Math 1010: One-Variable Analysis

Milan Capoor

Spring 2024

Chapter 1

The Real Numbers

Lecture 1 - Jan 24:

Preliminaries

1. Sets

   Definition: A *set* is a collection of objects.

   De Morgan's Laws:

   $$(A \cap B)^c = A^c \cup B^c$$
   $$(A \cup B)^c = A^c \cap B^c$$

   Proof: HW

2. Functions

   Definition: Given two sets A, B, a *function* $f : A \to B$ is a rule that assigns to each $a \in A$ a unique element $f(a) \in B$.

   The *domain* of f is A. The *range* of f is a subset of B.

   Examples:

   (a) Dirichlet Function:

   $$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

(Its domain is $\mathbb{R}$ and its range is $\{0, 1\}$)

(b) Absolute value function:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Properties:

$$|ab| = |a| \cdot |b|$$
$$|a + b| \leq |a| + |b| \qquad \text{(Triangle Inequality)}$$

3. Proofs

Types of Proofs:

- *Direct Proof* - Start with a valid statement (usually the hypothesis) and proceed by logical steps
- *Indirect Proof (Proof by Contradiction)* - Begin by negating the conclusion and proceed by logical steps to a contradiction.

Theorem: Let $a, b \in \mathbb{R}$. Then $a = b \iff \forall \varepsilon > 0, |a - b| < \varepsilon$

Proof: We have two statements:

- If $a = b \implies \forall \varepsilon > 0, |a - b| < \varepsilon$
- If $\forall \varepsilon > 0, |a - b| < \varepsilon \implies a = b$

Proof of first statement: Suppose $a = b$. Then $|a - b| = 0$. Thus, $\forall \varepsilon > 0,\ |a - b| < \varepsilon$.

Proof of second statement: Assume $a \neq b$. Then $\exists\, \varepsilon_0 > 0$ s.t. $|a - b| = \varepsilon_0$ But this is contradiction by hypothesis. ■

Proof by induction:

Example: Let $x_1 = 2$ and $\forall n \in \mathbb{N}$, define $x_{n+1} = \frac{x_n+5}{3}$, $n \geq 1$. Prove that x_n is increasing.

Proof:

(a) Base Case:

$$x_1 = 2 < x_2 = \frac{7}{3} \quad \checkmark$$

(b) Inductive Step: Assume $x_n \leq x_{n+1}$. Then

$$\underbrace{\frac{x_n + 5}{3}}_{x_{n+1}} \leq \underbrace{\frac{x_{n+1} + 5}{3}}_{x_{n+2}} \implies x_{n+1} \leq x_{n+2} \quad \blacksquare$$

Axioms for the real numbers

- **Field Axioms:** $\forall a, b, c \in \mathbb{R}$
  1. $(a + b) + c = a + (b + c)$ (Additive Associativity)
  2. $\exists\, 0 \in \mathbb{R}$ s.t. $a + 0 = a$ (Additive Identity)
  3. $\exists - a \in \mathbb{R}$ s.t. $a + (-a) = 0$ (Additive Inverse)
  4. $a \cdot b = b \cdot a$ (Commutativity)
  5. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (Multiplicative Associativity)
  6. $\exists\, 1 \in \mathbb{R}$ s.t. $a \cdot 1 = a$ (Multiplicative Identity)
  7. $\exists\, a^{-1} \in \mathbb{R}$ s.t. $a \cdot a^{-1} = 1$ (Multiplicative Inverse)
  8. $a \cdot (b + c) = a \cdot b + a \cdot c$ (Distributivity)
- **Order Axioms:** there exists a subset of positive numbers P such that
  10. exclusively either $a \in P$ or $-a \in P$ or $a = 0$ (Trichotomy)
  11. $a, b \in P \implies a + b \in P$ (Closure under addition)
  12. $a, b \in P \implies a \cdot b \in P$ (Closure under multiplication)
- **Completeness Axiom:** a least upper bound of a set A is a number x such that $x \geq y$ for all $y \in A$, and such that if z is also an upper bound of A, then

$z \geq x$.

13. Every nonempty set A which is bounded above has a least upper bound.

We will call Properties 1-12, and anything that follows from them, *elementary arithmetic*. These alone imply that $\mathbb{Q}$ is a subfield of $\mathbb{R}$ and basic properties of inequalities under addition and multiplication.

Adding Property 13 uniquely determines the real numbers. The standard proof is to identify each $x \in \mathbb{R}$ with the subset of rationals $\{y \in \mathbb{Q} : y < x\}$, *the Dedekind cut*. This can also construct the reals from the rationals.

Lecture 2 - Jan 30:

Axiom of Completeness

1. $\mathbb{R}$ is an ordered field.
2. There is a least upper bound and a greatest lower bound

Note: the axiom of completeness is only true for $\mathbb{R}$

Definition: Let $A \subseteq \mathbb{R}$ be a set. Then:

1. A is *bounded above* if $\exists\, b \in \mathbb{R}$ s.t. $a \leq b$ for all $a \in A$. Conversely, then b is an *upper bound* of A.
2. A is *bounded below* if $\exists\, l \in \mathbb{R}$ s.t. $a \geq l$ for all $a \in A$. Conversely, then l is a *lower bound* of A.

