APMA 2110: Real Analysis

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Definitions

Power set: $P(X) = \{E : E \subseteq X\}$

Limsup/Liminf: for $\{E_n\}_{n=1}^{\infty}$

$$\limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$
$$\liminf E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$$

Set differences: Let $E, F \subseteq X$. Then,

$$E \setminus F = \{x : x \in E \wedge \ x \notin F\} E \triangle F \qquad = (E \setminus F) \cup (F \setminus E) E^c = X \setminus E$$

De Morgan's Laws:

$$\left(\bigcup_{\alpha \in A} E_\alpha\right)^c = \bigcap_{\alpha \in A} E_\alpha^c$$
$$\left(\bigcap_{\alpha \in A} E_\alpha\right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

Relation: $R \subseteq X \times Y$ such that
$$xRy \iff (x, y) \in R$$

Equivalence Relation: $\sim$ is a relation in the case $X = Y$ such that

- $x \sim x \quad \forall x \in X$
- $x \sim y \iff y \sim x$
- $x \sim y \ \wedge \ y \sim z \implies x \sim z$

Function: $f : X \to Y$ is a relation such that $\forall x \in X$, there exists a *unique* $y \in Y$ such that xRy

Images: If $D \subseteq X, E \subseteq Y$, the *image* of D under $f : X \to Y$ is

$$f(D) = \{f(x) : x \in D\}$$
$$f^{-1}(E) = \{x : f(x) \in E\}$$

further, X is the *domain* of f and Y is the *codomain* of f. The *range* of f is $f(X)$.

Inverses:

$$f^{-1}\left(\bigcup_{\alpha\in A} E_\alpha\right) = q\bigcup_{\alpha\in A} f^{-1}(E_\alpha)$$
$$f^{-1}\left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A} f^{-1}(E_\alpha)$$
$$f^{-1}(E^c) = (f^{-1}(E))^c$$

Bijectivity:

- $f : X \hookrightarrow Y$ iff $f(x_1) = f(x_2) \implies x_1 = x_2$
- $f : X \twoheadrightarrow Y$ iff $\forall y \in Y, \exists x \in X$ s.t. $f(x) = y$
- $f : X \hookrightarrow\!\!\!\!\rightarrow Y$ iff f is both injective and surjective

If $f : X \hookrightarrow\!\!\!\!\rightarrow Y$, then f^{-1} is a function.

Partial Ordering: a relation R on $X \neq \emptyset$ is a partial ordering if

- $xRy \wedge yRx \implies x = y$
- $xRy \wedge yRz \implies xRz$
- xRx for all x

Total/Linear ordering: an ordering $\leq$ is a total ordering if $\forall x, y \in X$, either $x \leq y$ or $y \leq x$

Extrema: If X is partially ordered by $\leq$, $x \in X$ s.t. $x \leq y \implies y = x$ is a *maximal* element of X

Bounds: If $E \subseteq X$, $x \in X$ s.t. $y \leq x \quad \forall y \in E$ is an *upper bound.*

Well-ordered: A set is *well-ordered* if

1. It is linearly ordered by $\leq$
2. Every nonempty subset has a minimal element

Zorn's Lemma: If X is partially ordered by $\leq$ and every linearly ordered subset of X has an upperbound, then X has a maximal element.

Proof: Axiomatic

Well ordering principle: Every non-empty set X can be well-ordered

Proof: Let $\mathcal{W}$ be the set of all well-ordered subsets of S. Let $\mathcal{S}_\alpha$ be the set of all linear orderings of $E_\alpha \subseteq \mathcal{W}$.

Let $E_\infty = \bigcup_\alpha E_\alpha$ be equipped the partial ordering $\leq_\infty$ such that $\leq_\infty \big|_{E_\alpha} = \leq_\alpha$ for $\alpha \in A$.

By construction, E_∞ is an upper bound for any sequence of well-ordered sets in $\mathcal{W}$.

(**Subtlety:** need to show that E_∞ is an upper bound by defining a relation R by extension of linear orderings, showing that R is a partial ordering, and then showing that $\leq_\alpha R \leq_\infty$ is well-defined)

By Zorn's lemma, E_∞ has a maximal element $E_{\max}$. And we have $E_{\max} = X$ by maximality.

