Definitions

$$\begin{aligned}\limsup E_n &= \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} E_k \\ &= \{x : x \in E_n \text{ for infinitely many } n\} \\ &= -\liminf\{-E_n\} \\ \liminf E_n &= \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} E_n \\ &= \{x : x \in E_n \text{ for all but finitely many } n\} \\ &= -\limsup\{-E_n\}\end{aligned}$$

De Morgan's Laws:

$$\left(\bigcup_{\alpha\in A} E_\alpha\right)^c = \bigcap_{\alpha\in A} E_\alpha^c$$
$$\left(\bigcap_{\alpha\in A} E_\alpha\right)^c = \bigcup_{\alpha\in A} E_\alpha^c$$

Equivalence relation:

- Reflexive: $x \sim x$
- Symmetric: $x \sim y \iff y \sim x$
- Transitive: $x \sim y \wedge y \sim z \implies x \sim z$

Useful inverse properties:

$$f^{-1}\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} f^{-1}(E_\alpha)$$
$$f^{-1}\left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A} f^{-1}(E_\alpha)$$
$$f^{-1}(E^c) = (f^{-1}(E))^c$$

Partial Ordering:

- X nonempty
- Reflexivity: xRx
- Antisymmetry: $xRy \wedge yRx \implies x = y$
- Transitivity: $xRy \wedge yRz \implies xRz$

Linear ordering:

- Partial ordering
- $\forall x, y$, either xRy or yRx

Well-ordering:

- Linearly ordered
- Every nonempty subset has a least element

Choice function:

$$\prod_{\alpha\in A} X_\alpha = \left\{ f : A \to \bigcup_{\alpha\in A} X_\alpha \wedge f(\alpha \in X_\alpha)\,\forall\alpha\in A \right\}$$

Metric space: $\rho : X \times X \to [0, \infty]$

1. $\rho(x, y) = 0 \iff x = y$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

Open and Closed Sets:

- E open $\iff \forall x \in E,\ \exists\varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq E$
- E closed $\iff E^c$ open
- $\{U_x\}$ open $\implies \bigcup_{x\in A} U_x$ open
- $\{F_x\}$ closed $\implies \bigcap_{x\in A} F_x$ closed

Density:

- $E \subseteq X$ dense $\iff \overline{E} = X$
- E nowhere dense $\iff (\overline{E})^\circ = \emptyset$
- X separable $\iff \exists E \subseteq X$ countable and dense

Complete: $E \subseteq X$ complete $\iff \forall\{x_n\} \in E, x_n \to x \in E$

Distance:

- $\rho(x, E) = \inf\{\rho(x, y) : y \in E\}$
- $\rho(E, F) = \inf\{\rho(x, y) : x \in E, y \in F\}$
- diam $E = \sup\{\rho(x, y) : x, y \in E\}$
- E bounded $\iff$ diam $E < \infty$
- E totally bounded $\iff \forall\varepsilon > 0,\ E \subseteq \bigcup_{i=1}^N B_\varepsilon(x_i)$

Algebra: $\mathcal{A} \subseteq P(X)$ such that for $E_1, \ldots, E_n \subseteq \mathcal{A}$,

1. $\bigcup_{i=1}^n E_i \in \mathcal{A}$
2. $E \in \mathcal{A} \implies E^c \in \mathcal{A}$

Sigma Algebra:

- Algebra
- $\bigcup_{I=1}^\infty E_i \in \mathcal{A}$ for $\{E_i\} \in \mathcal{A}$
- $E \in \mathcal{A} \implies E^c \in \mathcal{A}$

Equivalently, $\mathcal{A}$ is a σ-algebra if for $\{E_i\} \in \mathcal{A}$, $E_1 \subseteq E_2 \subseteq \cdots \implies \bigcup_{j=1}^\infty E_j \in \mathcal{A}$.

Borel Algebra: the σ-algebra generated by the open sets of X

Measure: $\mu : \mathcal{M} \to [0, \infty]$ satisfies

1. $\mu(\emptyset) = 0$
2. $\{E_j\} \in \mathcal{M}$ pairwise disjoint, $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_k)$

Outer Measure:

1. $\mu^*(\emptyset) = 0$
2. $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$
3. $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$

Carathéodory Criterion: If $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$, A is μ^*-measurable

Lebesgue-Stieltjes Measure:

$$\mu^*(A) = \inf \left\{ \sum_n \rho(a_i, b_i) \middle| A \subseteq \bigcup_i (a_i, b_i) \right\}$$

for

- $\rho(a, b) = b - a$ on $\mathbb{R}$
- $\rho(\mathbf{a}, \mathbf{b}) = \prod_{j=1}^{n} (b_j - a_j)$ on $\mathbb{R}^n$

Hausdorff Measure:

$$\mathcal{H}_\alpha = \inf \left\{ \lim_{\varepsilon \to 0} \sum_k \text{diam } (A_k)^\alpha \middle| A \subseteq \bigcup_k A_k, \text{ diam } A_k \leq \varepsilon \right\}$$

Lebesgue-Stieltjes Measure:

$$\mu^*(A) = \inf \left\{ \sum_n \rho(a_i, b_i] \middle| A \subseteq \bigcup_i (a_i, b_i] \right\}$$

for $\rho(a, b] = f(b) - f(a)$ and f monotone increasing. If f is further right-continuous, $\mu^*(a, b] = \rho(a, b]$.

