

# APMA 2110: Real Analysis

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Fall 2024

# Chapter 1

## Analysis and Metric Spaces

### 1.1 Sept 05

Some basic notation:

$$\mathbb{N} := \{1, 2, 3, \dots\}$$

$$\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$$

$$\mathbb{Q} := \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \right\}$$

$$\mathbb{R} := \text{the set of real numbers}$$

$$\mathbb{C} := \text{the set of complex numbers}$$

Some basic logic:

- $(A \implies B) \iff (\neg B \implies \neg A)$  (contrapositive)
- $E \subset X \implies \forall x \in E, x \in X$

### Sets

Note that in this course,  $\subset$  includes the possibility of equality, while  $\subsetneq$  does not.

**Power Set:**  $P(X) = \{E : E \subseteq X\}$

*Example:*  $X = \{1, 2, 3\}$

$$P(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$$

**Sets:** Let  $\mathbb{E}$  be a collection of sets  $E$

- $\bigcup_{E \in \mathcal{E}} = \{x : x \in E, \text{ for some } E \in \mathcal{E}\}$
- $\bigcap_{E \in \mathcal{E}} = \{x : x \in E, \text{ for all } E \in \mathcal{E}\}$
- $\mathcal{E} = \{E_\alpha : \alpha \in A\} = \{E_\alpha\}_{\alpha \in A}$
- $E_\alpha \cap E_\beta = \emptyset$  for  $\alpha \neq \beta \iff E_\alpha$  and  $E_\beta$  are *disjoint*

**Limsup and Liminf:** For  $\{E_n\}_{n=1}^\infty$ ,

$$\limsup E_n = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n$$

$$\liminf E_n = \bigcup_{k=1}^\infty \bigcap_{n=k}^\infty E_n$$

**Exercise:** Prove that

$$\limsup E_n = \{x : x \in E_n \text{ for infinitely many } n\}$$

$$\liminf E_n = \{x : x \in E_n \text{ for all but finitely many } n\}$$

i.e. after first finite  $n$ ,  $x$  is in  $E_n$  for all  $n$ .

**Difference and Symmetric Difference:** Let  $E$  and  $F$  be two sets

$$E \setminus F = \{x : x \in E, x \notin F\}$$

$$E \triangle F = (E \setminus F) \cup (F \setminus E)$$

$$E^c = X \setminus E, \quad E \subseteq X$$

**De Morgan's Laws:**

$$\left( \bigcup_{\alpha \in A} E_\alpha \right)^c = \bigcap_{\alpha \in A} E_\alpha^c$$

$$\left( \bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

**Exercise:** Prove De Morgan's Laws.

**Cartesian Product:** If  $X$  and  $Y$  are sets, then  $X \times Y$  is the *ordered* set

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

## Relations

**Relations:** A *relation*  $R$  from  $X$  to  $Y$  is a subset of  $X \times Y$  such that

$$xRy \iff (x, y) \in R$$

**Equivalence relation:** A relation  $\sim$  is an equivalence relation in the special case  $Y = X$  if it is

- Reflexive:  $x \sim x \quad \forall x \in X$
- Symmetric  $x \sim y \iff y \sim x$
- Transitive  $x \sim y, y \sim z \implies x \sim z$

## Functions

**Mappings:** A mapping/function  $f : X \rightarrow Y$  is a relation  $R$  from  $X$  to  $Y$  such that  $\forall x \in X$ , there exists a *unique*  $y \in Y$  such that  $xRy$ . We write  $y = f(x)$ .

**Composition:** If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , then  $g \circ f : X \rightarrow Z$  is a function such that  $g \circ f(x) = g(f(x))$

**Images:** If  $D \subseteq X, E \subseteq Y$ , the *image* of  $D$  (and the *inverse image*/pre-image of  $E$ ) under  $f : X \rightarrow Y$  is

$$\begin{aligned} f(D) &= \{f(x) : x \in D\} \\ f^{-1}(E) &= \{x \in X : f(x) \in E\} \end{aligned}$$

For  $f : X \rightarrow Y$  we further call  $X$  the *domain* of  $f$  and  $Y$  the *codomain* of  $f$ . The *range/image* of  $f$  is  $f(X)$ .

**Inverses:**  $f^{-1}$  defines an operation on  $P(X)$  such that

$$\begin{aligned}f^{-1}\left(\bigcup_{\alpha \in A} E_{\alpha}\right) &= \bigcup_{\alpha \in A} f^{-1}(E_{\alpha}) \\f^{-1}\left(\bigcap_{\alpha \in A} E_{\alpha}\right) &= \bigcap_{\alpha \in A} f^{-1}(E_{\alpha}) \\f^{-1}(E^c) &= (f^{-1}(E))^c\end{aligned}$$

**Exercise:** Prove the above properties of inverses. Warning: in general,  $f$  also commutes with unions but *not* with intersections. Why?

**Bijectivity:**

- $f$  is *injective* iff  $f(x_1) = f(x_2) \implies x_1 = x_2$
- $f$  is *surjective* iff  $\forall y \in Y, \exists x \in X$  s.t.  $f(x) = y$
- $f$  is *bijective* iff it is both injective and surjective

In the case of a bijective mapping  $f$ , then  $f^{-1}$  is a function from  $Y$  to  $X$  (i.e.  $f^{-1}$  has a unique value for bijective  $f$ )

## Sequences

**Sequences:** A sequence in a set  $X$  is a function  $f : \mathbb{N} \rightarrow X$ . We  $\{x_n\}$  for  $x_n \in X$

**Subsequence:** A subsequence  $x_{n_k} \subseteq \{x_n\}$  with  $n_k \in \{1, \dots, \infty\}$

## Ordering

**Partial ordering:** a partial ordering on a nonempty set  $X$  is a relation  $R$  on  $X$  such that

- If  $xRy$  and  $yRz$ , then  $xRz$  (transitivity)
- If  $xRy$  and  $yRx$ , then  $x = y$  (antisymmetry)
- $xRx$  for all  $x$  (reflexivity)

*Example:* Let  $E$  be a set. Consider the relation  $\subseteq$ . Let  $E_1, E_2, E_3 \subseteq E$ .

- $E_1 \subseteq E_2$  and  $E_2 \subseteq E_3$  implies  $E_1 \subseteq E_3$  (transitivity ✓)
- $E_1 \subseteq E_2$  and  $E_2 \subseteq E_1$  implies  $E_1 = E_2$  (antisymmetry ✓)
- $E_1 \subseteq E_1$  (reflexivity ✓)

Therefore, inclusion (with equality) is a partial ordering. (Proof for first two by considering elements, proof for last by equality)

**Total ordering:** A total ordering/linear ordering is a partial ordering such that for all  $x, y \in X$ , either  $xRy$  or  $yRx$ .

*Example:* Inclusion is not a total ordering on  $P(X)$  since (in general)  $E_1 \not\subseteq E_2$  and  $E_2 \not\subseteq E_1$  for  $E_1 \neq E_2$ .

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**Recall:** a *partial ordering* is a relation that satisfies

1. if  $xRy$  and  $yRz$ , then  $xRz$
2. if  $xRy$  and  $yRx$ , then  $x = y$
3.  $xRx$  for all  $x$

*Examples:*

- In the real numbers,  $\leq$  is the typical ordering.
- For a set  $X$  and its power set  $P(X)$ ,  $\subseteq$  is a partial ordering.

**Warning:** In this class, we will use  $\leq$  to denote an abstract partial ordering.

**Total/Linear Ordering:** A total ordering is a partial ordering such that for all  $x, y \in X$ , either  $x \leq y$  or  $y \leq x$ .

**Extrema:** If  $X$  is partially ordered by  $\leq$ , a *maximal* (resp. *minimal*) element of  $X$  is an element  $x \in X$  such that  $x \leq y \implies y = x$

**Bounds:** If  $E \subseteq X$ , an *upper* (resp. *lower*) *bound* for  $E$  is an element  $x \in X$  such that  $y \leq x$  (resp.  $x \leq y$ ) for all  $y \in E$ .

**Zorn's Lemma (transfinite induction):** If  $X$  is partially ordered by  $\leq$ , assume every linearly ordered subset of  $X$  has an upper bound. Then  $X$  has a maximal element.

*Proof:* We regard this as axiomatic

**Well-Ordering:** A set  $X$  is *well-ordered* if

1. it is linearly ordered by  $\leq$
2. every nonempty subset of  $X$  has a minimal element.

**Well-ordering Principle:** Every non-empty set  $X$  can be well-ordered

*Proof:* Consider  $\mathcal{W} = \{\text{all well-ordered subsets of } X\}$ .

Suppose there exist well-ordered sets  $E_1, E_2 \subseteq W$ . Then each has a minimal element.

We know  $\mathcal{W}$  is non-empty because for all finite subsets of  $X$ , we can order them (using the normal linear order on  $\mathbb{R}$ ).

We will proceed by defining a relation  $R$  between the linear orderings  $\leq_1$  and  $\leq_2$  of  $E_1$  and  $E_2$  respectively. We will say  $\leq_1 R \leq_2$  if:

1.  $\leq_2$  extends  $\leq_1$  (i.e.  $E_1 \subseteq E_2$  and  $\leq_1 = \leq_2$  on  $E_1$ )
2. If  $x \notin E_1, x \in E_2$ , then  $y \leq_2 x$  for all  $y \in E_1$

**Exercise:** Prove that  $R$  is a partial ordering in  $\mathcal{W}$

Assume  $\mathcal{S} = \{\leq_\alpha; R\}$  is the set of linear orderings  $\leq_\alpha$  of  $E_\alpha \subseteq \mathcal{W}$  for  $\alpha \in A$ . Thus,  $\leq_\alpha R \leq_\beta$  for  $\alpha, \beta \in A$ .

*Claim:* Let

$$E_\infty = \bigcup_{\alpha \in A} E_\alpha$$

equipped with the partial ordering  $\leq_\infty$  such that  $\leq_\infty \upharpoonright_{E_\alpha} = \leq_\alpha$  for all  $\alpha \in A$ .

Clearly,  $\leq_\alpha R \leq_\infty$  for all  $\alpha \in A$ . Then for any sequence of well-ordered sets in  $\mathcal{W}$ ,  $E_\infty$  is an upper-bound.

**Exercise:** Verify that  $\leq_\alpha R \leq_\infty$  is well defined and that  $E_\infty$  is an upper bound for  $\mathcal{W}$

By Zorn's Lemma, there exists a maximal element  $E_{\max} \in \mathcal{W}$ . (Verify it's a well-ordering by extending  $\leq_{\max}$  to include any  $x_0 \in X \setminus E_{\max}$  such that  $x \leq x_0$  for all  $x \in E_{\max}$ ).

Consider  $E_{\max} \cup \{x_0\}$ . Clearly,  $E_{\max} \leq E_{\max} \cup \{x_0\}$ , so  $E_{\max} \cup \{x_0\}$  and by the extension above,  $E_{\max} \cup \{x_0\} \in \mathcal{W}$ . This contradicts the maximality of  $E_{\max}$ , so  $E_{\max} = X$ .

**Definition:** Let  $\prod_{\alpha \in A} X_\alpha$  be the set of all maps  $f : A \rightarrow \bigcup_{\alpha \in A} X_\alpha$  such that  $f(\alpha) \in X_\alpha$  for all  $\alpha \in A$ .

**Axiom of Choice:** If  $\{X_\alpha\}_{\alpha \in A}$  is a nonempty collection of nonempty sets,  $\prod_{\alpha \in A} X_\alpha$  is nonempty, i.e. there exists at least one choice function  $f$

*Proof:* Let  $X = \bigcup_{\alpha \in A} X_\alpha$ . Pick a well-ordering on  $X$  and  $\alpha \in A$ . Let  $f(\alpha)$  be the minimal element of  $X_\alpha$ . Then  $f \in \prod_{\alpha \in A} X_\alpha$

## Cardinality

**Definition:**

- $\text{card } X \leq \text{card } Y$  if there exists an injective function  $f : X \rightarrow Y$
- $\text{card } X = \text{card } Y$  if there exists a bijective function  $f : X \rightarrow Y$
- $\text{card } X \geq \text{card } Y$  if  $\text{card } X \leq \text{card } Y$  but  $\text{card } X \neq \text{card } Y$  there exists a surjective function  $f : X \rightarrow Y$

**Property:**  $\text{card } X \leq \text{card } Y$  iff  $\text{card } Y \geq \text{card } X$

*Proof:*  $\text{card } X \leq \text{card } Y$  implies there exists an injective  $f : X \rightarrow Y$ . Pick  $x_0 \in X$  and define  $g : Y \rightarrow X$  by

$$g(y) = \begin{cases} f^{-1}(y) & y \in f(X) \\ x_0 & \text{otherwise} \end{cases}$$



In the first case, we have injectivity of  $f$  so each  $f^{-1}(y)$  is unique. In the second case we ensure surjectivity.

