Math 1010 - Homework 1

To solve these problems, you may use any definition of theorem we have learned or proven in class, but remember to state what you use!

Problem 1 (De Morgan's Laws)

Let A and B be subsets of R.

1. If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subset A^c \cup B^c$.

   If $x \in (A \cap B)^c$, then $x \notin (A \cap B)$. This can mean that either x is in neither A nor B, or it is not in one of the two. Hence,

$$(x \notin A) \vee (x \notin B) \implies (x \in A^c) \vee (x \in B^c) \implies x \in (A^c \cup B^c) \implies (A \cap B)^c \subset A^c \cup B^c \quad \blacksquare$$

2. Prove the reverse inclusion $(A \cap B)^c \supset A^c \cup B^c$ and conclude that $(A \cap B)^c = A^c \cup B^c$.

$$\begin{aligned} x \in (A^c \cup B^c) &\implies (x \in A^c) \vee (x \in B^c) \\ &\implies (x \notin A) \vee (x \notin B) \\ &\implies x \notin (A \cap B) \\ &\implies x \in (A \cap B)^c \\ &\implies (A^c \cup B^c) \subset (A \cap B)^c \quad \blacksquare \end{aligned}$$

3. Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

$$\begin{aligned} x \in (A \cup B)^c &\implies x \notin (A \cup B) \\ &\implies (x \notin A) \wedge (x \notin B) \\ &\implies (x \in A^c) \wedge (x \in B^c) \\ &\implies x \in (A^c \cap B^c) \\ &\implies (A \cup B)^c \subset (A^c \cap B^c) \\ x \in (A^c \cap B^c) &\implies (x \in A^c) \wedge (x \in B^c) \\ &\implies (x \notin A) \wedge (x \notin B) \\ &\implies x \notin (A \cup B) \\ &\implies x \in (A \cup B)^c \\ &\implies (A^c \cap B^c) \subset (A \cup B)^c \end{aligned}$$

$$\begin{matrix} (A \cup B)^c \subset (A^c \cap B^c) \\ (A^c \cap B^c) \subset (A \cup B)^c \end{matrix} \implies (A \cup B)^c = A^c \cap B^c \quad \blacksquare$$

Problem 2

Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = (2y_n - 6)/3$.

1. Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.

   (a) Base case:

   $$y_2 = \frac{2y_1 - 6}{3} = \frac{2(6) - 6}{3} = 2 > -6 \quad \checkmark$$

   (b) Inductive step: Assume $y_n > -6$. Then

   $$y_{n+1} = \frac{2y_n - 6}{3} > \frac{2(-6) - 6}{3} = -6 \quad \blacksquare$$

2. Use another induction argument to show the sequence $(y_1, y_2, y_3, \dots)$ is decreasing.

   (a) Base case:

   $$y_2 = \frac{2y_1 - 6}{3} = \frac{2(6) - 6}{3} = 2 < y_1 = 6 \quad \checkmark$$

   (b) Inductive step: Assume $y_{n+1} \leq y_n$. Then

   $$\frac{2y_{n+1} - 6}{3} \leq \frac{2y_n - 6}{3} \implies y_{n+2} \leq y_{n+1} \quad \blacksquare$$

Problem 3

Let $A \subset \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. Define the set $cA = \{ca : a \in A\}$

1. If $c \geq 0$, show that $\sup(cA) = c \sup A$.

   We will proceed in two steps: (a) show that $c \sup A$ is an upper bound for cA, and (b) show that $c \sup A$ is the least upper bound for cA.

   (a) *$c \sup A$ is an upper bound for cA:* Let $x \in cA$. Then $x = ca$ for some $a \in A$. Since $a \leq \sup A$, $x = ca \leq c \sup A$. Thus, $c \sup A$ is an upper bound for cA.

   (b) *$c \sup A$ is the least upper bound for cA:* Suppose that b is an upper bound for cA such that $b < c \sup A$. If $c \neq 0$, then $\frac{b}{c} < \sup A$. Definitionally, $\frac{b}{c}$ is not an upper bound for A, so there exists $a \in A$ such that $\frac{b}{c} < a$. Thus $b < ca$, and b cannot be an upper bound for cA. This is a contradiction.

