Math 1010 - Homework 4

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1 Problem 1 (Calculating square roots)

Let $x_1 = 2$ and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

1. Show that x_n^2 is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

   First observe $x_1^2 = 4 > 2$. Then,

$$\begin{aligned} x_{n+1}^2 &= \left[\frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\right]^2 \\ &= \frac{1}{4}\left(x_n + \frac{2}{x_n}\right)^2 \\ &= \frac{1}{4}(x_n^2 + 4 + \frac{4}{x_n^2}) \end{aligned}$$

   Suppose $x_i^2 \geq 2$ for all $1 \leq i \leq n$. Then

$$x_{n+1}^2 \geq \frac{1}{4}((2)^2 + 4 + \frac{4}{(2)^2}) = \frac{9}{4} \geq 2$$

   Thus, by induction, $x_n^2 \geq 2$ for all $n \in \mathbb{N}$.

   Now consider

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ &= x_n - \frac{1}{2}x_n - \frac{1}{x_n} \\ &= \frac{x_n}{2} - \frac{1}{x_n} = \frac{x_n^2 - 2}{2x_n} \geq \frac{2-2}{2x_n} = 0 \end{aligned}$$

   Thus, $x_n - x_{n+1} \geq 0$ for all $n \in \mathbb{N}$.

   Since $x_n - x_{n+1} \geq 0$, the sequence (x_n) is decreasing. Since $x_1 = 2$, the sequence is bounded above by 2.

Since the sequence is bounded and monotone, it is convergent. Let $\lim x_n = L$. Then

$$\begin{aligned} L &= \lim \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ &= \frac{1}{2}\lim x_n + \lim \frac{1}{x_n} \end{aligned}$$

Define the sequence $y_n = 1$. Trivially, $(y_n) \to 1$. Then by the Algebraic Limit Theorem,

$$\lim \frac{1}{x_n} = \lim \frac{y_n}{x_n} = \frac{1}{L}$$

Then, substituting above

$$L = \frac{L}{2} + \frac{1}{L} = \frac{L^2+2}{2L} \implies 2L^2 = L^2 + 2 \implies L^2 = 2$$

Finally note that while (x_n) is decreasing, its terms are strictly positive and $x_1 = 2 > 0$ so $L = \lim x_n = \sqrt{2}$. ■

2. Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

   Let $(x_n) \to L$ given by sequence given by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

   Then, as above,

$$\begin{aligned} \lim x_{n+1} &= \lim \frac{L}{2} + \lim \frac{c}{2x_n} \\ L &= \frac{L}{2} + \frac{c}{2L} = \frac{L^2+c}{2L} \\ 2L^2 &= L^2 + c \\ L^2 &= c \implies \lim x_n = \sqrt{c} \quad ■ \end{aligned}$$

2 Problem 2 (Limit Superior)

Let (a_n) be a bounded sequence.

1. Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges. (You are allowed to use the fact that for two nonempty sets A, B bounded above with $A \subset B$, we have $\sup A \leq \sup B$.)

   Let $|a_n| \leq M$. Then $A = \{a_k : k \geq n\}$ is a nonempty set bounded above by M. Therefore, $\sup A$ exists. Now denote $B = \{a_k : k \geq 1\}$. Clearly, $\sup B \leq M$. But since $A \subset B$, we have $\sup A \leq \sup B \leq M$. Thus, $y_n = \sup A$ is a bounded sequence.

   Now notice

   $$\{a_k : k \geq n+1\} \subset \{a_k : k \geq n\}$$

   so

   $$y_n = \sup\{a_k : k \geq n\} \geq \sup\{a_k : k \geq n+1\} = y_{n+1}$$

   Therefore, (y_n) is decreasing sequence.

   Since it is monotonic and bounded, it is convergent. ■

2. The *limit superior* of (a_n), or $\limsup a_n$. is defined by

   $$\limsup a_n = \lim y_n,$$

   where y_n is the sequence from part 1). Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

   Denote

   $$y_n = \inf a_n = \inf\{a_k : k \geq n\}$$

   and consider $\lim y_n = \liminf\{a_k : k \geq n\}$.

   As $|a_n| \leq M$, $\{a_k : k \geq n\}$ is a nonempty set bounded below by $-M$. Therefore, $\inf\{a_k : k \geq n\}$ exists. Further,

   $$\{a_k : k \geq n\} \subset \{a_k : k \geq 1\} \implies \inf\{a_k : k \geq n\} \geq \inf\{a_k : k \geq 1\}$$

   so $\inf\{a_k : k \geq n\} \geq \inf a_n \geq -M \implies y_n \geq -M$.

