Math 1010: One-Variable Analysis

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Chapter 1

The Real Numbers

Lecture 1 - Jan 24:

Preliminaries

1. Sets

   Definition: A *set* is a collection of objects.

   De Morgan's Laws:

   $$(A \cap B)^c = A^c \cup B^c$$
   $$(A \cup B)^c = A^c \cap B^c$$

   Proof: HW

2. Functions

   Definition: Given two sets A, B, a *function* $f : A \to B$ is a rule that assigns to each $a \in A$ a unique element $f(a) \in B$.

   The *domain* of f is A. The *range* of f is a subset of B.

   Examples:

   (a) Dirichlet Function:

   $$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

(Its domain is $\mathbb{R}$ and its range is $\{0, 1\}$)

(b) Absolute value function:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Properties:

$$|ab| = |a| \cdot |b|$$
$$|a + b| \leq |a| + |b| \qquad \text{(Triangle Inequality)}$$

3. Proofs

Types of Proofs:

- *Direct Proof* - Start with a valid statement (usually the hypothesis) and proceed by logical steps
- *Indirect Proof (Proof by Contradiction)* - Begin by negating the conclusion and proceed by logical steps to a contradiction.

Theorem: Let $a, b \in \mathbb{R}$. Then $a = b \iff \forall \varepsilon > 0, |a - b| < \varepsilon$

Proof: We have two statements:

- If $a = b \implies \forall \varepsilon > 0, |a - b| < \varepsilon$
- If $\forall \varepsilon > 0, |a - b| < \varepsilon \implies a = b$

Proof of first statement: Suppose $a = b$. Then $|a - b| = 0$. Thus, $\forall \varepsilon > 0,\ |a - b| < \varepsilon$.

Proof of second statement: Assume $a \neq b$. Then $\exists\, \varepsilon_0 > 0$ s.t. $|a - b| = \varepsilon_0$ But this is contradiction by hypothesis. ■

Proof by induction:

Example: Let $x_1 = 2$ and $\forall n \in \mathbb{N}$, define $x_{n+1} = \frac{x_n+5}{3}$, $n \geq 1$. Prove that x_n is increasing.

Proof:

(a) Base Case:

$$x_1 = 2 < x_2 = \frac{7}{3} \quad \checkmark$$

(b) Inductive Step: Assume $x_n \leq x_{n+1}$. Then

$$\underbrace{\frac{x_n + 5}{3}}_{x_{n+1}} \leq \underbrace{\frac{x_{n+1} + 5}{3}}_{x_{n+2}} \implies x_{n+1} \leq x_{n+2} \quad \blacksquare$$

Axioms for the real numbers

- **Field Axioms:** $\forall a, b, c \in \mathbb{R}$
  1. $(a + b) + c = a + (b + c)$ (Additive Associativity)
  2. $\exists\, 0 \in \mathbb{R}$ s.t. $a + 0 = a$ (Additive Identity)
  3. $\exists - a \in \mathbb{R}$ s.t. $a + (-a) = 0$ (Additive Inverse)
  4. $a \cdot b = b \cdot a$ (Commutativity)
  5. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (Multiplicative Associativity)
  6. $\exists\, 1 \in \mathbb{R}$ s.t. $a \cdot 1 = a$ (Multiplicative Identity)
  7. $\exists\, a^{-1} \in \mathbb{R}$ s.t. $a \cdot a^{-1} = 1$ (Multiplicative Inverse)
  8. $a \cdot (b + c) = a \cdot b + a \cdot c$ (Distributivity)
- **Order Axioms:** there exists a subset of positive numbers P such that
  10. exclusively either $a \in P$ or $-a \in P$ or $a = 0$ (Trichotomy)
  11. $a, b \in P \implies a + b \in P$ (Closure under addition)
  12. $a, b \in P \implies a \cdot b \in P$ (Closure under multiplication)
- **Completeness Axiom:** a least upper bound of a set A is a number x such that $x \geq y$ for all $y \in A$, and such that if z is also an upper bound of A, then

$z \geq x$.

13. Every nonempty set A which is bounded above has a least upper bound.

We will call Properties 1-12, and anything that follows from them, *elementary arithmetic*. These alone imply that $\mathbb{Q}$ is a subfield of $\mathbb{R}$ and basic properties of inequalities under addition and multiplication.

Adding Property 13 uniquely determines the real numbers. The standard proof is to identify each $x \in \mathbb{R}$ with the subset of rationals $\{y \in \mathbb{Q} : y < x\}$, *the Dedekind cut*. This can also construct the reals from the rationals.

Lecture 2 - Jan 30:

Axiom of Completeness

1. $\mathbb{R}$ is an ordered field.
2. There is a least upper bound and a greatest lower bound

Note: the axiom of completeness is only true for $\mathbb{R}$

Definition: Let $A \subseteq \mathbb{R}$ be a set. Then:

1. A is *bounded above* if $\exists\, b \in \mathbb{R}$ s.t. $a \leq b$ for all $a \in A$. Conversely, then b is an *upper bound* of A.
2. A is *bounded below* if $\exists\, l \in \mathbb{R}$ s.t. $a \geq l$ for all $a \in A$. Conversely, then l is a *lower bound* of A.

Definition: $s \in \mathbb{R}$ is *least upper bound* of $A \subseteq \mathbb{R}$ if

1. s is an upper bound of A
2. if b is any upper bound for A, then $s \leq b$

s is called *the supremum of* A and is denoted $s := \sup A$. Further, it is unique.

Similarly, $\inf A$ (the *infimum*) is the greatest lower bound of A.

Example: $A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. Then $\sup A = 1$.

Proof:

1. $1 \geq \frac{1}{n}$ for all $n \in \mathbb{N}$ ✓

2. Assume b is another upper bound. Since $1 \in A$, $1 \leq b$ ■

Remark: $\sup A$ and $\inf A$ do not have to be elements of A.

- When $\sup A \in A$, we call it the *maximum*
- When $\inf A \in A$, we call it the *minimum*

Example: In the example above, $\inf A = 0 \notin A$.

Example:

$$(0,2) = \{x \in \mathbb{R} : \underbrace{0}_{\text{inf}} < x < \underbrace{2}_{\text{sup}}\}$$
$$[0,2] = \{x \in \mathbb{R} : \underbrace{0}_{\text{min}} \leq x \leq \underbrace{2}_{\text{max}}\}$$

Theorem: There is no rational number whose square is 2

Proof: Suppose $\exists, p, q \in \mathbb{Z}$ s.t. $(\frac{p}{q})^2 = 2$. We further assume that $q \neq 0$ and $\text{GCF}(p,q) = 1$.

Then

$$(\frac{p}{q})^2 = 2 \implies \frac{p^2}{q^2} = 2 \implies p^2 = 2q^2$$

Thus, p^2 is even so p is even (because the product of two odd numbers is odd).