Product map: Let $\prod \alpha \in AX_\alpha$ be the set of all functions $f : A \to \bigcup_{\alpha\in A} X_\alpha$ such that $f(\alpha) \in X_\alpha$.

Axiom of Choice: If $\{X_\alpha\}_{\alpha\in A} \neq \emptyset$, then $\prod_{\alpha\in A} X_\alpha \neq \emptyset$ (i.e. there exists a choice function)

Proof: Let $X = \bigcup_{\alpha\in A} X_\alpha$. Pick a well ordering on X and $\alpha \in A$. Let $f(\alpha)$ be the minimal element of X_α. Then

$$f \in \prod_{\alpha\in A} X_\alpha$$

Cardinality:

- card $X \leq$ card $Y \iff \exists f : X \hookrightarrow Y$
- card $X =$ card $Y \iff \exists f : X \hookrightarrow\!\!\!\rightarrow Y$
- card $X \geq$ card $Y \iff \exists f : X \twoheadrightarrow Y$

Property: card $X \leq$ card $Y \iff$ card $Y \geq$ card X

Proof: card $X \leq$ card $Y \implies \exists f : X \hookrightarrow Y$. Choose $x_0 \in X$ and define $g : Y \to X$ by

$$g(y) = \begin{cases} f^{-1}(y) & y \in f(X) \\ x_0 & y \notin f(X) \end{cases}$$

Conversely, if $\exists g : Y \twoheadrightarrow X$, consider $g^{-1}(x)$ for $x \in X$. Then $f \in \prod_{x\in X} g^{-1}(x)$ is an injection from X to Y.

Property: Either card $X \leq$ card Y or card $Y \leq$ card X

Proof: Let J be the set of all injections $f_E : E \to Y$ for $E \subseteq X$.

Repeating the argument from the Well-ordering principle, we can find an upper bound $E_{\max}$ for J. Then by Zorn's lemma, there exists a maximal element $f_{E_{\max}}$ (with respect to the extension partial ordering).

Case 1: $E_{\max} = X$. Then $f_{E_{\max}} : X \hookrightarrow Y$ and card $X \leq$ card Y.

Case 2: $E_{\max} \subsetneq X$. Then $X \setminus E_{\max} \neq \emptyset$ so $f(E_{\max}) = Y$ (or else $y_0 \in Y, y_0 \notin f(E_{\max})$ and $f_{E_{\max}} \cup \{(x_0, y_0)\}$ is a larger injection). Then $f_{E_{\max}}^{-1} : Y \hookrightarrow X$ and we are done.

Schröder-Bernstein Theorem: If $f : X \hookrightarrow Y$ and $g : Y \hookrightarrow X$, then $\exists h : X \hookrightarrow\!\!\!\rightarrow Y$

Proof: If $f(X) = Y$, then we are done.

Otherwise, consider $Y_1 = Y \setminus f(X)$. Then $f(Y_1) \subsetneq X$ so let $X_1 = f(Y_1)$. Now we have a bijection $X_1 \to Y_1$.

Assume we have $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ with bijections $X_n \to Y_n$.

Since $f(X_i) \subseteq Y_{i+1}$, define

$$Y_{n+1} = \left(Y \setminus \bigcup_{i=1}^{n} Y_i\right) \setminus f\left(X \setminus \bigcup_{i=1}^{n} X_i\right)$$

So inductively, we have a bijection on the full sets.

Corollary: If card $X \leq$ card Y and card $Y \leq$ card X, then card $X =$ card Y

Proposition: card $X <$ card $P(X)$

Proof: Clearly, $f : X \hookrightarrow P(x)$ by $f(x) = \{x\}$.

We claim $\nexists g : X \to P(X)$. Suppose there is such a g. Then define

$$Y = \{x \in X \text{ s.t. } x \notin g(x)\}$$

We claim $Y \notin g(X)$ so g not surjective. If not, $\exists x_0 \in X$ s.t. $g(x_0) = Y$.

Case 1: $x_0 \in Y \implies x_0 \notin g(x_0) = Y$. Contradiction.

Case 2: $x_0 \notin Y \implies x_0 \in g(x_0) = Y$. Contradiction.

Proposition:

1. X, Y countable $\implies X \times Y$ countable
2. A countable and X_α countable for $\alpha \in A$ implies $\bigcup_{\alpha \in A} X_\alpha$ countable

Proof: 1. card $X =$ card $Y \leq$ card $\mathbb{N}$ so it suffices to show card $\mathbb{N} \times \mathbb{N} =$ card $\mathbb{N}$.

