsilon$ by boundedness. Thus, $|a_n - \sup A| < \varepsilon$ for all $n \geq N$. Hence, $\lim a_n = \sup A$.

$(b \to c)$ Let a_n be a bounded sequence in $\mathbb{R}$. By (b), it suffices to show that every bounded sequence contains a bounded monotone subsequence. Boundedness of the subsequence is trivial. For monotonicity we proceed by cases:

1. If there exist infinitely many points $a_{n_k} \geq a_{n_i}$ for $n_k \leq n_i$, then a_{n_k} is a monotone decreasing subsequence and we are done.

2. If there exist finitely many points a_{n_k} such that $a_{n_k} \geq a_{n_i}$ for all $n_k \leq n_i$, then we may define $N = \max\{n_k : a_{n_k} \leq a_{n_i} \forall n_k \leq n_i\}$. Then for all a_{n_i} with $n_i > N$, there exists n_{i+1} such that $a_{n_{i+1}} > a_{n_i}$. Thus, a_{n_i} is a monotone increasing subsequence.

Thus, every bounded sequence has a bounded monotone subsequence. By (b), this subsequence converges.

($c \to d$) Let F be a closed and bounded set of real numbers. Let $\mathcal{O}$ be an open cover for F¿ If F is finite, then there trivially exist a finite subcover by selecting the $\{O_x \in \mathcal{O} : x \in F\}$ where O_x is an open set containing x.

Suppose F is infinite. By (c) and boundedness of F, every sequence in F has a convergent subsequence.

We claim that for all $\varepsilon > 0$, there exists a finite set $\{x_1, \dots, x_n\} \subseteq F$ such that

$$F \subseteq \bigcup_{i=1}^{n} B_\varepsilon(x_i)$$

where $B_\varepsilon(x_i)$ is the open ball of radius ε centered at x_i. We shall call $F(\varepsilon) = \{B_\varepsilon(x_1), \dots, B_\varepsilon(x_n)\}$.

If this claim is not true then there exists an $\varepsilon > 0$ for which $F(\varepsilon)$ does not exist. Then by induction, we can choose a sequence (x_n) such that $|x_i - x_j| \geq \varepsilon$ for $i \neq j$ (if we could not choose such a sequence, all of F would be contained in a finite union of open balls). But this sequence can have no convergent subsequence, a contradiction.

Now suppose that $\mathcal{O}$ does not admit a finite subcovering of F. By the above argument, for $n \geq 1$, there exists a finite set $F(1/n)$. Since $\mathcal{O}$ does not admit a finite subcover, at least one element $B_{1/n}(x_i)$ from $F(1/n)$ which cannot be covered by finitely many open sets in $\mathcal{O}$.

This gives us an infinite set $E = \{x \in F : x \in B_{1/n}(x_i)\}$ which is bounded because F is bounded. As E is infinite, we can select a sequence $(x_n) \in E$. By (c), (x_n) has a convergent subsequence $(x_{n_k}) \to x$. Since F is closed, $x \in F$ which implies $x \in O$ for some $O \in \mathcal{O}$. But O is open so $\forall \varepsilon > 0$, $B_\varepsilon(x) \subseteq O$.

Since $x_{n_k} \to x$, $\exists N \in \mathbb{N}$ such that for all $n_k \geq N$, $x_{n_k} \in B_{\varepsilon/2}(x)$. Choose an integer $n_m > N$ with $1/n_m < \varepsilon/2$. By construction,

$$B_{1/n_m}(x_{n_m}) \subseteq B_\varepsilon(x) \subseteq O$$

which contradicts the fact the $B_{1/n_m}(x_{n_m})$ cannot be covered by finitely many open sets in $\mathcal{O}$. Thus, $\mathcal{O}$ admits a finite subcovering of F

$(d \to e)$ Let X be a closed and bounded set of real numbers. Let $\mathcal{C}$ be the collection of closed sets F in X such that for $F_1, \dots, F_n \in \mathcal{C}$, $\bigcap_{i=1}^{n} F_i \neq \emptyset$. Assume one of the closed sets is bounded.

Suppose $\bigcap_{F \in \mathcal{C}} F = \emptyset$.

Lemma 1 (De Morgan's Laws):

$$\left(\bigcup_{E \in \mathcal{E}} E\right)^c = \bigcap_{E \in \mathcal{E}} E^c, \quad \left(\bigcap_{E \in \mathcal{E}} E\right)^c = \bigcup_{E \in \mathcal{E}} E^c$$

Proof:

1.
$$\begin{aligned} x \in \left(\bigcup_{E \in \mathcal{E}} E\right)^c &\implies x \notin \bigcup_{E \in \mathcal{E}} E \implies \forall E \in \mathcal{E}, x \notin E \implies \forall E \in \mathcal{E}, x \in E^c \\ &\implies x \in \bigcap_{E \in \mathcal{E}} E^c \end{aligned}$$

2.
$$\begin{aligned} x \in \left(\bigcap_{E \in \mathcal{E}} E\right)^c &\implies x \notin \bigcap_{E \in \mathcal{E}} E \implies \exists E \in \mathcal{E} \text{ s.t. } x \notin E \\ &\implies \exists E \in \mathcal{E} \text{ s.t. } x \in E^c \implies x \in \bigcup_{E \in \mathcal{E}} E^c \end{aligned}$$

Lemma 2 (Open and closed complements): O open iff O^c closed; F closed iff F^c open.

