Math 1010 - Final Exam Review

Milan Capoor

1 Definitions

Open Set: $O \subset \mathbb{R}$ is open if $\forall x \in O, \exists \varepsilon > 0$ such that $V_\varepsilon(x) \subset O$

Limit point: x is a limit point of A if every ε-neighborhood of x intersects A at a point other than x

Isolated point: a point which is not a limit point

Closed set: a set which contains all its limit points

Closure of a set: let L be the set of all limit points of $A \subseteq \mathbb{R}$. The closure of A is $A \cup L$

Compact set: a set K is compact if every sequence in K has a convergent subsequence with limit in K

Bounded set: $\exists M > 0$ such that $|a| < M$ for all $a \in A$

Open cover: a collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ such that

$$A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$$

Finite subcover: a finite subcollection of an open cover whose union still contains A

Functional limit: Let $f : A \to \mathbb{R}$ be a function with c a limit point of the domain of A. $\lim f(x) = L$ if $x \to c$

Continuity: $f : A \to \mathbb{R}$ is continuous at $c \in A$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$

Uniform Continuity: $f : A \to \mathbb{R}$ is uniformly continuous on A if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x, y \in A$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Differentiability: $f : A \to \mathbb{R}$ is differentiable at $c \in A$ if $g'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists

Convergence at infinity: Given $g : A \to \mathbb{R}$ and a limit point $c \in A$, we say that $\lim_{x \to c} g(x) = \infty$ if $\forall M > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \implies g(x) \geq M$

Pointwise convergence: (f_n) converges pointwise to f if $\forall x \in A$, $\lim_{n \to \infty} f_n(x) = f(x)$

Uniform convergence: f_n converges uniformly on A to f if $\forall \varepsilon > 0$, $\exists N$ such that $\forall n \geq N$ and $x \in A$,

$$|f_n(x) - f(x)| < \varepsilon$$

Series of functions convergence: The infinite series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + \cdots$$

converges pointwise on A if the sequence of partial sums converges pointwise on A

The series converges uniformly on A if the sequence of partial sums converges uniformly on A

Power series: a function of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

Partition: a partition $P \subseteq [a, b]$ is a finite set of points $P = \{x_0, x_1, \dots, x_n\}$ such that $a = x_0 < x_1 < \dots < x_n = b$

Refinement: a partition Q is a refinement of P if $P \subseteq Q$ (that is, Q contains all the points of P)

Upper/lower sums: For each subinterval $[x_{k-1}, x_k]$ of P, let

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$$
$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$$

The lower sum of f with respect to P is

$$L(f, P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$$

The upper sum of f with respect to P is

$$U(f, P) = \sum_{k=1}^{\infty} M_k (x_k - x_{k-1})$$

Upper/lower integrals: Let $\mathcal{P}$ be the collection of all possible partitions of $[a, b]$, the upper integral of f is

$$U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}$$

The lower integral of f is

$$L(f) = \sup\{L(f, P) : P \in \mathcal{P}\}$$

Riemann integrable: A bounded function f is Riemann integrable on $[a, b]$ if $U(f) = L(f)$, in which case

$$\int_a^b f = U(f) = L(f)$$

2 Theorems

Limit point $\iff$ limit of some sequence: A point x is a limit point of a set A iff exists $(a_n) \in A$ such that $(a_n) \to x$ and $a_n \neq x$ for all $n \in \mathbb{N}$

Proof: $(\impliedby)$ Pick $\varepsilon = \frac{1}{n}$ and $a_n \in V_{1/n}(x) \cap A$ such that $a_n \neq x$. Choose N so $\frac{1}{N} < \varepsilon \implies |a_n - x| < \varepsilon$.

$(\implies)$ Let $V_\varepsilon(x)$ be arbitrary. By convergence, $\exists N \in \mathbb{N}$ so $n \geq N \implies |a_n - x| < \varepsilon \implies a_n \in V_\varepsilon(x)$

Closed set $\iff$ all Cauchy have limit points in set: $F \subseteq \mathbb{R}$ is closed iff every Cauchy sequence in F has a limit in F

Proof: $(\implies)$: Let $(x_n) \in F$ be Cauchy. Since F closed, $\exists x \in F$ such that $(x_n) \to x$.