Clearly, $\forall n \in \mathbb{N}$, $f(n) \hookrightarrow (n, 1) \in \mathbb{N} \times \mathbb{N}$.

Now consider $g(m, n) \to 2^m 3^n \in \mathbb{N}$. By unique prime factorization of integers, $2^m 3^n = 2^{m'} 3^{n'} \implies m = m', n = n'$ so injective.

We have a bijection by Schroder-Bernstein.

2. A countable $\implies \exists f_\alpha : \mathbb{N} \to X_\alpha$. Define $F : \mathbb{N} \times A \to \bigcup_{\alpha \in A} X_\alpha$ by $F(n, \alpha) = f_\alpha(n)$ which is surjective because f_α is surjective.

By the previous part, card $\mathbb{N} \times A =$ card $\mathbb{N}$ so card $\bigcup_{\alpha \in A} X_\alpha \leq$ card $\mathbb{N}$. Hence, it is countable.

Corollary: $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

Proof:

1. $\mathbb{Z} = \mathbb{N} \cup \{-\mathbb{N}\} \cup \{0\}$
2. $f : \mathbb{Z}^2 \to \mathbb{Q}$ by

$$f(m, n) = \begin{cases} m/n & n \neq 0 \\ 0 & n = 0 \end{cases}$$

Proposition: Every open set in $\mathbb{R}$ is a countable disjoint union of open intervals

Proof: For all $x \in U$, $\exists (a, b) \subseteq U$ such that $x \in (a, b)$. By def of inf and sup, $x \in I_x := (\inf a, \sup b) \subseteq U$.

We claim that $\forall x, y \in U$, $I_x = I_y$ or $I_x \cap I_y = \emptyset$.

Suppose $I_x \cap I_y \neq \emptyset$. Then $x \in I_x \cup I_y$ but I_x is maximal so $I_x = I_x \cup I_y \implies I_x = I_y$.

Now $U = \bigcup_{x \in U} I_x$ which is countable by $f : U \hookrightarrow \mathbb{Q}$ by choosing a rational in each interval (by density of $\mathbb{Q}$)

Metric Space: a set X with a distance function $\rho : X \times X \to [0, \infty]$ such that

1. $\rho(x, y) = 0 \iff x = y$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

Open Set: E open $\iff \forall x \in E, \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq E$

Closed set: E closed $\iff$ E^c open

Properties:

- $\emptyset$ is open
- U_x open $\implies \bigcup_{x\in A} U_x$ open
- F_x closed $\implies \bigcap_{x\in A} F_x$ closed

Interior: $E \subseteq X$, the interior of E (the largest open set in E) is

$$\overset{\circ}{E} = \bigcup_{O \subseteq E} O$$

Closure:

$$\overline{E} = \bigcap_{F \supset E} F$$

(the smallest closed set containing E)

Proposition: Let (X, ρ) be a metric space with $E \subseteq X$ and $x \in X$. The following are equivalent:

1. $x \in \overline{E}$
2. $B(x, r) \cap E \neq \emptyset$ for all $r > 0$
3. $\exists \{x_n\} \subseteq E$ s.t. $x_n \to x$

Proof:

$(1 \to 2)$ Suppose $\exists r > 0$ such that $B(x, r) \cap E = \emptyset$. Then $E \subseteq (B(x, r))^c$ but $(B(x, r))^c$ is closed so $\overline{E} \subseteq (B(x, r))^c$ so $x \in B(x, r) \subseteq (\overline{E})^c$, contradiction.

$(2 \to 3)$ Let $r = 1/n$. By (1), $\exists x_n \in B(x, \frac{1}{n}) \cap E$. By construction, $\rho(x_n, x) < \frac{1}{n} \to 0 \implies x_n \to x$

$(3 \to 1)$ $x \notin \overline{E} \implies x \in (\overline{E})^c$. But $(\overline{E})^c$ closed so $\exists r > 0$ s.t. $B(x, r) \subseteq (\overline{E})^c \subseteq E^c$ so there cannot exist any sequence in E, a contradiction.

