APMA 2110: Homework 7

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1. Let $f \geq 0$ be a measurable function and define its distribution function for $\lambda \geq 0$,

$$d_f(\lambda) = \mu\{x : |f(x)| > \lambda\}$$

Show

$$\int |f| \; d\mu = \int_0^\infty d_f(\lambda) \; d\lambda$$

First, $f \geq 0$ so $|f| = f$.

Since f is measurable, we can take a sequence of simple functions $\phi_n \to f$ such that $0 \leq |\phi_n| \leq |\phi_{n+1}| \leq |f|$ for $n \geq 1$.

By the Monotone Convergence Theorem,

$$\int f \; d\mu = \int \lim phi_n \; d\mu = \lim \int \phi_n \; d\mu$$

Each ϕ_n is a simple function, i.e. $\phi_n = \sum_{i=1}^{m_n} a_i^{(n)} \mathbb{1}_{A_i^{(n)}}(x)$, so

$$\begin{aligned}\int \phi_n \; d\mu &= \sum_{i=1}^{m_n} a_i^{(n)} \mu(A_i^{(n)}) \\ &= \sum_{i=1}^{m_n} (a_i^{(n)} - a_{i-1}^{(n)}) \; \mu(x : \phi_n(x) > a_{i-1}^{(n)})\end{aligned}$$

By definition,

$$\begin{aligned}\int f \; d\mu &= \sup_{\phi \leq f} \int \phi \; d\mu \\ &= \lim_{n\to\infty} \int \phi_n \; d\mu && \text{(monotonicity of } \phi_n\text{)} \\ &= \lim_{n\to\infty} \sum_{i=1}^{m_n} (a_i^{(n)} - a_{i-1}^{(n)}) \; \mu(x : \phi_n(x) > a_{i-1}^{(n)})\end{aligned}$$

But since $\phi_n \to f$, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|f - \phi_n| < \varepsilon$. WLOG, consider

$$\phi := \phi_N = \sum_{i=1}^{m} a_i \mathbb{1}_{A_i}$$

so

$$\begin{aligned}\int \phi \; d\mu &= \sum_{i=1}^{\infty} (a_i - a_{i-1}) \mu\{x : \phi(x) > a_{i-1}\} \\ &= \sum_{i=1}^{\infty} (a_i - a_{i-1}) \left[\mu\{x : f(x) > a_{i-1}\} - \mu\{a_{i-1} : a_{i-1} < |f - \phi|\}\right] \\ &= \sum_{i=1}^{\infty} (a_i - a_{i-1}) \left[\mu\{x : f(x) > a_{i-1}\} - \mu\{a_{i-1} : a_{i-1} < \varepsilon\}\right] \\ &= \sum_{i=1}^{\infty} (a_i - a_{i-1}) \mu\{x : f(x) > a_{i-1}\} \\ &= \int_0^\infty \mu\{x : f(x) > \lambda\} \; d\lambda \quad \blacksquare\end{aligned}$$

2. Let $1 < a \in \mathbb{R}$. Determine the limit of the following Lebesgue integral:

$$\lim_{n\to\infty} \int_0^\infty (1 + \frac{t}{n})^n e^{at}\, dt$$

Let $f_n(t) = (1 + \frac{t}{n})^n e^{-at}$. Notice that $f_{n+1}(t) \geq f_n(t)$ for all $t \geq 0$

Proof: It suffices to show that

$$(1 + \frac{t}{n})^n \leq (1 + \frac{t}{n+1})^{n+1}$$

Consider

$$\begin{aligned}
(1 + \frac{t}{n+1})^{n+1} - (1 + \frac{t}{n})^n &= (1 + \frac{t}{n+1})(1 + \frac{t}{n+1})^n - (1 + \frac{t}{n})^n && (1)\\
&\geq (1 + \frac{t}{n+1})(1 + \frac{t}{n+1})^n - (1 + \frac{t}{n+1})^n && (2)\\
&= (1 + \frac{t}{n+1})^n[(1 + \frac{t}{n+1}) - 1] && (3)\\
&= (1 + \frac{t}{n+1})^n[\frac{t}{n+1}] && (4)\\
&\geq 1 \cdot 0 = 0 && (5)
\end{aligned}$$

