Math 1010: Problem set 6

Problem 1

Prove the converse of Theorem 3.2.5 in the book by showing that if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$, then x is a limit point of A.

Suppose $x = \lim a_n$ for some $(a_n) \in A$ with $a_n \neq x$. Let $V_\varepsilon(x)$ be an ε-neighborhood of x.

Since $(a_n) \to x$, $\exists N \in \mathbb{N}$ such that $n \geq N \implies a_n \in V_{ep}(x)$. However, $(a_n) \in A$ and $a_n \neq x$ so $V_\varepsilon(x)$ contains a point other than x which is in A. Therefore, x is a limit point of A. ■

Problem 2

Let $a \in A$. Prove that a is an isolated point of the set A if and only if there exists an ε-neighborhood of a, $V_\varepsilon(a)$, such that $V_\varepsilon(a) \cap A = \{a\}$.

Suppose a is an isolated point of A. Then it is not a limit point. Therefore, by definition, $\exists V_\varepsilon(a)$ which does not intersect A at any point other than a, i.e, $V_\varepsilon(a) \cap A = \{a\}$.

Now suppose there exists an ε-neighborhood of a, $V_\varepsilon(a)$, such that $V_\varepsilon(a) \cap A = \{a\}$. Then a cannot be a limit point of A because $V_\varepsilon(a)$ does not intersect A at any point other than a. Therefore, a is an isolated point of A. $\blacksquare$

Problem 3

Prove Theorem 3.2.8 from the book.

Theorem 3.2.8 says a set $F \subseteq \mathbb{R}$ is closed iff every Cauchy Sequence contained in F has a limit in F.

Suppose F is closed. Let $(x_n) \to x$ be a Cauchy sequence in F. Then by Theorem 3.2.5, x is a limit point of F because x is the limit of a sequence in F. Since F is closed, $x \in F$. Therefore, every Cauchy sequence contained in F has a limit in F.

Now suppose every Cauchy sequence contained in F has a limit in F. Assume F is not closed, i.e. $\exists x$, a limit point of F which is not contained in F. Since every ε-neighborhood of x intersects F at a point other than x, we can construct a sequence (x_n) by picking a point $x_n \in V_{1/n}(x) \cap F$ such that $x_n \neq x$. Clearly $(x_n) \to x$.

However, by construction, all the elements of the Cauchy sequence (x_n) are in F so by assumption x must be in F. This is a contradiction. Therefore, F is closed. $\blacksquare$

Problem 4

Let $x \in O$ where O is some open set. If (x_n) is a sequence converging to x, prove that all but a finite number of terms of (x_n) must lie in O.

Since O is open, $\exists V_\varepsilon(x) \subseteq O$. Since $(x_n) \to x$, $\exists N \in \mathbb{N}$ such that $n \geq N \implies x_n \in V_\varepsilon(x)$. Since (x_n) is an infinite sequence, there is an infinite number of terms of (x_n) which are in $V_\varepsilon(x)$. The only terms which could be outside O are the first $N-1$ terms of (x_n). $\blacksquare$

Problem 5

Prove the converse of Theorem 3.3.4 in the book, by showing that if a set $K \subset \mathbb{R}$ is closed and bounded then it is compact.

Suppose K is closed and bounded.

By definition, a set $K \subseteq \mathbb{R}$ is compact if every sequence in K has a convergent subsequence whose limit is in K.

Let (x_n) be a sequence in K. Since K is bounded, (x_n) is bounded. By the Bolzano-Weierstrass theorem, (x_n) has a convergent subsequence (x_{n_k}). Since K is closed, all convergent sequences in K have their limit is in K. Therefore, every sequence in K has a convergent subsequence whose limit is in K. $\blacksquare$