Math 1010: Homework 3

Problem 1 (Squeeze Theorem)

Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $y_n = l$ as well.

By the Order Limit Theorem, since $y_n \leq z_n$ for all $n \in \mathbb{N}$, then $\lim y_n \leq \lim z_n$. Similarly, since $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $\lim x_n \leq \lim y_n$. Thus,

$$\lim x_n \leq \lim y_n \leq \lim z_n \implies l \leq \lim y_n \leq l \implies \lim y_n = 1 \quad \blacksquare$$

Problem 2

Give an example of each of the following or state that such request is impossible by referencing the proper theorem(s):

1. sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

   Let $x_n = n$ and $y_n = -n$. Then, (x_n) and (y_n) both diverge since they are clearly not bounded, but $(x_n + y_n) = 0$ for all n, so $(x_n + y_n)$ converges.

2. sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges and $(x_n + y_n)$ converges;

   Let $\lim_{n\to\infty} x_n = x$ for $x < \infty$. Since (y_n) diverges, we can write $\lim_{n\to\infty} y_n = \infty$ to mean that for $\varepsilon > 0$ there does not exist $N \in \mathbb{N}$ such that $|y_n - y| < \varepsilon$ for all $n > N$ and any $y \in \mathbb{R}$.

   Then by the algebraic limit theorem,

   $$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = x + \infty = \infty$$

   Therefore $(x_n + y_n)$ diverges for any choice of (x_n) and (y_n).

3. a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

   Let $b_n = \sum_{m=1}^{n} \frac{1}{m}$. Since all terms $1/n > 0$, $b_n > 0$ for all n. However, (b_n) is simply the sequence of partial sums of the harmonic series, which is known to diverge. Thus, (b_n) diverges.

4. an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded.

   As (a_n) is unbounded, we can write $\lim_{n\to\infty} a_n = \infty$ to mean that (a_n) is not bounded so there is not finite value of convergence. Let $\lim_{n\to\infty} b_n = b$ for $b < \infty$. Then by the Algebraic Limit Theorem,

   $$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = \infty - b = \infty$$

   Therefore, $(a_n - b_n)$ is unbounded for any choice of (a_n) and (b_n).

Problem 3 (Cesaro Means)

1. Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

Let $\varepsilon > 0$. Since $(x_n) \to x$, $\exists N_x \in \mathbb{N}$ such that for all $n > N_x$, $|x_n - x| < \frac{\varepsilon}{2}$. Then,

$$\begin{aligned}
|y_n - x| &= \left| \frac{x_1 + x_2 + \cdots + x_n}{n} - x \right| \\
&= \left| \frac{x_1 + x_2 + \cdots + x_n - nx}{n} \right| \\
&\leq \frac{|x_1 - x| + |x_2 - x| + \cdots + |x_n - x|}{n} \\
&= \frac{1}{n} \sum_{i=1}^{N_x} |x_i - x| + \frac{1}{n} \sum_{i=N_x}^{n} |x_i - x| \\
&< \frac{1}{n} \sum_{i=1}^{N_x} |x_i - x| + \frac{1}{n} \sum_{i=N_x}^{n} \frac{\varepsilon}{2} \\
&\leq \frac{N_x}{n} \sup_{x_i :\, i \leq N_x} |x_i - x| + \frac{n - N_x}{n} \cdot \frac{\varepsilon}{2}
\end{aligned}$$

Now we can choose $N_y > \frac{2}{\varepsilon} \cdot N_x \sup_{x_i :\, i \leq N_x} |x_i - x|$ so that for all $n > N_y$,

$$|y_n - x| < \frac{N_y}{n} \cdot \frac{\varepsilon}{2} + \frac{n - N_x}{n} \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

Therefore, $\lim_{n \to \infty} y_n = x$. ■

2. Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Let $x_n = (-1)^n$. Clearly, (x_n) is divergent. However,

$$y_n = \frac{(-1)^1 + (-1)^2 + \cdots + (-1)^n}{n} = \frac{1 - 1 + 1 - 1 + \cdots + 1}{n} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -1/n & \text{if } n \text{ is odd} \end{cases}$$