Thus, we can write $p = 2r,\ r \in \mathbb{Z}$. Substituting,

$$(2r)^2 = 2q^2 \implies 4r^2 = 2q^2 \implies 2r^2 = q^2$$

By similar logic, q is even. But this contradicts our assumption that $\text{GCF}(p,q) = 1$. ■

This allows us to show that $\mathbb{Q}$ has gaps (it is incomplete). Consider:

$$S = \{r \in \mathbb{Q} : r^2 < 2\}$$

A sensible upper bound is $\sqrt{2} \approx 1.4142\ldots$. Since $\sqrt{2} \notin \mathbb{Q}$, we need to approximate it with rational numbers. We can get infinitely close,

$$\frac{3}{2}, \frac{142}{100}, \frac{1415}{1000}, \ldots$$

but because we need infinitely many terms, we do not have a least upper bound (the next term will always be closer).

Lemma: Let $s \in \mathbb{R}$ be an upper bound for a set $A \subseteq \mathbb{R}$. Then $s = \sup A$ iff $\forall \varepsilon > 0\, \exists a \in A$ s.t. $s - \varepsilon < a$

Proof:

1. Suppose $s = \sup A$. Consider any $s - \varepsilon$ with $\varepsilon > 0$. From the definition of supremum, $s - \varepsilon$ is not an upper bound for A (because $s - \varepsilon < \sup A$). Thus, $\exists a \in A$ s.t. $s - \varepsilon < a$

2. Suppose $\forall \varepsilon > 0\, \exists\, a \in A$ s.t. $s - \varepsilon < a$.

   Since $s - \varepsilon < a$, it cannot be an upper bound by definition. Thus, for any $b < s$, b is not an upper bound. Therefore, any upper bound b' must satisfy $s \leq b'$. This is precisely the definition of $\sup A$. ■

Lecture 3 - Feb 1:

Recall

- $\mathbb{R}$ is an ordered field satisfying the Axiom of Completeness
- $\mathbb{Q}$ is an ordered field but does not satisfy the Axiom of Completeness
- $\mathbb{Z}$ satisfies the AOC but is not a field (so we ignore it in analysis)
- $s = \sup A \implies a \leq b$ for any other upper bound b

Consequences of Completeness

Theorem (Nested interval property): For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that I_n contains I_{n+1}. Then the resulting nested sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ has a nonempty intersection $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$

Proof:

Let $A = \{a_n : n \in \mathbb{N}\}$ be the set of all left endpoints of the intervals I_n. Then A is nonempty and bounded above by the b (right) endpoints.

Consider $x = \sup A$. We know $a_n \leq x \leq b_n$ for all $n \in \mathbb{N}$ by the fact that x is an upper bound for A and that it is the *least* upper bound for A.

And indeed, this is exactly the intersection of the intervals. ■

Note that the theorem does not hold for $\mathbb{Q}$! Imagine the series of intervals centered at $\frac{1}{\sqrt{2}}$ – all are non-empty but their intersection is empty (because there are rational numbers infinitely close to $\frac{1}{\sqrt{2}}$ but that final interval would be empty).

Theorem (Archimedian Property): Given any number $x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ satisfying $n > x$. (i.e. $\mathbb{N}$ is *not* bounded above)

Proof by contradiction:

Suppose $\mathbb{N}$ is bounded above. By the axiom of completeness, $\mathbb{N}$ has a least upper bound $\alpha = \sup \mathbb{N}$. By definition of supremum, $\alpha - 1 < n \implies \alpha < n+1$. But $n + 1 \in \mathbb{N}$, so α is not an upper bound. ■

Consequence: Given any real number $y > 0$, $\exists n \in \mathbb{N}$ satisfying $\frac{1}{n} < y$.

Proof: Let $x = \frac{1}{y}$. By the Archimedean Property, $\exists n \in \mathbb{N}$ satisfying $n > x$. Then $n > \frac{1}{y} \implies y < \frac{1}{n}$

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$): For every two real numbers a and b with $a < b$, $\exists r \in \mathbb{Q}$ s.t. $a < r < b$

Proof:

We want to show that $\exists m \in \mathbb{Z}, n \in \mathbb{N} : a < \frac{m}{n} < b$.

First note that we can choose $m \in \mathbb{Z}, n \in \mathbb{N}$ to bound a. We choose n such that

$$\frac{m-1}{n} < a < \frac{m}{n}$$

and m to be the smallest integer greater than na:

$$m - 1 \leq na < m$$

The RHS inequality gives $a < \frac{m}{n}$.

By Archimedean property, we can pick $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$. Equivalently, $a < b - \frac{1}{n}$.

The LHS gives

$$m \leq na + 1 < n(b - \frac{1}{n}) + 1 = nb \implies m < nb \implies \frac{m}{n} < b$$

Thus,

$$a < \frac{m}{n} < b \quad \blacksquare$$

Corollary: Density of Irrationals ($\mathbb{I}$) in $\mathbb{R}$

Cardinality

Definition: *Cardinality* is the size of a set

Definition:

- A function $f : A \to B$ is *injective* (or *one-to-one*) if $a_1 \neq a_2$ in A implies $f(a_1) \neq f(a_2)$.
- A function $f : A \to B$ is *surjective* (*onto*) if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$ (all elements in B have a pre-image in A)
- A function $f : A \to B$ is *bijective* (has a "1-to-1 correspondence") if it is both injective and surjective

Definition: The set A has the same cardinality as the set B if there exists a bijection $f : A \to B$.

Example: $E = \{2, 4, 6, 8, \ldots\}$. We create an equivalence relation $\mathbb{N} \sim E$ induced by $f : \mathbb{N} \to E$ given by $f(n) = 2n$. Thus $\mathbb{N}$ and E have the same cardinality.

Example: $\mathbb{N} \sim \mathbb{Z}$. Consider

$$f(n) = \begin{cases} \frac{n-1}{2} & n \text{ is odd} \\ -\frac{n}{2} & n \text{ is even} \end{cases}$$

Proof of bijection is left as an exercise.

Example: $(a, b) \sim \mathbb{R}$

Lecture 4 - Feb 6:

Countable Sets

Definition: A set A is *countable* if $A \sim \mathbb{N}$ (it has the same cardinality as $\mathbb{N}$)

Theorem: $\mathbb{Q}$ is countable

Proof: It suffices to construct a bijection $\phi : \mathbb{N} \to \mathbb{Q}$.

Consider $A_1 = \{0\}$ and for each $n \geq 2$,

$$A_n = \{\pm\frac{p}{q} : p, q \in \mathbb{N} \quad \text{with p/q in lowest term with } p + q = n\}$$

i.e., $A_2 = \{1, -1\},\ A_3 = \{\frac{1}{2}, -\frac{1}{2}, 2, -2\},\ A_4 = \{\pm\frac{1}{3}, \pm 3\}$

We know that each A_n is finite. Further, every rational number appears *exactly* once in these sets.