So, by the Monotone Convergence Theorem,

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty (1 + \frac{t}{n})^n e^{-at}\, dt &= \int_0^\infty \lim_{n\to\infty} (1 + \frac{t}{n})^n e^{-at}\, dt\\
&= \int_0^\infty e^t e^{-at}\, dt && \text{(by definition of exp)}\\
&= \int_0^\infty e^{-(a-1)t}\, dt\\
&= \left[\frac{e^{-(a-1)t}}{1-a}\right]_0^\infty\\
&= \frac{0}{1-a} - \frac{1}{1-a} = \boxed{\frac{1}{a-1}}
\end{aligned}$$

3.

(a) Let $\{r_n\}_1^\infty$ be an enumeration of $\mathbb{Q} \cap [0,1]$. Let $0 < a_n < \infty$ with $\sum_{n=1}^\infty a_n < \infty$.

Prove that the series

$$h(t) = \sum_{n=1}^\infty \frac{a_n}{\sqrt{|t - r_n|}}$$

converges a.e. (WRT the Lebesgue measure) for $t \in [0,1]$

We want to show that $\mu\{t : h(t) = \infty\} = 0$.

CASE 1. $t \in \mathbb{Q} \cap [0,1]$. Then, $\exists r_n$ such that $t = r_n$ so $\sqrt{|t - r_n|} = 0$ and $\frac{a_n}{\sqrt{|t-r_n|}} = \infty \implies h(t) = \infty$.

However, $Q \cap [0,1]$ is countable so $\mu\{t : t \in \mathbb{Q} \cap [0,1] \wedge h(t) = \infty\} = 0$.

CASE 2. $t \in [0,1] \setminus \mathbb{Q}$. Let $\varepsilon > 0$.

Define

$$E_n(\varepsilon) = \left\{ t \in [0,1] \setminus \mathbb{Q} \middle| \frac{a_n}{\sqrt{|t - r_n|}} > \varepsilon \right\}$$

We WTS that $\mu(E_n(\varepsilon)) = 0$.

Certainly, if $t \in E_n(\varepsilon)$, then $|t - r_n| < (\frac{a_n}{\varepsilon})^2$. By the faithfulness of the Lebesgue measure, $\mu(E_n(\varepsilon)) \leq 2(\frac{a_n}{\varepsilon})^2$.

We know that $\lim_{\varepsilon \to \infty} 2(\frac{a_n}{\varepsilon})^2 = 0$ so $\frac{a_n}{\sqrt{|t-r_n|}} < \infty$ a.e.

It only remains to show that $\sum_{n=1}^\infty \frac{a_n}{\sqrt{|t-r_n|}} < \infty$ a.e.

Let

$$E(\varepsilon) = \{t \in [0,1] \setminus \mathbb{Q} \mid h(t) > \varepsilon\}$$

Then, $E(\varepsilon) \subseteq \bigcup_n E_n(\varepsilon)$, so

$$\begin{aligned}
\mu(E(\varepsilon)) &\leq \mu\left(\bigcup_n E_n(\varepsilon)\right) \\
&\leq \sum_n \mu(E_n(\varepsilon)) \\
&\leq \sum_n 2(\frac{a_n}{\varepsilon})^2 \\
&= 2\varepsilon^{-2} \sum_n a_n^2
\end{aligned}$$

Lemma: $\sum_n a_n^2 \leq (\sum_n a_n)^2$

Proof:

$$\left(\sum_n a_n\right)^2 = \sum_n \sum_m a_n a_m = \sum_n a_n^2 + \sum_n \sum_{n \neq m} a_n a_m \geq \sum_n a_n^2$$

So $\mu(E(\varepsilon)) \leq 2\varepsilon^{-2}\left(\sum_n a_n\right)^2 < \infty$ for all fixed $\varepsilon > 0$. And indeed,

$$\lim_{\varepsilon \to \infty} \mu\{t \in [0,1] \setminus \mathbb{Q} \mid h(t) > \varepsilon\} = \mu(E(\varepsilon)) = 0$$

so $h(t) < \infty$ a.e.