Define $z_n = -1/n$. Then $|z_n| < 1$ and

$$z_{n+1} = -\frac{1}{n+1} > -\frac{1}{n} = z_n$$

so it is bounded and monotone. Thus, (z_n) converges.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > 1/\varepsilon$. Then for all $n > N$, $|z_n| < \varepsilon$. Therefore, $\lim_{n \to \infty} z_n = 0$. Then for all $n > N$, $|y_n - 0| = |y_n| \leq \left| -\frac{1}{n} \right| < \varepsilon$. Therefore, (y_n) converges. ■

Problem 4

Let (x_n) and (y_n) be given and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \cdots, x_n, y_n, \cdots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Suppose (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Let $l = \lim x_n = \lim y_n$. Then, for any $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that for all $n > N_1$, $|x_n - l| < \epsilon$ and for all $n > N_2$, $|y_n - l| < \epsilon$. Let $N = \max\{N_1, N_2\}$.

We can write

$$z_n = \begin{cases} x_{(n+1)/2} & \text{if } n \text{ is odd} \\ y_{n/2} & \text{if } n \text{ is even} \end{cases}$$

Then, for all $n > 2N + 1$, $\frac{n+1}{2} > \frac{n}{2} > N$ so $z_n = x_n = y_n$ and

$$|z_n - l| = |x_n - l| = |y_n - l| < \varepsilon$$

So (z_n) is convergent with $\lim z_n = l$.

For the other direction, suppose $(z_n) \to z$. Then $\exists N \in \mathbb{N}$ such that $|z_n - z| < \varepsilon$ for all $n > N$. Since

$$z_n = \begin{cases} x_{(n+1)/2} & \text{if } n \text{ is odd} \\ y_{n/2} & \text{if } n \text{ is even} \end{cases}$$

and $n > (n+1)/2 > n/2$,

$$\left|x_{(n+1)/2} - z\right| < \varepsilon \text{ and } \left|y_{n/2} - z\right| < \varepsilon$$

Therefore, (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n = z$. ∎

Problem 5

1. Prove that the sequence defined by $x_1 = 3$ and
$$x_{n+1} = \frac{1}{4 - x_n}$$
converges.

   By the Monotone Convergence Theorem, a sequence converges if it is bounded and monotone.

   We will show (x_n) is decreasing by induction. First, $x_1 = 3$ and $x_2 = \frac{1}{4-3} = 1 < x_1$. Now suppose $x_{n+1} < x_n$ for some $n \in \mathbb{N}$. Then,
$$x_{n+2} = \frac{1}{4 - x_{n+1}} < \frac{1}{4 - x_n} = x_{n+1}$$
   Thus, (x_n) is decreasing.

   Further, since (x_n) is decreasing, it is bounded above by $x_1 = 3$ since $x_n \leq 3$ for all $n \in \mathbb{N}$. Below, $x_n \geq -3$ for all $n \in \mathbb{N}$ since if $x_n < -3$, then $x_{n+1} < \frac{1}{4-(-3)} > 0$, which is a contradiction of the monotonicity of (x_n).

   Thus, (x_n) is bounded and decreasing, so it converges. ■

2. Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and be equal to the same value.

   Let $\lim x_n = l$, then by definition, $\forall \varepsilon > 0$, $|x_n - l| < \varepsilon$ for all n greater than some $N \in \mathbb{N}$. Since $n + 1 > n > N$, $|x_{n+1} - l| < \varepsilon$. But this is precisely the statement that $\lim x_{n+1} = l$. Thus, $\lim x_{n+1} = \lim x_n$ ■.

3. Take the limit of each side of the recursive equation in part (1) to explicitly compute $\lim x_n$.

$$\begin{aligned}
\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{4 - x_n} = \lim_{n\to\infty} x_n &\implies \lim_{n\to\infty} x_n^2 - 4x_n + 1 = 0 \\
&\implies \lim_{n\to\infty} x_n = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}
\end{aligned}$$

   But since $x_1 = 3$ and (x_n) is decreasing, $\boxed{\lim x_n = 2 - \sqrt{3}}$.