Dense:

- E is dense in X if $\overline{E} = X$
- E is nowhere dense if $(\overline{E})^\circ = \emptyset$

Separable: there exists a countable dense subset $E \subseteq X$

Continuity: Let $(X_1, \rho_1), (X_2, \rho_2)$. $f : X_1 \to X_2$ is continuous at $x \in X_1$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\rho_1(x, y) < \delta_x \implies \rho_2(f(x), f(y)) < \varepsilon$$

Uniform Continuity: f uniformly continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\rho_1(x, y) < \delta_x \implies \rho_2(f(x), f(y)) < \varepsilon$$

for all $x \in X_1$.

Proposition: $f : X_1 \to X_2$ is continuous iff $f^{-1}(U) \subseteq X_1$ is open for all open $U \subseteq X_2$

Proof: $f^{-1}(U) = \emptyset$ is open so take $x \in f^{-1}(U)$ so $f(x) = y \in U$.

Since U is open, take $B_2(y, \varepsilon_y) = B_2(f(x), \varepsilon_y) \subseteq U$. By continuity,

$$z \in B_1(x, \delta_2) \implies f(z) \in B_2(y, \varepsilon_y) \implies z \in f^{-1}(U)$$

so $f^{-1}(U)$ is open.

Conversely, take $y = f(x) \in X_2$. $B_2(y, \varepsilon)$ is open so $f^{-1}(B_2(y, \varepsilon))$ is open by assumption. Now

$$B_1(x, \delta_x) \subseteq f^{-1}(B_2(y, \varepsilon)) \implies f(B_1(x, \delta_x)) \subseteq B_2(y, \varepsilon)$$

which is the definition of continuity

Cauchy Sequence: $\{x_n\} \in (X, \rho)$ is Cauchy if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$, $\rho(x_m, x_n) < \varepsilon$

Complete: $E \subseteq X$ is complete if every Cauchy sequence $x_n \in E$ has a limit $x \in E$

Set Distance:

- Let $x \in X$ and $E \subseteq X$.
$$\rho(x, E) = \inf\{\rho(x, y) : y \in E\}$$
- Let $E, F \subseteq X$
$$\rho(E, F) = \inf\{\rho(x, y) : x \in E,\ y \in F\}$$

Diameter: $\text{diam } E = \sup\{\rho(x, y) : x, y \in E\}$

Bounded: E bounded $\iff$ $\text{diam } E < \infty$

Totally bounded: $\forall \varepsilon > 0$, E can be covered by finitely many ε-balls

Characterization of compactness: The following are equivalent definitions of *compactness*:

1. E is complete and totally bounded
2. Every sequence in E has a convergent subsequence with its limit in E
3. Every open cover has a finite subcover

Proof:

$(1 \to 2)$ Let x_n be a sequence in E. Inductively define a sequence of open balls B_k of radius $1/2^k$ that each contain infinitely many points of x_n (guaranteed by completeness).

For each ball, define an index set $N_k = \{n \in \mathbb{N} : x_n : B_k\}$. Using the AC, pick $n_1 \in N_1, n_2 \in N_2, \ldots$ such that $n_1 < n_2 < \ldots$.

By construction, $\{x_{n_k}\}$ is a Cauchy sequence ($\rho(x_{n_k}, x_{n_j}) < \frac{1}{2^{1-k}}$ for $j > k$). Since E is complete, $\{x_{n_k}\}$ converges to $x \in E$.

$(2 \to 3)$

Product metric: For (X, ρ_1) and (Y, ρ_2) metric spaces, the product metric on $(X_1 \times X_2, \rho_1 \times \rho_2)$ is

$$\rho_1 \times \rho_2 = \sqrt{\rho_1^2(x_1, y_1) + \rho_2^2(x_2, \rho_2)}$$

Property: $\rho_1 \times \rho_2 \to 0 \iff \rho_1 \to 0 \wedge \rho_2 \to 0$

Proof: $\rho_1^2, \rho_2^2 > 0$ so

$$\sqrt{\rho_1^2(x_1,y_1) + \rho_2^2(x_2,y_2)} = 0 \implies \rho_1^2(x_1,y_1) = -\rho_2^2(x_2,y_2) \implies \rho_1, \rho_2 = 0$$

Other direction, clear.