   If $c = 0$, then $cA = \{0\}$, and 0 is the least upper bound for cA. Further, $c \sup A = 0$ so $c \sup A$ is the least upper bound for cA.

   Thus, $c \sup A$ is the least upper bound for cA.

   Since $c \sup A$ is the least upper bound for cA, $\sup(cA) = c \sup A$. ■

2. Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$. Hint: this might have something to do with $\inf A$!

   Claim: Let $A \subset \mathbb{R}$ be nonempty and bounded below, and let $c \in \mathbb{R}$. Define the set $cA = \{ca : a \in A\}$. If $c < 0$, then $\sup(cA) = c \inf A$.

   Proof: Exactly analogous to the above, we first show that $c \inf A$ is an upper bound for cA, and then show that $c \inf A$ is the lowest upper bound for cA.

   Let $ca \in cA$ for some $a \in A$. Since $\inf A \leq a$ and $c < 0$, $ca \leq c \inf A$. Thus, $c \inf A$ is an upper bound for cA so $\sup(cA) \leq c \inf A$

   Now let $b \in A$. By definition,

   $$cb \leq \sup(cA) \implies b \geq \frac{\sup(cA)}{c} \quad \forall b \in A$$

   thus, $\sup(cA)/c$ is a lower bound for A. Since $\inf A$ is the greatest lower bound for A, $\sup(cA)/c \leq \inf A$. Thus, $\sup(cA) \geq c \inf A$.

   Since we now have $c \inf A \geq \sup(cA)$ and $c \inf A \leq \sup(cA)$, we conclude that $\sup(cA) = c \inf A$. ■

Remark: See Example 1.3.7 in the book.

Problem 4

Prove that if a is an upper bound for A, and if a is also an element of A, then it must be true that $a = \sup A$.

Suppose that $a \neq \sup A$. Since a is an upper bound for A, a cannot be less than $\sup A$. Thus, $a > \sup A$. However, since $a \in A$, there exists at least one element of A which is greater than $\sup A$ so $\sup A$ is not an upper bound. This is a contradiction. Thus, $a = \sup A$. ■

Problem 5

Recall that $\mathbb{I}$ stands for the set of irrational numbers.

1. Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in I$ and $at \in I$ as long as $a \neq 0$

   (a) $a+t \in \mathbb{I}$: Suppose that $a+t \in \mathbb{Q}$. $\mathbb{Q}$ is a field so it is closed under addition. Thus, $(a+t) - a \in \mathbb{Q} \implies t \in \mathbb{Q}$, which is a contradiction. Thus, $a + t \in \mathbb{I}$.

   (b) $at \in \mathbb{I}$: Suppose that $at \in \mathbb{Q}$. $\mathbb{Q}$ is a field so it is closed under multiplication. Thus (if $a \neq 0$), $\frac{at}{a} = t \in \mathbb{Q}$, which is a contradiction. Thus, $at \in \mathbb{I}$. $\blacksquare$

2. Prove that $\mathbb{I}$ is dense in $\mathbb{R}$ by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$

   We want to show that for any $a, b \in \mathbb{R}$ with $a < b$, there exists an irrational number t such that $a < t < b$.

   Without loss of generality, consider the real numbers $a - \sqrt{2}, b - \sqrt{2}$ with $a, b \in \mathbb{R}$ and $a < b$.

   Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number q such that

$$a - \sqrt{2} < q < b - \sqrt{2} \implies a < q + \sqrt{2} < b$$

   By a lemma proved in class, $\sqrt{2} \notin \mathbb{Q}$. Then by part 1, $q + \sqrt{2} \in \mathbb{I}$

   Thus, for any $a, b \in \mathbb{R}$, $\exists t \in \mathbb{I}$ such that $a < t < b$. Therefore, $\mathbb{I}$ is dense in $\mathbb{R}$. $\blacksquare$

Hint: you are allowed to use the theorem on the density of $\mathbb{Q}$ in $\mathbb{R}$ which we proved in class