Math 1010 - Homework 5

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Problem 1

Assume (a_n) and (b_n) are Cauchy sequences. Use a triangle inequality argument to prove $c_n = |a_n - b_n|$ is Cauchy

Let $\varepsilon > 0$. Since (a_n) is Cauchy, $\exists N \in \mathbb{N}$ such that $m, n \geq N$ implies $|a_m - a_n| < \frac{\varepsilon}{2}$. Similarly, since (b_n) is Cauchy, $\exists M \in \mathbb{N}$ such that $m, n \geq M$ implies $|b_m - b_n| < \frac{\varepsilon}{2}$. Let $K = \max\{N, M\}$. Then $m, n \geq K$ implies

$$\begin{aligned}
|c_n - c_m| &= |(a_n - b_n) - (a_m - b_m)| \\
&= |a_n - a_m - b_n + b_m| \\
&\leq |a_n - a_m| + |b_n - b_m| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

Therefore, (c_n) is Cauchy. $\blacksquare$

Problem 2

If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion: (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

1. Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

   Let $\varepsilon > 0$. Since (x_n) is Cauchy, $\exists N \in \mathbb{N}$ such that $m, n \geq N$ implies $|x_m - x_n| < \frac{\varepsilon}{2}$. Similarly, since (y_n) is Cauchy, $\exists M \in \mathbb{N}$ such that $m, n \geq M$ implies $|y_m - y_n| < \frac{\varepsilon}{2}$. Let $K = \max\{N, M\}$. Then $m, n \geq K$ implies

   $$|(x_n + y_n) - (x_m + y_m)| = |x_n - x_m + y_n - y_m| \leq |x_n - x_m| + |y_n - y_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

   Therefore, $(x_n + y_n)$ is Cauchy. $\blacksquare$

2. the same for the product $(x_n \cdot y_n)$.

   Let $\varepsilon > 0$. Since (x_n) and (y_n) are Cauchy, they are bounded. Let $B = \max\{|x_n|, |y_n|\}$.

   Further, since (x_n) is Cauchy, $\exists N \in \mathbb{N}$ such that $m, n \geq N$ implies $|x_m - x_n| < \frac{\varepsilon}{2B}$ Similarly, since (y_n) is Cauchy, $\exists M \in \mathbb{N}$ such that $m, n \geq M$ implies $|y_m - y_n| < \frac{\varepsilon}{2B}$. Let $K = \max\{N, M\}$. Then $m, n \geq K$ implies

   $$\begin{aligned} |x_n y_n - x_m y_m| &= |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \\ &= |x_n(y_n - y_m) + y_m(x_n - x_m)| \\ &\leq |x_n| \, |y_n - y_m| + |y_m| \, |x_n - x_m| \\ &< |x_n| \frac{\varepsilon}{2B} + |y_m| \frac{\varepsilon}{2B} \\ &= \frac{\varepsilon}{2B}(|x_n| + |y_m|) \\ &\leq \frac{\varepsilon}{2B}(B + B) = \varepsilon \end{aligned}$$

   Therefore $(x_n y_n)$ is Cauchy. $\blacksquare$

Problem 3

Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1$$

then the series converges absolutely.

1. Let r' satisfy $r < r' < 1$ (Why must such an r' exist?) Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$

   (r' exists by density of $\mathbb{R}$)

   Suppose $\lim \left|\frac{a_{n+1}}{a_n}\right| = r < r' < 1$. Then $\exists N \in \mathbb{N}$ such that $n \geq N$ implies $\left|\frac{a_{n+1}}{a_n}\right| < r'$ by definition of convergence. Then

   $$\left| \frac{a_{n+1}}{a_n} \right| = |a_{n+1}| \cdot \left| \frac{1}{a_n} \right| = |a_{n+1}| \cdot \frac{1}{|a_n|}$$

   Since $\left|\frac{a_{n+1}}{a_n}\right| < r'$,

   $$|a_{n+1}| \leq |a_n| r' \quad \blacksquare$$

2. Why does $|a_N| (r')^n$ necessarily converge?

   Since $|a_N|$ is a constant and $r' < 1$, $\sum_{n=1}^{\infty} |a_N| (r')^n$ is a geometric series and converges. Since the series converges, $|a_N| (r')^n$ converges. $\blacksquare$

3. Now show that $\sum |a_n|$ converges

   From part 1, for all $n > N$, $|a_{n+1}| \leq |a_n| r'$.

