Math 1010 - Homework 8

Problem 1

Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

We proceed by cases. If $K = \emptyset$, then certainly it is closed.

Suppose K is non-empty and finite. Then $K = \{x_1, x_2, \ldots, x_n\} = \bigcup_{i=1}^{\infty} \{x_i\}$. Since $\{x_i\}$ is a singleton set, it is closed. (*Proof:* $\{x_i\}^c = (-\infty, x_i) \cup (x_i, \infty)$ and the union of an arbitrary collection of open sets is open so $\{x_i\}$ is closed). Then, the union of a finite collection of closed sets is closed, so K is closed.

Now suppose K is non-empty and infinite. Let $(x_n) \in K$ and suppose $x_n \to x$. By continuity of f, $f(x_n) \to f(x)$. Since $x_n \in K$ for all n, $f(x_n) = 0 \implies f(x) = 0$. Therefore, $x \in K$. Since every convergent sequence in K has its limit in K, K is closed. ■

Problem 2 (Contraction Mapping Theorem)

Let f be a function defined on all of $\mathbb{R}$ and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c\,|x - y|$$

for all $x, y \in \mathbb{R}$.

1. Show that f is continuous on $\mathbb{R}$.

   Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{c}$ Suppose $|x - y| < \delta$. Then

   $$|f(x) - f(y)| \leq c\,|x - y| < c\delta = \varepsilon$$

   Therefore, $f(x)$ is continuous at all $y \in \mathbb{R}$. ■

2. Pick some point $y_1 \in \mathbb{R}$ and construct the sequence

   $$(y_1, f(y_1), f(f(y_1)), \cdots).$$

   In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.

   We seek to show that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$,

   $$|y_m - y_n| < \varepsilon$$

   In the case $m = n$, $|y_m - y_n| = 0 < \varepsilon$ and we are done.

   Notice that

   $$\begin{aligned} |y_{n+1} - y_n| &= |f(y_n) - f(y_{n-1})| \\ &\leq c\,|y_n - y_{n-1}| \\ &\leq c^2\,|y_{n-1} - y_{n-2}| \\ &\leq \cdots \\ &\leq c^{n-1}\,|y_2 - y_1| \end{aligned}$$

   Since $0 < c < 1$, $\lim_{n\to\infty} c^{n-1}\,|y_2 - y_1| = 0$.

   Thus for all $\varepsilon_0 > 0$, choose N such that $|y_{n+1} - y_n| \leq c^{N-1}\,|y_2 - y_1| < \varepsilon$. Then $\forall m, n \geq N$,

   $$\begin{aligned} |y_m - y_n| &= |y_m - y_{m-1} + y_{m-1} - y_{m-2} + \cdots + y_{n+1} - y_n| \\ &\leq |y_m - y_{m-1}| + |y_{m-1} - y_{m-2}| + \cdots + |y_{n+1} - y_n| \\ &\leq \varepsilon_0 + \varepsilon_0 + \cdots + \varepsilon_0 \\ &= (m - n + 1)\varepsilon_0 \end{aligned}$$

   Let $\varepsilon > 0$. Since ε_0 is arbitrary, we can say $\varepsilon = \frac{\varepsilon_0}{m-n+1}$ so $|y_m - y_n| < \varepsilon$. Therefore, (y_n) is a Cauchy sequence. ■

3. Prove that y is a fixed point of f (i.e. $f(y) = y$) and that it is unique in this regard.

   From (2), $(y_n) \to y$. Notice, though, that

   $$f(y_n) = (f(y_1), f(f(y_1)), f(f(f(y_1))), \ldots)$$

is a subsequence of (y_n) so $f(y_n) \to y$.

By continuity of f, $(y_n) \to y \implies f(y_n) \to f(y)$. Since the limit of a sequence is unique, $f(y) = y$.

Now suppose that $\exists x \neq y$ for which $f(x) = x$. Then

$$|f(x) - f(y)| = |x - y| \leq c\,|x - y|$$

by definition of f. However, this implies $c = 1$ which is a contradiction of the condition $0 < c < 1$. Therefore, y is the only fixed point of f. ■

4. Finally, prove that if x is any arbitrary point in $\mathbb{R}$, then the sequence $(x, f(x), f(f(x)), \cdots)$ converges to y defined in (2).

   Let $x \neq y_1 \in \mathbb{R}$. Repeating the argument in (2), we see that $(x_n) = (x, f(x), f(f(x)), \ldots)$ is a Cauchy sequence with $\lim x_n = x$. However, by the same subsequence argument in (3), $f(x) = x$. By the continuity of f and the uniqueness of the fixed point y in (3), $x = y$. Therefore, the sequeunce $(x, f(x), f(f(x)), \ldots) \to y$ for all $x \in \mathbb{R}$. ■

Problem 3

1. Show that $f(x) = x^3$ is continuous on all of $\mathbb{R}$

   Let $\varepsilon > 0$. Denote $g(x) = x$. Choose $\delta = \varepsilon$. Suppose $|x - y| < \delta$. Then

   $$|g(x) - g(y)| = |x - y| < \delta = \varepsilon$$

   Therefore, $g(x)$ is continuous at all $y \in \mathbb{R}$.