Definition: $s \in \mathbb{R}$ is *least upper bound* of $A \subseteq \mathbb{R}$ if

1. s is an upper bound of A
2. if b is any upper bound for A, then $s \leq b$

s is called *the supremum of* A and is denoted $s := \sup A$. Further, it is unique.

Similarly, $\inf A$ (the *infimum*) is the greatest lower bound of A.

Example: $A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. Then $\sup A = 1$.

Proof:

1. $1 \geq \frac{1}{n}$ for all $n \in \mathbb{N}$ ✓

2. Assume b is another upper bound. Since $1 \in A$, $1 \leq b$ ■

Remark: $\sup A$ and $\inf A$ do not have to be elements of A.

- When $\sup A \in A$, we call it the *maximum*
- When $\inf A \in A$, we call it the *minimum*

Example: In the example above, $\inf A = 0 \notin A$.

Example:

$$(0,2) = \{x \in \mathbb{R} : \underbrace{0}_{\text{inf}} < x < \underbrace{2}_{\text{sup}}\}$$
$$[0,2] = \{x \in \mathbb{R} : \underbrace{0}_{\text{min}} \leq x \leq \underbrace{2}_{\text{max}}\}$$

Theorem: There is no rational number whose square is 2

Proof: Suppose $\exists, p, q \in \mathbb{Z}$ s.t. $(\frac{p}{q})^2 = 2$. We further assume that $q \neq 0$ and $\text{GCF}(p,q) = 1$.

Then
$$(\frac{p}{q})^2 = 2 \implies \frac{p^2}{q^2} = 2 \implies p^2 = 2q^2$$

Thus, p^2 is even so p is even (because the product of two odd numbers is odd).

Thus, we can write $p = 2r,\ r \in \mathbb{Z}$. Substituting,

$$(2r)^2 = 2q^2 \implies 4r^2 = 2q^2 \implies 2r^2 = q^2$$

By similar logic, q is even. But this contradicts our assumption that $\text{GCF}(p,q) = 1$. ■

This allows us to show that $\mathbb{Q}$ has gaps (it is incomplete). Consider:

$$S = \{r \in \mathbb{Q} : r^2 < 2\}$$

A sensible upper bound is $\sqrt{2} \approx 1.4142\ldots$. Since $\sqrt{2} \notin \mathbb{Q}$, we need to approximate it with rational numbers. We can get infinitely close,

$$\frac{3}{2}, \frac{142}{100}, \frac{1415}{1000}, \ldots$$

but because we need infinitely many terms, we do not have a least upper bound (the next term will always be closer).

Lemma: Let $s \in \mathbb{R}$ be an upper bound for a set $A \subseteq \mathbb{R}$. Then $s = \sup A$ iff $\forall \varepsilon > 0\ \exists a \in A$ s.t. $s - \varepsilon < a$

Proof:

1. Suppose $s = \sup A$. Consider any $s - \varepsilon$ with $\varepsilon > 0$. From the definition of supremum, $s - \varepsilon$ is not an upper bound for A (because $s - \varepsilon < \sup A$). Thus, $\exists a \in A$ s.t. $s - \varepsilon < a$

2. Suppose $\forall \varepsilon > 0\ \exists\, a \in A$ s.t. $s - \varepsilon < a$.

   Since $s - \varepsilon < a$, it cannot be an upper bound by definition. Thus, for any $b < s$, b is not an upper bound. Therefore, any upper bound b' must satisfy $s \leq b'$. This is precisely the definition of $\sup A$. ■

Lecture 3 - Feb 1:

Recall

- $\mathbb{R}$ is an ordered field satisfying the Axiom of Completeness
- $\mathbb{Q}$ is an ordered field but does not satisfy the Axiom of Completeness
- $\mathbb{Z}$ satisfies the AOC but is not a field (so we ignore it in analysis)
- $s = \sup A \implies a \leq b$ for any other upper bound b

Consequences of Completeness

Theorem (Nested interval property): For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that I_n contains I_{n+1}. Then the resulting nested sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ has a nonempty intersection $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$

Proof:

Let $A = \{a_n : n \in \mathbb{N}\}$ be the set of all left endpoints of the intervals I_n. Then A is nonempty and bounded above by the b (right) endpoints.

Consider $x = \sup A$. We know $a_n \leq x \leq b_n$ for all $n \in \mathbb{N}$ by the fact that x is an upper bound for A and that it is the *least* upper bound for A.

And indeed, this is exactly the intersection of the intervals. ■

Note that the theorem does not hold for $\mathbb{Q}$! Imagine the series of intervals centered at $\frac{1}{\sqrt{2}}$ – all are non-empty but their intersection is empty (because there are rational numbers infinitely close to $\frac{1}{\sqrt{2}}$ but that final interval would be empty).