We can then define $\phi : \mathbb{N} \to \mathbb{Q}$ by the one-to-one correspondence between the natural numbers and each element of the A_n's

$$\begin{array}{lccccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \\ \mathbb{Q}: & \underbrace{0}_{A_1} & \underbrace{1 & -1}_{A_2} & \underbrace{\frac{1}{2} & -\frac{1}{2} & 2 & -2}_{A_3} & \dots \end{array}$$

The correspondence is onto: every rational will appear. (e.g. $\frac{22}{7} \in A_{29}$)

The correspondence is 1-1: each rational appears exactly once. ■

Theorem: $\mathbb{R}$ is uncountable

Proof: Assume $\mathbb{R}$ is countable. Then $\mathbb{R} = \{x_1, x_2, \dots\}$

Let I_1 be a closed interval which does not contain x_1. Then $I_2 \subseteq I_1$ and does not contain x_2. By induction, $I_{n+1} \subseteq I_n,\ x_n \notin I_n$

Consider $\bigcap_{n=1}^{\infty} I_n$. If x_{n_0} is in the list, $\exists\, I_{n_0}$ s.t. $x_{n_0} \notin I_{n_0}$. But then

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

However, by the nested interval property, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem: If $A \subseteq B$ and B is countable, then A is countable or finite

Proof: HW

Theorem:

1. If $A_1, A_2, \ldots, A_m$ are countable, then $\bigcup_{n=1}^{m} A_n$ is countable
2. If $A_1, A_2, \ldots$ are countable, then $\bigcup_{n=1}^{\infty} A_n$ is countable

Proof: HW

Chapter 2

Sequences and Series

Lecture 1 - Feb 6 (Continued):

The Limit of a Sequence

Definition: A *sequence* is a function whose domain is $\mathbb{N}$

Examples:

- $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots) = (\frac{1}{n})_{n\in\mathbb{N}}$
- $(\frac{1+n}{n})_{n=1}^{\infty} = (2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots)$
- $x_1 = 2,\ x_{n+1} = \frac{x_n+1}{2}$

Definition (convergence of a sequence): A sequence (a_n) *converges* to a real number a if, for every positive number ε, there exists a $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - a| < \varepsilon$:

$$\begin{aligned} \lim_{n\to\infty} a_n = a &\iff a_n \to a \\ &\iff \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } n \geq N \implies |a_n - a| < \varepsilon \end{aligned}$$

Definition (ε-neighborhood): The *ε-neighborhood* of $a \in \mathbb{R}$ (given $\varepsilon > 0$) is the set $V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$

Here, ε is the radius about the center a.

Definition: A sequence (a_n) converges to a if, given any ε-neighborhood $V_\varepsilon(a)$ if a, there exists a point in the sequence after which all the terms are in $V_\varepsilon(a)$

Lecture 2 - Feb 08:

Convergence

Example: Let $a_n = \frac{1}{\sqrt{n}}$. Show $\lim_{n\to\infty} a_n = 0$.

First we try a few values of epsilon:

- $\varepsilon = \frac{1}{10}$: $(0 - \frac{1}{10}, 0 + \frac{1}{10}) = (-\frac{1}{10}, \frac{1}{10})$

  When $n = 100 \implies a_{100} = \frac{1}{10}$. So the first element in the interval is a_{101}.

- $\varepsilon = \frac{1}{50}$: $(-\frac{1}{50}, \frac{1}{50})$

  Here, the first element in the interval is a_{2501}.

Now for the rigorous version: Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon^2} \implies \frac{1}{\sqrt{N}} < \varepsilon$.

Let $n \geq N$. Then

$$n > \frac{1}{\varepsilon^2} \implies \frac{1}{\sqrt{N}} < \varepsilon \implies \left|\frac{1}{\sqrt{n} - 0}\right| < \varepsilon$$

A template for convergence proofs:

1. Let $\varepsilon > 0$
2. Demonstrate a choice for $N \in \mathbb{N}$
3. Verify N
4. With N well chosen, it should be possible to get $|x_n - x| < \varepsilon$

Example: Prove that $\lim \frac{n+1}{n} = 1$

We want $\left|\frac{n+1}{n} - 1\right| < \varepsilon$. This is equivalent to $\left|\frac{1}{n}\right| < \varepsilon$. So we choose $N \in \mathbb{N} > \frac{1}{\varepsilon}$.

The actual proof then reads: Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ s.t. $N\frac{1}{\varepsilon}$. Let $n \geq N$.

$$n > \frac{1}{\varepsilon} \implies \frac{1}{n} < \varepsilon \implies \left|\frac{n+1}{n} - 1\right| < \varepsilon$$

Theorem (Uniqueness of limits): The limit of a sequence, when it exists, is unique

Proof: HW

The algebraic and order limit theorems

Definition: A sequence (x_n) is bounded if there exists a number $M > 0$, such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem: Every convergent sequence is bounded

Proof: Assume (x_n) converges to l.

Given $\varepsilon > 0$,

$$\exists\, N \in \mathbb{N} \text{ s.t. } x_n \in (l - \varepsilon, l + \varepsilon)\ \forall n \geq N$$

Since we do not know if l is positive or negative, we can only say

$$|x_n| < |l| + \varepsilon$$

From this we know x is bounded for $n \geq N$. Now we check the case $n < N$. Luckily, this is a finite number of cases.

By construction, $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |l| + 1\}$. Then $|x_n| \leq M$ for all $n \in \mathbb{N}$. ■

Theorem (Algebraic Limit Theorems): Let $\lim a_n = a$, $\lim b_n = b$

1. $\lim(ca_n) = ca, \quad \forall c \in \mathbb{R}$
2. $\lim(a_n + b_n) = a + b$
3. $\lim(a_n \cdot b_n) = a \cdot b$
4. $\lim \frac{a_n}{b_n} = \frac{a}{b}$, provided $b \neq 0$

Proof:

1. Let $\varepsilon > 0$. We want to show $|ca_n - ca| < \varepsilon$. Notice

$$|ca_n - ca| = |c| \cdot |a_n - a|$$

Since a_n is convergent, we can make $|a_n - a|$ arbitrarily small.