$(\impliedby)$: Assume $\exists x$, a limit point of F not in F. Construct (x_n) such that $x_n \subset V_{1/n}(x) \cap F$ with $x_n \neq x$. Then $(x_n) \to x$. $(x_n) \in F \implies x \in F$ (contradiction) so F is closed.

Complements of open and closed sets:

1. O open $\iff O^c$ closed
2. F closed $\iff F^c$ open

Proof:

1. $(\implies)$ if x is a limit point of O^c, then every ε-neighborhood of x intersects O^c. Thus $V_\varepsilon(x) \not\subset O \implies x \notin O$. Thus $x \in O^c$.

   $(\impliedby)$ O^c closed $\implies x$ not a limit point of $O^c \implies \exists V_\varepsilon(x)$ which does not intersect O^c. Thus $V_\varepsilon(x) \subset O$
2. $(E^c)^c = E$. The rest follows from part 1.

Characterization of compactness on $\mathbb{R}$: $K \subseteq \mathbb{R}$ is compact $\iff K$ is closed and bounded

Proof: $(\implies)$ K not bounded implies (x_n) with no convergent subsequence (contradiction of compactness). K closed because K compact implies $\exists (x_{n_k}) \to x \implies x \in K \implies K$ closed.

$(\impliedby)$ K bounded implies $(x_n) \subset K$ bounded. By Bolzano-Weierstrass and K closed, $\exists (x_{n_k}) \to x \in K$.

Nested compact set: If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ is a nested sequence of nonempty compact sets, then $\bigcap_{n=1}^\infty K_n \neq \emptyset$

Proof: By compactness of K_n, $\exists (x_n) \in K_n \implies (x_n) \in K_1 \implies \exists (x_{n_k}) \to x \in K_1$. Given $n_0 \in \mathbb{N}$, $(x_n) \in K_{n_0}$ if $n > n_0$. Ignoring the finite terms $n_k < n_0$, $(x_{n_k}) \in K_{n_0} \implies \lim x_{n_k} = x \in K_{n_0}$. Since n_0 arbitrary,

$$x \in \bigcap_{n=1}^\infty K_n$$

Heine-Borel: For $K \subseteq \mathbb{R}$, K is compact $\iff K$ is closed and bounded $\iff$ every open cover of K has a finite subcover

Proof: $(i) \iff (ii)$ by characterization of compactness.

$((ii) \Longleftarrow (iii))$ K is bounded because it is contained in a finite collection of sets. K is closed because $\exists (y_n) \to y$ with $y \notin K$ implies $\exists y_N$ such that $\forall x \in K$,

$$|y_N - y| < \min\{\frac{x_i - y}{2} : 1 \leq i \leq n\} \implies y_N \notin V_{|x-y|/2}(x) \implies y \notin \bigcup_{i=1}^{N} V_{|x_i-y|/2}(x_i) \implies \text{ no finite subcover}$$

which is a contradiction so $y \in K$ and K is closed.

$((ii) \implies (iii))$ HW

Sequential criterion for functional limits: gIVEN $f : A \to \mathbb{R}$ and c is a limit point of A, then the following are equivalent:

1. $\lim_{x \to c} f(x) = L$
2. For every sequence $(x_n) \in A$ with $x_n \neq c$ and $(x_n) \to c$, $(f(x_n)) \to L$

Proof: $(i) \implies (ii)$: $(x_n) \to c \implies x_n \in V_\delta(c)$ for all $n \geq N$. So $f(x_n) \in V_\varepsilon(L)$

$(ii) \implies (i)$: Argue contrapositive by $\delta_n = \frac{1}{n}$ so $\exists x_n \in V_{\delta_n}(c)$ with $f(x_n) \notin V_\varepsilon(L)$. Then $(x_n) \to c$ but $(f(x_n)) \not\to L$