Conversely, if  $g : Y \rightarrow X$  is surjective, consider  $g^{-1}(\{x\})$  for  $x \in X$ . These sets are non-empty and disjoint because  $g$  is a map (each  $x$  can map to a single  $y$ ). Then any  $f \in \prod_{x \in X} g^{-1}(\{x\})$  is an injection from  $X$  to  $Y$ .

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**Property:** For any sets  $X$  and  $Y$ , either  $\text{card } X \leq \text{card } Y$  or  $\text{card } Y \leq \text{card } X$

*Proof Sketch:* Consider the (non-empty) set

$$J = \{\text{all injections } f_E : E \rightarrow Y \text{ with respect to } E \subseteq X\}$$

Define a relation  $R$  on  $J$  such that  $f_{E_1} R f_{E_2}$  if  $E_1 \subseteq E_2$  and  $f_{E_2}|_{E_1} = f_{E_1}$ , i.e.  $f_{E_2}$  is an extension of  $f_{E_1}$ .

Repeating the argument of the Well-Ordering Principle,  $R$  is a partial ordering.

Then we can find an upper bound for  $J$  by considering the union of all  $E \in J$  and extending the injections.

By Zorn's Lemma, there exists a maximal element  $f_{E_{\max}} \in J$  with respect to the ordering  $R$ .

*Case 1:* Suppose  $E_{\max} = X$ . Then  $f_{E_{\max}}$  is an injection from  $X$  to  $Y$  so  $\text{card } X \leq \text{card } Y$

*Case 2:* Suppose  $E_{\max} \subsetneq X$ . Then  $\exists x_0 \in X \setminus E_{\max}$ . Consider the image  $f(E_{\max})$ . We claim  $f(E_{\max}) = Y$  so  $f_{E_{\max}}^{-1}$  is defined on all of  $Y$  and is injective  $Y \rightarrow X$  and we are done. Thus, it only remains to show  $f(E_{\max}) = Y$ .

If the claim is not true,  $\exists y_0 \in Y$  but  $y_0 \notin f(E_{\max})$  but this is a contradiction to maximality (as in the Well-Ordering Principle proof).

**Schröder-Bernstein Theorem:** If  $\text{card } X \leq \text{card } Y$  and  $\text{card } Y \leq \text{card } X$ , then  $\text{card } X = \text{card } Y$

*Note:* This seems trivial but in fact the two functions are not necessarily the same so we must construct our own bijection.

*Proof:* Denote the cardinality injections  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$ .

If  $f(X) = Y$ , then  $f$  is a bijection and we are done.

If  $f(X) \neq Y$  (i.e.  $f(X) \subsetneq Y$ ), then consider  $Y_1 = Y \setminus f(X)$  and  $g(Y_1)$ . Then  $f(Y_1) \subsetneq X$ , so call  $X_1 = f(Y_1)$ . We now have a bijection  $X_1 \rightarrow Y_1$ .

Let's repeat.  $f(X \setminus X_1) \subsetneq Y \setminus Y_1$  so define  $Y_2 = (Y \setminus Y_1) \setminus f(X \setminus X_1)$ .

Now we know  $f(X_1) \subseteq Y_2$  and  $f^{-1}(Y_1) \subseteq X_1$  so we can define a bijection  $X_2 \rightarrow Y_2$ .

Assume  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$  are constructed. WLOG assume that this procedure can be repeated infinitely (or else we would already have a bijection).

Define

$$\left( Y \setminus \bigcup_{i=1}^n Y_i \right) \setminus f \left( X \setminus \bigcup_{i=1}^n X_i \right) = Y_{n+1}$$

since  $f(X_i) \subseteq Y_{i+1}$ .

**Exercise:** Verify that

$$g : \bigcup_{i=1}^{\infty} Y_i \rightarrow \bigcup_{i=1}^{\infty} X_i$$

is a bijection and further that

$$f : \left( X \setminus \bigcup_{i=1}^{\infty} X_i \right) \rightarrow \left( Y \setminus \bigcup_{i=1}^{\infty} Y_i \right)$$

is also a bijection.

Together, these steps show that we have a bijection on the full sets  $X$  and  $Y$ .

**Proposition:** For any set  $X$ ,  $\text{card } X < \text{card } P(X)$

*Proof:* Clearly,  $\forall x \in X$ , we have an injection  $f : X \hookrightarrow P(X)$  defined by  $f(x) = \{x\}$ .

We claim there is no surjection  $g : X \rightarrow P(X)$  and proceed by contradiction.

Let  $g : X \rightarrow P(X)$ . Define

$$Y = \{x \in X \text{ s.t. } x \notin g(x)\}$$

We claim  $Y \notin g(X)$ . If not, assume  $x_0 \in X$  such that  $g(x_0) = Y$ .

*Case 1:* If  $x_0 \in Y$ , then  $x_0 \notin g(x_0) = Y$  - contradiction

*Case 2:* If  $x_0 \notin Y$ , then  $x_0 \in g(x_0) = Y$  - contradiction

Therefore,  $Y \notin g(X)$  so  $g$  is not surjective.

**Countable:** A set  $X$  is *countably infinite* if  $\text{card } X \leq \text{card } \mathbb{N}$ .

**Proposition:**

- (a) If  $X$  and  $Y$  are countable, so is  $X \times Y$ .
- (b) If  $A$  is countable and  $X_\alpha$  is countable for every  $\alpha \in A$ , then  $\bigcup_{\alpha \in A} X_\alpha$  is countable.

*Proof:*

- (a)  $\text{card } X = \text{card } Y = \text{card } \mathbb{N}$  so it suffices to show  $\mathbb{N} \times \mathbb{N} = \text{card } \mathbb{N}$

$\forall n \in \mathbb{N}$ , define  $f(n) \hookrightarrow (n, 1) \in \mathbb{N} \times \mathbb{N}$ .

Consider  $g((m, n)) \rightarrow 2^m 3^n \in \mathbb{N}$ . Is this injective? Consider  $g(m_1, n_1) = 2^{m_1} 3^{n_1}$ . By the unique prime factorization of integers,  $2^{m_1} 3^{n_1} = 2^m 3^n$  iff  $(m_1, n_1) = (m, n)$  so  $g$  is injective.

Now we can use Schröder-Bernstein and we are done.

- (b) As  $A$  is countable,  $\forall \alpha \in A$ ,  $\exists f_\alpha : \mathbb{N} \rightarrow X_\alpha$ . So we can define  $F : \mathbb{N} \times A \rightarrow \bigcup_{\alpha \in A} X_\alpha$  by

$$F(n, \alpha) = f_\alpha(n)$$

which is surjective

**Corollary:**  $\mathbb{Z}$  and  $\mathbb{Q}$  are countable

*Proof:*  $\mathbb{Z} = \mathbb{N} \cup \{-\mathbb{N}\} \cup 0$

We can define  $f : \mathbb{Z}^2 \rightarrow \mathbb{Q}$  by

$$f(m, n) = \begin{cases} \frac{m}{n} & n \neq 0 \\ 0 & n = 0 \end{cases}$$

**Convention for this course:** We will use  $\mathbb{R}$  to denote the standard reals and will define the *extended reals*  $\overline{\mathbb{R}}$  by  $\mathbb{R} \cup \pm\infty$

Under this notation, we can state that for any  $E \subseteq \overline{\mathbb{R}}$ ,  $\sup \overline{E}$  and  $\inf \overline{E}$  are always well-defined, i.e. all sets are bounded above by  $\infty$  and below by  $-\infty$ .

We define the following rules:

- $X \pm \infty = \pm\infty$
- $\infty + \infty = \infty$
- $-\infty - \infty = -\infty$
- $\infty - \infty$  is undefined
- $x(\pm\infty) = \pm\infty$  for  $x > 0$  and  $x(\pm\infty) = \mp\infty$  for  $x < 0$
- $0 \cdot (\pm\infty) = 0$

**Note:** this last point does *not* talk about limits, it is just notation

**Proposition:** Every open set in  $\mathbb{R}$  is a countable disjoint union of open intervals

*Proof Sketch:* For all  $x \in U$ , there exists an open interval  $I_{\alpha, \beta} = (\alpha, \beta) \subseteq U$  with  $\alpha < x < \beta$ .

Let  $\mathcal{J}_x = \{x \in I_{\alpha, \beta} \mid I_{\alpha, \beta} \in U\}$ .

Take  $\alpha_{\inf} = \inf \alpha$  and  $\beta_{\sup} = \sup \beta$ .

**Exercise:** Check that  $x \in (\alpha_{\inf}, \beta_{\sup}) \subseteq U$

We call  $I_x = (\alpha_{\inf}, \beta_{\sup})$  for all  $x \in U$

We claim  $\forall x, y \in U$ , either  $I_x \cap I_y = \emptyset$  or  $I_x = I_y$ .

Suppose  $I_x \cap I_y \neq \emptyset$ . Then  $I_x \cup I_y$  is an open interval containing  $x$ , so  $I_x \cup I_y \in \mathcal{J}_x$  but  $I_x$  is maximal so this is a contradiction unless  $I_x = I_y$ .

Now we can write

$$U = \bigcup_{x \in U} I_x$$

Why is this countable? We can define an injection  $U \rightarrow \mathbb{Q}$  by choosing a rational number in each  $I_x$  (exist by density of  $\mathbb{Q}$ ).

## Metric Spaces

**Definition:** A *metric space* is a set  $X$  together with a *distance function*  $\rho : X \times X \rightarrow [0, \infty)$  such that

1.  $\rho(x, y) = 0 \iff x = y$
2.  $\rho(x, y) = \rho(y, x)$
3.  $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

**Examples:**

- $\mathbb{R}^n$  with  $\rho(x, y) = |x - y|$
- Set of continuous functions  $f$  over  $[0, 1]$  with  $\rho_1(f, g) = \int_0^1 |f(x) - g(x)| dx$  (or alternatively  $\rho_\infty = \sup_{0 \leq x \leq 1} |f(x) - g(x)|$ )

**Exercise:** Check the above are metric spaces

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### Closed and Open Sets

**Open ball:** Let  $(X, \rho)$  be a metric space. If  $x \in X$ ,  $r > 0$ , we define the *open ball*  $B(x, r) = \{y \in X \text{ s.t. } \rho(x, y) < r\}$

**Open set:** a set  $E$  is open iff  $\forall x \in E, \exists r > 0 \text{ s.t. } B(x, r) \subseteq E$

**Closed set:** a set  $E$  is closed iff  $E^c$  is open

*Example:*  $B(x, r)$  is open. Consider  $y \in B(x, r)$ . Then  $\rho(x, y) = s < r$ . By the triangle inequality,  $B(y, r - s) \subseteq B(x, r)$

**Exercise:** Prove that  $B(x, r)$  is open

**Properties:**

- $\emptyset$  is open
- If  $U_x$  are open sets,  $\bigcup_{x \in A} U_x$  is open (as is the finite intersection)
- If  $F_x$  are closed sets,  $\bigcap_{x \in A} F_x$  is closed (as is the finite union)

**Interior:** Let  $E \subseteq X$ . The *interior* of  $E$  is

$$\overset{\circ}{E} = \bigcup_{O \subseteq E} O$$

(this is the largest open set in  $E$ )

**Closure:** The *closure* of  $E$  is

$$\overline{E} = \bigcap_{E \subseteq F} F$$

(this is the smallest closed set containing  $E$ )

**Proposition:** Let  $(X, \rho)$  be a metric space. Let  $E \subseteq X$  and  $x \in X$ . Then the following are equivalent:

- (a)  $x \in \overline{E}$
- (b)  $B(x, r) \cap E \neq \emptyset$  for all  $r > 0$
- (c)  $\exists (x_n) \subseteq E$  such that  $x_n \rightarrow x$

*Proof:* ((a)  $\rightarrow$  (b)) Let  $x \in \overline{E}$ . Suppose  $\exists r_0 > 0$  such that  $B(x, r_0) \cap E = \emptyset$ . Then  $E \subseteq (B(x, r_0))^c$ . But  $(B(x, r_0))^c$  is closed so  $\overline{E} \subseteq (B(x, r_0))^c$  so  $x \in B(x, r_0) \subseteq (\overline{E})^c$  but this implies  $x \in (\overline{E})^c$  which is a contradiction.

