Math 1010: Problem Set 2

Problem 1

Prove the following theorem: If $A_1, A_2, \cdots A_m$ are each countable sets, then the union $A_1 \cup A_2 \cup \cdots \cup A_m$ is countable.
For your proof, use the following outline:

(a) First, prove the statement for two countable sets A_1 and A_2. Example 1.5.3 (ii) might be a useful reference. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?)

Let A_1 and A_2 be countable sets. Let $B_2 = A_2 \setminus A_1$. Then $A_1 \cup B_2 = A_1 \cup A_2$ and $A_1 \cap B_2 = \emptyset$. Therefore, it suffices to show that $A_1 \cup B_2$ is countable.

The simplest case is when B_2 is finite. Then $B_2 = \{b_1, b_2, \ldots, b_n\}$ for some $n \in \mathbb{N}$ while $A_1 = \{a_1, a_2, \ldots\}$. Then $A_1 \cup B_2 = \{b_1, b_2, \ldots, b_n, a_1, a_2, \ldots\}$. We induce a natural bijection $A_1 \cup B_2 \to \mathbb{N}$ by

$$\begin{array}{rcccccccc} A_1 \cup B_2: & b_1 & b_2 & \cdots & b_n & a_1 & a_2 & \cdots \\ & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{N}: & 1 & 2 & \cdots & n & n+1 & n+2 & \cdots \end{array}$$

As $A_1 \cap B_2 = \emptyset$, every element of $A_1 \cup B_2$ is uniquely mapped to an element of $\mathbb{N}$ and vice versa. Therefore, $A_1 \cup B_2$ is countable.

Now suppose B_2 is infinite. We will re-index $A_1 \cup B_2 = \{a_0, a_1, b_1, a_2, b_2, \ldots\}$

Now consider

$$f(n) = \begin{cases} (n-1)/2 & \text{if } n \text{ is odd} \\ -n/2 & \text{if } n \text{ is even} \end{cases}$$

from Example 1.5.3. This is a bijection from $\mathbb{N}$ to $\mathbb{Z}$. We can introduce a second bijection $g : \mathbb{Z} \to A_1 \cup B_2$ by

$$g(n) = \begin{cases} a_0 & \text{if } n = 0 \\ a_{|n|} & \text{if } n < 0 \\ b_{|n|} & \text{if } n > 0 \end{cases}$$

Then $g \circ f : \mathbb{N} \to A_1 \cup B_2$ is a bijection given by

$$\begin{array}{rcccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{Z}: & 0 & -1 & 1 & -2 & 2 & -3 & 3 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ A_1 \cup B_2: & a_0 & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & \cdots \end{array}$$

As we have a bijection from $\mathbb{N}$ to $A_1 \cup B_2$, we conclude that $A_1 \cup B_2$ is countable. ■

(b) Now explain how the more general statement follows.

Let $A_1, A_2, \ldots, A_m$ be countable sets.

Denote

$$B_n = A_n \setminus \bigcup_{n=1}^{n-1} A_n$$

From part 1, $A_1 \cup B_2 = A_1 \cup (A_2 \setminus A_1) = A_1 \cup A_2$ is countable.

Suppose $A = A_1 \cup \cdots \cup A_{n-1}$ is countable for $n < m$. Then

$$A \cup B_n = A_1 \cup \cdots \cup A_{n-1} \cup \left(A_n \setminus \bigcup_{n=1}^{n-1} A_n \right) = A_1 \cup \cdots \cup A_n$$

and $A \cap B_n = \emptyset$. By part 1, $A \cap B_n$ is countable so $A_1 \cup \cdots \cup A_n$ is countable.

Therefore, by induction, $A_1 \cup \cdots \cup A_m$ is countable. $\blacksquare$

Problem 2

Show that $(a, b) \sim \mathbb{R}$ for any interval (a, b).
We seek to show that $(a, b) \sim \mathbb{R}$ by constructing a bijection $f : (a, b) \to \mathbb{R}$.
Consider

$$f(x) = \tan(\frac{\pi}{b-a}(x-a) - \frac{\pi}{2})$$

Conveniently, $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ so by rescaling the argument to (a, b) we intuitively have a map $(a, b) \to \mathbb{R}$.
To confirm this is, in fact, a bijection it suffices to construct an inverse:

$$f^{-1}(x) = \frac{b-a}{\pi}(\arctan(x) + \frac{\pi}{2}) + a$$

and indeed

$$f^{-1}(f(x)) = \frac{b-a}{\pi}(\arctan(\tan(\frac{\pi}{b-a}(x-a) - \frac{\pi}{2})) + \frac{\pi}{2}) + a = x$$
$$f(f^{-1}(x)) = \tan(\frac{\pi}{b-a}(\frac{b-a}{\pi}(\arctan(x) + \frac{\pi}{2}) + a - a) - \frac{\pi}{2}) = x$$

so we are done. $\blacksquare$

Problem 3

Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

1. $\lim \frac{1}{6n^2+1} = 0.$

   Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \sqrt{\frac{1-\varepsilon}{6\varepsilon}}$. Let $n \geq N$. Then

$$n > \sqrt{\frac{1-\varepsilon}{6\varepsilon}} \implies 6n^2 > \frac{1-\varepsilon}{\varepsilon} \implies 6n^2 + 1 > \frac{1}{\varepsilon} \implies \left|\frac{1}{6n^2+1}\right| < \varepsilon \quad \blacksquare$$

2. $\lim \frac{3n+1}{2n+5} = \frac{3}{2}.$

   Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > -\frac{13}{4\varepsilon} - \frac{10}{4}$. Let $n \geq N$. Then

$$\begin{aligned} n > -\frac{13}{4\varepsilon} - \frac{10}{4} &\implies 4n + 10 > -\frac{13}{\varepsilon} \\ &\implies \frac{-13}{4n+10} < \varepsilon \\ &\implies \frac{1 - \frac{15}{2}}{2n+5} < \varepsilon \\ &\implies \frac{3n + 1 - 3n - \frac{15}{2}}{2n+5} < \varepsilon \\ &\implies \frac{3n+1}{2n+5} - \frac{3n + \frac{15}{2}}{2n+5} < \varepsilon \\ &\implies \left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \varepsilon \quad \blacksquare \end{aligned}$$

3. $\lim \frac{2}{\sqrt{n+3}} = 0.$

   Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \frac{4}{\varepsilon^2} - 3$. Let $n \geq N$. Then

$$n > \frac{4}{\varepsilon^2} - 3 \implies n + 3 > \frac{4}{\varepsilon^2} \implies \sqrt{n+3} > \frac{2}{\varepsilon} \implies \left|\frac{2}{\sqrt{n+3}}\right| < \varepsilon \quad \blacksquare$$

Problem 4

Prove the theorem: the limit of a sequence, when it exists, is unique. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.
Let $(a_n) \to a$ and $(a_n) \to b$.
Then, for $\varepsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1$ implies $|a_n - a| < \frac{\varepsilon}{2}$ and $n \geq N_2$ implies $|a_n - b| < \frac{\varepsilon}{2}$.
Then, for $n \geq \max\{N_1, N_2\}$, we have

$$\begin{aligned}
|b - a| &= |b - a_n + a_n - a| \\
&\leq |b - a_n| + |a_n - a| \\
&= |-(a_n - b)| + |a_n - a| \\
&= |a_n - b| + |a_n - a| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

so for $\varepsilon > 0$,

$$|b - a| < \varepsilon$$

Since ε is arbitrary, we have $b - a = 0 \implies a = b$. $\blacksquare$.

Problem 5

Let $x_n \geq 0$ for all $n \in \mathbb{N}$.

1. If $(x_n) \to 0$, show that $\sqrt{x_n} \to 0$.

   Let $\lim_{n\to\infty} x_n = 0$. Suppose $\lim_{n\to\infty} \sqrt{x_n}$ exists and denote the value of the limit x. Then by the Algebraic Limit Theorem,

   $$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (\sqrt{x_n})(\sqrt{x_n}) = 0 \implies x \cdot x = 0 \implies x = 0 \implies \lim_{n\to\infty} \sqrt{x_n} = 0 \quad \blacksquare$$

2. If $(x_n) \to x$, show that $\sqrt{x_n} \to \sqrt{x}$.

   By similar argument, suppose that $\lim_{n\to\infty} \sqrt{x_n} = y$. Then by the Algebraic Limit Theorem,

   $$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{x_n} \cdot \sqrt{x_n} = x \implies y^2 = x \implies y = \sqrt{x} \implies \lim_{n\to\infty} \sqrt{x_n} = \sqrt{x} \quad \blacksquare$$