Math 1010 - Homework 10

Problem 1

Let
$$f_n(x) = \frac{nx}{1+nx^2}$$

1. Find the pointwise limit of (f_n) for all $x \in (0, \infty)$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+nx^2} = \lim_{n\to\infty} \frac{x}{\frac{1}{n}+x^2} = \frac{x}{x^2} = \frac{1}{x}$$

2. Is the convergence uniform on $(0, \infty)$?

Let $\varepsilon > 0$.

$$\begin{aligned} |f_n(x) - f(x)| &= \left|\frac{nx}{1+nx^2} - \frac{1}{x}\right| \\ &= \left|\frac{nx^2}{x(1+nx^2)} - \frac{1+nx^2}{x(1+nx^2)}\right| \\ &= \left|-\frac{1}{x(1+nx^2)}\right| = \frac{1}{x(1+nx^2)} \end{aligned}$$

Therefore, any choice of n to make $|f_n(x) - f(x)|$ will depends on x so the convergence is not uniform on $(0, \infty)$. ■

3. Is the convergence uniform on $(0, 1)$?

Let $\varepsilon = 1$. Then
$$|f_n(x) - f(x)| = \frac{1}{x+nx^3}$$

If we choose $x = \frac{1}{4}$ then
$$\left|f_n(\frac{1}{4}) - f(\frac{1}{4})\right| = \frac{1}{1/4 + n(1/64)} = \frac{64}{16+n}$$
so we need $n > 48$ to make $\left|f_n(\frac{1}{4}) - f(\frac{1}{4})\right| < 1$.

However, if we chose $x = \frac{1}{2}$, then
$$\left|f_n(\frac{1}{2}) - f(\frac{1}{2})\right| = \frac{1}{1/2 + n/8} = \frac{8}{4+n}$$
so we just need $n > 4$ to make $\left|f_n(\frac{1}{2}) - f(\frac{1}{2})\right| < 1$.

Therefore, the convergence is not uniform on $(0, 1)$. ■

4. Is the convergence uniform on $(1, \infty)$?

$x \geq 1$ implies that
$$|f_n(x) - f(x)| = \frac{1}{x+nx^3} < \frac{1}{1+n} < \frac{1}{n}$$

Therefore, to bound $|f_n(x) - f(x)| < \varepsilon$, we need $n > \frac{1}{\varepsilon}$ which is independent of x. The convergence is uniform on $(1, \infty)$. ■

Problem 2

Using the Cauchy Criterion for convergent sequences of real numbers (Thm. 2.6.4 in the book), supply a proof for the Cauchy Criterion for Uniform Convergence (Thm. 6.2.5 in the book). First define a candidate for $f(x)$ and then argue that $f_n \to f$ uniformly.

The Cauchy Criterion for uniform convergence says a sequence of functions (f_n) defined on $A \subseteq \mathbb{R}$ converges uniformly on A iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \varepsilon$ whenever $n, m \geq N$ and $x \in A$.

The Cauchy Criterion for convergent sequences of real numbers says a sequence converges iff it is a Cauchy sequence.

Suppose $(f_n) \to f$ uniformly. Then $\exists N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ whenever $n \geq N$ and $x \in A$. Choose $m > N$. Then

$$|f_n(x) - f_m(x)| = |f_n(x) - f_m(x) - f(x) + f(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

so (f_n) is a Cauchy sequence.

Suppose now that $\exists N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$ whenever $n, m \geq N$ and $x \in A$. Then, by the Cauchy Criterion for Convergent Sequences, (f_n) converges to some limit L.

Define $f(x) = L$. then

$$|f_n(x) - f(x)| = |f_n(x) - f(x) + f_m(x) - f_m(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)|$$

By assumption, $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$. By pointwise convergence of (f_n), if we choose m sufficiently large, we have $|f_m(x) - f(x)| = |f_m(x) - L| < \frac{\varepsilon}{2}$. Therefore, $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and $x \in A$ so (f_n) converges uniformly to f. $\blacksquare$

Problem 3 (Arzela-Ascoli Theorem)

A sequence of functions (f_n) defined on a set $E \subset \mathbb{R}$ is called *equicontinuous* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon$ for all $n \in \mathbb{N}$ and $|x - y| < \delta$ in E.