((b)  $\rightarrow$  (c)) Let  $r = \frac{1}{n}$ . By (b),  $B(x, \frac{1}{n}) \cap E \neq \emptyset$ . Choose  $x_n \in B(x, \frac{1}{n}) \cap E$ . Certainly  $\rho(x_n, x) < \frac{1}{n}$  so  $\lim \rho(x_n, x) = 0$  and  $x_n \rightarrow x$

((c)  $\rightarrow$  (a)) If  $x \notin \overline{E}$ ,  $x \in (\overline{E})^c$  but  $(\overline{E})^c$  is open so  $\exists r > 0$  s.t.  $B(x, r) \subseteq$

$(\overline{E})^c \subseteq E^c$ . Then there cannot exist any sequence in  $E$ . But this contradicts  $x_n \rightarrow x$

## Density

**Dense:**  $E$  is dense in  $X$  if  $\overline{E} = X$  (examples  $\mathbb{R}^n, \mathbb{Q}^n$ )

**Nowhere dense:**  $E$  is nowhere dense if  $(\overline{E})^\circ = \emptyset$  (example: emptyset)

**Separable:**  $X$  is separable if there exists a countable dense subset  $E \subseteq X$

**Limits:** In this class,  $x_n \rightarrow x$  iff  $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$

## Continuity

Let  $\mathcal{C} = \{\text{continuous functions on } [0, 1]\}$ .

**Continuity at a point:** If  $(X_1, \rho_1)$  and  $(X_2, \rho_2)$  are metric spaces,  $f : X_1 \rightarrow X_2$  is continuous at  $x \in X_1$  if  $\forall \varepsilon > 0, \exists \delta_x > 0$  such that  $\forall y \in X_1$  such that  $\rho_1(x, y) < \delta_x$  (i.e.  $y \in B_1(x, \delta_x)$ ),

$$\rho_2(f(x), f(y)) < \varepsilon$$

(i.e.  $f(y) \in B_2(f(x), \varepsilon)$ )

**Continuity on a set:**  $f$  is continuous in  $X$  iff  $f$  is continuous at every  $x \in X$

**Uniform Continuity:**  $f$  is uniformly continuous if  $\delta$  is independent of  $x$ , i.e.  $\forall \varepsilon > 0, \exists \delta > 0$  such that

$$\rho_1(x, y) < \delta \implies \rho_2(f(x), f(y)) < \varepsilon$$

for all  $x \in X$ .

**Proposition:**  $f : X_1 \rightarrow X_2$  is continuous iff  $f^{-1}(U) \subseteq X_1$  is open for all open  $U \subseteq X_2$

*Proof:* Let  $f$  be continuous and  $U \subseteq X_2$  be open.  $f^{-1}(U) = \emptyset$  is open so take  $x \in f^{-1}(U)$ . Then  $f(x) = y \in U$ .

Since  $U$  is open,  $\exists \varepsilon_y > 0$  s.t.  $B_2(y, \varepsilon_y) = B_2(f(x), \varepsilon_y) \subseteq U$ .

By continuity,  $\exists \delta_x > 0$  such that  $\forall z \in B_1(x, \delta_x)$ ,

$$\rho_2(f(x), f(z)) < \varepsilon_y \implies f(z) \in B_2(y, \varepsilon_y) \subseteq U \implies z \in f^{-1}(U)$$

so  $B_1(x_1, \delta_x) \subseteq f^{-1}(U)$  and  $f^{-1}(U)$  is open.

Conversely, suppose  $f^{-1}(U)$  is open for all open  $U \subseteq X_2$ . Let  $\varepsilon > 0$ . Consider  $y = f(x)X_2$ . Then  $B_2(y, \varepsilon)$  is open so  $f^{-1}(B_2(y, \varepsilon))$  is open by assumption.

Let  $x \in f^{-1}(B_2(y, \varepsilon))$ . Then  $\exists \delta_x$  such that  $B_1(x, \delta_x) \subseteq f^{-1}(B_2(y, \varepsilon))$ .

Then  $f(B_1(x, \delta_x)) \subseteq B_2(y, \varepsilon)$  which is precisely the definition of continuity.

## Cauchy Sequences

**Cauchy Sequence:** A sequence  $(x_n)$  in a metric space  $(X, \rho)$  is Cauchy if  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  such that  $\forall m, n \geq N$ ,

$$\rho(x_m, x_n) < \varepsilon$$

**Completeness:** A subset  $E \subseteq X$  is *complete* if every Cauchy sequence  $x_n \in E$  has a limit  $x \in E$

*Examples:*

- In  $\mathbb{R}^n$ , any bounded closed subset is complete.
- $(\mathcal{C}, \rho_\infty)$  is complete

**Exercise:** Prove that  $(\mathcal{C}, \rho_\infty)$  is complete for

$$\rho_\infty(x, y) = \sup_{x \in [0, 1]} |f(x) - g(x)|$$

(though in general this is not true for other metrics)

**Proposition:** A closed subset  $(X, \rho)$  of a complete metric space is complete and complete subsets of a metric space must be closed

*Proof:*

**Exercise**

**Set Distance:**



- Let  $x \in X$  and  $E \subseteq X$ . The *distance* from  $x$  to  $E$  is

$$\rho(x, E) = \inf\{\rho(x, y) : y \in E\}$$

- For  $E, F \subseteq X$ ,

$$\rho(E, F) = \inf\{\rho(x, y) : x \in E, y \in F\}$$

**Diameter:**  $\text{diam } E = \sup\{\rho(x, y) : x, y \in E\}$

**Bounded:**  $E$  is bounded iff  $\text{diam } E < \infty$

**Open cover:** Let  $\{V_\alpha\}_{\alpha \in A}$  be a family of sets.  $\{V_\alpha\}$  *covers*  $E$  if

$$E \subseteq \bigcup_{\alpha \in A} V_\alpha$$

**Total boundedness:**  $E$  is *totally bounded* if  $\forall \varepsilon > 0$ ,  $E$  can be covered by finitely many balls of radius  $\varepsilon$

*Example:*  $\mathbb{R}^n$  is totally bounded. *Proof:* consider a hypercube of side length  $R$ . Clearly we can divide this into  $\varepsilon$ -cubes and then take slightly larger balls to cover the whole space.

**Theorem (Characterization of Compactness):** The following are equivalent:

1.  $E$  is complete and totally bounded
2. Every sequence in  $E$  has a convergent subsequence with its limit in  $E$
3. If  $\{V_\alpha\}_{\alpha \in A}$  is an open cover of  $E$ , then there exists a finite set  $F \subseteq A$  such that  $\{V_\alpha\}_{\alpha \in F}$  covers  $E$

*Proof:* HW

## 1.5 Sept 19

**Products of Metric Spaces:** Let  $(X, \rho_1)$  and  $(Y, \rho_2)$  be metric spaces. Define the *product metric* on  $X \times Y$  by  $(X_1 \times X_2, \rho_1 \times \rho_2)$  where

$$\rho_1 \times \rho_2 = \sqrt{\rho_1^2(x_1, y_1) + \rho_2^2(x_2, y_2)}$$

(so called *Euclidean Metric*)

Though many other metrics are possible, such as  $\max(\rho_1, \rho_2)$  and  $\rho_1 + \rho_2$ .

In general, we will simply take the Euclidean metric because all these metrics are equivalent in the sense that  $\exists C_1, C_2$  such that

$$C_1(\rho_1 \times \rho_2)_1 \leq C_2(\rho_1 \times \rho_2)_2 \leq C_2(\rho_1 \times \rho_2)_3$$

*Properties:*

- $\rho_1 \times \rho_2 \rightarrow 0 \iff \rho_1 \rightarrow 0 \text{ and } \rho_2 \rightarrow 0$

# Chapter 2

## Measure Theory

### 2.1 Sept 19

#### Measure Theory Motivation

**Riemann Integral:** Let  $f : [a, b] \rightarrow \mathbb{R}$ . We subdivide  $[a, b]$  by

$$a = x_0 < x_1 < \cdots < x_n = b$$

and define subintervals  $[x_i, x_{i+1}]$ .

Then

$$\int f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \cdot (x_{i+1} - x_i)$$

**Convergence:** Many times, we are interested in the question:

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) \, dx \stackrel{?}{=} \int_0^1 f(x) \, dx$$

for  $f_n(x) \rightarrow f(x)$ .

This is easy when  $f_n \rightarrow f$  uniformly but in general, we need something else.

In Riemann integration, we divide the domain into intervals and sum the function over these intervals.

In Lebesgue integration, we instead divide *the range*, i.e. we take a set

$$E_i = \{x : a_n \leq f(x) \leq a_{n+1}\}$$

**Measure:** We define  $\mu(E)$ , the *measure* of a subset, by:

1. (Countable Additivity)  $\{E_n\}$  such that  $E_i \cap E_j = \emptyset$  for  $i \neq j$  then  $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$
2. (Translation invariance)  $\mu(E + r) = \mu(\{x + r : x \in E\}) = \mu(E)$
3.  $\mu([0, 1]) = 1$

**Proposition:** There is no measure  $\mu$  satisfying the above properties which is defined for all subsets of  $[0, 1]$

*Proof:* Step 1. Let  $\mathbb{Q}_1 = \mathbb{Q} \cap [0, 1)$ . Define an equivalence relation  $x \sim y$  iff  $x - y \in \mathbb{Q}_1$ .

Now consider the equivalence class  $\mathcal{E}_x = \{y \in [0, 1) : y \sim x\}$ . (As it is an equivalence class:  $\mathcal{E}_x \cap \mathcal{E}_y \neq \emptyset \implies \mathcal{E}_x = \mathcal{E}_y$ )

And clearly,

$$[0, 1) = \bigcup_{x \in [0, 1)} \mathcal{E}_x$$

By the Axiom of Choice, choose a unique element  $e_x \in \mathcal{E}_x$ . Define  $N = \{e_x\}$ . Now  $e_x - e_y \notin \mathbb{Q}_1$ .

Step 2.  $\forall r \in \mathbb{Q}_1$ , define

$$N_r = \{e_x + r : e_x \in N \cap [0, 1 - r)\} \cup \{e_x + r - 1, e_x \in N \cap [1 - r, 1)\}$$

(the first set is the points that don't leave the interval under translation, the second set is the pullback of the points that do)

Step 3. We claim

$$[0, 1) = \bigcup N_r, \quad N_r \cap N_s = \emptyset \text{ for } r \neq s$$

*Proof:*

1. (Subset)  $\forall y \in [0, 1), \exists e_x \in N$  such that  $y - e_x \in \mathbb{Q}_1$ .

If  $y \geq e_x$ ,  $r = e_x - y + 1$ . Otherwise,  $r = e_x - y$ .

2. (Disjoint Union) Suppose  $N_r \cap N_s \neq \emptyset$ . Let  $r \neq s$ . Select  $y \in N_r \cap N_s$  so  $y - s \in N$  and  $y - r \in N$

Case 1.  $y - s \neq y - r$ . But then

$$(y - r) - (y - s) = s - r \in \mathbb{Q}_1$$

which is a contradiction of the construction of  $N$ .

Case 2.  $y - s \neq y - r + 1$ . Contradiction again by rational difference.

Step 4. By the definition of a measure,

$$\begin{aligned}\mu(N_r) &= \mu(N_r \cap (0, 1 - r)) + \mu(N_r \cap [1 - r, 1]) \\ &= \mu(N)\end{aligned}$$

**Exercise:** Check that  $\mu(N_r) = \mu(N)$

But by countable Additivity,

$$1 = \mu([0, 1)) = \sum_{r \in \mathbb{Q}_1} \mu(N_r) = \begin{cases} 0 \\ \infty \end{cases}$$

which is a contradiction.