   By the Algebraic Continuity theorem, $x^3 = g(x)g(x)g(x)$ is continuous at all $x \in \mathbb{R}$. $\blacksquare$

2. Argue, using the sequential criterion for absence of uniform continuity, that f is not uniformly continuous on $\mathbb{R}$

   By the Sequential Criterion for Absence of Uniform Continuity, f fails to be uniformly continuous if $\exists \varepsilon > 0$ and $(x_n), (y_n) \in \mathbb{R}$ such that $\forall \delta > 0$, $|x - y| \to 0$ but $|f(x) - f(y)| \geq 0$.

   Consider $x_n = \frac{1}{n}$ and $y_n = -\frac{1}{n}$. Clearly, $|x_n - y_n| = \frac{2}{n} \to 0$. However,

   $$|f(x_n) - f(y_n)| = \left| \frac{1}{n^3} + \frac{1}{n^3} \right| = \frac{2}{n^3}$$

   Therefore, we can choose an ε_0 for which $|f(x_n) - f(x_n)| \geq \varepsilon_0$ so f is not uniformly continuous. $\blacksquare$

3. Show that f is uniformly continuous on any bounded subset of $\mathbb{R}$

   Let A be a bounded subset of $\mathbb{R}$. Then there exists $M > 0$ such that $|x| < M$ for all $x \in A$.

   Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{3M^2}$. Suppose $|x - y| < \delta$. Then

   $$\begin{aligned} |f(x) - f(y)| &= \left|x^3 - y^3\right| \\ &= |x - y| \left|x^2 + xy + y^2\right| \\ &\leq |x - y| \left|M^2 + M^2 + M^2\right| \\ &= 3M^2 |x - y| \\ &< 3M^2 \delta \\ &= \varepsilon \end{aligned}$$

   Therefore, $f(x)$ is uniformly continuous on A. $\blacksquare$

Problem 4 (Lipschitz Functions)

A function $f : A \to \mathbb{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x \neq y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

1. Show that if $f : A \to \mathbb{R}$ is Lipschitz, then it is uniformly continuous on A.

   Let $\varepsilon > 0$. If $f : A \to \mathbb{R}$ is Lipschitz, then

   $$|f(x) - f(y)| \leq M\,|x - y|$$

   Choose $\delta = \frac{\varepsilon}{M}$. If $|x - y| < \delta$,

   $$|f(x) - f(y)| \leq M\,|x - y| < M\delta = \varepsilon$$

   Therefore, $f(x)$ is uniformly continuous on A. ■

2. Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

   No. We proceed by contradiction: if $f : A \to \mathbb{R}$ is uniformly continuous, then for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\forall x, y \in A$,

   $$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

   In the case $|x - y| < \delta$, certainly

   $$\frac{|f(x) - f(y)|}{|x - y|} = \left| \frac{f(x) - f(y)}{x - y} \right| < \frac{\varepsilon}{\delta} := M$$

   However, when $|x - y| \geq \delta$, the Lipschitz condition is not necessarily satisfied. Since the condition may not hold for all $x \neq y \in A$, we cannot conclude that all uniformly continuous functions are necessarily Lipschitz. ■

Problem 5

Solve ONE of the following two:

1. Finish the proof of the Intermediate Value Theorem using the Axiom of Completeness, started on page 138

   The IVT states: Let $f : [a, b] \to \mathbb{R}$ be continuous. If $L \in \mathbb{R}$ satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists $c \in (a, b)$ such that $f(c) = L$.

   Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < L < f(b)$. Let $K = \{x \in [a, b] : f(x) \leq L\}$. K is bounded above by b and $a \in K$ so K is non-empty. By the Axiom of Completeness, $c = \sup K$ exists.

   There are now three cases:

   (a) $f(c) > L$: Let $\varepsilon = f(c) - L > 0$¿ Since f is continuous, $\exists \delta_0 > 0$ such that $|x - c| < \delta_0 \implies |f(x) - f(c)| < \varepsilon$.

   Therefore, we can choose a δ such that $0 < \delta < \delta_0$ so

   $$|(c - \delta) - c| < \delta_0 \implies |f(c - \delta) - f(c)| < \varepsilon$$

   But

   $$\begin{aligned}
   |f(c - \delta) - f(c)| = |-f(c - \delta) + f(c)| < \varepsilon \\
   &\implies -\varepsilon < f(c) - f(c - \delta) < \varepsilon \\
   &\implies f(c) - f(c - \delta) < f(c) - L \\
   &\implies -f(c - \delta) < -L \\
   &\implies f(c - \delta) > L
   \end{aligned}$$

   But then $c - \delta$ is an upper bound for K less than c which contradicts the fact that c is the least upper bound. This case is impossible.

   (b) $f(c) < L$: Again since f is continuous, $f(x) \in V_\varepsilon(f(c))$ for all $\varepsilon > 0$. Therefore, $f(c + \delta) \in V_\varepsilon(f(c))$ for some $\delta > 0$. We can choose ε such that $f(c + \delta) < L$ but then $c + \delta \in K$ which contradicts the fact that c is an upper bound for K. This case is impossible.

   (c) $f(c) = L$. Clearly b is an upper bound for K and $L < f(b)$ so $a < c < b$ and we are done. ■

2. Finish the proof of the Intermediate Value Theorem using the Nested Interval Property, started on page 138-139.