For each $n \in \mathbb{N}$, let f_n be a function defined on $[0, 1]$. If (f_n) is bounded on $[0, 1]$ – that is, there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in [0, 1]$ – and if the collection of functions (f_n) is equicontinuous, follow these steps to show that (f_n) contains a uniformly convergent subsequence.

1. Explain why the construction in Exercise 6.2.13 (see below) produces a subsequence (f_{n_k}) that converges at every rational point in $[0, 1]$. To simplify notation, set $g_k = f_{n_k}$. It remains to show that (g_k) converges uniformly on all of $[0, 1]$.

   We proved in class that $\mathbb{Q}$ is countable. Let $A = \{x \in [0, 1] \cap \mathbb{Q}\}$. Since $A \subseteq \mathbb{Q}$, A is countable (not finite by density of $\mathbb{Q}$ in $\mathbb{R}$).

   Further, since $A \subseteq [0, 1]$, (f_n) is defined on A and is bounded on A by assumption. By Exercise 6.2.13, there exists a subsequence (f_{n_k}) that converges pointwise on A.

2. Let $\varepsilon > 0$. By equicontinuity, there exists a $\delta > 0$ such that

   $$|g_k(x) - g_k(y)| < \frac{\varepsilon}{3}$$

   for all $|x - y| < \delta$ and $k \in \mathbb{N}$. Using this δ, let $r_1, r_2, \cdots, r_m$ be a *finite* collection of rational points with the property that the union of the neighborhoods $V_\delta(r_i)$ contains $[0, 1]$.

   Explain why there must exist an $N \in \mathbb{N}$ such that

   $$|g_s(r_i) - g_t(r_i)| < \frac{\varepsilon}{3}$$

   for all $s, t \geq N$ and r_i in the finite subset of $[0, 1]$ just described. Why does having the set $\{r_1, r_2, \cdots, r_m\}$ be finite matter?

   Since r_i is a rational point in $[0, 1]$, we have that (g_k) converges at r_i. Therefore, it is Cauchy so $\exists N_i \in \mathbb{N}$ such that $|g_s(r_i) - g_t(r_i)| < \frac{\varepsilon}{3}$ for all $s, t \geq N$.

   We want the sequence to converge for all r_i, so we take $N = \max\{N_1, N_2, \cdots, N_m\}$. However, taking a maximum of a set requires that the set be finite. The supremum would not suffice because our domain is limited to $\mathbb{N}$ (axiom of completeness).

3. Finish the argument by showing that, for an arbitrary $x \in [0, 1]$,

   $$|g_s(x) - g_t(x)| < \varepsilon$$

   for all $s, t \geq N$.

   By the triangle inequality,

   $$\begin{aligned} |g_s(x) - g_t(x)| &= |g_s(x) - g_t(x) + g_s(r_i) - g_s(r_i) + g_t(r_i) - g_t(r_i)| \\ &= |g_s(x) - g_s(r_i)| + |g_t(x) - g_t(r_i)| + |g_s(r_i) - g_t(r_i)| \end{aligned}$$

   for any r_i in the finite subset of $[0, 1]$.

   By (2), for all $s, t \geq N$, we have $|g_s(r_i) - g_t(r_i)| < \frac{\varepsilon}{3}$.

   By the definition of $\{r_i\}_{i=1}^m$, we have that any $x \in [0, 1]$ is in $V_\delta(r_i)$ for some r_i. Therefore, $|x - r_i| < \delta$. By equicontinuity, $|g_s(x) - g_s(r_i)| < \frac{\varepsilon}{3}$ and $|g_t(x) - g_t(r_i)| < \frac{\varepsilon}{3}$.

Thus,

$$|g_s(x) - g_t(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \blacksquare$$

Result from Exercise 6.2.13: Let $A = \{x_1, x_2, x_3, \cdots\}$ be a countable set. For each $n \in \mathbb{N}$, let f_n be defined on A and assume there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in A$. Then there exists a subsequence of (f_n) that converges pointwise on A. (This a version of the Bolzano–Weierstrass Thm. for bounded sequences of functions).

Problem 4

Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}.$$

1. Show (g_n) converges uniformly on $[0, 1]$ and find $g = \lim g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0, 1]$.

   Let $\varepsilon > 0$. Since $x \leq 1$, $g_n(x) \leq \frac{1}{n}$. Therefore, $N > \frac{1}{\varepsilon}$ will suffice to make $|g_n(x) - 0| < \varepsilon$ for all $n \geq N$ and $x \in [0, 1]$. Thus, $\lim g_n = 0$.