Proposition: There is no measure μ which satisfies Countable Additivity, Translation invariance, and Faithfulness on all subsets of $[0,1)$

Proof:

Define $x \sim y \iff x - y \in \mathbb{Q} \cap [0,1)$. Clearly

$$[0,1) = \bigcup_{x \in [0,1)} \{y \in [0,1) : y \sim x\}$$

Using AC, select a unique element e_x in each equivalence class and take $N = \{e_x : x \in [0,1)\}$. By construction, $e_x - e_y \notin \mathbb{Q} \cap [0,1)$

Pick $r \in \mathbb{Q} \cap [0,1)$ and define

$$N_r = \{e_x + r : e_x \in N \cap [0, 1-r)\} \cup \{e_x + (r-1) : e_x \in N \cap [1-r, 1]\}$$

(the points that don't leave the interval under translation and those that do)

First notice, $N_r \cap N_s = \emptyset$ (or else contradiction by difference being rational)

Then $[0,1) = \bigcup N_r$ because $\forall y \in [0,1)$, $\exists e_x \in N$ such that $y - e_x \in \mathbb{Q} \cap [0,1)$

Now because they are disjoint,

$$\begin{aligned} \mu(N_r) &= \mu(N_r \cap [0, 1-r)) + \mu(N_r \cap [1-r, 1)) \\ &= \mu(N) \end{aligned}$$

By by countable additivity,

$$1 = \mu([0,1)) = \sum_{r \in \mathbb{Q} \cap [0,1)}^{\infty} \mu(N_r) = \begin{cases} 0 \\ \infty \end{cases}$$

which is a contradiction

Algebra: $\mathcal{A} \subseteq P(X)$ such that for $E_1, \ldots, E_n \subseteq \mathcal{A}$,

1. $\bigcup_{i=1}^n E_i \in \mathcal{A}$
2. $E \in \mathcal{A} \implies E^c \in \mathcal{A}$

Sigma Algebra:

1. $\bigcup_{i=1}^\infty E_i \in \mathcal{A}$ for $E_i \in \mathcal{A}$
2. $E \in \mathcal{A} \implies E^c \in \mathcal{A}$

Generated σ-algebra: The smallest σ-algebra containing $\mathcal{E} \subseteq P(X)$ is the σ-algebra generated by $\mathcal{E}$,

$$M(\mathcal{E}) = \bigcap_{\mathcal{E} \subseteq \mathcal{A}} \mathcal{A}$$

Lemma: $\mathcal{E} \subseteq M(\mathcal{F}) \implies M(\mathcal{E}) \subseteq M(\mathcal{F})$

Borel Algebra: $\mathcal{B}_X$, the σ-algebra generated by the open sets of X

Proposition: $\mathcal{B}_{\mathbb{R}}$ is generated by

1. $\{(a, b)\}$
2. $\{[a, b]\}$
3. $\{(a, b]\}$ and $\{[a, b)\}$
4. $\{(a, \infty)\}$ and $\{(-\infty, a)\}$

Proof: Follows from

$$(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$$
$$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$$

Proposition: $\mathcal{B}_{\mathbb{R}^n}$ is the Borel set generated by $\otimes_{i=1}^{n} \mathcal{B}_{\mathbb{R}}$

Proof:

Let

$$\bigoplus_{i=1}^{n} O_i = O_1 \times O_2 \times \cdots \times O_n$$

for O_i open sets in X_i. It is not hard to show that $\bigoplus_{i=1}^{n} O_i$ is open in the $X_1 \times X_2 \times \cdots \times X_n$ topology. Let $\bigotimes_{i=1}^{n} \mathcal{B}_{x_i}$ be the Borel set generated by $\bigoplus_{i=1}^{n} O_i$.

Lemma: If X_i is separable, then

$$\bigoplus_{i=1}^{n} \mathcal{B}_{X_i} = \mathcal{B}_{X_1 \times X_2 \times \cdots \times X_n}$$

Proof: It suffices to show that for all $\mathbf{x} \in \bigoplus_{i=1}^{n} O_i$ and $\forall \varepsilon > 0$,

$$B_\varepsilon(\mathbf{x}) \subseteq \bigotimes_{\mathbf{i=1}}^{\mathbf{n}}$$

Since $\mathbb{Q} \subseteq \mathbb{R}$ and $\mathbb{Q}$ is dense, $\mathbb{R}$ is separable. Hence, by the Lemma,

$$\bigotimes_{i=1}^{n} \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^n}$$

Let $\mathcal{C}_i \subseteq X_i$ be a countable subset such that $\overline{\mathcal{C}_i} = X_i$.