   Then, since y_n is non-empty, $\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\} \leq M$, so $|y_n| \leq M$ and y_n is bounded.

   Finally,

   $$\{a_k : k \geq n{+}1\} \subset \{a_k : k \geq n\} \implies \inf\{a_k : k \geq n{+}1\} \geq \inf\{a_k : k \geq n\} \implies y_n \leq y_{n+1} \implies y_n \text{ increasin}$$

   Since the sequence is bounded and monotonic, it is convergent and so $\lim y_n = \liminf a_n$ exists. ■

3. Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

   Let $A_n = \{a_k : k \geq n\}$. By definition of bounds,

   $$\inf A_n \leq A_n \leq \sup A_n \implies \inf a_n \leq \sup a_n$$

From parts 1) and 2), we have that $\liminf a_n$ and $\limsup a_n$ exist since (a_n) is bounded. Thus, we can take limits of the inequality to get

$$\liminf a_n \leq \limsup a_n \quad \blacksquare$$

4. Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Suppose $\lim a_n = a$ so $\exists N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$. Then, for all $n \geq N$,

$$|a_n - a| = |a_n - \limsup a_n + \limsup a_n - a| \leq |a_n - \limsup a_n| + |\limsup a_n - a| < \varepsilon$$

Since $|a_n - \limsup a_n| > 0$, we have that $|\limsup a_n - a| < \varepsilon$ so $\limsup a_n = a$.

Similarly,

$$|a_n - a| = |a_n - \liminf a_n + \liminf a_n - a| \leq |a_n - \liminf a_n| + |\liminf a_n - a| < \varepsilon$$

and $|a_n - \liminf a_n| > 0 \implies |\liminf a_n - a| < \varepsilon$ so $\liminf a_n = a$.

Therefore, $\liminf a_n = \limsup a_n = a$.

Now, suppose $\liminf a_n = \limsup a_n = a$. Then, for $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n > N$,

$$|\inf\{a_k : k \geq n\} - a| < \varepsilon, \qquad |\sup\{a_k : k \geq n\} - a| < \varepsilon$$

We want to show that $|a_n - a| < \varepsilon$ for all $n > N$. Let $M > N$, then for all $m > M$,

$$\{a_k : k \geq n\} = a_m \implies \inf\{a_k : k \geq n\} = a_m = \sup\inf\{a_k : k \geq n\}$$

so $\lim a_m = a$. $\quad \blacksquare$

3 Problem 3

Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

Let $(a_{n_k}) \to a$ and $(a_{n_j}) \to a$ be two subsequences of (a_n). Let $\varepsilon > 0$. We want to show that $\exists N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$.

Since (a_{n_k}) converges, we can say that $\exists K \in \mathbb{N}$ such that $|a_{n_k} - a| < \frac{\varepsilon}{2}$ for all $k \geq K$. Similarly, $\exists J \in \mathbb{N}$ such that $\left|a_{n_j} - a\right| < \frac{\varepsilon}{2}$ for all $j \geq J$. Let $N = \max K, J$. Pick $m \in \{n_k\}$ and $n \in \{n_j\}$ such that $m > n \geq N$.

$$|a_m - a_n| = |a_m - a - a_n + a| \leq |a_m - a| + |a_n - a| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

Therefore, (a_n) is Cauchy and so converges.

Since it is convergent and all subsequences converge to a, $(a_n) \to a$. $\blacksquare$

4 Problem 4

Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}.$$

Show there exists a subsequence (a_{n_k}) converging to $s = \sup S$. This is a direct proof of the Bolzano-Weierstrass Theorem using the Axiom of Completeness.

By the axiom of completeness, $s = \sup S$ exists and is the least upper bound for S. That is, for all $x < a_n$, $x \le s < a_n$.

Let $\varepsilon > 0$. Then $s + \varepsilon \notin S$ so $s + \varepsilon \ge a_n$ for infinitely many terms a_n. Therefore, we can create a subsequence (a_{n_k}) from the set $\{a_n \mid s < a_n < s + \varepsilon\}$ with elements chosen such that $n_1 < n_2 < \cdots$.

Now we want to show that $(a_{n_k}) \to s$. By the choice of (a_{n_k}), we have that for all $n_k \ge 1$,

$$|a_{n_k} - s| < |(s + \varepsilon) - s| = |\varepsilon| = \varepsilon$$

Therefore, $(a_{n_k}) \to s$. ■