Algebraic limit theorem for functional limits: Let f and g be functions on a domain $A \subset \mathbb{R}$ and assume $\lim_{x \to c} f(x) = L, \lim_{x \to c} g(x) = M$. Then

1. $\lim_{x \to c} kf(x) = kL$
2. $\lim_{x \to c}(f(x) + g(x)) = L + M$
3. $\lim_{x \to c}(f(x)g(x)) = LM$
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$

Proof: Omitted

Divergence criterion: If $f : A \to \mathbb{R}$ with c a limit point of f and $\exists (x_n) \to c, (y_n) \to c \in A$ but $\lim_{x_n \to c} f(x_n) \neq \lim_{y_n \to c} f(y_n)$, then $\lim_{x \to c} f(x)$ does not exist

Proof: Omitted

Characterization of continuity: Let $f : A \to \mathbb{R}$ and $c \in A$. The following definitions of continuity of f at c are equivalent:

1. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$
2. $\forall V_\varepsilon(f(c))$, $\exists \delta > 0$ such that $x \in V_\delta(c) \implies f(x) \in V_\varepsilon(f(c))$
3. $\forall (x_n) \in A$ with $(x_n) \to c$, we have $f(x_n) \to f(c)$

Proof: Omitted

Criterion for discontinuity: Let $f : A \to \mathbb{R}$ with c a limit point of f. If $\exists (x_n) \in A$ with $(x_n) \to c$ but $(f(x_n)) \not\to f(c)$, then f is discontinuous at c

Proof: Omitted

Algebraic continuity theorem: Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be continuous at $c \in A$. Then

1. $kf(x)$ is continuous at c for all $k \in \mathbb{R}$
2. $f(x) + g(x)$ is continuous at c
3. $f(x)g(x)$ is continuous at c
4. $\frac{f(x)}{g(x)}$ is continuous at c if $g(c) \neq 0$

Proof: Omitted

Composition of continuous functions: Let $f : A \to \mathbb{R}$ be continuous at c. Let $g : B \to \mathbb{R}$ be continuous at $f(c)$ with $f(A) \subseteq B$. Then $g \circ f$ is continuous at c

Proof: Omitted

Preservation of compact sets: Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact.

Proof: Find a subsequence (y_{n_k}) of $(y_n) \in f(K)$ which converges to a limit contained in $f(K)$ using compactness and continuity (i.e. existence of $(x_{n_k}) \in K$) of f.

Extreme Value Theorem: If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attains a maximum and minimum on K

Proof: Since $f(K)$ is compact, $\alpha = \sup f(K)$ and $\beta = \inf f(K)$ are in $f(K)$ so $\exists x_1, x_2 \in K$ such that $f(x_1) = \alpha$ and $f(x_2) = \beta$

Sequential criterion for absence of uniform continuity: $f : A \to \mathbb{R}$ fails to be uniformly continuous iff $\exists \varepsilon_0 > 0$ and $(x_n), (y_n) \in A$ such that $\forall \delta > 0$, $|x - y| \to 0$ but $|f(x) - f(y)| \geq \varepsilon_0$

Proof: ($\implies$) By definition, we have $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$ for some $\varepsilon_0 > 0$. Just construct $(x_n), (y_n)$ such that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$

($\impliedby$) Trivial by $|f(x_n) - f(y_n)| \geq \varepsilon_0$

Uniform continuity on compact sets: A function that is continuous on a compact set K is uniformly continuous on K

Proof: Contradiction with the Criterion for absence of uniform continuity using convergent subsequences

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Intermediate Value Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. If $L \in \mathbb{R}$ satisfies $f(a) < L < f(b)$ (or $f(a) > L > f(b)$), then $\exists c \in (a, b)$ such that $f(c) = L$

Proof: Omitted

Differentiability implies continuity: If $f : A \to \mathbb{R}$ is differentiable at $c \in A$, then f is continuous at c

Proof: $\lim_{x \to c} g(x) = g(c)$ by ALT and differentiability of f at c

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Algebraic differentiability theorem: Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be differentiable at $c \in A$. Then