Useful Results

Archimedean Property: $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{Z}$ s.t. $x < n$

Zorn's Lemma: If X is partially ordered and every linearly ordered subset of X has an upper bound, X has a maximal element

Well-ordering principle: Every non-empty set can be well-ordered

Axiom of Choice: If $\{X_{\alpha \in A}\} \neq \emptyset$, $\prod_{\alpha \in A} X_\alpha \neq \emptyset$

Cardinality Results:

- card $X \leq$ card $Y \iff$ card $Y \geq$ card X
- Either card $X \leq$ card Y or card $Y \leq$ card X
- card $X <$ card $P(X)$

Schröder-Berstein Theorem: If $f : X \hookrightarrow Y$ and $g : Y \hookrightarrow X$, then $\exists h : X \hookrightarrow\!\!\!\rightarrow Y$

Countability Results:

- X countable $\iff$ card $X \leq$ card $\mathbb{N} \iff \mu(X) = 0$
- X, Y countable $\implies X \times Y$ countable
- $A, \{X_\alpha\}_{\alpha \in A}$ countable $\implies \bigcup_{\alpha \in A} X_\alpha X_\alpha$ countable
- Every open set in $\mathbb{R}$ is a countable disjoint union of open intervals

Metric space results:

- $E^\circ = \bigcup_{O \subseteq E} O \subseteq E$
- $E \subseteq \overline{E} = \bigcap_{F \supseteq E} F$
- $\lim_{n \to \infty} \rho(\mathbf{x}, \mathbf{y}) = 0 \iff \lim_{n \to \infty} \rho(x_i^n, y_i^n) = 0$

Characterization of Closure: For all $x \in X$, $E \subseteq X$,

$\iff$ E closed

$\iff$ $x \in \overline{E}$

$\iff$ $B_\varepsilon(x) \cap E \neq \emptyset \quad \forall \varepsilon > 0$

$\iff$ $\exists \{x_n\} \subseteq E$ s.t. $x_n \to x$

Continuity results:

- $\{f_n\} \in \mathbb{R}$ continuous and $f_n \to f$ uniformly $\implies$ f continuous
- f continuous on bounded set $\implies$ f attains a max/min on the set

Ascoli-Arzela: With (X, ρ) bounded and separable, $\{f_n\} \in \mathcal{F}$ equicontinuous (i.e. $\forall x \in X \exists O_x > 0$ s.t. $\sigma(f_n(x), f_n(y)) < \varepsilon \; \forall y \in O_x$), and the closure of $\{f_n(x) : 0 \leq n < \infty\}$ compact for all $x \in X$, $\exists f_{n_k} \to f$ pointwise for f continuous.

Characterization of compactness:

$\iff$ E compact

$\iff$ E complete and totally bounded

$\iff$ every sequence has convergent subsequence with limit in E

$\iff$ every open cover has finite subcover

Generated σ-algebras: $\mathcal{E} \subseteq M(\mathcal{F}) = \bigcap_{\mathcal{F} \subseteq \mathcal{A}} \mathcal{A} \implies M(\mathcal{E}) \subseteq M(\mathcal{F})$

Generating Borel sets: If $\{X_i\}$ are separable,

$$\bigoplus_{i=1}^{n} \mathcal{B}_{X_i} = \mathcal{B}_{X_1} \times \mathcal{B}_{X_2} \times \cdots \times \mathcal{B}_{X_n} = \mathcal{B}_{X_1 \times X_2 \times \cdots \times X_n}$$

Properties of Measures:

1. Monotonicity: $E \subseteq F \in \mathcal{M} \implies \mu(E) \leq \mu(F)$
2. Subadditivity: $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$
3. Continuity from below: $E_1 \subseteq E_2 \subseteq \cdots \implies \mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$
4. Continuity from above: $E_1 \supseteq E_2 \supseteq \cdots \wedge \mu(E_1) < \infty \implies \mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$
5. $E, F \in \mathcal{M} \implies \mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$
6. $\mu(\liminf E_j) \leq \liminf \mu(E_k)$
7. If $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) < \infty$, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$

Property of Outer measure: If $\sum_{k=1}^{\infty} \mu^*(E_k) < \infty$, $\mu^*(\limsup E_k) = 0$

Carathéodory Process: $\mu(E) = \mu^*(E)\big|_{\mathcal{M}}$ is a measure if μ^* is an outer measure and $\mathcal{M}$ is the collection of μ^* measurable sets (a sigma algebra)

Properties of the Lebesgue Measure:

- for any interval $I \subseteq \mathbb{R}$, $\mu(I) = \rho(I)$.
- $\mu^*(E + a) = \mu^*(E)$
- $\mu^*(rE) = |r|\,\mu^*(E)$

Property of the Hausdorff Measure: If $\mathcal{H}_\alpha(A) < \infty$, $\mathcal{H}_\beta(A) = 0$ for $\beta > \alpha$.

Approximation of Measurable Sets:

- $\exists O$ open such that $E \subseteq O$ and $\mu(O) \geq \mu(E) \geq \mu(O) - \varepsilon$
- $\exists K$ closed such that $\mu(K) \leq \mu(E) \leq \mu(K) + \varepsilon$