**Conclusion:** it is not always possible to define a measure so we need to be careful.

## Algebras

**Algebra:** Given a set  $X$ , an *algebra* is a collection of subsets  $\mathcal{A} \subseteq P(X)$  such that if  $E_1, \dots, E_n \subseteq \mathcal{A}$ ,

1.  $\bigcup_{i=1}^n E_i \in \mathcal{A}$
2.  $E \in \mathcal{A} \implies E^c \in \mathcal{A}$

Property 2 gives us that  $X \in \mathcal{A}$  and  $\emptyset \in \mathcal{A}$  ( $E \cup E^c = X$ ,  $X^c = \emptyset$ )

**Sigma Algebra:** An algebra  $\mathcal{A}$  is a  *$\sigma$ -algebra* if it is closed under countable unions and complements, i.e. for  $E_1, E_2, \dots \in \mathcal{A}$ ,

1.  $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$
2.  $E \in \mathcal{A} \implies E^c \in \mathcal{A}$

*Remark:* It suffices to demand closure for disjoint countable unions since

$$\bigcup_{n=1}^{\infty} E_i = \bigcup_{n=1}^{\infty} F_i$$

for  $F_k = E_k \setminus \bigcup_{i=1}^{k-1} E_i$  and  $F_i \cap F_{i+1} = \emptyset$

*Examples:*

- $P(X)$
- $\phi, X$
- $\mathcal{A} = \{E \subseteq X : E \text{ countable or } E^c \text{ countable}\}$

**Proposition:** Let  $\mathcal{A}_1, \mathcal{A}_2$  be two  $\sigma$ -algebras on  $X$ . Then  $\mathcal{A}_1 \cap \mathcal{A}_2$  is also a  $\sigma$ -algebra

**Exercise:** Prove this proposition (easy using definition)

**Generated  $\sigma$ -algebra:** Given a collection of subsets  $\mathcal{E} \subseteq P(X)$ , there exists a smallest  $\sigma$ -algebra containing  $\mathcal{E}$ , denoted

$$M(\mathcal{E}) = \bigcap_{\mathcal{A} \supseteq \mathcal{E}} \mathcal{A}$$

**Lemma:**  $\mathcal{E} \subseteq M(\mathcal{F}) \implies M(\mathcal{E}) \subseteq M(\mathcal{F})$

*Proof:* Omitted

## Metric Spaces

**Borel  $\sigma$ -algebra:** Let  $(X, \rho)$  be a metric space. We call the  $\sigma$ -algebra generated by the open sets of  $X$ , the *Borel  $\sigma$ -algebra*  $B_x$  on  $X$ .

This is a  $\sigma$ -algebra because  $X, \emptyset, \bigcup_{i=1}^{\infty} U_i$  are all open since their union is open and we have complements from the generating set.

We define

$$\bigcup_{n=1}^{\infty} F_n = F_{\sigma}$$
$$\bigcap_{n=1}^{\infty} O_n = G_{\delta}$$

for  $F_n$  closed and  $O_n$  open.

**Example:** The Borel set of  $\mathbb{R}$ ,  $B_{\mathbb{R}}$  can be generated by any of the following:

1. open intervals  $\mathcal{E}_1 = \{(a, b) : a < b\}$
2. closed intervals  $\mathcal{E}_2 = \{[a, b] : a < b\}$
3. the half-open intervals  $\mathcal{E}_3 = \{(a, b] : a < b\}$ ,  $\mathcal{E}_4 = \{[a, b) : a < b\}$
4. open rays  $\mathcal{E}_5 = \{(a, \infty) : a \in \mathbb{R}\}$ ,  $\mathcal{E}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$
5. closed rays

**Exercise:**

1. Prove that  $(a, b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n})$  and  $(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$
2. Prove that the above methods all generate  $B_{\mathbb{R}}$

**Conclusion:** any open set in  $\mathbb{R}$  is the countable union of open intervals

## 2.2 Sept 24

Recall last time, we were trying to characterization the Borel  $\sigma$ -algebra of  $\mathbb{R}$ ,  $\mathcal{B}_{\mathbb{R}}$ .

**Proposition:** We claim that  $\mathcal{B}_{\mathbb{R}}$  is generated by:

1. open intervals  $\mathcal{E}_1 = \{(a, b) : a < b\}$
2. closed intervals  $\mathcal{E}_2 = \{[a, b] : a < b\}$
3. the half-open intervals  $\mathcal{E}_3 = \{(a, b] : a < b\}$ ,  $\mathcal{E}_4 = \{[a, b) : a < b\}$
4. open rays  $\mathcal{E}_5 = \{(a, \infty) : a \in \mathbb{R}\}$ ,  $\mathcal{E}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$
5. closed rays

*Proof:*

1. Open intervals.

Let  $\mathbb{E}_1 = \{(a, b) : a < b\}$ . Clearly  $B_{\mathbb{E}_1} \subseteq B_{\mathbb{R}}$  because any open set  $O \subseteq \mathcal{B}_{\mathbb{R}}$ .

For the other direction, we also have

$$O = \bigcup_{i=1}^{\infty} (a_i, b_i)$$

(a countable union), so  $B_{\mathbb{R}} \subseteq B_{\mathbb{E}_1}$

2. Closed intervals.

We claim

$$(a, b) = \bigcup_{n=1}^N \left[ a + \frac{1}{n}, b - \frac{1}{n} \right]$$

for  $N$  sufficiently large.

*Proof:* HW

Now  $\forall y \in (a, b)$ ,

$$y \in \bigcup_{n=1}^N \left[ a + \frac{1}{n}, b - \frac{1}{n} \right] \implies a < y < b$$

for  $N$  sufficiently large.



For the other direction, take  $[a, b] \in \mathcal{E}_2$ . Then

$$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$$

*Proof:*  $[a, b] \subseteq \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$  is clear.

For the other direction, let  $y \in \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$ . Suppose  $a \leq y \leq b$  is false. Then  $y \notin (a - \frac{1}{N}, b + \frac{1}{N})$  so it cannot be in the intersection

**Exercise:** Prove the last two versions: half intervals and rays

**Recall:** For a cartesian product  $X_1 \times X_2 \times \cdots \times X_n$  of metric spaces with  $(X_i, \rho_i)$ , we define the *product metric* by  $(X_1 \times X_2 \times \cdots \times X_n, \rho)$  where

$$\rho(\bar{x}, \bar{y}) = \sqrt{\rho_1^2(x_1, y_1) + \cdots + \rho_n^2(x_n, y_n)}$$

where  $\bar{x} = (x_1, x_2, \dots, x_n)$  with  $x_i \in X_i$  (and similarly for  $\bar{y}$ )

**Proposition:**

$$\lim_{m \rightarrow \infty} \rho(\bar{x}, \bar{y}) = 0 \iff \lim_{m \rightarrow \infty} \rho_i(x_i^m, y_i^m) = 0$$

*Proof:* Omitted

In this way, we can consider  $\mathbb{R}^n$  as a metric space with this Euclidean metric. What is the Borel set of  $\mathbb{R}^n$ ?

**Proposition:**  $B_{\mathbb{R}^n}$  is

*Proof:* First take  $O_i$  open set in  $X_i$

$$\bigoplus_{i=1}^n O_i = O_1 \times O_2 \times \cdots \times O_n$$

We claim that this is an open set in the  $X_1 \times X_2 \times \cdots \times X_n$  topology.

*Proof:* Take  $\bar{x} \in \bigoplus_{i=1}^n O_i$  with  $x_i \in O_i$ .

It suffices to show  $\exists \varepsilon_0 > 0$  such that  $B_{\varepsilon_0}(\bar{x}) \subseteq \bigoplus_{i=1}^n O_i$  where

$$B_{\varepsilon_0}(\bar{x}) = \{\bar{y} : \rho(\bar{x}, \bar{y}) < \varepsilon_0\}$$

so  $\bar{y} \in B_{\varepsilon}(\bar{x})$  iff  $\rho_i(x_i, y_i) < \varepsilon$  for all  $i$ .

Hence  $y_i \in B_{\varepsilon_0}(x_i) \subseteq O_i$

Let  $\bigotimes_{i=1}^n \mathcal{B}_{x_i}$  be the Borel set generated by  $\bigoplus_{i=1}^n O_i$

Clearly,  $\bigoplus_{i=1}^n \mathcal{B}_{x_i} \subseteq \mathcal{B}_{x_1 \times x_2 \times \cdots \times x_n}$

**Lemma:** If  $x_i$  is separable then

$$\bigotimes_{i=1}^n \mathcal{B}_{x_i} = \mathcal{B}_{x_1 \times x_2 \times \cdots \times x_n}$$

In particular:

$$\bigotimes_{i=1}^n \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^n}$$

*Proof:* It suffices to show that  $\forall \bar{x}, \varepsilon$ ,

$$\mathcal{B}_\varepsilon(\bar{x}, \varepsilon) \subseteq \bigotimes_{i=1}^n B_{x_i}$$

Let  $\mathcal{C}_i$  be a countable subset of  $X_i$  such that  $\overline{\mathcal{C}_i} = X_i$  for all  $1 \leq i \leq n$

We claim

$$B_\varepsilon(\bar{x}) \subseteq \bigcup_{\substack{c_i \in \mathcal{C}_i \\ r_i \in \mathbb{Q}}} \bigotimes_{i=1}^n B_{r_i}(c_i)$$

for  $\sqrt{r_1^2 + \dots + r_n^2} < \varepsilon$

(And this has cardinality  $\mathbb{N}^{2n}$  so countable)

Pick

$$\bar{y} \in B_\varepsilon(\bar{x}) \subseteq \bigcup_{\substack{c_i \in \mathcal{C}_i \\ r_i \in \mathbb{Q}}} \bigotimes_{i=1}^n B_{r_i}(c_i) \subseteq \bigotimes_{i=1}^n \mathcal{B}_{x_i}$$

Then

$$\sigma(\bar{x}, \bar{y}) = \sqrt{\rho_1^2(y_1, x_1), \dots, \rho_n^2(y_n, x_n)} < \varepsilon$$

but each  $\rho_i^2(y_i, x_i)$  is fixed so we may choose  $c_i \in \mathcal{C}_i, r_i \in \mathbb{Q}$  such that

$$\rho_i(y_i, c_i) < r_i = \rho_i(y_i, x_i) - [\rho(y_i, x_i) - \rho(y_i, c_i)]$$

by density (from separability)

Since  $\mathbb{Q}^n \subseteq \mathbb{R}^n$  which is countable and dense,  $\mathbb{R}^n$  is separable and we are done.

## Measure Spaces

Recall that we could not always define a measure except on a  $\sigma$ -algebra. Therefore, we limit our attention.

**Measure space:**  $(X, \mathcal{M})$  where  $X$  is a set and  $\mathcal{M}$ , a  $\sigma$ -algebra, is the “measurable sets”

**Measure:** For a measure space  $(X, \mathcal{M})$ , we define a *measure*  $\mu : \mathcal{M} \rightarrow [0, \infty]$  such that

1.  $\mu(\emptyset) = 0$
2. (Countable additivity) if  $\{E_j\}_1^\infty$  is a sequence of pairwise disjoint sets in  $\mathcal{M}$ , then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

Intuitively, this countable additivity property lets us pull out the limits:

$$\mu\left(\lim_{n \rightarrow \infty} \bigcup_1^n E_j\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(E_j)$$

**$\sigma$ -finite:** If  $\mu(X) = \infty$  but

$$X = \bigcup_{i=1}^{\infty} X_i$$

where  $\mu(X_i) < \infty$  for all  $i$ , then we call  $X$   *$\sigma$ -finite*

**Example:** Let  $(X, P(X))$  be a measure space. Let  $f : X \rightarrow [0, \infty]$ . For each  $E \in P(X)$ , we define

$$\mu(E) = \sum_{x \in E} f(x) = \sup\left\{\sum_{x \in F} f(x) : F \subseteq E \wedge F \text{ finite}\right\}$$

**Exercise:** Prove that  $\mu$  is a measure on  $P(X)$

In particular:

- $f(x) = 1$  for all  $x$ , then  $\mu(E)$  is the *counting measure*
- Take  $x_0 \in X$  and define

$$f(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$

is the *Dirac-Delta Mass* at  $x_0$

**Example:** Let  $X$  be an uncountable set. Let  $\mathcal{M} = \{E \text{ is finite or } E^c \text{ is finite}\}$

Define

$$\mu(E) = \begin{cases} 0 & E \text{ is countable} \\ 1 & E^c \text{ is countable} \end{cases}$$

**Exercise:** Check that  $\mathcal{M}$  is a  $\sigma$ -algebra and that  $\mu$  is a measure

## 2.3 Sept 26

**Theorem (Properties of Measures):** Let  $(X, \mathcal{M}, \mu)$  be a measure space. Then:

1. (Monotonicity) with  $E \subseteq F$  with  $E, F \in \mathcal{M}$ , then

$$\mu(E) \leq \mu(F)$$

2. (Subadditivity) If  $\{E_j\}_1^\infty \in \mathcal{M}$ , then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$$

3. (Continuity from Below) If  $\{E_j\}_1^\infty \subseteq \mathcal{M}$  and  $E_1 \subseteq E_2 \subseteq \dots$ , then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j \rightarrow \infty} \mu(E_j)$$

4. (Continuity from Above) If  $\{E_j\}_1^\infty \subseteq \mathcal{M}$  and  $E_1 \supset E_2 \supset \dots$  and  $\mu(E_1) < \infty$ , then

$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j \rightarrow \infty} \mu(E_j)$$

*Proof:* (1) Take  $E \subseteq F \in \mathcal{M}$ . We want to use finite additivity. Consider  $F \setminus E = F \cap E^c$ . Certainly,  $E \cap (F \cap E^c) = \emptyset$  so

$$\mu(F) = \mu(E \cup F \setminus E) = \mu(E) + \mu(F \setminus E)$$

but the measure is nonnegative so  $\mu(E) \leq \mu(F)$ .