We choose $N \in \mathbb{N}$ s.t. $|a_n - a| < \frac{\varepsilon}{|c|}$ so $\forall n > N$,

$$|ca_n - ca| < |c| \frac{\varepsilon}{|c|} = \varepsilon \quad \checkmark$$

2. Let $\varepsilon > 0$. We want to show $|a_n + b_n - (a+b)| < \varepsilon$. We can say $|a_n - a + b_n - b| \leq |a_n - a| + |b_n - b|$ by the Triangle inequality. Then since a_n and b_n are convergent, we note that

$$\exists N_1 \in \mathbb{N},\ \text{s.t. } \forall n \geq N_1 : \quad |a_n - a| < \frac{\varepsilon}{2}$$
$$\exists N_2 \in \mathbb{N},\ \text{s.t. } \forall n \geq N_2 : \quad |b_n - b| < \frac{\varepsilon}{2}$$

Choose $N = \max\{N_1, N_2\}$ so

$$\forall n \geq N : \quad |(a_n + b_n) - (a+b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \checkmark$$

3. Let $\varepsilon > 0$. We want to show that $|a_n \cdot b_n - a \cdot b| < \varepsilon$. We can say

$$\begin{aligned} |a_n b_n - ab_n + ab_n - ab| &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n| \cdot |a_n - a| + |a| \cdot |b_n - b| \end{aligned}$$

Since a_n and b_n are convergent, $\exists N_1 \in \mathbb{N},$ s.t. $\forall n \geq N_1 : \quad |b_n - b| < \frac{\varepsilon}{2|a|}$. Note then that b_n is convergent so bounded: $|b_n| \leq M$. Then $\exists N_2,$ s.t. $\forall n \geq N_2 : \quad |a_n - a| < \frac{\varepsilon}{2M}$

So with $N = \max N_1, N_2$, $\forall n \geq N$, we have

$$|a_n b_n - ab| \leq M \cdot \frac{\varepsilon}{2M} + |a| \cdot \frac{\varepsilon}{2\,|a|} = \varepsilon$$

4. Let $\varepsilon > 0$. We want to show that $\left| \frac{a_n}{b_n} - \frac{a}{b} \right| < \varepsilon$. This is the same as showing $a_n \cdot \frac{1}{b_n} \to a \cdot \frac{1}{b}$ so it suffices to show that $\frac{1}{b_n} \to \frac{1}{b}$ and apply the multiplicative limit theorem.

Observe:

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_n \cdot b}\right| = \frac{|b - b_n|}{|b_n| \cdot |b|}$$

Intuitively, finding a lower bound for b_n gives an upper bound for $1/b_n$. *Trick:* Choose a large n such that $|b_n - b| > |b_n - 0| \implies |b_n| > \frac{|b|}{2}$. By convergence of (b_n), $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N : |b_n - b| < \frac{|b|}{2}$. Then $|b_n| > \frac{|b|}{2}$.

Now bound $|b_n - b| < \frac{\varepsilon |b|^2}{2}$ by convergence at $N_2 \in \mathbb{N}$.

Finally, let $N = \max\{N_1, N_2\}$ then for $n > N$,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b_n| \cdot |b|} < \frac{\varepsilon |b|^2}{2} \cdot \frac{2}{|b|} \cdot \frac{1}{|b|} = \varepsilon \quad \blacksquare$$

Lecture 3 - Feb 15:

Theorem (Order Limit Theorem): Assume $(a_n) \to a$, $(b_n) \to b$.

1. If $a_n \geq 0 \quad \forall n \in \mathbb{N}$, then $a \geq 0$
2. If $a_n \leq b_n \quad \forall n \in \mathbb{N}$, then $a \leq b$
3. If $\exists c \in \mathbb{R}$ s.t. $c \leq b_n \quad \forall n \in \mathbb{N}$, then $c \leq b$

Proof:

1. Suppose $a < 0$. Consider $\varepsilon = |a|$ so $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N : |a_n - a| < |a|$. However, since $a < 0$, this tells us

$$a < a_n - a < -a \implies a_n < 0$$

   But this contradicts the fact that $a_n \geq 0$.

2. By the Algebraic limit theorem, $(b_n - a_n) \to b - a$. Since $a_n \leq b_n$ for all $n \in \mathbb{N}$, $b_n - a_n \geq 0$, by part 1, $b - a \geq 0 \implies b \geq a$
3. Take $a_n = c \quad \forall n \in \mathbb{N}$. Then $(a_n) \to c$. The result follows from part 2. $\blacksquare$

Monotone Convergence Theorem

Definition: A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. It is *decreasing* if $a_n \geq a_{n+1} \quad \forall n \in \mathbb{N}$.

A sequence is *monotone* if it is either increasing or decreasing for all $n \in \mathbb{N}$.

Theorem (Monotone Convergence Theorem): If a sequence is monotone and bounded, then it is convergent

Proof: Let (a_n) be monotone and bounded. Assume WLOG that (a_n) is increasing. Consider the set $A = \{a_n : n \in \mathbb{N}\}$. SInce (a_n) is bounded, $supA$ exists.

We claim $\lim_{n\to\infty} a_n = \sup A$. Let $\varepsilon > 0$. Since $\sup A$ is the least upper bound, $\sup A - \varepsilon$ is not an upper bound. Thus, $\exists N \in \mathbb{N}$ s.t. $a_N > \sup A - \varepsilon$. Since a_n is monotone, $a_n > \sup A - \varepsilon \quad \forall n \geq N$. Further, $a_n \leq \sup A + \varepsilon$ so

$$|a_n - \sup A| < \varepsilon$$

Series Introduction

Definition (Convergence of Series): Let (b_n) be a sequence. A *infinite series* is an expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \dots$$

The series *converges* to S if the sequence of *partial sums* (S_n) given by

$$S_m = \sum_{n=1}^{m} b_n = b_1 + \cdots + b_m$$

converges to S.

Example: Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$S_m = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{m^2}$$

We seek an upper bound for (S_m). Notice

$$\begin{aligned}
S_m &= \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \cdots + \frac{1}{m\cdot m} \\
&< \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(m-1)\cdot m} \\
&= 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{m-1} - \frac{1}{m}) \\
&= 2 - \frac{1}{m} < 2
\end{aligned}$$

Since (S_m) has an upper bound and is increasing, it is convergent to some limit s.

Example (Harmonic Series): Consider $\sum_{n=1}^{\infty} \frac{1}{n}$. Taking partial sums,

$$S_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

Is S_m bounded? No!

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 2$$

But

$$S_8 > 2 + \frac{1}{2}$$

and

$$S_{2^k} > 1 + k(\frac{1}{2})$$

and this is unbounded!

Lecture 4 - Feb 22:

Theorem (Cauchy Condensation Test): Suppose (b_n) is decreasing and $b_n \geq 0 \quad \forall n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} b_n$ converges iff $\sum_{n=1}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + \ldots$ converges

Proof: Omitted.

Remark: This is a mostly useless theorem used only for showing the harmonic series diverges.

Corollary: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

Subsequences

Definition: Let (a_n) be a sequence and let $n_1 < n_2 < n_3 < \dots$ be an increasing sequence of natural numbers. Then the sequence $(a_{n_1}, a_{n_2}, a_{n_3}, \dots)$ is a *subsequence* of (a_n) and is denoted by (a_{n_k}) where $k \in \mathbb{N}$ is the index.