   Then,

   $$|a_{n+2}| \leq |a_{n+1}| r' \leq |a_n| (r')(r')$$

   By induction,

   $$|a_{n+k}| \leq |a_n| (r')^k$$

   Then, for $n > N$,

   $$|a_n| \leq |a_N| (r')^{n-N}$$

   From part 2, $\sum_{n=1}^{\infty} |a_N| (r')^n$ converges and an identical argument shows that $\sum_{n=1}^{\infty} |a_N| (r')^{n-N}$ converges. Thus, by the series comparison test, $\sum_{n=1}^{\infty} |a_n|$ converges. $\blacksquare$

Problem 4

1. **(Summation by parts)** Let (x_n) and (y_n) be sequences, and let $s_n = x_1 + x_2 + \cdots + x_n$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m+1}^{n} x_j y_j = s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j (y_j - y_{j+1})$$

$$\begin{aligned}
\sum_{j=m+1}^{n} x_j y_j &= \sum_{j=m+1}^{n} (s_j - s_{j-1}) y_j \\
&= \sum_{j=m+1}^{n} s_j y_j - \sum_{j=m+1}^{n} s_{j-1} y_j \\
&= \sum_{j=m+1}^{n} s_j y_j - \left(-s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n+1} s_{j-1} y_j \right) \\
&= s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j y_j - \sum_{j=m+1}^{n} s_j y_{j+1} \\
&= s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j (y_j - y_{j+1}) \quad \blacksquare
\end{aligned}$$

2. **(Dirichlet's Test)** Dirichlet's Test for convergence states that if the partial sums of $\sum_{n=1}^{\infty} x_n$ are bounded (but not necessarily convergent), and if (y_n) is a sequence satisfying $y_n \geq y_2 \geq \cdots \geq 0$ and $\lim y_n = 0$, then $\sum_{n=1}^{\infty} x_n y_n$ converges.

   (a) Let $M > 0$ be an upper bound for the partial sums of $\sum_{n=1}^{\infty} x_n$. Use part 1 to show that

$$\left| \sum_{j=m+1}^{n} x_j y_j \right| \leq 2M \left| y_{m+1} \right|$$

$$\begin{aligned}
\left| \sum_{j=m+1}^{n} x_j y_j \right| &= \left| s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j (y_j - y_{j+1}) \right| \\
&\leq \left| s_n y_{n+1} - s_m y_{m+1} \right| + \left| \sum_{j=m+1}^{n} s_j (y_j - y_{j+1}) \right| \\
&\leq M \left| y_{n+1} - y_{m+1} \right| + \left| \sum_{j=m+1}^{n} M (y_j - y_{j+1}) \right| \\
&= M \left| y_{n+1} - y_{m+1} \right| + M \left| (y_{m+1} - y_{m+2}) + (y_{m+2} - y_{m+3}) + \cdots + (y_n - y_{n+1}) \right| \\
&= M \left| y_{n+1} - y_{m+1} \right| + M \left| y_{m+1} - y_{n+1} \right| \\
&= 2M \left| y_{m+1} - y_{n+1} \right|
\end{aligned}$$

Since $y_n > 0$, $2M \left| y_{m+1} - y_{n+1} \right| \leq 2M \left| y_{m+1} \right|$. $\blacksquare$

(b) Prove Dirichlet's Test

Since $\lim y_n = 0$, $\exists N \in \mathbb{N}$ such that $m, n \geq N$ implies $|y_n| < \frac{\varepsilon}{2M}$.

Now let $S_n = \sum_{j=1}^{n} x_j y_j$ so

$$|S_n - S_m| = \left| \sum_{j=m+1}^{n} x_j y_j \right| \leq 2M \left| y_{m+1} \right| < 2M \frac{\varepsilon}{2M} = \varepsilon$$

Then by the Cauchy Criterion, (S_n) converges so $\sum_{k=m+1}^{n} x_j y_j$ converges. ■