And in fact, if  $\mu(F) < \infty$ , then  $\mu(F) - \mu(E) = \mu(F \setminus E)$ .

(2) Once again, we would like to take advantage of finite additivity by expressing  $\bigcup_{j=1}^{\infty} E_j$  as a countable disjoint union.

Let

$$F_k = E_k \setminus \bigcup_{i=1}^{k-1} E_i = \bigcup_{j=1}^{\infty} F_k$$

so

$$\mu \left( \bigcup_{j=1}^{\infty} E_j \right) = \mu \left( \bigcup_{j=1}^{\infty} F_j \right) = \sum_{j=1}^{\infty} \mu(F_j) \leq \sum_{j=1}^{\infty} \mu(E_j)$$


---

(3) Let  $E_1 \subseteq E_2 \subseteq \dots \subseteq E_n \subseteq \dots$ . Denote  $E_0 = \emptyset$ . Then

$$\begin{aligned} E_1 &= E_1 \setminus \emptyset \\ E_2 &= E_1 \cup (E_2 \setminus E_1) \\ E_3 &= E_2 \cup (E_3 \setminus E_2) \\ &= E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2) \\ &= (E_1 \setminus \emptyset) \cup (E_2 \setminus E_1) \cup (E_3 \setminus E_2) \end{aligned}$$

and each of these sets are disjoint.

Inductively define

$$E_n = \bigcup_{k=0}^{n-1} E_{k+1} \setminus E_k$$

We claim

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=0}^{\infty} (E_n \setminus E_{n-1})$$

By additivity,

$$\begin{aligned} \mu \left( \bigcup_{i=1}^{\infty} E_n \right) &= \sum_{n=0}^{\infty} \mu(E_{n+1} \setminus E_n) &= \lim_{N \rightarrow \infty} \sum_{n=0}^N \mu(E_{n+1} \setminus E_n) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \end{aligned}$$


---

(4) Let  $E_1 \supset E_2 \supset \dots$  and  $\mu(E_1) < \infty$ . Define  $F_j = E_1 \setminus E_j$ .

Clearly,  $F_n \subseteq F_{n+1}$ . By part 3,

$$\mu \left( \bigcup_{n=1}^{\infty} F_n \right) = \lim_{n \rightarrow \infty} \mu(F_n)$$

and

$$F_j = E_1 \setminus E_j \implies \bigcup_{n=1}^{\infty} F_j = E_1 \setminus \bigcap_{n=1}^{\infty} E_j$$

(by  $E_1 \supset E_2 \supset \dots$ )

So

$$\mu \left( E_1 \setminus \bigcap_{j=1}^{\infty} E_j \right) = \lim_{n \rightarrow \infty} \mu(E_1 \setminus \bigcap_{j=1}^n E_j)$$

By Part 1,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu(E_1 \setminus \bigcap_{j=1}^n E_j) &= \lim_{n \rightarrow \infty} \left[ \mu(E_1) - \mu\left(\bigcap_{j=1}^n E_j\right) \right] \\ &= \lim_{n \rightarrow \infty} [\mu(E_1) - \mu(E_n)] \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \end{aligned}$$

## Constructing Measures

We have showed that finding measures is hard in general. Let's construct them instead.

**Outer Measure:** Let  $X$  be a set and  $\mu^* : P(X) \rightarrow [0, \infty]$  be an outer measure if

1.  $\mu^*(\emptyset) = 0$
2. (Monotonicity)  $E \subseteq F \implies \mu^*(E) \leq \mu^*(F)$
3. (Subadditivity)  $\mu^* \left( \bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{\infty} \mu^*(E_j)$

(note: this is *almost* a measure and would be if we allowed additivity rather than subadditivity)



**Proposition:** Let  $\mathcal{E} \subseteq P(X)$  such that  $X, \emptyset \in \mathcal{E}$ . Define  $\rho : \mathcal{E} \rightarrow [0, \infty]$  with  $\rho(\emptyset) = 0$ .  $\forall A \subseteq P(X)$ , let

$$\mu^*(A) = \inf \left\{ \sum_{i=0}^{\infty} \rho(E_j) \mid E_j \in \mathcal{E}, A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}$$

(i.e. take the inf of the sum of all coverings of  $A$ ). Then  $\mu^*$  is an outer measure.

*Proof:*

First note that  $\mu^*$  is well-defined: certainly  $A \subseteq X$  so the set will not be empty and the inf is well defined.

Clearly,  $\mu^*(\emptyset) = 0$  because  $\rho(\emptyset) = 0$ .

(Monotonicity) Let  $A \subseteq B$  and  $\{E_j \in \mathcal{E}\}_1^{\infty}$  be any covering of  $B$ . Since  $A \subseteq B$ ,

$$\mu^*(A) \leq \sum_{n=0}^{\infty} \rho(E_j)$$

Taking the inf,

$$\mu^*(A) \leq \inf \left\{ \sum_{i=1}^{\infty} \rho(E_j) \right\} \subseteq \mu^*(B)$$

(Subadditivity) Take  $\bigcup_{j=1}^{\infty} A_j$  for all  $A_j \in P(X)$ .

By definition of inf,  $\forall \varepsilon > 0$  there exists  $E_{jk} \subseteq \mathcal{E}$  such that

$$\sum_{i=1}^{\infty} \rho(E_{jk}) \leq \mu^*(A_j) + \frac{\varepsilon}{2^j}$$

so

$$\bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{jk} E_{jk}$$

for  $E_{jk} \in \mathcal{E}$ .

Then

$$\begin{aligned}\mu^* \left( \bigcup_{j=1}^{\infty} A_j \right) &\leq \sum_{j,k}^{\infty} \rho(E_{jk}) \\ &\leq \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon\end{aligned}$$

Then certainly,

$$\mu^* \left( \bigcup_{j=1}^{\infty} A_j \right) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$$

**$\mu^*$ -measurable (Carathéodory Criterion):** a collection  $\mathcal{M}$  of subsets of  $X$  is  $\mu^*$ -measurable if, given  $A \in \mathcal{M}$ , for all  $E \subseteq P(X)$ ,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

And in fact, it suffices to show

$$\mu^*(E) \geq \mu^*(A \cup E) + \mu^*(E \cap A^c)$$

## 2.4 Oct 01

**Carathéodory Procedure:** If  $\mu^*$  is an outer measure and  $\mathcal{M}$  are  $\mu^*$ -measurable sets, then  $\mathcal{M}$  is a  $\sigma$ -algebra and  $\mu^*|_{\mathcal{M}}$  is a measure on  $\mathcal{M}$

*Proof:*

STEP 1.  $A \in \mathcal{M}$ ,  $A^c \in \mathcal{M}$  by definition

---

STEP 2. Let  $A, B \in \mathcal{M}$  and  $A \cup B \in \mathcal{M}$ . Then

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B), \quad A \cap B = \emptyset$$

Since  $\mu^* < \infty$  by definition, it suffices to show

$$\begin{aligned}\mu^*(E) &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)\end{aligned}$$

Since  $E \in \mathcal{M}$  satisfies the Carathéodory Criterion (by assumption),

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c)\end{aligned}$$

Now consider  $A \cup B$ . Using set algebra,

$$A \cup B = (A \cap B) \cup (A \cap B^c) \cup (B \cap A^c)$$

(and this matches the first three terms above very nicely)

Then

$$E \cap (A \cup B) = (E \cap A \cap B) \cup (E \cap A \cap B^c) \cup (E \cap B \cap A^c)$$

By subadditivity of the outer measure,

$$\mu^*(E \cap (A \cup B)) \leq \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap B \cap A^c)$$

Further,

$$\begin{aligned}\mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ \geq \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap B \cap A^c)\end{aligned}$$

which is just

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$$

Now take  $\mu^*(A \cup B)$ . Using the above,

$$\mu^*(A \cup B) = \mu^*(A \cup B \cap A) + \mu^*(A \cup B \cap A^c) = \mu^*(A) + \mu^*(B)$$

which is what we wanted to show.

Now we can inductively extend this pairwise additivity to a finite union.

Let  $A_i \in \mathcal{M}$  and

$$\bigcup_{i=1}^N A_i \in \mathcal{M}$$

with  $A_i \cap A_j = \emptyset$  for  $i \neq j$

By induction,

$$\mu^* \left( \bigcup_{i=1}^N A_i \right) = \sum_{i=1}^N \mu^*(A_i)$$

---

STEP 3 (Countable Additivity): Let  $A_i \in \mathcal{M}$  and  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$  with  $A_i \cap A_j = \emptyset$  for  $i \neq j$ .

Define

$$B_n = \bigcup_{j=1}^n A_j$$

Take a test set  $E$  with  $\mu^*(E) < \infty$ .

By induction on the Carathéodory Criterion,

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap B_j) + \mu^*(E \cap B_j^c) \\
&= \mu^*\left(E \cap \bigcup_{j=1}^n A_j\right) + \mu^*\left(E \cap \bigcap_{j=1}^n A_j^c\right) \\
&= \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*\left(E \cap \bigcap_{j=1}^n A_j^c\right) \quad \text{Step 2} \\
&\geq \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*\left(E \cap \bigcap_{j=1}^{\infty} A_j^c\right) \\
&\geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*\left(E \cap \bigcap_{j=1}^n A_j^c\right) \\
&\geq \mu^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) + \mu^*\left(E \cap \bigcap_{i=1}^{\infty} A_j^c\right) \quad \text{Subadditivity}
\end{aligned}$$


---

STEP 4 (Completeness) Let  $\mu^*(A) = 0$ , then  $A \in \mathcal{M}$ .

Take any  $E \subseteq P(X)$ . We want to show

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

But by monotonicity,

$$\mu^*(E \cap A) \leq \mu^*(A) = 0 \implies \mu^*(E \cap A) = 0$$

Using monotonicity again, we get the inequality. Hence, every set with outer-measure 0 is in  $\mathcal{M}$ .

Now take  $A_1 \subseteq A$ . By monotonicity,  $\mu^*(A_1) = 0 \in \mathcal{M}$

**Completeness:** A measure space  $(X, \mathcal{M}, \mu)$  is complete if  $\forall A \in \mathcal{M}$  with  $\mu(A) = 0$ , then  $B \in \mathcal{M}$  for all  $B \subseteq A$ .

**Exercise:** Use the Carathéodory Procedure to produce the Hausdorff Measure (HW 4)

## Lebesgue Measure

On the real numbers, it would be very nice to have a measure  $\mu$  such that  $\mu((a, b)) = \rho(a, b) = b - a$ .

**Lemma:** If  $A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i)$ ,

$$\mu^*(A) = \inf \sum_{n=1}^{\infty} \rho(a_i, b_i)$$

and  $\mu^*((a, b)) = b - a$ .

Then using the Carathéodory process, we get the Lebesgue Measure on  $(\mathbb{R}, \mathcal{M}, \mu)$

**Proposition (Faithfulness of the Lebesgue Measure):** Let  $I$  be any interval (closed, open, half-open, etc.) on  $\mathbb{R}$ . Then  $\mu(I) = \rho(I)$

*Proof:*

STEP 1. Suppose  $I = [a, b]$  is closed and finite.  $\forall \varepsilon > 0$ , consider  $(a - \varepsilon, b + \varepsilon) \supset [a, b]$ .

By definition of inf,

$$\mu^*([a, b]) \leq \rho((a - \varepsilon, b + \varepsilon)) = b - a + 2\varepsilon$$

But by arbitrariness of  $\varepsilon$ ,  $\mu^*([a, b]) \leq b - a$

On the other hand, take  $\bigcup_{i=1}^{\infty} (a_i, b_i) \supseteq [a, b]$ . By Heine-Borel, there exists a finite cover for  $[a, b]$  so we can take

$$[a, b] \subseteq \bigcup_{i=1}^N (a_i, b_i)$$

We want to show that

$$\sum_{i=1}^N (b_i - a_i) \geq b - a \implies \mu^*([a, b]) \geq b - a$$

$a$  must be in some open interval, so call it  $a \in (a_1, b_1)$ . WLOG, suppose  $b_1 \leq b$ . But then  $b_1 \in (a_2, b_2)$ . If  $b_2 > b$ , then  $(a_1, b_1) \cup (a_2, b_2)$  would cover  $[a, b]$  and

$$b_2 - a_2 + b_1 - a_1 \geq b - a$$

Therefore, assume  $b_2 \leq b$ . Inductively define  $b_n \in (a_{n+1}, b_{n+1})$ . But because we have a finite cover, this process is not infinite, i.e. for some  $N$ ,  $b_N > b$ .