(b) If $g = h$ a.e., prove that g is unbounded in every subinterval of $[0,1]$.

Suppose $g = h$ a.e. and let $I \subseteq [0,1]$ be a subinterval.

Suppose g is bounded on I. Then, $\exists M \in \mathbb{R}$ s.t. $\forall t \in I,\ g(t) \leq M$.

Once again, we proceed by cases.

CASE 1. $\{t \in I \cap \mathbb{Q} : h(t) = g(t)\} \neq \emptyset$. Then, we can take $t = r_n \in I \cap \mathbb{Q}$ such that $h(t) = g(t) = \infty$ which is a contradiction.

CASE 2. $\{t \in I \setminus \mathbb{Q} : h(t) = g(t)\} \neq \emptyset$.

By the density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists r_n \in I \cap \mathbb{Q}$ such that $|t - r_n| < \delta$ for some $\delta > 0$.

But then, for all $t \in I \setminus \mathbb{Q}$, $\frac{a_n}{\sqrt{|t-r_n|}} > \frac{a_n}{\sqrt{\delta}}$ so by taking $\delta \to 0$, we have $\frac{a_n}{\sqrt{|t-r_n|}} \to \infty$. But this implies that $h(t) = \infty$ which is a contradiction of Part 1.

4. Assume $f_n \xrightarrow{\mu} f$.

- Prove that $\liminf_n \int |f_n| \ d\mu \geq \int |f| \ d\mu$

  Since $f_n \xrightarrow{\mu} f$, we have (by a Theorem from class), that there exists a subsequence $f_{n_k} \to f$ a.e.

  Then, $|f_{n_k}| \to |f|$ a.e. so by Fatou's lemma,

$$\liminf_n \int |f_{n_k}| \ d\mu \geq \int \liminf_n |f_{n_k}| \ d\mu = \int |f| \ d\mu$$

  By monotonicity of the integral, we also have

$$\liminf_n \int |f_n| \ d\mu \geq \liminf_n \int |f_{n_k}| \ d\mu$$

  Therefore,

$$\liminf_n \int |f_n| \ d\mu \geq \int |f| \ d\mu$$

- Further assume $|f_n| \leq g \in \mathcal{L}^1$. Prove that $f_n \to f$ in $\mathcal{L}^1$.

  As before, we have that $f_{n_k} \to f$ a.e. and in particular, $|f_{n_k} - f| \to 0$ a.e.

  By assumption, $|f_{n_k}| \leq g \in \mathcal{L}^1$ so by the Dominated Convergence Theorem,

$$\lim \int |f_{n_k} - f| \ d\mu = \int \lim |f_{n_k} - f| \ d\mu = \int 0 \ d\mu = 0$$

  Therefore, $f_{n_k} \to f$ in $\mathcal{L}^1$.

  Suppose now that $f_n \not\to f$ in $\mathcal{L}^1$. Then, $\forall \varepsilon > 0$, $\exists f_{n_i}$ for infinitely many n_i such that

$$\int |f_{n_i} - f| \ d\mu \geq \varepsilon$$

  But $f_{n_i} \xrightarrow{\mu} f$ so it also has a subsequence $f_{n_{i_j}} \to f$ in $\mathcal{L}^1$. But then $\left|f_{n_{i_j}} - f\right| \to 0$ a.e. and $\left|f_{n_{i_j}}\right| \leq g \in \mathcal{L}^1$ so $\left|f_{n_{i_j}} - f\right| \leq 2g$, hence

$$\liminf \left|f_{n_{i_j}} - f\right| \ d\mu = 0$$

  which is a contradiction.