Example:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	$\dots$
1	5	-3	10	0	-8	12	$\dots$

If we choose $n_1 = 3, n_2 = 4, n_3 = 6, \dots$ then $(a_{n_k}) = (-3, 10, -8, \dots)$

Note: The order of the terms in the subseq is the same as in the original sequence. Further, no repetitions are allowed.

Examples: $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots)$

- $(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8})$ is a subsequence
- $(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \dots)$ is a subsequence
- $(\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{5}, \dots)$ is *not* a subsequence
- $(1, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \dots)$ is *not* a subsequence

Theorem: A subsequence of a convergent sequence converges to the same limit as the original sequence

Proof: Assume $(a_n) \to a$. Let (a_{n_k}) be a subsequence. Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N : |a_n - a| < \varepsilon$. Since $n_k \geq k \quad \forall k$, the same N will suffice for the subsequence. Then,

$$|a_{n_k} - a| < \varepsilon \quad \forall k \geq N \quad \blacksquare$$

Example: Let $0 < b < 1$. Then $b > b^2 > b^3 > \dots > 0$. Therefore, (b^n) is decreasing and bounded below. By the Monotone Convergence Theorem, $(b^n) \to l$. (b^{2n}) is a subsequence so by the Theorem above, $(b^{2n}) \to l$. However,

$$b^{2n} = b^n \cdot b^n \to l \cdot l \implies l^2 = l \implies l = 0$$

Therefore, $(b_n) \to 0$.

Example: Consider the sequence $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \dots)$. Does it converge?

Consider:

- $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \dots) \to \frac{1}{5}$
- $(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \dots) \to -\frac{1}{5}$

Since the subsequences do not converge to the same limit, the original sequence does not converge.

Theorem (Bolzano-Weierstrass): Every bounded sequence contains a convergent subsequence

Proof: Let (a_n) be a bounded subsequence. $\exists M > 0$ s.t. $|a_n| \leq M \quad \forall n \in \mathbb{N}$.

Split $[-M, M]$ into equal intervals $[-M, 0]$ and $[0, M]$. At least one these intervals must contain infinitely many terms of (a_n). Call this interval I_1. WLOG, suppose $I_1 = [-M, 0]$.

Let (a_{n_1}) to be some term of (a_n) which lies in I_1. Now we repeat: $I_1 = [-M, \frac{M}{2}] \cup [-\frac{M}{2}, 0]$. Label the interval with infinite terms I_2 and pick (a_{n_2}) from I_2 with $n_2 > n_1$.

In general, construct the closed I_k by taking the half of I_{k-1} containing infinitely many terms of (a_n). Select $n_k > n_{k-1} > n_{k-2} > \cdots > n_1$ such that $a_{n_k} \in I_k$.

Notice that the sets $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ are nested and closed. By the Nested Interval Property, $\exists x \in \mathbb{R}$ which lies in every I_k. Intuitively, this is a good limit candidate.

Now we seek to show that $(a_{n_k}) \to x$. Let $\varepsilon > 0$. By construction, each I_k has length $M(\frac{1}{2})^{k-1} \to 0$. $\exists N \in \mathbb{N}$ s.t. $\forall k \geq N$, the length of I_k is less than ε. Since $x \in I_k$ and $a_{n_k} \in I_k$, $|a_{n_k} - x| < \varepsilon$.

Therefore, (a_{n_k}) is a convergent subsequence of the bounded sequence (a_n). ■

Lecture 5 - Feb 27:

Recall:

- A *subsequence* of (a_n) is a sequence (a_{n_k}) where $n_1 < n_2 < n_3 < \dots$
- Any subsequence of a convergent sequence converges to the same limit as the original sequence

- If two convergent subsequences converge to different limits, the original sequence diverges
- *Bolzano-Weierstrass Theorem:* Every bounded sequence contains a convergent subsequence

The Cauchy Criterion

Definition: A sequence (a_n) is called a Cauchy sequence if $\forall \varepsilon > 0$,

$$\exists N \in \mathbb{N} \text{ s.t. } |a_n - a_m| < \varepsilon \quad \forall n, m \geq N$$

Theorem: Every convergent sequence is a Cauchy sequence

Proof: Assume (x_n) converges to x. To prove (x_n) is a Cauchy sequence, we need to find a point in the sequence after which $|x_n - x_m| < \varepsilon$.

Since $(x_n) \to x$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2}$.

$$|x_n - x_m| = |(x_n - x) + (x - x_m)| \leq |x_n - x| + |x_m - x| < \varepsilon$$

Lemma: Cauchy sequences are bounded

Proof: Set $\varepsilon = 1$. Then $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$,

$$|x_n - x_m| < 1 \implies |x_n| < |x_N| + 1 \quad \forall n \geq N$$

Then

$$M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, |x_N| + 1\}$$

is a bound for the sequence. $\blacksquare$

Theorem (Cauchy Criternion): A sequence converges iff it is a Cauchy sequence

Proof: The first direction follows from the fact that every convergent sequence is Cauchy.

For the other direction, assume (x_n) is a Cauchy sequence. Then (x_n) is bounded by the Lemma. By the Bolzano-Weierstrass Theorem, (x_n) contains a convergent subsequence $(x_{n_k}) \to x$.

Since (x_n) is Cauchy, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|x_n - x_m| < \frac{\varepsilon}{2} \quad \forall n, m \geq N$.

Since $(x_{n_k}) \to x$, choose x_{n_k} with $n_k \geq N$. Then,

$$|x_{n_k} - x| > \frac{\varepsilon}{2}$$

Now

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \blacksquare$$

Properties of Infinite Series

Recall:

- For a sequence $(a_1, a_2, a_3, \ldots)$, the sequence of partial sums is given by

$$(S_m) = (S_1, S_2, S_3, \ldots,) = (a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots)$$

- A series $\sum_{n=1}^{\infty} a_n$ converges to A if $\lim(S_m) = A$

Theorem (Algebraic Limit Theory for Series): If $\sum_{k=1}^{\infty} = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

1. $\sum_{k=1}^{\infty} ca_k = cA, \quad \forall c \in \mathbb{R}$
2. $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$

Proof:

1. Since $\sum_{k=1}^{\infty} a_k = A$, $(S_m) = \sum_{k=1}^{m} a_k \to A$. Then $\lim(cS_m) = c \lim S_m = cA$ by the Algebraic Limit Theorem for Sequences (ALT). Then, by definition, $\sum_{k=1}^{\infty} ca_k = cA$
2. Let $S_m = \sum_{k=1}^{m} a_k$ and $T_m = \sum_{k=1}^{m} b_k$. Then $S_m + T_m = \sum_{k=1}^{m}(a_k + b_k)$. Since $(S_m) \to A$ and $(T_m) \to B$, $(S_m + T_m) \to A + B$ by the ALT. Then $\sum_{k=1}^{\infty}(a_k + b_k) = A + B \quad \blacksquare$

Theorem (Cauchy Criterion for Series): The series $\sum_{k=1}^{\infty} a_k$ converges iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall m \geq n \geq N$ we have

$$\left| \sum_{k=m+1}^{n} a_k \right| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon$$

Proof: Define $S_n = a_1 + a_2 + \cdots + a_n$. Observe that

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff (S_n) \text{ converges} \overset{*}{\iff} (S_n) \text{ Cauchy seq}$$

where $\overset{*}{\iff}$ follows from the Cauchy Criterion for sequences.