Then it suffices to show

$$b_N - a_N + b_{N-1} - a_{N-1} + \cdots + b_1 - a_1 \geq b - a$$

and by construction, each  $-a_i + b_{i-1} > 0$

---

STEP 2. Now take any interval  $I$ . Consider

$$[a + \varepsilon, b - \varepsilon] \subseteq I \subseteq (a - \varepsilon, b + \varepsilon)$$

But this is an open cover so

$$\mu^*(I) \leq b - a + 2\varepsilon$$

But in part 1, we showed that

$$b - a - 2\varepsilon \leq \mu^*(I)$$

So by arbitrariness of  $\varepsilon$ ,  $\mu^*(I) = b - a$

## 2.5 Oct 03

**Corollary:** If  $A$  is a countable subset of  $\mathbb{R}$ ,  $\mu^*(A) = 0$ . Furthermore,  $[0, 1]$  is not countable.

*Proof:*  $\forall x \in \mathbb{R}$ ,  $\{x\} = (x - \varepsilon, x + \varepsilon)$ , so

$$\mu^*(\{x\}) \leq 2\varepsilon \implies \mu^*(\{x\}) = 0$$

Now suppose  $A = \bigcup_{n=1}^{\infty} a_n$  with  $a_n \in \mathbb{R}$ .

By subadditivity,

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(\{a_n\}) = 0$$

---

Since  $\mu^*([0, 1]) = 1 \neq 0$ ,  $[0, 1]$  is not countable.

**Proposition:**  $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}$

*Proof:* By the characterization of the Borel set on  $\mathbb{R}$ , it suffices to show that  $(a, \infty) \in \mathcal{M}$  for all  $a \in \mathbb{R}$ .

$\forall E \in \mathcal{M}$  with  $\mu^*(E) < \infty$ , we want to show

$$\begin{aligned} \mu(E) &\geq \mu^*(E \cap (a, \infty)) + \mu^*(E \cap (a, \infty)^c) \\ &= \mu^*(E \cap (a, \infty)) + \mu^*(E \cap (-\infty, a]) \end{aligned}$$

For notational convenience, let

$$\begin{aligned} E_1 &= E \cap (a, \infty) \\ E_2 &= E \cap (-\infty, a] \end{aligned}$$

Let  $\varepsilon > 0$ . By the sharpness of the outer measure,  $\exists \bigcup_{n=1}^{\infty} I_n$  such that

$$E \subseteq \bigcup_{n=1}^{\infty} I_n$$



and

$$\sum_{n=1}^{\infty} |I_n| < \mu^*(E) + \varepsilon$$

Then

$$E_1 \subseteq \bigcup_{n=1}^{\infty} I_n \cap (a, \infty)$$
$$E_2 \subseteq \bigcup_{n=1}^{\infty} I_n \cap (-\infty, a]$$

so

$$\mu^*(E_1) \leq \sum_{n=1}^{\infty} \mu^*(I_n \cap (a, \infty))$$
$$\mu^*(E_2) \leq \sum_{n=1}^{\infty} \mu^*(I_n \cap (-\infty, a])$$

Now by the faithfulness of the Lebesgue Measure,

$$\mu(I_n) = \mu^*(I_n \cap (a, \infty)) + \mu^*(I_n \cap (-\infty, a])$$

so

$$\mu^*(E_1) + \mu^*(E_2) \leq \sum_{n=1}^{\infty} \mu(I_n) \leq \mu^*(E)$$

## Transformations

**Definitions:** Given  $E \subseteq \mathbb{R}$ , we define

- (Translation)  $E + a := \{x + a : x \in E\}$
- (Dilation)  $rE := \{rx : x \in E\}$

**Lemma:** For the Lebesgue outer measure and  $E \in \mathcal{M}$ ,

1.  $\mu^*(E + a) = \mu^*(E)$
2.  $\mu^*(rE) = |r| \mu^*(E)$

*Proof Sketch:* Let  $E \subseteq \bigcup_{n=1}^{\infty} I_n$ .

Certainly,

$$\begin{aligned} E + a &\subseteq \bigcup_{n=1}^{\infty} \{I_n + a\} \\ rE &\subseteq \bigcup_{n=1}^{\infty} \{|r| I_n\} \end{aligned}$$

Then

$$\sum_{n=1}^{\infty} \rho(I_n) = \sum_{n=1}^{\infty} \rho(I_n + a) \geq \mu^*(E + a)$$

And taking the infimum,

$$\mu(E) \geq \mu^*(E + a)$$

For dilation, notice  $|I_n| = \frac{1}{|r|} |rI_n|$ . The result follows similarly.

The other direction is exactly the same.

## Approximation of Measurable Sets

**Lemma:**

1. (Approximation from Above)  $\forall E \subseteq P(X)$  and  $\forall \varepsilon > 0$ , then exists an open set  $O$  such that  $E \subseteq O$  and

$$\mu(O) \geq \mu^*(E) \geq \mu(O) - \varepsilon$$

2. (Approximation from Below)  $\forall E \subseteq \mathcal{M}$  and  $\forall \varepsilon > 0$ ,  $\exists K$  closed such that

$$\mu(K) \leq \mu(E) \leq \mu(K) + \varepsilon$$

*Proof:*

1. By definition of  $\mu^*(E)$ ,  $\exists O = \bigcup_{n=1}^{\infty} I_n \supset E$  such that

$$\mu^*(E) \geq \sum_{n=1}^{\infty} \rho(I_n) - \varepsilon$$

and by subadditivity,  $\mu^*(E) \geq \mu^*(O) - \varepsilon$ .

2. Assume  $E \subseteq [a, b]$ . Consider  $E^c \cap [a, b]$ . By part 1,  $\exists O \supset E^c \cap [a, b]$  such that

$$\mu^*(E^c \cap [a, b]) \geq \mu^*(O) - \varepsilon$$

and with some algebra,

$$|b - a| - \mu^*(E^c) \leq |b - a| - \mu^*(O) + \varepsilon$$

By measurability,

$$\begin{aligned} \mu(E) &\leq |b - a| - \mu^*(O \cap [a, b]) - \mu^*(O \cap [a, b]^c) + \varepsilon \\ &\leq |b - a| - \mu^*(O \cap [a, b]) \\ &= \mu^*([a, b] \cap O^c) + \varepsilon \end{aligned}$$

**Exercise:** Complete the case  $E \not\subseteq [a, b]$

# Chapter 3

## Measurable functions

### 3.1 Oct 08

**Measurable function:** Let  $(X, \mathcal{M}, \mu)$  be a measure space. Let  $f : X \rightarrow \mathbb{R}$ .  $f$  is *measurable* iff  $\forall \alpha \in \mathbb{R}$ ,

$$\{x \in X, f(x) > \alpha\} \in \mathcal{M}$$

Equivalently,  $f$  is measurable iff  $\forall \alpha \in \mathbb{R}$ ,  $\{f \geq \alpha\}$  is measurable.

**Proposition:** If  $f, g$  are measurable, so is

(a)  $f + c$

(b)  $f + g$

(c)  $fg$

(d)  $cf$

for  $c \in \mathbb{R} \setminus \{0\}$

*Proof:* (a.) For all  $\alpha \in \mathbb{R}$ ,

$$\{x : f(x) + c > \alpha\} = \{x : f(x) > \alpha - c\} \in \mathcal{M}$$

,

(b.) Consider  $\{x : f(x) + g(x) > \alpha\}$ . We claim

$$\{x : f(x) + g(x) > \alpha\} = \bigcup_{q \in \mathbb{Q}} \{f(x) > \alpha - q\} \cap \{g(x) > q\}$$

*Proof:* Certainly,

$$\{x : f(x) + c > \alpha\} = \{x : g(x) > \alpha - f(x)\}$$

and for fixed  $x$ , we can invoke the density and countability of the rational numbers...

(c.) Consider  $\{f^2 > \alpha\} = \{f > \sqrt{\alpha}\} \cup \{f < -\sqrt{\alpha}\}$ . So  $f^2$  is measurable. But then

$$fg = \frac{1}{2} [(f + g)^2 - (f - g)^2]$$

The rest follows from (b.)

(d.) Let  $g(x) = c$  and the result follows from (c.)

**Proposition:** If  $\{f_n\}$  are measurable, so are

- (a)  $\max_i f_i(x)$
- (b)  $\min_i f_i(x)$
- (c)  $\sup_n f_n$
- (d)  $\inf_n f_n$
- (e)  $\limsup_n f_n$
- (f)  $\liminf_n f_n$

*Proof:*

(a.) Suppose  $n < \infty$ . Then  $\max_i f_i(x) > \alpha \iff \exists 1 \leq i_0 \leq n$  such that  $f_{i_0}(x) > \alpha$ .

Hence

$$\{x : \max_i f_i(x) > \alpha\} = \bigcup_{i=1}^n \{f_i(x) > \alpha\}$$

which are measurable by assumption.

(b.) Analogously,

$$\{x : \min_i f_i(x) > \alpha\} = \bigcap_{i=1}^n \{f_i(x) > \alpha\}$$

(c.) Now suppose we have countably many  $f_i$ . We claim

$$\{\sup_n f_n(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{f_n(x) > \alpha\}$$

*Proof:* If  $x \in \bigcup_{i=1}^{\infty} \{f_i(x) > \alpha\}$ , then  $\exists f_{i_0}(x) > \alpha$  so  $\sup_n f_n(x) > \alpha$ .

Now we want to show that if  $\sup_n f_n(x) > \alpha$ ,  $\exists i_0$  such that  $f_{i_0}(x) > \alpha$ .  
Suppose not.

Then  $f_{i_0}(x) \leq \alpha \implies \sup_n f_n(x) \leq \alpha$ , a contradiction.

(d.) Now we want to show  $\inf_n f_n > \alpha$

**Warning:** when taking an infinite sequence, the strict inequality may not be preserved

**Fact (Sup/Inf Parity):**

$$\inf_n f_n = -\sup_n \{-f_n\}$$

$$\sup_n f_n = -\inf_n \{-f_n\}$$

and

$$\limsup_n f_n = -\liminf_n \{-f_n\}$$

$$\liminf_n f_n = -\limsup_n \{-f_n\}$$

**Exercise:** Prove the fact above

The proof of (d.) follows from (c.)

## Preparations for Integration

**Characteristic Function:** Given  $E \in \mathcal{M}$ , we define

$$\chi_E = \mathbb{1}_E = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

**Homework:** Show  $\mathbb{1}_E$  is measurable

And now we can define an integral

$$\int_x \chi_E dx = \mu(E)$$

**Simple functions:** Let  $E_1, \dots, E_n \in \mathcal{M}$ . We define *simple functions*

$$\phi = a_1 \chi_{E_1} + \dots + a_n \chi_{E_n}$$

with *standard representation*

$$\phi = \sum_{i=1}^n a_i \chi_{E_i}$$

where  $a_i < \infty$  and  $a_i \neq a_j$  for  $E_i \cap E_j = \emptyset$

Now we define a new integral

$$\int \phi := \sum_{i=1}^n a_i \mu(E_i)$$

**Remark:** The integral  $\int_x \chi_E dx = \mu(E)$  corresponds to the Riemann integral and works by dividing the domain.  $\int \phi$  is the *Lebesgue integration* and partitions the range of the function instead

**Theorem:** Let  $(X, \mathcal{M})$  be a measurable space

- (a) If  $f : X \rightarrow [0, \infty]$  is measurable, then  $\exists \{\phi_n\}$  of simple functions such that

$$0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f$$

Further,  $\phi_n \rightarrow f$  pointwise on  $X$  and uniformly on any set which  $f$  is bounded.

- (b) If  $f : X \rightarrow \mathbb{R}$  is measurable, then  $\exists \{\phi_n\}$  of simple functions such that

$$0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq |f|$$

and  $\phi_n \rightarrow f$  pointwise on  $X$  and uniformly on any set where  $f$  is bounded.

*Proof:*

- (a) (Proof by construction) Fix  $n$  and choose  $0 \leq k \leq 2^{2n} - 1$ .

Define

$$E_n^k = \{x : \frac{k}{2^n} < f(x) \leq \frac{k+1}{2^n}, \quad 0 \leq k \leq 2^{2n} - 1\}$$

$$F_n = \{x : f(x) \geq 2^n\}$$



So we can choose

$$\phi_n(x) = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \chi_{E_n^k} + 2^n \chi_{F_n}$$

(Each  $n$  subdivides the range and then rounds down to each subdivision. As we iterate, each range is further divided in such a way that  $\phi_{n+1} > \phi_n$  for all  $n$  because the value on  $E_{n+1}$  is rounded down to a higher value than  $E_n$ . )

So by construction,

$$0 \leq \phi_n \leq \phi_{n+1} \leq f(x)$$

If we fix any  $x < \infty$ , for  $f(x) < 2^{n_0}$  and  $n \geq n_0$ , we have

$$|\phi_n - f(x)| \leq |(k+1)2^{-n} - k2^{-n}| = \frac{1}{2^n}$$

i.e., we have uniform convergence on any set where  $f$  is bounded.

