Math 1010 - Homework 9

1 Problem 1

Let $f_a(x) = \begin{cases} x^a & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$

1. For which values of a is f continuous at zero?

   Let $\varepsilon > 0$. We will proceed by cases.

   If $x \leq 0$, $f_a(x) = 0$ so when $|x - 0| < \delta$, $|f_a(x) - f_a(0)| = |0 - 0| = 0 < \varepsilon$ for all $a \in \mathbb{R}^\times$.

   If $x > 0$, $f_a(x) = x^a$. Suppose $|x - 0| < \delta$. Then

   $$|f_a(x) - f_a(0)| = |x^a - 0^a| = |x^a| = |x|^a$$

   If $a > 0$, then $|x|^a < \delta^a$.

   Therefore, if $\delta^a < \varepsilon$, f is continuous at zero. We can write

   $$\log \delta = \frac{\log \varepsilon}{a} \implies \delta = \varepsilon^{1/a}$$

   $\delta > 0$ for all $a > 0$ so $|x| < \delta \implies |f_a(x) - f_a(0)| < \varepsilon$ as desired.

   If $a = 0$, then $|f_a(x) - f_a(0)| = |x^0 - 0^0|$ is not defined

   Therefore, f is continuous at zero for all $a > 0$. $\blacksquare$

2. For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

   We can explicitly calculate

   $$f_a'(x) = \begin{cases} ax^{a-1} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

   As above, if $x \leq 0$, $f_a'(x) = 0$ so $f_a'(x)$ is continuous for all $a \in \mathbb{R}$.

   If $x > 0$, $f_a'(x) = ax^{a-1}$. Let $\varepsilon > 0$. Suppose $|x - 0| < \delta$. Then

   $$|f_a'(x) - f_a'(0)| = \left|ax^{a-1} - 0\right| = \left|ax^{a-1}\right| = |a| \left|x^{a-1}\right| = |a| \, |x|^{a-1} < |a| \, \delta^{a-1}$$

   For f to be continuous, we need $|a| \, \delta^{a-1} < \varepsilon \implies (a-1) \log \delta = \log \frac{\varepsilon}{|a|}$. We can therefore choose

   Rearranging, we have

   $$\delta = a^{\frac{-1}{a-1}} \cdot \varepsilon^{\frac{1}{a-1}}$$

   Clearly this is not defined at $a \leq 1$ so f is differentiable at 0 for $a > 1$.

   As f_a is differentiable at 0, we have

   $$f_a'(0) = \lim_{x \to 0} \frac{f_a(x) - f_a(0)}{x}$$

Immediately,

$$f_a'(0) = 0 = \lim_{x\to 0} \frac{f_a(x) - 0}{x} = \frac{\lim_{x\to 0} f_a(x)}{\lim_{x\to 0} x}$$

By L'Hopital's Rule,

$$\frac{\lim_{x\to 0} f_a(x)}{\lim_{x\to 0} x} = \frac{\lim_{x\to 0} f_a'(x)}{\lim_{x\to 0} 1} = \lim_{x\to 0} f_a'(x)$$

So

$$f_a'(0) = \lim_{x\to 0} f_a'(x)$$

which implies the derivative function is continuous at 0. ■

3. For which values of a is f twice-differentiable?

$$f_a''(x) = \begin{cases} a(a-1)x^{a-2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

To be differentiable we need,

$$\begin{aligned} f_a''(0) &= \lim_{x\to 0} \frac{ax^{a-1} - a(0)^{a-1}}{x} && (\implies a > 1) \\ &= \lim_{x\to 0} \frac{ax^{a-1}}{x} \\ &= \lim_{x\to 0} a(a-1)x^{a-2} && \text{(L'hopital)} \end{aligned}$$

However, (supposing the derivative exists), we can explicitly calculate $f_a''(0) = 0$. So f is twice differentiable iff

$$\lim_{x\to 0} a(a-1)x^{a-2} = 0$$

This is true for all $a > 2$. ■

2 Problem 2

Review the definition of uniform continuity. Given a differentiable function $f : A \to \mathbb{R}$, let's say that f is uniformly differentiable on A if, given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| < \varepsilon \qquad \text{whenever } 0 < |x-y| < \delta.$$

1. Is $f(x) = x^2$ uniformly differentiable on $\mathbb{R}$? How about $g(x) = x^3$?

   Let $\varepsilon > 0$. Suppose $|x-y| < \delta$. Then

$$\begin{aligned}
\left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| &= \left|\frac{x^2-y^2}{x-y} - 2y\right| \\
&= \left|\frac{(x-y)(x+y)}{x-y} - 2y\right| \\
&= |x+y-2y| \qquad (\text{since } 0 < |x-y|) \\
&= |x-y| < \delta
\end{aligned}$$

   Let $\delta = \varepsilon$. Then $0 < |x-y| < \delta$ implies $\left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| < \varepsilon$ so $f(x) = x^2$ is uniformly differentiable on $\mathbb{R}$.

