Math 1530: Homework 7

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7 November 2023

Problem 6.3

In the dihedral group D_n, let R be a clockwise rotation by $2\pi/n$ radians, and let F be a flip.

(a) Prove that the subgroup $\{e, R, R_2, ..., R_{n-1}\}$ is a normal subgroup of D_n.

Let $H = \{e, R, R_2, ..., R_{n-1}\}$. H is normal if $\forall a \in D_n$, $\forall h \in H$,

$$a^{-1}ha \in H$$

First, notice that if a and a^{-1} are rotations, then $a, a^{-1} \in H$ and the product is in H by group closure.

If a and a^{-1} are not rotations, then they are flips and thus not part of the subgroup. Example 6.8 very neatly gives the formula

$$\text{flip} \cdot \text{rotation} \cdot \text{flip} = \text{rotation}$$

which means that for $a, a^{-1} \in D_n$ (and not in H),

$$a^{-1}ha \in H \quad \forall h \in H$$

Thus, the subgroup of rotations is normal because conjugations by all members of the dihedral group are in the subgroup. ■

(b) Let $n > 3$. Prove that the subgroup $\{e, F\}$ is not a normal subgroup of D_n

Suppose $H = \{e, F\}$ were normal. Then, $a^{-1}Fa \in H \quad \forall a \in D_n$. However, with $n > 3$, flips in D_n are not in $Z(D_n)$, i.e., they do not commute with every element in D_n. Thus, for some $g \in D_n$

$$g^{-1}Fg \neq Fg^{-1}g = F$$

Further, $g^{-1}Fg \neq e$ because

$$g^{-1}Fg = e \implies Fg = g \implies F = e \quad \text{contradiction}$$

so we have found an element in D_n for which $g^{-1}Fg \notin H$ so H is not normal.
■

6.5

Let G be a group, and let $H \subset G$ be a subgroup of index 2; i.e., there are exactly two cosets of H. Prove that H is a normal subgroup of G. (Hint. For every $g \in G$, prove that the left coset gH is equal to the right coset Hg.)

Let $g \in G$. If $g \in H$, then $g \in eH = H = He$. Here, left and right cosets are equal so H is normal.

In the other case, if $g \notin H$, then we know that it is in gH because H has only two cosets and cosets are disjoint. Equivalently, $gH = \{g \in G : g \notin H\}$.

However, we can make exactly the same argument for right cosets. If $g \notin H$, then it must be in $Hg = \{g \in G : g \notin H\}$. Clearly,

$$\{g \in G : g \notin H\} = Hg = \{g \in G : g \notin H\} = gH$$

so for all $g \in G$,

$$gH = Hg$$

and H is normal. $\blacksquare$

6.6abc

. Let G be a group, let $H \subset G$ and $K \subset G$ be subgroups, and assume that K is a normal subgroup of G.

(a) Prove that $HK = \{hk : h \in H, k \in K\}$ is a subgroup of G.

(a) *Closure*: Let $h_1, h_2 \in H$ and $k_1, k_2 \in K$. We want to show that

$$h_1 k_1 \cdot h_2 k_2 \in HK$$

Since K is normal, we can say that $k_2 = h_2^{-1} k_3 h_2$ for some $k_3 \in K$. Thus

$$h_1 k_1 \cdot h_2 k_2 = h_1 k_1 \cdot h_2 (h_2^{-1} k_3 h_2) = h_1 k_1 \cdot k_3 h_2$$

Let $k = k_1 k_3 \in K$ by closure. So

$$h_1 k h_2$$

Then by closure of h, we can write $h_1 = h \cdot h_2^{-1}$ for some $h \in H$. But by normality of K, this gives us

$$h \cdot h_2^{-1} k h_2 = h \cdot k' \quad (k' \in K)$$

Hence, $h_1 k_1 \cdot h_2 k_2 \in HK$.