Further, if and only if (S_n) is Cauchy, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n > m \geq N$,

$$|S_n - S_m| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon \quad \blacksquare$$

Lecture 6 - Feb 29:

Theorem: If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.

Proof: Pick $n = m + 1$ in previous theorem: for $m > N$,

$$|a_{m+1}| < \varepsilon$$

Remark: The converse is *not* true! Consider the harmonic series: $a_n = \frac{1}{n} \to 0$ but $\sum_{n=1}^{\infty} a_n = \infty$

Theorem (Comparison Test): Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. Then

1. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof: Apply Cauchy Criterion for series and observe that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |b_{m+1} + \cdots + b_n|$$

Example (Geometric Series): A series is called a *geometric series* if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots$$

If $r \geq 1$ and $a \neq 0$, then the series diverges. If $r \neq 1$, we use the identity

$$(1-r)(1+r+r^2+r^3+\dots+r^{m-1}) = 1-r^m$$

Then for partial sums

$$S_m = a + ar + ar^2 + \dots + ar^{m-1} = a(1+r+r^2+\dots+r^{m-1}) = a\frac{1-r^m}{1-r}$$

If $|r| < 1$, $a\frac{1-r^m}{1-r} \to \frac{a}{1-r}$. Therefore, for $|r| < 1$,

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

Theorem (Absolute Convergence Test): If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof: Since $\sum_{n=1}^{\infty} |a_n|$ converges, by Cauchy Criterion, given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n > m \geq N$,

$$|a_{m+1}| + |a_{m+1}| + \dots + |a_n| < \varepsilon$$

By triangle inequality,

$$|a_{m+1} + a_{m+2} + \dots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \dots + |a_n| < \varepsilon$$

Remark: The converse is not true! Consider the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges,} \quad \sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

Theorem (Alternating Series Test): Let (a_n) be a sequence satisfying

(a) $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \ldots$ (Decreasing)

(b) $(a_n) \to 0$ (Converges to 0)

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Proof: From conditions (i) and (ii), we have that $a_n \geq 0$. We want to show that the sequence of partial sums (S_n) converges by showing that (S_n) is Cauchy. Let $\varepsilon > 0$ be arbitrary. We need to find an N such that $n > m \geq N$ implies $|S_n - S_m| < \varepsilon$.

$$|S_n - S_m| = |a_{m+1} - a_{m+2} + a_{m+3} - \cdots \pm a_n|$$

Since (a_n) is decreasing and all the terms are positive, we can use an induction argument to show $|S_n - S_m| \leq |a_{m+1}|$ for all $n > m$.

Sketch:

$$|a_{m+3}| \leq |a_{m+2}| \leq |a_{m+1}| \implies a_{m+1} - a_{m+2} + a_{m+3} \leq a_{m+1}$$

Since $(a_n) \to 0$, we can choose N such that $m \geq N$ implies $|a_m| < \varepsilon$. Then

$$|S_n - S_m| \leq |a_{m+1}| < \varepsilon$$

Therefore, (S_n) is Cauchy so it converges ■

Definition:

- If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ *converges absolutely*
- If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ *converges conditionally*

Definition: Let $\sum_{n=1}^{\infty} a_n$ be a series. A series $\sum_{n=1}^{\infty} b_n$ is called a rearrangement of the original series if there exists $f: \mathbb{N} \hookrightarrow\!\!\!\!\rightarrow \mathbb{N}$ such that $b_{f(n)} = a_n$ for all $n \in \mathbb{N}$.

Note: the bijectivity means that every term eventually appears and there are no repetitions.

Theorem: If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then every rearrangement of the series converges to the same limit.

Proof: Omitted

Chapter 3

Basic Topology on $\mathbb{R}$

March 05:

Recall: an *ε-neighborhood* of a point $x \in \mathbb{R}$ is the set

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$$

Definition: A set $O \subseteq \mathbb{R}$ is *open* if for all points $a \in O$, there exists an ε-neighborhood of a such that $V_\varepsilon(a) \subseteq O$.

Examples:

- $\mathbb{R}$ is open
- $\emptyset$ is open
- $(c, d) = \{x \in \mathbb{R} : c < x < d\}$ is open (*Proof:* Let $x \in (c, d)$. Then $V_{\min\{x-c, d-x\}}(x) \subseteq (c, d)$)

Theorem:

1. The union of an arbitrary collection of open sets is open
2. The intersection of a finite collection of open sets is open

Proof:

1. Let $\{O_\lambda : \lambda \in \Lambda\}$ be a collection of open sets.

Let $O = \bigcup_{\lambda \in \Lambda} O_\lambda$. We need an ε-neighborhood of an arbitrary $a \in O$ to be completely contained in O.

Notice that $a \in O \implies a \in O_{\lambda'}$ for some $\lambda' \in \Lambda$. Since $O_{\lambda'}$ is open, $\exists \varepsilon > 0$ such that $V_\varepsilon(a) \subseteq O_{\lambda'} \subseteq O$.

2. Let $\{O_1, O_2, \ldots, O_n\}$ be a finite collection of open sets. Denote $O = \bigcap_{k=1}^{n} O_k$. We need to show that O is open.

   Let $a \in O$. Then $a \in O_k$ for all $k = 1, 2, \ldots, n$. Since O_k is open, $\exists \varepsilon_k > 0$ such that $V_{\varepsilon_k}(a) \subseteq O_k$ for all k.

   Now, we have different ε-neighborhoods in each O_k. We want an ε-neighborhood which is contained in *every* O_k.

   Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then $V_\varepsilon(a) \subseteq O_k$ for all $k = 1, 2, \ldots, n$. Therefore, $V_\varepsilon(a) \subseteq \bigcap_{k=1}^{n} O_k$. ■

Definition: A point x is a *limit point* (cluster point/accumulation point) of a set A if every ε-neighborhood of x intersects A at some point other than x.