   Suppose $|x-y| < \delta$. Then

$$\begin{aligned}
\left|\frac{g(x)-g(y)}{x-y} - g'(y)\right| &= \left|\frac{x^3-y^3}{x-y} - 3y^2\right| \\
&= \left|\frac{(x-y)(x^2+xy+y^2)}{x-y} - 3y^2\right| \\
&= \left|x^2+xy+y^2-3y^2\right| \qquad (\text{since } 0 < |x-y|) \\
&= \left|x^2+xy-2y^2\right| \\
&= |(x+2y)(x-y)| \\
&= |x+2y|\,|x-y| \\
&< \delta\,|x+2y|
\end{aligned}$$

   If we choose $\varepsilon = 1$ and $x = y = 1$, then $|x-y| = 0 < \delta$ but $\left|\frac{g(x)-g(y)}{x-y} - g'(y)\right| < \varepsilon$ only if $\delta = \frac{1}{3}$. For $x = y = 2$, again $|x-y| = 0 < \delta$ but $\left|\frac{g(x)-g(y)}{x-y} - g'(y)\right| = 2$ which is not less than ε. Therefore, $g(x) = x^3$ is not uniformly differentiable on $\mathbb{R}$. $\blacksquare$

2. Show that if a function is uniformly differentiable on an interval A, then the derivative must be continuous on A.

   Let $\varepsilon > 0$. Suppose $|x-y| < \delta$. Then we want to show

$$|f'(x) - f'(y)| < \varepsilon$$

   Since f is uniformly differentiable, we have

$$0 < |x-y| < \delta_1 \implies \left|\frac{f(x)-f(y)}{x-y} - f'(y)\right| < \frac{\varepsilon}{2}$$

However, since uniform differentiability holds for all $x, y \in A$, we can choose δ_2 such that

$$0 < |y - x| < \delta_2 \implies \left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < \frac{\varepsilon}{2}$$

Therefore, if $|x - y| < \delta = \min\{\delta_1, \delta_2\}$, we have

$$\begin{aligned}
|f'(x) - f'(y)| &= \left| f'(x) - f'(y) + \frac{f(x) - f(y)}{x - y} - \frac{f(x) - f(y)}{x - y} \right| \\
&= \left| f'(x) - \frac{f(x) - f(y)}{x - y} - f'(y) + \frac{f(x) - f(y)}{x - y} \right| \\
&\leq \left| f'(x) - \frac{f(x) - f(y)}{x - y} \right| + \left| -f'(y) + \frac{f(x) - f(y)}{x - y} \right| \\
&= \left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| + \left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

so f' is continuous on A. ■

3. Is there a theorem analogous to the theorem "Uniform Continuity on Compact Sets" (Thm 4.4.7 from the book) for differentiation? Are functions that are differentiable on a closed interval $[a, b]$ necessarily uniformly differentiable?

No. Consider

$$f(x) = x^2 \sin(\frac{1}{x})$$

on $[-\frac{1}{\pi}, \frac{1}{\pi}]$.

We can calculate

$$f'(x) = x^2 \cos\frac{1}{x} \cdot \frac{-1}{x^2} + 2x \cdot \sin\frac{1}{x} = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$$

so the derivative exists but is not continuous at $x = 0$.

From (2), if f were uniformly differentiable, the derivative would be continuous. Therefore, $\exists f$ which is differentiable on a compact set but not uniformly differentiable. ■

3 Problem 3

Assume that g is differentiable on $[a, b]$ and satisfies $g'(a) < 0 < g'(b)$.

1. Show that there exists a point $x \in (a, b)$ where $g(a) > g(x)$, and a point $y \in (a, b)$ where $g(y) < g(b)$.

   We have
   $$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} < 0$$

   Since a is a lower bound, $x \geq a$ so
   $$\lim_{x \to a} g(x) - g(a) < 0 \implies \lim_{x \to a} g(x) < \lim_{x \to a} g(a) = g(a)$$

   (if $x \neq 0$) so
   $$\lim_{x \to a} g(x) < g(a)$$

   therefore, $\exists x \in (a, b)$ with $g(a) > g(x)$.