1. $(f + g)'(c) = f'(c) + g'(c)$

2. $(kf)'(c) = kf'(c)$
3. $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$
4. $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c)-f(c)g'(c)}{(g(c))^2}$ if $g(c) \neq 0$

Proof:

1. Omitted
2. Omitted
3.

$$\begin{aligned}\lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x-c} &= \lim_{x\to c} \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x-c} \\ &= \lim_{x\to c} f(x)\frac{g(x)-g(c)}{x-c} + g(c)\frac{f(x)-f(c)}{x-c} \\ &= f(c)g'(c) + g(c)f'(c)\end{aligned}$$

4. Omitted

Chain rule: Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ with $f(A) \subseteq B$. If f is differentiable at $c \in A$ and g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c with

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

Proof:

$$g'(f(c)) = \lim_{y\to f(c)} \frac{g(y) - g(f(c))}{y - f(c)}$$

Then let $y = f(t)$ and apply the ALT.

Interior Limit Theorem: Let f be differentiable on (a, b). If f attains a max at $c \in (a, b)$, then $f'(c) = 0$

Proof: Construct $(x_n) \to c, (y_n) \to c$ with $x_n < c < y_n$. Using the order limit theorem,

$$f'(c) = \lim n \to \infty \frac{f(y_n) - f(c)}{y_n - c} \leq 0$$
$$f'(c) = \lim n \to \infty \frac{f(x_n) - f(c)}{x_n - c} \geq 0$$

so $f'(c) = 0$

Darboux's Theorem: If f is differentiable on $[a, b]$ and if $f'(a) < L < f'(b)$, then $\exists c \in (a, b)$ such that $f'(c) = L$

Proof: $g(x) = f(x) - L$ has $g'(a) < 0 < g'(b)$ so g attains a max at $c \in (a, b)$ so $g'(c) = 0 \implies f'(c) = L$

Rolle's Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then $\exists c \in (a, b)$ such that $f'(c) = 0$

Proof: If the extrema are on the endpoints, f is constant. Otherwise, interior limit theorem gives the result.

Mean Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) then $\exists c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof: Omitted

Corollary 1 of MVT: If $g : A \to \mathbb{R}$ is differentiable on A and $g'(x) = 0$ for all $x \in A$, then $g(x) = k$ with $k \in \mathbb{R}$

Proof: By MVT on $[x, y]$,

$$g'(c) = \frac{g(y) - g(x)}{y - x} = 0 \implies g(y) - g(x) = 0 \implies g(y) = g(x) = k$$

Corollary 2 of MVT: If f and g are differentiable functions on A and $f'(x) = g'(x)$ for all $x \in A$, then $f(x) = g(x) + k$ for some $k \in \mathbb{R}$

Proof: Let $h(x) = f(x) - g(x)$. Then $h'(x) = 0$ so $h(x) = k$

Generalized MVT: If f and g are continuous on $[a, b]$ and differentiable on (a, b) then $\exists c \in (a, b)$ such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

Proof: Omitted

L'Hopital's Rule: Assume f, g are differentiable on (a, b) and $g'(x) \neq 0$ for all $x \neq c \in (a, b)$. If $f(c) = g(c) = 0$ or $\lim_{x \to a} g(x) = \pm\infty$, then

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to c} \frac{f(x)}{g(x)} = L$$

Proof: Omitted

Cauchy Criterion for Uniform Convergence: (f_n) on $A \subset \mathbb{R}$ converges uniformly on A iff $\forall \varepsilon > 0$, $\exists N$ such that $\forall n, m \geq N$ and $x \in A$,

$$|f_n(x) - f_m(x)| < \varepsilon$$

Proof: Cauchy criterion for sequences of real numbers and bounding pointwise convergence

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Continuous Limit Thm: Let $(f_n) \to f$ uniformly on A. If each f_n is continuous at $c \in A$, then f is continuous at c

Proof: By uniform convergence we can choose $N \in \mathbb{N}$ so $|f_N(x) - f(x)| < \frac{\varepsilon}{3}$. Since f_N is continuous at c, $|f_N(x) - f_N(c)| < \frac{\varepsilon}{3}$ when $|x - c| < \delta$.