Theorem: A point x is a limit point of a set A iff there exists a sequence (a_n) in A such that $(a_n) \to x$ and $a_n \neq x$ for all $n \in \mathbb{N}$

Proof:

Assume x is a limit point of A. We need a sequence (a_n) in A such that $(a_n) \to x$. By definition, every ε-neighborhood of x intersects A at some point other than x. Pick $\varepsilon = \frac{1}{n}$. Then for all $n \in \mathbb{N}$, pick

$$a_n \in V_{1/n}(x) \cap A, \quad a_n \neq x$$

Now we want $(a_n) \to x$. Given $\varepsilon > 0$ choose N such that $\frac{1}{N} < \varepsilon$ so $|a_n - x| < \varepsilon$ for all $n \in N$

Now, suppose there exists a sequence (a_n) in A such that $(a_n) \to x$ and $a_n \neq x$ for all $n \in \mathbb{N}$. We need to show that x is a limit point of A.

Let $V_\varepsilon(x)$ be an arbitrary ε-neighborhood. By definition of convergence, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - x| < \varepsilon$. Then $a_n \in V_\varepsilon(x)$ for all $n \geq N$. ■

Definition: A point $a \in A$ is an isolated point of A if it is *not* a limit point of A

Note: An isolated point is *always* a point in the set. A limit point does not necessarily belong to the set.

Definition: a set $F \subseteq \mathbb{R}$ is closed if it contains its limit points.

Theorem: A set $F \subseteq \mathbb{R}$ is closed iff every Cauchy sequence contained in F has a limit in F

Proof: HW

Example: Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Show each point in A is isolated.

Given $\frac{1}{n} \in A$, choose $\varepsilon = \frac{1}{n} - \frac{1}{n+1}$. Therefore, $V_\varepsilon(\frac{1}{n}) \cap A = \{\frac{1}{n}\}$ so $\frac{1}{n}$ is an isolated point and not a limit point.

Further, the limit of A is 0. Therefore, $\forall \varepsilon > 0$, $V_\varepsilon(0)$ contains points in A. Since $0 \notin A$, A is not closed.

However, we can create a closed set $F = A \cup \{0\}$. This is the *closure* of A.

Example: Show $[c, d] = \{x \in \mathbb{R} : c \leq x \leq d\}$ is closed.

If x is a limit point, then $\exists (x_n) \in [c, d]$ with $(x_n) \to x$. We want to show that $x \in [c, d]$. Since $c \leq x_n \leq d$, by the Order Limit Theorem,

$$c \leq \lim x_n \leq d \implies \lim x_n \in [c, d] \implies x \in [c, d]$$

so the set is closed.

Example: $\mathbb{Q} \subseteq \mathbb{R}$. The set of all limit point in $\mathbb{Q}$ is $\mathbb{R}$.

Proof: Let $y \in \mathbb{R}$. Consider any neighborhood $V_\varepsilon(y) = (y - \varepsilon, y + \varepsilon)$. From the density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists r \neq y$ such that $y - \varepsilon < r < y + \varepsilon$. Therefore, $r \in V_\varepsilon(y)$ so y is a limit point of $\mathbb{Q}$.

Lecture 1 - March 7:

Definition: given a set $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. The *closure* of A is the set $\overline{A} = A \cup L$.

Example:

- $\overline{\mathbb{Q}} = \mathbb{R}$
- $A = (a, b) \implies \overline{A} = [a, b]$

- If A is closed, $\overline{A} = A$

Theorem: For any $A \subseteq \mathbb{R}$, the closure $\overline{A}$ is a closed set and it is the smallest closed set containing A

Proof: Let L be the set of limit points of A. Then $\overline{A} = A \cup L$ is closed (it contains all its limit points, obviously). Any closed set containing A must contain L. Therefore $\overline{A}$ is the smallest closed set containing A. ∎

Complement: Recall that $A^c = \{x \in \mathbb{R} : x \notin A\}$

Theorem:

1. A set O is open $\iff$ O^c is closed
2. A set F is closed $\iff$ F^c is open

Proof:

1. Let $O \subseteq \mathbb{R}$ be open. We want to show O^c is closed. By definition, if x is a limit point of O^c, then every ε-neighborhood of x contains some point of O^c. Thus, any ε-neighborhood of x cannot be a subset of O so $x \notin O$. Since $x \in O^c$, O^c is closed.

   Now assume O^c is closed. We want to show that O is open, i.e. for any $x \in O$, $\exists V_\varepsilon(x) \subseteq O$. By definition, O^c is closed so x is not a limit point of O^c. Therefore, $\exists V_\varepsilon(x)$ which does not intersect O^c. Then $V_\varepsilon(x) \subseteq O$.
2. $(E^c)^c = E$. The rest of the proof follows from 1).

Theorem:

1. The union of a finite collection of closed sets is closed
2. The intersection of an arbitary collection of closed sets is closed

Proof: Follows from previous theorem and de Morgan's laws:

$$\left(\bigcup_{\lambda \in \Lambda} E_\lambda\right)^c = \bigcap_{\lambda \in \Lambda} E_\lambda^c, \quad \left(\bigcap_{\lambda \in \Lambda} E_\lambda\right)^c = \bigcup_{\lambda \in \Lambda} E_\lambda^c$$

Compact Sets

Motivation: Bring "finite" quality to infinite arguments.

Definition: A set $K \subseteq \mathbb{R}$ is *compact* if every sequence in K has a convergent subsequence whose limit is in K.

Example: $[c, d]$ is compact. *Proof:* if $(a_n) \in [c, d]$, then it is bounded so by Bolzano-Weierstrass, $\exists (a_{n_k})$ which converges to a. Further $a \in [c, d]$ since $[c, d]$ is closed.

Definition: A set $A \subseteq \mathbb{R}$ is bounded if $\exists M > 0$ such that $|a| < M$ for all $a \in A$.

Theorem (Characterization of compactness in $\mathbb{R}$): A set $K \subseteq \mathbb{R}$ is compact iff it is closed and bounded

Proof: Assume K is compact. Suppose K is not bounded. Since K is not bounded:

$$\forall n \in \mathbb{N}: \quad \exists x_n \in K, \text{ s.t. } |x_n| > n$$

Since K is compact, (x_n) should have a convergent subsequence. However, (x_n) is unbounded so (x_{n_k}) is unbounded. Therefore, there is no convergent subsequence in (x_n). This is a contradiction of compactness so K is bounded.

Now we want to show K is closed. Let $x = \lim x_n$ with $(x_n) \in K$. It suffices to show $x \in K$. By definition, K is compact so (x_n) has a convergent subsequence (x_{n_k}) which converges to x and lies in K. $(x_{n_k}) \to x \implies x \in K \implies$ K is closed.

It remains to prove that K is compact if it is closed and bounded. This is left for HW.