Proof:

1. Let $O \subseteq \mathbb{R}$ be open. If x is a limit point of O^c, then $\forall \varepsilon > 0$, $\exists y \in O^c$ with $y \in B_\varepsilon(x)$. But then $B_\varepsilon(x) \not\subset O$ so x is not a limit point of O.

Thus, O^c contains all its limit points so O^c is closed.

2. $(E^c)^c = E$ so the result follows.

Lemma 3: The intersection of an arbitrary collection of closed sets is closed. The union of an arbitrary collection of open sets is open.:

Proof: Follows from Lemmas 1 and 2.

Then

$$\left(\bigcap_{F\in\mathcal{C}} F\right)^c = \bigcup_{F\in\mathcal{C}} F^c = X$$

by Lemma 1.

By Lemma 2, F^c is open for all $F \in \mathcal{C}$ closed. Then by Lemma 3, $\bigcup_{F\in\mathcal{C}} F^c$ is an open cover for X. By (d), there exists a finite subcover $\{F_1^c, \ldots, F_n^c\}$.

But then

$$X \subseteq \bigcup_{i=1}^{n} F_i^c \implies \bigcap_{i=1}^{n} F_i = \emptyset$$

again by Lemma 1. But this contradicts the assumption that all finite intersections of $F \in \mathcal{C}$ are nonempty. Thus, $\bigcap_{F\in\mathcal{C}} F \neq \emptyset$. ■

$(e \to a)$ Let A be an arbitrary non-empty subset of $\mathbb{R}$ with an upper bound, b_1. Pick an $a_1 < b_1 \in A$ and define $I_1 = [a_1, b_1]$.

Consider

$$m_1 = \frac{a_1 + b_1}{2}$$

If m_1 is an upper bound for A, let $b_2 = m_1$ and $a_2 = a_1$. Otherwise, let $a_2 = m_1$ and $b_2 = b_1$. Define $I_2 = [a_2, b_2]$.

Now we iterate. For any n, let $m_n = \frac{a_n+b_n}{2}$. Define

$$\begin{cases} b_{n+1} = m_n,\ a_{n+1} = a_n & \text{if } m_n \text{ is an upper bound for } A \\ a_{n+1} = m_n,\ b_{n+1} = b_n & \text{otherwise} \end{cases}$$

and $I_{n+1} = [a_{n+1}, b_{n+1}]$.

This gives us nested closed sets $I_1 \subseteq I_2 \subseteq \cdots$. Because they are nested, any finite intersection is nonempty. By (e), $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Let $b \in \bigcap_{n=1}^{\infty} I_n$. We claim that $b = \sup A$.

First suppose b is not an upper bound for A. Then $\exists a \in A$ such that $a > b$. Let $\varepsilon_0 = a - b > 0$. By construction, $a \leq b_n$ for all n because b_n is a sequence of upper bounds for A. But then for any $N > \frac{1}{\varepsilon_0}$,

$$b_N - b \geq a - b = \varepsilon_0$$

But this is impossible because $b \in I_N = [a_N, b_N]$ which has length

$$\frac{1}{2^N} < \frac{1}{N} < \varepsilon_0$$

Thus, b is an upper bound for A. Now it remains to show that it is the least upper bound. Let $\varepsilon > 0$. Let $N > \frac{1}{\varepsilon}$.

Again no a_n is an upper bound for A so $\exists a \in A$ such that

$$a_N < a \leq b_N$$

and $a, b \in [a_N, b_N]$ which has length $1/2^N$ so

$$|b - a| \leq \frac{1}{2^N} < \frac{1}{N} < \varepsilon$$

so $b - a < \varepsilon$ (because $b - a > 0$ by construction). Thus, $b - \varepsilon < a$ for arbitrary $\varepsilon > 0$ so b is the least upper bound. ∎

2. Prove that (c) is equivalent to: any Cauchy sequence in $\mathbb{R}$ is convergent.

Assume that every bounded sequence in $\mathbb{R}$ has a convergent subsequence. Let (a_n) be a Cauchy sequence and choose $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|a_n - a_m| < \varepsilon$

WLOG, let $m = N$ so

$$\begin{aligned} |a_n - a_N| < \varepsilon &\implies |a_n| - |a_N| < |a_n - a_N| < \varepsilon \\ &\implies |a_n| - |a_N| < \varepsilon \\ &\implies |a_n| < 1 + |a_N| \end{aligned}$$

Thus

$$|a_n| < \max\{|a_0|, |a_1|, \cdots, |a_N| + 1\}$$

so (a_n) is bounded. By (c), (a_n) has a convergent subsequence $(a_{n_k}) \to a$.

Therefore, for large enough N and $n_k, n \geq N$, we can say

$$|a_n - a_{n_k}| < \frac{\varepsilon}{2}$$

by Cauchy and

$$|a_{n_k} - a| < \frac{\varepsilon}{2}$$

by convergent subsequence.