$$|f(x) - f(c)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| < \varepsilon$$

Differentiable Limit Theorems:

1. Let $f_n \to f$ is a sequence of differentiable functions which converge pointwise on $[a, b]$. If $(f_n') \to g$ uniformly, $f' = g$
2. Let (f_n) be a sequence of differentiable functions on $[a, b]$. If $(f_n') \to g$ uniformly and $\exists x_0 \in [a, b]$ where $f_n(x_0) \to L$, then $(f_n) \to f$ uniformly.
3. Let (f_n) be a sequence of differentiable functions on $[a, b]$ with $(f_n') \to g$ uniformly. If $(f_n(x_0)) \to f(x_0)$ for some $x_0 \in [a, b]$, then $(f_n) \to f$ uniformly and $f' = g$.

Proof: Omitted

Term-by-term Continuity Thm for Series: Let f_n be continuous function. If $\sum f_n$ converges uniformly on A to f, then f is continuous on A

Proof: Apply continuous limit theorem to (S_k)

Term-by-term Differentiability Thm for Series: Let f_n be differentiable functions and assume $\sum f_n'$ converges uniformly to g. If $\exists x_0 \in A$ where $\sum f_n(x_0)$ converges, then $\sum f_n(x)$ converges uniformly to f with $f' = g$

Proof: Apply differentiable limit theorem to (S_k)

Cauchy Criterion for Uniform convergence of series: $\sum f_n$ converges uniformly on A iff $\forall \varepsilon > 0$, $\exists N$ such that $\forall n, m \geq N$ and $x \in A$,

$$|f_{m+1} + f_{m+2} + \cdots + f_n(x)| < \varepsilon$$

Proof: Omitted

Weierstrass M-Test: If $|f_n(x)| \leq M_n$ for $M_n > 0$, if $\sum M_n$ converes, then $\sum f_n(x)$ converfes uniformly on A

Proof: Cauchy Criterion for uniform convergence of series

Convergence of Power Series: If $\sum a_n x^n$ converges at $x_0 \in \mathbb{R}$ then it converges absolutely for $|x| < |x_0|$

Proof: Comparison test with geometric series from $|a_n x_0^n| < M$

Uniform convergence of Power Series: If $\sum a_n x^n$ converges absolutely at x_0, then it converges uniformly on $[-|x_0|, |x_0|]$

Proof: Omitted

Abel's Thm: Let $g(x) = \sum a_n x^n$ converge at $x = R > 0$. Then $g(x)$ converges uniformly on $[0, R]$

Proof: Omitted

Convergence of Power Series on Compact sets: If $\sum a_n x^n$ converges pointwise on $A \subseteq \mathbb{R}$, it converges uniformly on compact subsets of A

Proof: Abel's theorem and existence of extrema on compact sets

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Differentiation of Power Series: If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$, then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ converges on $(-R, R)$

Proof: Uniform convergence on compact sets and boundedness of ns^{n-1} for $0 < s < 1$

Taylor's Formula: Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ on some nontrivial interval centered at 0. Then $a_n = \frac{f^{(n)}(0)}{n!}$

Proof: Omitted

Lagrange Remainder Thm: Let f be $N+1$ times differentiable on $(-R, R)$. Given $x \neq 0$ in $(-R, R)$, $\exists c$ with $|c| < |x|$ such that

$$E_N(x) = f(x) - S_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{n+1}$$

Proof: Successive application of generalized MVT

Two lemmas on partitions:

1. If $P \subset Q$, then $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq U(f, Q)$

   Proof: Induction on k considering the refinement $\{z\} \cup [x_{k-1}, x_k]$

2. If P_1 and P_2 are partitions of $[a, b]$, then $L(f, P_1) \leq U(f, P_2)$

   Proof: Apply the previous lemma to the common refinement $Q = P_1 \cup P_2$

Integrability Criterion: A bounded function f is integrable on $[a, b]$ iff $\forall \varepsilon > 0$, $\exists P_\varepsilon$ (a partition of $[a, b]$) such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < 0$$

Proof: ($\Longleftarrow$): If such a partition exists, $U(f) = L(f)$ so f is integrable

($\Longrightarrow$) $U(f)$ is the greatest lower bound of upper sums so $U(f, P_1) < U(f) + \frac{\varepsilon}{2}$ and $L(f, P_2) > L(f) - \frac{\varepsilon}{2}$. Let $P_\varepsilon = P_1 \cup P_2$

Continuity implies integrability: If f is continuous on $[a, b]$, then it is integrable.