Theorem (Nested Compact Set Property): If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ is a nested sequence of nonempty compact sets, then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$

Proof: Use compactness of K_n to produce a sequence that belongs to each set. $\forall n \in \mathbb{N}$, pick $x_n \in K_n$. Therefore, $(x_n) \in K_1 \implies \exists (x_{n_k}) \in K_1$ with $\lim x_{n_k} = x \in K_1$.

Given an $n_0 \in \mathbb{N}$, the terms of (x_n) are contained in K_{n_0} as long as $n > n_0$. We now ignore the finite number of terms for which $n_k < n_0$. Therefore,

$(x_{n_k}) \in K_{n_0}$ so $\lim x_{n_k} = x \in K_{n_0}$. Since n_0 was arbitrary,

$$x \in \bigcap_{n=1}^{\infty} K_n$$

March 12:

Definition: Let $A \subseteq \mathbb{R}$. An *open cover* of A is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ such that

$$A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$$

Given an open cover for A, a *finite subcover* is a finite collection of open sets from the original open cover, whose union still contains A

Example: Find an open cover for $(0, 1)$.

$\forall x \in (0, 1)$, let O_x be the open interval $(\frac{x}{2}, 1)$ so we have the infinite collection

$$\{O_x : x \in (0, 1) \text{ covering } (0, 1)\}$$

However, it is impossible to find a finite subcover for $(0, 1)$ using this open cover: Construct $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$ and set $x' = \min\{x_1, \ldots, x_n\}$. But then any $y \in \mathbb{R}$ with $0 < y \leq \frac{x'}{2}$ is not in $\bigcup_{i=1}^{n} O_{x_i}$

Example: Find an open cover for $[0, 1]$.

Naturally, we can use the same open cover as $(0, 1)$. However, this does not include the endpoints. Now let $\varepsilon > 0$ and define $O_0 = \{-\varepsilon, \varepsilon\}, O_1 = (1 - \varepsilon, 1 + \varepsilon)$. Then

$$\{O_0, O_1, O_x : x \in (0, 1)\}$$

is an open cover if $[0, 1]$.

To find a finite subcover, choose x' such that $\frac{x'}{2} < \varepsilon$:

$$\{O_0, O_1, O_{x'}\}$$

Theorem (Heine-Borel): For $K \subseteq \mathbb{R}$, then the following are equivalent:

(i) K is compact

(ii) K is closed and bounded

(iii) Every open cover of K has a finite subcover

Proof: (i) $\iff$ (ii) follows from the Characterization of compactness in $\mathbb{R}$.

It suffices to show (ii) $\iff$ (iii):

Assume that every open cover of K has a finite subcover. We want to show that K is closed and bounded. Let $O_x = \{|x-a| < 1 : a \in \mathbb{R}\} = V_1(x)$. Since $\{O_x : x \in K\}$ must have finite subcover, $\exists x_1, x_2, \ldots, x_n \in K$ such that $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$ is a finite subcover of K.

Since K is contained in a finite collection of sets, it is bounded.

To show K is closed, let (y_n) be a Cauchy sequence is K with $(y_n) \to y$. Suppose $y \notin K$, i.e. $\forall x \in K$, x lies some positive distance away from y.

Construct an open cover by taking O_x to be the interval of radius $\frac{|x-y|}{2}$ around $x \in K$. By (iii), we have a finite subcover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$.

Let $\varepsilon_0 = \min\left\{\frac{|x_i - y|}{2} : 1 \leq i \leq n\right\}$. Since $(y_n) \to y$, $\exists y_N$ such that $|y_N - y| < \varepsilon_0$.

This means that y_N must be excluded from each O_x so certainly, $y \notin \bigcup_{i=1}^n O_{x_i}$. Therefore, this finite collection cannot be a subcover since it does not contain all of K. This is a contradiction so K contains every limit point, and therefore K is closed.

The other direction, (ii) $\implies$ (iii), is left for homework. $\blacksquare$

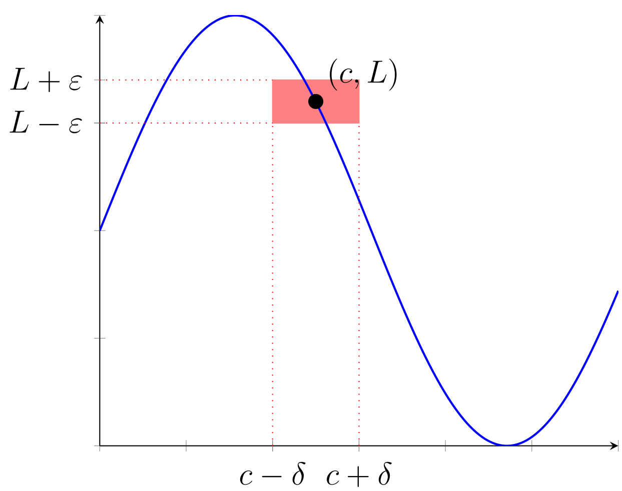
Chapter 4

Functional Limits and Continuity

March 12 (Continued)

Definition (Functional limit): Let $f : A \to \mathbb{R}$ be a function and let c be a limit point of the domain A. We say $\lim f(x) = L$ if $x \to c$.

Then $\forall \varepsilon > 0$, $\exists \delta > 0$, such that whenever $0 < |x - c| < \delta$ and $x \in A$, we have $|f(x) - L| < \varepsilon$.



Topological Definition: Let c be a limit point in A of $f : A \to \mathbb{R}$. We say that

$$\lim_{x \to c} f(x) = L$$

if $\forall V_\varepsilon(L)$, there exists $V_\delta(c)$ such that $\forall x \in V_\delta(c),\ f(x) \in V_\varepsilon(L)$

Example: Show $\lim_{x\to 2} f(x) = 7$ with $f(x) = 3x + 1$.

Let $\varepsilon > 0$. We need to produce a $\delta > 0$ such that $0 < |x - 2| < \delta$ implies $|f(x) - 7| < \varepsilon$.

$$|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3\,|x - 2|$$

Choose $\delta = \frac{\varepsilon}{3}$ so $0 < |x - 2| < \delta \implies |f(x) - 7| < 3\delta = \varepsilon$.

Example: Show $\lim_{x\to 2} g(x) = 4,\ g(x) = x^2$.

Let $\varepsilon > 0$. We want $|g(x) - 4| < \varepsilon$ by restricting $|x - 2| < \delta$.

Notice

$$|g(x) - 4| = \left|x^2 - 4\right| = |x + 2|\,|x - 2|$$

So we construct a δ-neighborhood around $c = 2$ with radius no bigger than $\delta = 1$:

$$|x + 2| \leq |3 + 2| = 5$$

Choose $\delta = \min\{1, \frac{\varepsilon}{5}\}$. Then when $0 < |x - 2| < \delta$, we have

$$|g(x) - 4| < \varepsilon$$