   Clearly, g is differentiable since $g(x) = 0$ for all $x \in [0, 1]$. Therefore, $g'(x) = 0$ for all $x \in [0, 1]$. ■

2. Now, show that (g'_n) converges on $[0, 1]$. Is the convergence uniform? Set $h = \lim g'_n$ and compare h and g'. Are they the same?

   $$\lim_{n\to\infty} g'_n = \lim_{n\to\infty} x^{n-1} = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

   Notice

   $$\left| g'_n(\frac{1}{2}) - h(\frac{1}{2}) \right| < \frac{1}{3} \implies n \geq 3$$
   $$\left| g'_n(\frac{9}{10}) - h(\frac{9}{10}) \right| < \frac{1}{3} \implies n \geq 12$$

   so the convergence is not uniform.

   We have $g'(x) = 0$ for all $x \in [0, 1]$ and $h = \mathbf{1}_{x=1}(x)$ so $h \neq g'$. ■

Problem 5

Use the Mean Value Theorem to supply a proof for Theorem 6.3.2 from the book. To get started, observe that the triangle inequality implies that, for any $x \in [a, b]$ and $m, n\mathbb{N}$,

$$|f_n(x) - f_m(x)| \le |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|.$$

Theorem 6.3.2: Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a, b]$ and assume (f_n') converges uniformly to g on $[a, b]$. If there exists a point $x_0 \in [a, b]$ such that $(f_n(x_0))$ converges, then (f_n) converges uniformly. Moreover, $f = \lim f_n$ is differentiable and $f'(x) = g(x)$ for all $x \in [a, b]$.

Proof: Let (f_n) be differentiable on $[a, b]$ and assume $(f_n') \to g$ uniformly on $[a, b]$. Suppose $\exists x_0 \in [a, b]$ so $(f_n(x_0))$ converges.

Let $\varepsilon > 0$. We want to show that (f_n) converges uniformly, i.e. $\exists K \in \mathbb{N}$ for which $\forall n, m \ge K$,

$$|f_n(x) - f_m(x)| < \varepsilon$$

Using the triangle inequality,

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f_m(x) + f_n(x_0) - f_m(x_0) + f_n(x_0) - f_m(x_0)| \\ &\le |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| \end{aligned}$$

By assumption, $\exists x_0 \in [a, b]$ so $\exists N \in \mathbb{N}$ for which $\forall n, m \ge N$,

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}$$

Now let $h(x) = f_n(x) - f_m(x)$. Since $f_n(x)$ is differentiable for all $x \in [a, b]$ and $n \in \mathbb{N}$, $h'(x) = f_n'(x) - f_m'(x)$ by the Algebraic Differentiability Theorem.

Since $(f_n') \to g$ uniformly, by the Cauchy Criterion, $\exists M \in \mathbb{N}$ such that $\forall m, n \ge M$,

$$|f_n'(x) - f_m'(x)| < \varepsilon_0$$

for all $x \in [a, b]$.

Therefore, $|h'(x)| < \varepsilon_0$.

By the mean value theorem, $\exists c \in [x_0, x]$ such that

$$h'(c) = \frac{h(x) - h(x_0)}{x - x_0} = \frac{(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))}{x - x_0}$$

Therefore,

$$\begin{aligned} |f_n(x) - f_m(x)| &\le |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| \\ &= |h'(c)(x - x_0)| + |f_n(x_0) - f_m(x_0)| \end{aligned}$$

Using our earlier convergence bounds, $\forall n, m \ge \max\{N, M\}$,

$$|f_n(x) - f_m(x)| \le |\varepsilon_0(x - x_0)| + \frac{\varepsilon}{2}$$

Since $x, x_0 \in [a, b]$, $|x - x_0| \le b - a$. If we let $\varepsilon_0 = \frac{\varepsilon}{2(b-a)}$, then

$$|f_n(x) - f_m(x)| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

so by the Cauchy Criterion, (f_n) converges uniformly.

Since $(f_n) \to f$ uniformly, it converges pointwise on $[a, b]$ and by assumption $(f_n') \to g$ uniformly. Therefore, by the Differentiable Limit Theorem $f = \lim f_n$ is differentiable and $f'(x) = g(x)$ for all $x \in [a, b]$. $\blacksquare$