We claim

$$B_\varepsilon(\mathbf{x}) \subseteq \bigcup_{r_i \in \mathbb{Q}} \bigcup_{c_i \in \mathcal{C}_i} \bigotimes_{i=1}^{n} B_{r_i}(c_i) \subseteq \bigotimes_{i=1}^{n} \mathcal{B}_{x_i}$$

for $\sqrt{r_1^2 + r_2^2 + \cdots + r_n^2} < \varepsilon$.

Further, this has cardinality $\mathbb{N}^{2n}$ so is countable.

Pick a $\mathbf{y} \in B_\varepsilon(\mathbf{x})$ so

$$\sigma(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} \rho_i^2(y_i, x_i)} < \varepsilon$$

but each $\rho_i^2(y_i, x_i)$ is fixed so for $c_i \in \mathcal{C}$, $r_i \in \mathbb{Q}$,

$$\rho_i(y_i, c_i) < r_i = \rho_i(y_i, x_i) - [\rho(y_i, x_i) - \rho(y_i, c_i)]$$

by density.

Measure: For a measure space $(X, \mathcal{M})$, we define $\mu : \mathcal{M} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$
2. If $\{E_j\}_1^\infty \in \mathcal{M}$ pairwise disjoint,
$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

σ-finite: If $\mu(X) = \infty$ but $X = \bigcup_{i=1}^{\infty} X_i$ and $\mu(X_i) < \infty$ for all i, then X is σ-finite

Properties of Measures: Let $(X, \mathcal{M}, \mu)$ be a measure space. Then

1. $E, F \in \mathcal{M} \wedge E \subseteq F \implies \mu(E) \leq \mu(F)$
2. $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$
3. If $E_1 \subseteq E_2 \subseteq \dots$, then
$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \lim_{j\to\infty} \mu(E_j)$$
4. If $E_1 \supseteq E_2 \supseteq \dots$ and $\mu(E_1) < \infty$, then
$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$$

Proof: todo

Outer Measure: Let $\mu^* : P(X) \to [0, \infty]$ be an outer measure if

1. $\mu^*(\emptyset) = 0$
2. $\mu^*(A) \leq \mu^*(B)$ for $A \subseteq B$
3. $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$

Carathéodory Criterion (μ^*-measurable): $\mathcal{M} \subseteq P(X)$ is μ^*-measurable if, given $A \in \mathcal{M}$, for all $E \subseteq P(X)$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

(by subadditivity, it suffices to show $\geq$)

Proof:

Carathéodory Extension: Let $\mathcal{M}$ be the μ^*-measurable sets. Then $\mu : \mathcal{M} \to [0, \infty]$ defined by $\mu(E) = \mu^*(E)\big|_{\mathcal{M}}$ is a measure

Proof: TODO

Completeness: $(X, \mathcal{M}, \mu)$ is complete if $\forall A \in \mathcal{M}$ with $\mu(A) = 0$, $B \subseteq A$ implies $B \in \mathcal{M}$

Lebesgue measure: On $(\mathbb{R}, \rho)$ with $\rho(a, b) = b - a$,

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \rho(a_i, b_i) \;\middle|\; A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i)\right\}$$

which gives the Lebesgue measure on $(\mathbb{R}, \mathcal{M}, \mu)$ via the Carathéodory process.

Faithfulness of the Lebesgue measure: For $I \subseteq \mathbb{R}$ an interval, $\mu(I) = \rho(I)$.

Proof:

STEP 1. Suppose $I = [a, b]$. Then

$$\mu^*(I) \leq \rho((a - \varepsilon, b + \varepsilon)) = b - a + 2\varepsilon \to b - a$$

Now take $I \subseteq \bigcup_{i=1}^{N} (a_i, b_i)$ (finite by Heine Borel).