(b) *Associativity*:

$$(h_1 k_1 \cdot h_2 k_2) \cdot (h_3 k_3) = h_1 k_1 h_2 k_2 h_3 k_3$$
$$h_1 k_1 \cdot (h_2 k_2 \cdot h_3 k_3) = h_1 k_1 h_2 k_2 h_3 k_3$$

(c) *Identity*: H, K subgroups implies that $e \in G$ is in H and K so

$$ee = e \in HK$$

(d) Inverses:

$$(hk)(hk)^{-1} = hkk^{-1}h^{-1}$$

By closure and normality, $hkk^{-1}h^{-1} \in K$ and $e \in H$ so

$$e \cdot (hkk^{-1}h^{-1}) = hkk^{-1}h^{-1} = e \in HK$$

(b) Prove that $H \cap K$ is a normal subgroup of H and that K is a normal subgroup of HK.

Showing that $K \subset HK$ is easy because $K \subset G$ is normal and from part (a), $HK \subset G$. Then because $HK = \{hk : h \in H, k \in K\}$, it is clear that $K = \{ek : e \in H, k \in K\} \subset HK$. So since K is a group, it is a subgroup of HK. Finally, because K is normal in G, it must be normal in HK since every element of HK is in G. $\blacksquare$

Now it just remains to show that $H \cap K$ is a normal subgroup of H. First note that $H \cap K \subseteq H$. Then from HW 2, Exercise 2.30, we know the intersection of subgroups is itself a subgroup. Finally, since every element of $H \cap K$ is itself a member of K, $g^{-1}kg \in K \subset H \cap K, \quad \forall g \in G, k \in K$. Thus, $H \cap K$ is normal. $\blacksquare$.

(c) Prove that HK/K is isomorphic to $H/(H \cap K)$.[1] (Hint. What is the kernel of the surjective homomorphism $H \to HK/K$?)

By the first isomorphism theorem for groups, the map $\alpha : G_1/N \to G_2$ is an isomorphism if N (a normal subgroup) is the kernel of a surjective homomorphism $\phi : G_1 \to G_2$. In our case, $\phi : H \to HK/K$ and (if it exists), $\alpha : H/(H \cap K) \to HK/K$ following this commutative diagram:

$$\begin{tikzcd}
H \arrow[r, "\phi"] \arrow[d, "\pi"'] & HK/K \\
H/(H \cap K) \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

So, we just need to show that $\ker(\phi) = H \cap K$. Observe:

$\ker(\phi) = \{h \in H : \phi(h) = eK\} = \{h \in H, k \in K : hkK = K\} = \{h \in H : hK = K\} = \{h \in H :$

[1]This is known as the second isomorphism theorem for groups (also sometimes called the diamond or parallelogram theorem)

Thus, $\alpha : H/(H \cap K) \to HK/K$ is an isomorphism. $\blacksquare$

6.7c

Let G be a group, let $K \subseteq H \subseteq G$ be subgroups, and assume that K is a normal subgroup of G. Prove that H is a normal subgroup of G if and only if H/K is a normal subgroup of G/K. (Note. You may take the statements in parts (a) and (b) of this problem as given.)

Suppose H is a normal subgroup of G. We would like to show that H/K is a normal subgroup of G/K. Equivalently, $\forall hK \in H/K$,

$$(gK)(hK)(gK)^{-1} \in H/K \quad (gK \in G/K)$$

Since K is a normal subgroup of G, "the formula" is well-defined so

$$(gK)(hK)(gK)^{-1} = (gK)(hK)(g^{-1}K) = ghg^{-1} \cdot K$$

Using the normality of H,

$$ghg^{-1} = h' \in H$$

so

$$ghg^{-1} \cdot K = h'K \in H/K$$

Hence, H/K is a normal subset of G/K.