- (b) For a general function  $f$ , we can write  $f = f^+ - f^-$  (so  $|f| = f^+ + f^-$ ) where  $f^+ = \max\{f, 0\}$  and  $f^- = \max\{-f, 0\}$ . Since  $f^+$  and  $f^-$  are nonnegative, we can apply part (a.) to each.

In particular,

$$0 \leq |f_n| \leq |f_{n+1}| \leq |f(x)|$$

and the rest follows.

**Conclusion:** these simple functions can approximate *any* measurable function in this very strong sense.

## 3.2 Oct 10

### Integration of Non-negative functions

In this section, assume  $f \geq 0$ .

For any  $f : E_i \rightarrow [0, \infty]$ , we consider the simple functions  $\phi = \sum_{i=1}^n a_i \chi_{E_i}$  in standard representation ( $a_i, a_j$  distinct when  $E_i, E_j$  disjoint)

Then, as above, the integral is defined as

$$\int \phi \, d\mu = \sum_{i=1}^n a_i \mu(E_i)$$

with respect to measure  $\mu$

**Proposition:** If  $\phi$  and  $\psi$  are two simple functions,

1. If  $c \geq 0$ ,  $\int c\phi \, d\mu = c \int \phi \, d\mu$
2.  $\int \phi + \psi \, d\mu = \int \phi \, d\mu + \int \psi \, d\mu$
3. If  $\phi \leq \psi$ , then  $\int \phi \, d\mu \leq \int \psi \, d\mu$

*Proof:*

1) is trivial

2) Let  $\phi = \sum_{i=1}^n a_i \chi_{E_i}$  and  $\psi = \sum_{j=1}^m b_j \chi_{F_j}$ . Then

$$\begin{aligned} \int \phi \, d\mu + \int \psi \, d\mu &= \sum_{i=1}^n a_i \mu(E_i) + \sum_{j=1}^m b_j \mu(F_j) \\ &= \sum_{i=1}^n a_i \mu(E_i \cap (\bigcup_{j=1}^m F_j)) + \sum_{j=1}^m b_j \mu(F_j \cap E_i) \\ &= \sum_{i=1}^n a_i \mu(E_i \cap F_j) + \sum_{j=1}^m b_j \mu(E_i \cap F_j) \\ &= \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j) \mu(E_i \cap F_j) \end{aligned}$$

Now we can cut  $\phi = \sum_{i=1}^n a_i \chi_{E_i \cap \bigcup_{j=1}^n F_j}$  and  $\psi$  similarly.

**Lemma:**  $\chi_E + \chi_F = \chi_{E \cup F}$

*Proof:* Exercise

So

$$\begin{aligned}\psi + \phi &= \sum_{i,j} a_i \chi_{E_i \cap F_j} + \sum_{i,j} b_j \chi_{E_i \cap F_j} \\ &= \sum_{i,j} (a_i + b_j) \chi_{E_i \cap F_j}\end{aligned}$$

However, integration is only defined in standard representation. But  $E_i, E_j$  are pairwise disjoint so  $(a_i + b_i)$  distinct.

By definition,

$$\psi + \phi = \sum (a_i + b_j) \mu(E_i \cap F_j) = RHS$$

3) **Claim:**  $\phi \leq \psi$  iff  $a_i \leq b_i$  on  $\chi_{E_i \cap E_j}$

Exercise

The result follows.

**Integrals on subsets:** Let  $A \in \mathcal{M}$ . Then

$$\int_A \phi \, d\mu = \int_X \chi_A \cdot \phi \, d\mu$$

with  $\chi_{A_1} \cdot \chi_{A_2} = \chi_{A_1 \cap A_2}$

**Proposition:** For  $A \in \mathcal{M}$ , the mapping  $A \rightarrow \int_A \phi \, d\mu$  is a measure on  $\mathcal{M}$

*Proof:*

The hardest part is to show that  $\int_A \phi \, d\mu$  is countably additive, that is

$$\int_{\bigcup A_i} \phi = \sum_{i=1}^{\infty} \int_{A_i} \phi$$

By definition,

$$\begin{aligned}\int_{\bigcup A_i} \phi &= \int \chi_{\bigcup A_i} \cdot \phi \\ &= \sum_{i=1}^n a_i \mu(E_i \cap \bigcup_{j=1}^{\infty} A_j)\end{aligned}$$

So it suffices to show that

$$\sum_{i=1}^n a_i \mu(E_i \cap \bigcup_{j=1}^{\infty} A_j) = \sum a_n \left[ \sum_{i=1, k=1}^{\infty} \mu(E_i \cap A_j) \right]$$

and in fact all we need is

$$\mu(E_i \cap \bigcup_{j=1}^{\infty} A_j) = \sum_{ij} \mu(E_i \cap A_j)$$

which follows from countable additivity of the measure.

**Integral of general functions:** For a general function  $f \geq 0$  on  $(X, \mathcal{M}, \mu)$ , we define

$$\int f \, d\mu = \sup_{\phi \leq f} \int \phi \, d\mu$$

for all simple functions  $\phi$

**Properties:** For  $f \geq 0$ , we have the following properties:

1.  $c \int f \, d\mu = \int cf \, d\mu$
2. If  $f \leq g$ , then  $\int f \, d\mu \leq \int g \, d\mu$
3.  $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$

*Proof:*

1. We have a 1-1 correspondence between  $cf$  and  $f$  so the result follows from the same proof as above
2. Follows from sup and definition

3. The third one is much harder and we will need to come back to it.

**Theorem (Monotone Convergence Theorem):** Assume  $0 \leq f_n \leq f_{n+1}$  for  $f$  measurable. Then

$$\int \lim_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$$

*Proof:*

$\lim_{n \rightarrow \infty} f_n$  is increasing so we can say  $\lim_{n \rightarrow \infty} f_n = f$  in the extended reals and the LHS is well defined.

From monotonicity,

$$\lim_{n \rightarrow \infty} \int f_n \leq \int f$$

The other direction is harder.

Take any simple function  $\phi \leq f$  and pick  $0 < \alpha < 1$ . Consider

$$E_n = \{x : f_n(x) \geq \alpha\phi, \phi > 0\}$$

for fixed  $x$ .

Since  $f_n$  is non-decreasing,  $E_n \subseteq E_{n+1}$ . We claim

$$\bigcup_{n=1}^{\infty} E_n = \{f(x) > \alpha\phi(x), \phi > 0\} := E$$

$\subseteq$  is clear. For  $\supseteq$ , notice  $\forall x, f_n \rightarrow f > \alpha\phi(x) \implies \exists n_0$  such that for  $n \geq n_0$ ,  $f_n(x) > \alpha\phi(x)$ .

Now take

$$\int f_n \geq \int f_n \chi_{E_n} = \int_{E_n} f_n > \alpha \int_{E_n} \phi$$

By the proposition above, we can take  $\nu(E_n) = \int_{E_n} \phi$  to be a measure so

$$\alpha \int_{E_n} \phi = \alpha \nu(E_n) \xrightarrow{n \rightarrow \infty} \alpha \nu\left(\bigcup_{n=1}^{\infty} E_n\right)$$

Taking the limit, we cannot keep the strict inequality, but we do have

$$\int f_n \geq \int f_n \chi_{E_n} \geq \alpha \nu \left( \bigcup_{n=1}^{\infty} E_n \right) \alpha \nu(E)$$

and the integral of  $E$  is just

$$\alpha \nu(E) = \alpha \int \phi \chi_{\{f(x) > \alpha \phi(x), \phi > 0\}}$$

Since  $\phi \leq f$ , for any  $x$ ,  $\phi(x) > 0$  so  $\alpha \phi(x) < f(x)$ . Note that for  $\phi(x) = 0$ ,  $\int_{\phi=0} \phi = 0$  so

$$\alpha \int \phi \chi_{\{f(x) > \alpha \phi(x), \phi > 0\}} = \alpha \int \phi$$

**Exercise:** Check  $\int \phi \chi_{\{f(x) > \alpha \phi(x), \phi > 0\}} = \alpha \int \phi$  using the standard expression

But by definition,

$$\int f \geq \sup_{\phi \leq f} \int \phi = \int f$$

**Fatou's Lemma:** Let  $0 \leq f_n$ . Then

$$\int \liminf f_n d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu$$

*Proof:* By definition,

$$\int \liminf f_n = \sup_n \int \inf_{k \geq n} f_k$$

Note  $\inf_{k \geq n} f_k$  is increasing in  $n$ .

We claim

$$\sup_n \int \inf_{k \geq n} f_k = \lim_{n \rightarrow \infty} \int \inf_{k \geq n} f_k$$

**Exercise:** Check the above claim

And then by the Monotone Convergence Theorem,

$$\int \lim f_n = \int \lim_{n \rightarrow \infty} \inf_{k \geq n} f_k = \lim_{n \rightarrow \infty} \int \inf_{k \geq n} f_k$$

For any  $m \geq n$ ,

$$\int \inf_{k \geq n} f_k \leq \int f_m$$

but this is true for all  $n$  so

$$\int \inf_{k \geq n} f_k \leq \inf_{k \geq n} \int f_k$$

Taking limits,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int \inf_{k \geq n} f_k &\leq \lim_{n \rightarrow \infty} \inf_{k \geq n} \int f_k \\ &= \liminf \int f_k \end{aligned}$$

### 3.3 Oct 17

**Corollary of MCT:** If  $f_n \geq 0$ ,

$$\sum_{n=1}^{\infty} \int f_n d\mu = \int \sum_{n=1}^{\infty} f_n d\mu$$

*Proof:* Define  $F_N = \sum_{n=1}^N f_n$ . Clearly  $F_N$  is increasing so by MCT,

$$\lim_{N \rightarrow \infty} \int F_N d\mu = \int \lim_{N \rightarrow \infty} F_N d\mu$$

and we are done.

Now at last we can verify that the integral is additive for general functions by

just taking the finite sum and applying this corollary.

**Theorem (Lebesgue Dominated Convergence):** Assume  $f_n \geq 0$ ,  $f_n \leq g$  for  $g$  fixed function with  $\int g \, d\mu < \infty$ . If  $\lim_{n \rightarrow \infty} f_n = f$ , then

$$\lim_{n \rightarrow \infty} \int f_n \, d\mu = \int f \, d\mu$$

*Proof:* By Fatou,

$$\int \liminf f_n \, d\mu \leq \liminf \int f_n \, d\mu$$

but  $\liminf f_n = f$  so

$$\int f \, d\mu \leq \liminf \int f_n \, d\mu$$

So it suffices to show

$$\limsup \int f_n \, d\mu = \liminf \int f_n \, d\mu = \int f \, d\mu$$

Further,  $g - f_n \geq 0$ , so again by Fatou

$$\int \liminf (g - f_n) \, d\mu \leq \liminf \int (g - f_n) \, d\mu$$

but  $\liminf f_n = f$  so

$$\int \liminf (g - f_n) \, d\mu = \int (g - f) \, d\mu$$

On the RHS,

$$\begin{aligned} \liminf \int (g - f_n) \, d\mu &= \liminf \left[ \int g \, d\mu - \int f_n \, d\mu \right] \\ &\stackrel{*}{=} \int g \, d\mu + \liminf \left[ - \int f_n \, d\mu \right] \end{aligned}$$

**Exercise:** Verify (\*) above



Now using the parity of the  $\liminf$ ,

$$RHS = \int g \, d\mu - \limsup \int f_n \, d\mu$$

Then,

$$\begin{aligned} \int g - \int f &\leq \int g - \limsup \int f_n \\ \int f &\geq \limsup \int f_n \end{aligned}$$

**Warning:** that subtraction only works because  $\int g$  is finite by assumption

All together,

$$\limsup \int f_n \, d\mu \leq \int f \, d\mu \leq \limsup \int f_n \, d\mu$$

**Integrable function:**  $f$  is *integrable* if  $\int_X f \, d\mu < \infty$

## Zero Measure Sets

**Almost Everywhere:** We say a statement  $S$  is valid *almost everywhere* if  $\exists E, \mu(E) = 0$  such that  $S$  is valid on  $X \setminus E$

**Lemma:**

1. If  $\mu(E) = 0$ , then  $\int_E f = 0$  for any  $f \geq 0$
2. If  $\int_X f \, d\mu = 0$  for  $f \geq 0$ , then  $f = 0$  almost everywhere
3. If  $\int_X f \, d\mu < \infty$ , then  $f < \infty$  almost everywhere

*Proof:*

1. Choose any simple function  $\phi \leq f$ . Notice

$$\begin{aligned}
\int_E \phi \, d\mu &= \int_X \mathbb{1}_E \cdot \phi \, d\mu \\
&= \int_X \mathbb{1}_E \sum_{i=1}^m a_i \mathbb{1}_{E_i} \, d\mu \\
&= \int_X \sum_{i=1}^m a_i \mathbb{1}_{E_i \cap E} \, d\mu \\
&= \sum_{i=1}^m a_i \mu(E_i \cap E) \\
&\leq \mu(E) = 0
\end{aligned}$$

So

$$\int_E f = \sup_{\phi \leq f} \int_E \phi = 0$$

for  $\phi$  simple function.