Proof: By integrability criterion, it suffices to bound $U(f, P) - L(f, P) < \varepsilon$.

Integrability with discontinuity at an endpoint: If $f : [a, b] \to \mathbb{R}$ is bounded and integrable on $[c, b]$ for all $c \in (a, b)$, then f is integrable on $[a, b]$

Proof: Produce a P such that $U(f, P) - L(f, P) < \varepsilon$. Let $P = \{a\} \cup P$ so

$$U(f, P) - L(f, P) \leq 2M(x_1 - a) + U(f, P_1)L(f, P_2) < \varepsilon$$

Integrable on [a, b] $\iff$ integrable on [a, c] and [c, b]: Let $f : [a, b] \to \mathbb{R}$ be bounded and $c \in (a, b)$. f is integrable on $[a, b]$ iff f is integrable on $[a, c]$ and $[c, b]$, in which case

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof: ($\Longrightarrow$) f integrable on $[a, b]$ implies $U(f, P) - L(f, P) < \varepsilon$. let $P_1 = P \cap [a, c]$ and $P_2 = P \cap [c, b]$. Then

$$U(f, P_1) - L(f, P_1) < \varepsilon, \quad U(f, P_2) - L(f, P_2) < \varepsilon$$

($\Longleftarrow$) $U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2}$ and $U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2}$ so $P = P_1 \cup P_2$ so

$$U(f, P) - L(f, P) < \varepsilon$$

Algebraic integrability Thm: Assume f, g are integrable on $[a, b]$. Then

1. $\int_a^b f + g = \int_a^b f + \int_a^b g$
2. $\int_a^b kf = k\int_a^b f$
3. $m \leq f(x) \leq M \implies m(b-a) \leq \int_a^b f \leq M(b-a)$
4. $f(x) \leq g(x) \implies \int_a^b f \leq \int_a^b g$
5. $\left|\int_a^b f\right| \leq \int_a^b |f|$

Proof: Omitted

Integrable Limit Thm: Let $f_n \to f$ uniformly on $[a, b]$ and suppose each f_n is integrable on $[a, b]$. Then f is integrable on $[a, b]$ and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

Proof: (Integrability of f) Bound $|U(f, P) - U(f_N, P)| = |\sum(M_k - N_k)\delta x_k|$ by $\varepsilon/3(b-a)$ using uniform convergence.

($\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$): $f_n \to f$ uniformly so $|f_n - f| < \frac{\varepsilon}{b-a}$ for large enough n.

$$\left|\int_a^b f_n - \int_a^b f\right| \leq \int_a^b |f_n(x) - f(x)| < \int_a^b \frac{\varepsilon}{b-a} = \varepsilon$$

Fundamental Thm of Calculus:

1. If $f : [a, b] \to \mathbb{R}$ is integrable and $F : [a, b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ then

$$\int_a^b f = F(b) - F(a)$$

2. Let $g : [a, b] \to \mathbb{R}$ be integrable and define $G(x) = \int_a^x f$. Then G is continuous on $[a, b]$. If g is continuous at $c \in [a, b]$, then G is differentiable at c with $G'(c) = g(c)$

Proof: Omitted

Final

1. Mostly problems from 11
2. definition of compact set
3. uniform continuity definition
4. pointwise convergent definition
5. Focus on Chapter 6, Continuity/uniform continuity
6. 30/120 points theorems and definitions
7. Examples in class should be good for 3/4 questions
8. The homework question is combined from two similar homeworks
9.