For the other direction, suppose H/K is a normal subgroup of G/K. Then for every $hK \in H/K$ and every $gK \in G/K$,

$$(gK)(hK)(gK)^{-1} = h'K \qquad (h' \in H)$$

As above, because K is normal,

$$(gK)(hK)(gK)^{-1} = ghg^{-1} \cdot K = h' \cdot K \implies ghg^{-1} = h' \in H$$

Since H is a subgroup of G, this is exactly the condition for H to be normal. ■

6.10

Let G be a group, let X be a set, and let S_X be the symmetry group of X as defined in Example 2.19. Let

$$\alpha : G \to S_X$$

be a function from G to S_X , and for $g \in G$ and $x \in X$, let $g \cdot x = \alpha(g)(x)$. Prove that this defines a group action if and only if the function α is a group homomorphism.

If α is a group action, then it must satisfy $e \cdot x = \alpha(e)(x) = x$ and

$$(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) \implies \alpha(g_1 g_2)(x) = \alpha(g_1)\alpha(g_2)(x)$$

The first equation implies that $\alpha(e) = e$. The second equation implies that

$$\alpha(g_1 g_2) = \alpha(g_1)\alpha(g_2).$$

These are precisely the conditions for a map to be a homomorphism, so α is a group homomorphism.

Conversely, if α is a group homomorphism, then

$$\alpha(g_1 g_2) = \alpha(g_1)\alpha(g_2) \quad g_1, g_2 \in G$$

Consider $\alpha(e)(x)$. Since α is a homomorphism, $\alpha(e) = e$ so $\alpha(e)(x) = e \cdot x = x$.

Now observe that

$$\alpha(g_1 g_2)(x) = \alpha(g_1)\alpha(g_2) = g_1 \cdot g_2 \cdot x$$

and

$$\alpha(g_1)(\alpha(g_2)(x)) = g_1 \cdot \alpha(g_2)(x) = g_1 \cdot g_2 \cdot x$$

So α is associative and respects identity. Hence, it is a group action on X. ■

6.11

(a) Prove that G acts transitively on X if and only if there is at least one $x \in X$ such that $Gx = X$.

(See Definition 6.20 in the textbook for the definition of a transitive action.)

By Definition 6.20, if G acts transitively on X, then $Gx = X \quad \forall x \in X$. So clearly, there must be at least one $x \in X$ for which $Gx = X$.

For the other direction, suppose there is at least one $x \in X$ for which $Gx = X$, i.e. the orbit of x is X. This means that every $y \in X$ can be written $gx = y$ for some $g \in G$. Since G is a group, we have closure and equivalently with some $g' \in G$,

$$g'gx = g'y = y' \in X$$

But $g'g$ is defined for all $g', g \in G$ so the product $g'y$ exists for any g' and any y. That is, $Gy = X$ for all $y \in X$. So G acts transitively on X. ■

(b) Prove that G acts transitively on X if and only if for every pair of elements $x, y \in X$ there exists a group element $g \in G$ such that $gx = y$.

From part (a), if G acts transitively, then for all $x \in X, Gx = X$, which means that there is a $g \in G$ such that $gx = y$ for any $y \in X$.

Conversely, if for all $x, y \in X$, $\exists g \in G : gx = y$, then we know that there is an equivalence relation on X such that $x \sim y$ and the equivalence class of x is the orbit of x. In other words, for any possible $y \in X$, $y \in Gx$. So, for a given x, the orbit of x is the entire set:

$$Gx = X$$

As this is true for all $x \in X$, G acts transitively on X. ■

(c) If G acts transitively on X, prove that $\#X$ divides $\#G$

If G acts transitively on X, $Gx = X$ so $\#Gx = \#X$. Proposition 6.19 says

$$\#Gx = \frac{\#G}{\#G_x}$$

Rearranging,

$$\#G_x = \frac{\#G}{\#Gx} = \frac{\#G}{\#X}$$

Clearly, $\#G_x$ is an integer so $\#X \mid \#G$. ■