**Remark:** This depends on our convention that  $0 \cdot \infty = 0$

2. Consider  $E = \{x : f(x) > 0\}$ . It suffices to show  $\mu(E) = 0$  (because  $E^c = \{x : f = 0\}$ ).

We can write

$$E = \bigcup_{n=1}^{\infty} \{x : f(x) \geq \frac{1}{n}\} = \bigcup_{n=1}^{\infty} E_n$$

so

$$\int_X f \, d\mu \geq \int_X \mathbb{1}_{E_n} \, d\mu = \int_{E_n} f \, d\mu$$

and by monotonicity,

$$\int_{E_n} f \, d\mu \geq \int_{E_n} \frac{1}{n} d\mu = \frac{1}{n} \int_{E_n} d\mu = \frac{1}{n} \mu(E_n)$$

But  $\int_X f \, d\mu = 0$  so  $\mu(E_n) = 0$  for all  $n$  and

$$\mu(E) = \lim_{n \rightarrow \infty} \mu(E_n) = 0$$

3. This statement is equivalent to the claim that  $F = \{f(x) : f(x) = \infty\}$  has measure zero.

Let  $F_n = \{x : f(x) \geq n\}$  so  $F \subseteq F_n$  for all  $n$ , and in fact,  $F = \bigcap_{n=1}^{\infty} F_n$

By exactly the same argument as before,

$$\begin{aligned} \int_X f \, d\mu &\geq \int_X \mathbb{1}_{F_n} f \, d\mu \\ &= \int_{F_n} f \, d\mu \\ &\geq n\mu(F_n) \end{aligned}$$

so

$$\mu(F_n) = \lim_{n \rightarrow \infty} \mu(F_n) = 0$$

Note that this gives a powerful fact: for all  $\int f < \infty$ ,

$$\int f \, d\mu = \int_{F^c} f \, d\mu = \int_{\{x: f < \infty\}} f \, d\mu$$

Let  $G = \{x : f < \infty\}$ . we know

$$G = \bigcup_{n=1}^{\infty} \{x : f(x) \leq n\} = \bigcup_{n=1}^{\infty} G_n$$

and  $G_n \uparrow G$  so  $f \mathbb{1}_{G_n} \rightarrow f \mathbb{1}_G$  (pointwise) and  $\mathbb{1}_{G_n} \leq \mathbb{1}_{G_{n+1}}$  so

$$\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int f \mathbb{1}_{G_n} \, d\mu = \lim_{n \rightarrow \infty} \int_{\{x: f \leq n\}} f \, d\mu$$

**Exercise:** Prove that for MCT, Fatou, and LDC it suffices to take the condition to be true almost everywhere.

For example, the MCT holds if  $f_n \leq f_{n+1}$  almost everywhere.

## Integration for General Functions

For any function  $f$ , we can write  $f = f^+ - f^-$  where  $f^+ = \max\{f, 0\}$  and  $f^- = \max\{-f, 0\}$ . This is powerful because both  $f^+$  and  $f^-$  are nonnegative and measurable.

**$\mathcal{L}^1$  Space:** We say  $f \in \mathcal{L}^1$  (i.e.  $f$  is integrable) if

$$\int |f| \, d\mu < \infty$$

or equivalently,

$$\int f^+ \, d\mu + \int f^- \, d\mu < \infty$$

In this case, we define

$$\int f \, d\mu = \int (f^+ - f^-) \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$$

**Proposition:** If  $f, g \in \mathcal{L}^1$  and  $a, b \in \mathbb{R}$ , then

1.  $af + bg \in \mathcal{L}^1$
2.  $\int af + bg \, d\mu = a \int f \, d\mu + b \int g \, d\mu$

*Proof:* By the triangle inequality

$$|af + bg| \leq |a| |f| + |b| |g|$$

so

$$\int |af + bg| \, d\mu \leq |a| \int |f| \, d\mu + |b| \int |g| \, d\mu < \infty$$

First let us check the easier linearity condition  $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$ .

Define  $h = f + g$  so  $f^+ - f^- + g^+ - g^- = h^+ - h^-$ . We would like to rearrange this so we have two nonnegative functions. But we need to be careful that each of these is finite. And in fact, this is true a.e. since  $f$  and  $g$  are integrable.

So

$$f^+ g^+ + h^- = h^+ + f^- + g^- \geq 0$$

which means we can integrate. Let  $X \setminus E$  be the set where all six terms are finite. Then

$$\int_{X \setminus E} h \, d\mu = \int_{X \setminus E} f \, d\mu + \int_{X \setminus E} g \, d\mu$$

## 3.4 Oct 22

### Complex-Valued Functions

$$\int f = \int \operatorname{Re}(F) + i \int \operatorname{Im}(F)$$

**Sign Functions:** Define

- $\operatorname{sign} x = x/|x|$  for  $x \in \mathbb{R} \setminus \{0\}$
- $\operatorname{sign} z = z/|z|$  for  $z \in \mathbb{C} \setminus \{0\}$

so we have the useful property  $\overline{\operatorname{sign} z} z = |z|$

**Lemma:** If  $f \in \mathcal{L}^1$ , then  $|\int f \, d\mu| \leq \int |f| \, d\mu$

*Proof:* Assume  $\int f \, d\mu \neq 0$ . For  $f$  real,

$$\left| \int f \, d\mu \right| = \left| \int f^+ \, d\mu - \int f^- \, d\mu \right|$$

Since these integrals are real numbers,

$$\left| \int f^+ \, d\mu - \int f^- \, d\mu \right| \leq \left| \int f^+ \, d\mu + \int f^- \, d\mu \right| = \left| \int (f^+ + f^-) \, d\mu \right| = \int |f| \, d\mu$$

---

If  $f$  is complex, we want to normalize it to the real case. By the property

above, we can write  $\alpha = \overline{\text{sign} \int f \, d\mu}$  so

$$\begin{aligned}
\left| \int f \, d\mu \right| &= \alpha \cdot \int f \, d\mu \\
&= \int \alpha f \, d\mu \\
&= \text{Re} \left( \int \alpha f \, d\mu \right) \\
&= \int \text{Re}(\alpha f) \, d\mu \\
&\leq \int |\alpha f| \, d\mu \\
&= \int |\alpha| |f| \, d\mu
\end{aligned}$$

Now we can invoke the LDC for non-negative functions ( $f_n = f_n^+ - f_n^-$  with  $|f_n^+| \leq |g| \implies \int f_n \, d\mu \rightarrow \int f \, d\mu$ ) and we are done.

**Lemma:** Assume  $\sum_{n=1}^{\infty} \int |f_n| \, d\mu < \infty$ . Then  $\sum_{n=1}^{\infty} f_n$  converges and

$$\sum_{n=1}^{\infty} \int f_n \, d\mu = \int \sum_{n=1}^{\infty} f_n \, d\mu$$

*Proof:* Reduces to the positive case as in the proof above.

## Approximation by $\mathcal{L}^1$ functions

**Theorem:** If  $f \in L^1(d\mu)$ ,  $\forall \varepsilon > 0$ , there exists a simple function  $\phi = \sum a_j \chi_j$  such that

$$\int |f - \phi| \, d\mu < \varepsilon$$

*Proof:* Recall the dyadic (partitioning the range) approximation  $\phi_n \rightarrow f$  pointwise for  $|\phi_n| \leq |f|$ .

By LDC,  $\int \phi_n \rightarrow \int f$  and

$$|\phi_n - f| \leq |\phi_n| + |f| \leq \int |f| \in \mathcal{L}^1$$

so

$$\lim_{n \rightarrow \infty} \int |\phi_n - f| \, d\mu = 0$$

Now  $\forall \varepsilon > 0$ ,

$$\phi_n = \sum_{i=1}^{i_n} a_i E_i, \quad E_i \cap E_j = \emptyset, \quad a_i \neq 0$$

And if  $\mu(E_j) < \infty$ ,

$$|a_j| \, \mu(E_j) = \int_{E_j} |\phi_n| \leq \int |f| \, d\mu < \infty$$

**Theorem (Reduction to smooth functions in  $\mathbb{R}$ ):** Let  $\mu$  be the Lebesgue-Stieltjes measure on  $\mathbb{R}$ . Then  $E_J$  (in the previous approximation)  $\phi_n$  can be taken as a finite union of open intervals. Moreover, there exists a  $C^\infty$  function  $\phi$  that vanishes outside a bounded interval such that

$$\int |f - \phi| \, d\mu < \varepsilon$$

*Proof:*  $\mu(E_j)$  can be approximated by an open set so

$$\int O_j \mathbb{1} < \infty = \mu(O_j) = \sum_{k=1}^{\infty} \mu(I_k)$$

with  $I_k$  disjoint.

But since  $\phi_n$  is approximated by finitely many open intervals,

$$\sum_{k=1}^N \mu(I_k) \rightarrow \mu(O_j) \rightarrow \mu(E_j)$$

But this is equivalent to

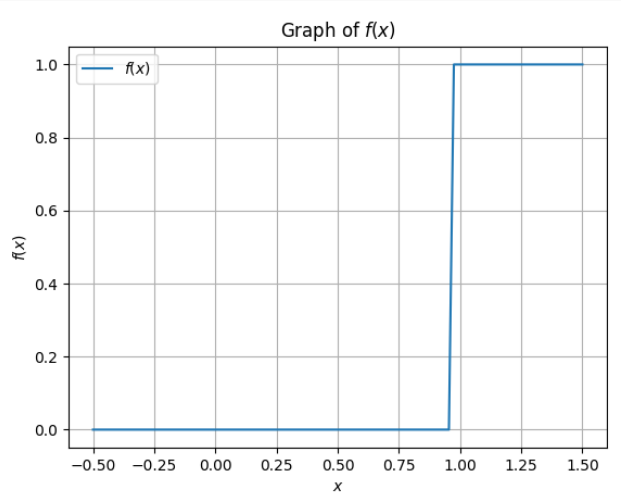
$$\sum_{k=1}^N \mu(I_k) = \sum_{I_k} \int_{I_k} d\mu \rightarrow \mu(E_j)$$

Therefore, it suffices to find a smooth function  $h$  such that

$$\int_{\mathbb{R}} |h - \mathbb{1}_{[a,b]}| < \varepsilon$$

Define the *One-sided Mollifier function*

$$f(x) = \begin{cases} \exp\left(-\frac{1}{e^{\frac{x}{1-x}} - 1}\right) & 0 < x < 1 \\ 0 & x \leq 0 \\ 1 & x \geq 1 \end{cases}$$



And with a little analysis, we can show that this is smooth:  $\partial^m f \Big|_{x=0, x=1} = 0$  for all  $m \in \mathbb{N}$ .

Define  $h_\varepsilon(x) = f(\frac{x}{\varepsilon})$  so  $0 \leq x/\varepsilon \leq 1$ . Given  $\mathbb{1}_{[a,b]}$ ,

$$g_\varepsilon(x) = \begin{cases} h(x - a - \varepsilon) & x \leq a + \varepsilon \\ 1 & a + \varepsilon \leq x \leq b \\ h_\varepsilon(b - x) & x \geq b \end{cases}$$



Clearly,  $g_\varepsilon(x) \rightarrow \mathbb{1}_{[a,b]}$  as  $\varepsilon \rightarrow 0$ .

And further,  $0 \leq g_\varepsilon(x) \leq \mathbb{1}_{[a-1,b+1]}$  for  $\varepsilon \ll 1$  so by the LDC,

$$\int g_\varepsilon(x) d\mu \rightarrow \int \mathbb{1}_{[a,b]} d\mu$$

or

$$\int |g_\varepsilon - \mathbb{1}_{[a,b]}| d\mu \rightarrow \varepsilon$$

**Remark:** we could also define a (more standard) *Symmetric Mollifier function*

$$\rho(x) = \begin{cases} \exp(-\frac{1}{1-|x|^2}) & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

so

$$j = \frac{\rho}{\int \rho d\mu} \int j d\mu = 1$$