Take $a \in (a_1, b_1)$ with $b_1 \leq b$. Inductively define $\{(a_i, b_i)\}_1^N$ by $b_n \in (a_{n+1}, b_{n+1})$. Eventually $b_N > b$ so

$$\begin{aligned}
\sum_{i=1}^{N} \rho(a_i, b_i) &= b_N - a_N + b_{N_1} - a_{N-1} + \cdots + b_1 - a_1 \\
&= b_N + (-a_N + b_{N_1}) + (-a_{N-1} + b_{N-2}) + \cdots + (-a_2 + b_1) - a_1 \\
&= \underbrace{b_N}_{>b} + \underbrace{(-a_N + b_{N_1})}_{>0} + \underbrace{(-a_{N-1} + b_{N-2})}_{>0} + \cdots + \underbrace{(-a_2 + b_1)}_{>0} - \underbrace{a_1}_{<a} \\
&\geq b - a
\end{aligned}$$

STEP 2. Now suppose I is any interval in $\mathbb{R}$.

$$[a + \varepsilon, b - \varepsilon] \subseteq I \subseteq (a - \varepsilon, b + \varepsilon)$$

so by Step 1,

$$b - a - 2\varepsilon \leq \mu^*(I) \leq b - a + 2\varepsilon \implies \mu^*(I) = b - a$$

Lemma: If $A \subseteq \mathbb{R}$ with card $A \leq$ card $\mathbb{N}$, $\mu^*(A) = 0$

Proof:

$$\mu^*(A) \leq \sum_{i=1}^{\infty} \mu^*(\{a_n\}) \leq \sum_{i=1}^{\infty} \mu^*(\{a_n - \varepsilon, a_n + \varepsilon\}) \leq \sum_{i=1}^{\infty} 2\varepsilon = 0$$

Corollary: $\mu^*([0, 1]) = 1 \neq 0$ so $[0, 1]$ is not countable.

Proposition: $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}$

Proof: It suffices to show that $(a, \infty) \in \mathcal{M}$ by the characterization of $\mathcal{B}_{\mathbb{R}}$.

For all $E \in P(\mathbb{R})$,

$$\begin{aligned}\mu^*(E \cap (a, \infty)) + \mu^*(E \cap (-\infty, a]) &\leq \sum_{n=1}^{\infty} \mu^*(I_n \cap (a, \infty)) + \mu^*(I_n \cap (-\infty, a]) \\ &= \sum_{n=1}^{\infty} \mu^*(I_n) \\ &\leq \mu^*(E)\end{aligned}$$

for $E \subseteq \bigcup_{i=1}^{\infty} I_n$ with $\sum_{n=1}^{I_n} \mu^*(I_n) < \mu^*(E) + \varepsilon$

Lemma: for the Lebesgue outer measure,

1. $\mu^*(E + a) = \mu^*(E)$
2. $\mu^*(rE) = |r| \, \mu^*(E)$

Proof:

If $E \subseteq \bigcup_{n=1}^{\infty} I_n$,

$$E + a \subseteq \bigcup_{n=1}^{\infty} \{I_n + a\}$$
$$rE \subseteq \bigcup_{n=1}^{\infty} \{|r| \, I_n\}$$

so

$$\sum_{n=1}^{\infty} \rho(I_n) = \sum_{n=1}^{\infty} \rho(I_n + a) \geq \mu^*(E + a) \implies \mu^*(E) \geq \mu^*(E + a)$$
$$\sum_{n=1}^{\infty} \rho(I_n) = \sum_{n=1}^{\infty} \frac{1}{|r|} \rho(rI_n) \geq \mu^*(rE) \implies \mu^*(E) \geq \mu^*(rE)$$

The other direction is the same.

Approximation of Measurable Sets:

1. $\forall E \subseteq P(X)$ and $\forall \varepsilon > 0$, $\exists O$ open such that $E \subseteq O$ and
$$\mu(O) \geq \mu(E) \geq \mu(O) - \varepsilon$$
2. $\forall E \subseteq \mathcal{M}$ and $\forall E > 0$, $\exists K$ closed such that
$$\mu(K) \leq \mu(E) \leq \mu(K) + \varepsilon$$

Proof:

1. For $E \subseteq O = \bigcup_{n=1}^{\infty} I_n$,

$$\mu(O) - \varepsilon \leq \sum_{n=1}^{\infty} \rho(I_n) - \varepsilon \leq \mu(E)$$

2. By part 1, $E \subseteq [a,b] \implies \exists O \supseteq E^c \cap [a,b]$ such that

$$\mu(E^c \cap [a,b]) \geq \mu(O) - \varepsilon \implies |b-a| - \mu(E^c) \leq |b-a| - \mu(O) + \varepsilon$$

so by measurability,

$$\mu(E) = \mu([a,b] \cap O^c) + \varepsilon$$

Now suppose $E \notin [a,b]$.