So,

$$|a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \varepsilon$$

Let (a_n) be a bounded sequence. Assume that every Cauchy sequence in $\mathbb{R}$ is convergent.

If (a_n) is Cauchy, then it is convergent. All convergent sequences trivially contain a convergent subsequence (a_{n_k}) given by $n_k = \mathbb{N}$.

If (a_n) is not Cauchy (but bounded), then it still contains a bounded monotone subsequence (a_{n_k}) by the proof for $(1.b \to c)$. By the Monotone Convergence Theorem, (a_{n_k}) is convergent. $\blacksquare$

3. Let $\limsup a_n = a$; where a is a finite number in $\mathbb{R}$: Then

   1. for any $\varepsilon > 0$; there are all but finitely many n such that $a_n < a + \varepsilon$;

      We will argue by contrapositive. Suppose that $a_n \geq a + \varepsilon$ for infinitely many $n \in \mathbb{N}$.

      Then certainly $\sup\{a_k : k \geq n\} \geq a + \varepsilon$ for all $n \in \mathbb{N}$ because

      $$\sup\{a_k : k \geq n\} \geq a_n \geq a + \varepsilon$$

      So

      $$\limsup a_n \geq \lim(a + \varepsilon) \implies \limsup a_n > a$$

      so $\limsup a_n \neq a$, as desired.

   2. there are infinitely many a_n such that $a_n > a - \varepsilon$

      Again by contrapositive, suppose tha there is an N such that for all $n > T$, $a_n \leq a - \varepsilon$. Then there would be finitely many a_n such that $a_n > a - \varepsilon$.

      But for all n, $\sup\{a_k : k \geq n\} \geq a_n$ so for $n > N$,

      $$a_n \leq \sup\{a_k : k \geq n\} \leq a - \varepsilon \implies \limsup a_n \leq \lim(a - \varepsilon)$$

      so $\limsup a_n < a$ as desired. ■

4. Prove that given any real number x; there is an integer n such that $x < n$ (Axiom of Archimedes). Furthermore, for any $x < y$; there is a rational number $q \in \mathbb{Q}$ such that $x < q < y$.

Suppose $x \geq n$ for all $n \in \mathbb{Z}$. Then x is an upper bound of $\mathbb{Z}$ so by the Axiom of Completeness, $\mathbb{Z}$ has a least upper bound Z. Since $Z - 1 < Z$, $Z - 1$ is not an upper bound of $\mathbb{Z}$. Thus, there exists $n \in \mathbb{Z}$ such that $n > Z - 1$. But then $Z < n + 1$ (an integer) which is a contradiction. Therefore, $\mathbb{N}$ is not bounded above.

We want to show that $\exists m \in \mathbb{Z}$, $n \in \mathbb{N}$ such that $x < \frac{m}{n} < y$.

By the Archimedian property, $\exists m \in \mathbb{Z} \forall n \in \mathbb{N}, x \in \mathbb{R}$ such that

$$nx < m$$

Then we can bound nx below by

$$m - 1 < nx < m$$

The RHS inequality gives $x < \frac{m}{n}$ as desired. So we need to show that $\frac{m}{n} < y$.

By the Archimedian property again, we can pick $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x \implies x < y - \frac{1}{n}$

The LHS inequality above then gives

$$m \leq nx + 1 < n(y - \frac{1}{n}) + 1 = ny \implies m < ny \implies \frac{m}{n} < y$$

5. If $\{f_n\}$ is a sequence of continuous functions in $\mathbb{R}$ and $\{f_n\}$ converges to f uniformly, then f is continuous.

Let $\varepsilon > 0$. Since $f_n \to f$ uniformly, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in \mathbb{R}$.

Choose $c \in \mathbb{R}$. Since f_N is continuous, $\exists \delta > 0$ such that $|x - c| < \delta \implies |f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$.

Then for all $x, c \in \mathbb{R}$ such that $|x - c| < \delta$,

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

Thus, f is continuous. $\blacksquare$

6. If f is a continuous function over a bounded closed set in $\mathbb{R}$ then f attains a maximum and minimum on the set.

Let K be a closed and bounded set in $\mathbb{R}$. By the Heine-Borel theorem, K is compact.

Lemma: If $f : A \to \mathbb{R}$ is continuous on A, $f(K)$ is compact for compact $K \subseteq A$

Proof: Let $(y_n) \in f(K)$. Then $\forall n \in \mathbb{N}, \exists (x_n) \in K$ s.t. $f(x_n) = y_n$. Since K is compact, $\exists x_{n_k} \to x \in K$. Since f is continuous, $f(x_{n_k}) \to f(x)$. Thus,

$$f(x) = \lim f(x_{n_k}) = \lim y_{n_k} \in f(K)$$

Then by the Lemma, $f(K)$ is compact so $\exists \alpha = \sup f(K)$ and we know $\alpha \in f(K)$ (closed). Therefore, $\exists x \in K$ such that $f(x) = \alpha$. Minimum follows by similar argument. ■