   By similar argument,
   $$\begin{aligned} g'(b) = \lim_{x \to b} \frac{g(x) - g(b)}{x - b} &> 0 \\ \implies \frac{\lim_{x \to b} g(x) - \lim_{x \to b} g(b)}{\lim_{x \to b}(x - b)} &> 0 \\ \implies \lim_{x \to b} g(x) - \lim_{x \to b} g(b) &< 0 \\ \implies \lim_{x \to b} g(x) &< \lim_{x \to b} g(b) \\ \implies \lim_{x \to b} g(x) &< g(b) \end{aligned}$$

   So $\exists y \in (a, b)$ such that $g(y) < g(b)$. $\blacksquare$

2. Now complete the proof of Darboux's Theorem, started on page 152 in the book.

   Darboux's Theorem states that if g is differentiable on $[a, b]$ and $f'(a) < \alpha < f'(b)$ (or $f'(a) > \alpha > f'(b)$), then $\exists c \in (a, b)$ such that $f'(c) = \alpha$.

   Let $g(x) = f(x) - \alpha x$ on $[a, b]$. Since f is differentiable, g is differentiable by the Algebraic Differentiability Theorem:
   $$g'(x) = f'(x) - \alpha$$

   Then, in terms of g, we want to show that $\exists c \in (a, b)$ such that $g'(c) = 0$ given $g'(a) < 0 < g'(b)$.

   From part (1), we have that $\exists x \in (a, b)$ where $g(a) > g(x)$ and $\exists y \in (a, b)$ where $g(y) < g(b)$. Therefore, a and b are not minima for g on (a, b). Therefore, since g is differentiable on (a, b), by the Interior Extremum Theorem, $\exists c \in (a, b)$ which is the minimum on (a, b) so $g'(c) = 0$.

   Therefore, $f'(c) = \alpha$ for some $c \in (a, b)$. $\blacksquare$

4 Problem 4

Recall from the previous problem set that a function $f : A \to \mathbb{R}$ is Lipschitz on A if there exists an $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M \qquad \text{for all } x \neq y \text{ in } A.$$

1. Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

   Since f is differentiable on $[a, b]$, it is continuous on $[a, b]$ and differentiable on $(a, b) \subset [a, b]$.

   By the mean value theorem, $\exists c \in (a, b)$ such that

   $$f'(c) = \frac{f(b) - f(a)}{b - a} \implies \left|\frac{f(b) - f(a)}{b - a}\right| = |f'(c)|$$

   Notice that $[a, b] \subset \mathbb{R}$ is compact because it is closed and bounded. Since f' is continuous on $[a, b]$, by the Extreme Value Theorem, f' attains a maximum and minimum on $[a, b]$. Let $M = \max_{x \in [a,b]} |f'(x)|$. Then $\left|\frac{f(b)-f(a)}{b-a}\right| \leq M$ for all $x \in [a, b]$. Therefore, f is Lipschitz on $[a, b]$. ■

2. Review the definition of a contractive function from the previous problem set. If we add the assumption that $|f'(x)| < 1$ on $[a, b]$, does it follow that f is contractive on this set?

   A contractive function is one such that

   $$|f(x) - f(y)| \leq c\,|x - y|$$

   with $0 < c < 1$ for all $x, y \in [a, b]$.

   Adding the condition $|f'(x)| < 1$ for $x \in [a, b]$ tells us

   $$|f'(x)| = \left|\frac{f(x) - f(y)}{x - y}\right| < 1$$

   for all $x, y \in [a, b]$

   Therefore, let $c = \left|\frac{f(x)-f(y)}{x-y}\right|$

   So

   $$|f(x) - f(y)| \leq \left|\frac{f(x) - f(y)}{x - y}\right| |x - y| = |f(x) - f(y)|$$

   with $0 < c < 1$ as desired. Therefore, f is contractive. ■

5 Problem 5

Recall that a fixed point of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Let f be differentiable (so continuous) on $[a, b]$. Let $x_1 \in (a, b)$ be a fixed point of f. Suppose there exists $x_2 \in (a, b)$ with $x_2 \neq x_1$ such that $f(x_2) = x_2$.

Suppose WLOG that $x_1 < x_2$. clearly, $[x_1, x_]] \subseteq [a, b]$ so f is differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$.

By the Mean Value Theorem, there exists $x \in (x_1, x_2)$ such that

$$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

However, this is a contradiction of the assumption that $f'(x) \neq 1$. Therefore, f can have at most one fixed point. $\blacksquare$