Exercises

Prove De Morgan's Laws

Proof:

Prove that

$$f^{-1}\left(\bigcup_{\alpha \in A} E_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(E_\alpha)$$
$$f^{-1}\left(\bigcap_{\alpha \in A} E_\alpha\right) = \bigcap_{\alpha \in A} f^{-1}(E_\alpha)$$
$$f^{-1}(E^c) = (f^{-1}(E))^c$$

Note: In general, f also commutes with unions but not intersections. Why?

Proof:

Define the relation R such that $\leq_1 R \leq_2$ for linear orderings $\leq_1, \leq_2$ if

1. $E_1 \subseteq E_2 \wedge \leq_2 \big|_{E_1} = \leq_1$ (i.e. $\leq_2$ extends $\leq_1$)
2. $x \notin E_1 \wedge x \in E_2 \implies y \leq_2 x$ for all $y \in E_1$ (i.e. E_2 is an upper bound for E_1)

Show that R is a partial ordering.

Proof:

Verify that

$$g : \bigcup_{i=1}^{\infty} Y_i \to \bigcup_{i=1}^{\infty} X_i$$

is a bijection. Further, show that

$$f : \left(X \setminus \bigcup_{i=1}^{\infty} X_i\right) \to \left(Y \setminus \bigcup_{i=1}^{\infty} Y_i\right)$$

is a bijection.

Proof:

Show that the following are metric spaces:

- $(\mathbb{R}^n, \rho_1)$ where $\rho_1(x,y) = |x-y|$
- $(C^1[0,1], \rho_2)$ where $C^1[0,1]$ is the space of continuous functions on $[0,1]$ and $\rho_2(f,g) = \int_0^1 |f(x) - g(x)|\ dx$
- $(C^1[0,1], \rho_\infty)$ where $\rho_2(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$

Proof:

Prove that $B(x,r)$ is open

Proof:

Prove that $(\mathcal{C}, \rho_\infty)$ is complete for

$$\rho_\infty(x,y) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

Proof:

Prove that a closed subset (X,ρ) of a complete metric spae is complete and complete subsets of a metric space must be closed

Proof:

Prove that for $\mathcal{A}_1, \mathcal{A}_2$ σ-algebras on X, $\mathcal{A}_1 \cap \mathcal{A}_2$ is a σ-algebra

Proof: Certainly, any $\forall E \in \mathcal{A}_1 \cap \mathcal{A}_2$, $E^c \in \mathcal{A}_1 \cap \mathcal{A}_2$ because $E^c \in \mathcal{A}_1$ and $E^c \in \mathcal{A}_2$ as they are σ-algebras.

Now take any $E_1, E_2, \ldots$ in $\mathcal{A}_1 \cap \mathcal{A}_2$. Since $\mathcal{A}_1$ is a σ-algebra,

For $f : X \to [0,\infty]$, show that

$$\mu(E) = \sum_{x\in E} f(x) = \sup\{\sum_{x\in F} f(x) : F \subseteq E \wedge F \text{ finite}\}$$

is a measure on $P(X)$

Proof:

Let X be uncountable. Let $\mathcal{M} = \{E \text{ is finite or } E^c \text{ is finite}\}$. Define

$$\mu(E) = \begin{cases} 0 & E \text{ is countable} \\ 1 & E^c \text{ is countable} \end{cases}$$

Check that $\mathcal{M}$ is a σ-algebra and that μ is a measure

Proof:

1. Let $E \in \mathcal{M}$. Then, by definition E finite or E^c finite.

If E finite, then $(E^c)^c = E$ is finite so $E^c \in \mathcal{M}$. If E^c finite, then $E^c \in \mathcal{M}$. So $\mathcal{M}$ is closed under complements.

Take $E_1, E_2 \in \mathcal{M}$.

Case 1: Both finite. Then clearly, $E_1 \cup E_2$ is finite, so $E_1 \cup E_2 \in \mathcal{M}$.

Case 2: One finite (WLOG E_1). Then $E_1 \cup E_2^c$ is finite so $E_1 \cup E_2 \in \mathcal{M}$.

Case 3: Both infinite. Then E_1^c and E_2^c are finite so $(E_1 \cup E_2)^c$ is finite so $E_1 \cup E_2 \in \mathcal{M}$.