Theorem (Archimedian Property): Given any number $x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ satisfying $n > x$. (i.e. $\mathbb{N}$ is *not* bounded above)

Proof by contradiction:

Suppose $\mathbb{N}$ is bounded above. By the axiom of completeness, $\mathbb{N}$ has a least upper bound $\alpha = \sup \mathbb{N}$. By definition of supremum, $\alpha - 1 < n \implies \alpha < n+1$. But $n+1 \in \mathbb{N}$, so α is not an upper bound. ■

Consequence: Given any real number $y > 0$, $\exists n \in \mathbb{N}$ satisfying $\frac{1}{n} < y$.

Proof: Let $x = \frac{1}{y}$. By the Archimedean Property, $\exists n \in \mathbb{N}$ satisfying $n > x$. Then $n > \frac{1}{y} \implies y < \frac{1}{n}$

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$): For every two real numbers a and b with $a < b$, $\exists r \in \mathbb{Q}$ s.t. $a < r < b$

Proof:

We want to show that $\exists m \in \mathbb{Z}, n \in \mathbb{N} : a < \frac{m}{n} < b$.

First note that we can choose $m \in \mathbb{Z}, n \in \mathbb{N}$ to bound a. We choose n such that

$$\frac{m-1}{n} < a < \frac{m}{n}$$

and m to be the smallest integer greater than na:

$$m - 1 \leq na < m$$

The RHS inequality gives $a < \frac{m}{n}$.

By Archimedean property, we can pick $n \in \mathbb{N}$ such that $\frac{1}{n} < b-a$. Equivalently, $a < b - \frac{1}{n}$.

The LHS gives

$$m \leq na + 1 < n(b - \frac{1}{n}) + 1 = nb \implies m < nb \implies \frac{m}{n} < b$$

Thus,

$$a < \frac{m}{n} < b \quad \blacksquare$$

Corollary: Density of Irrationals ($\mathbb{I}$) in $\mathbb{R}$

Cardinality

Definition: *Cardinality* is the size of a set

Definition:

- A function $f : A \to B$ is *injective* (or *one-to-one*) if $a_1 \neq a_2$ in A implies $f(a_1) \neq f(a_2)$.
- A function $f : A \to B$ is *surjective* (*onto*) if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$ (all elements in B have a pre-image in A)
- A function $f : A \to B$ is *bijective* (has a "1-to-1 correspondence") if it is both injective and surjective

Definition: The set A has the same cardinality as the set B if there exists a bijection $f : A \to B$.

Example: $E = \{2, 4, 6, 8, \dots\}$. We create an equivalence relation $\mathbb{N} \sim E$ induced by $f : \mathbb{N} \to E$ given by $f(n) = 2n$. Thus $\mathbb{N}$ and E have the same cardinality.

Example: $\mathbb{N} \sim \mathbb{Z}$. Consider

$$f(n) = \begin{cases} \frac{n-1}{2} & n \text{ is odd} \\ -\frac{n}{2} & n \text{ is even} \end{cases}$$

Proof of bijection is left as an exercise.

Example: $(a, b) \sim \mathbb{R}$

Lecture 4 - Feb 6:

Countable Sets

Definition: A set A is *countable* if $A \sim \mathbb{N}$ (it has the same cardinality as $\mathbb{N}$)

Theorem: $\mathbb{Q}$ is countable

Proof: It suffices to construct a bijection $\phi : \mathbb{N} \to \mathbb{Q}$.

Consider $A_1 = \{0\}$ and for each $n \geq 2$,

$$A_n = \{\pm\frac{p}{q} : p, q \in \mathbb{N} \quad \text{with p/q in lowest term with } p + q = n\}$$

i.e., $A_2 = \{1, -1\},\ A_3 = \{\frac{1}{2}, -\frac{1}{2}, 2, -2\},\ A_4 = \{\pm\frac{1}{3}, \pm 3\}$

We know that each A_n is finite. Further, every rational number appears *exactly* once in these sets.

We can then define $\phi : \mathbb{N} \to \mathbb{Q}$ by the one-to-one correspondence between the natural numbers and each element of the A_n's

$$\begin{array}{lccccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \\ \mathbb{Q}: & \underbrace{0}_{A_1} & \underbrace{1 & -1}_{A_2} & \underbrace{\frac{1}{2} & -\frac{1}{2} & 2 & -2}_{A_3} & \dots \end{array}$$

The correspondence is onto: every rational will appear. (e.g. $\frac{22}{7} \in A_{29}$)

The correspondence is 1-1: each rational appears exactly once. $\blacksquare$

Theorem: $\mathbb{R}$ is uncountable

Proof: Assume $\mathbb{R}$ is countable. Then $\mathbb{R} = \{x_1, x_2, \dots\}$

Let I_1 be a closed interval which does not contain x_1. Then $I_2 \subseteq I_1$ and does not contain x_2. By induction, $I_{n+1} \subseteq I_n,\ x_n \notin I_n$

Consider $\bigcap_{n=1}^{\infty} I_n$. If x_{n_0} is in the list, $\exists\, I_{n_0}$ s.t. $x_{n_0} \notin I_{n_0}$. But then

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

However, by the nested interval property, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem: If $A \subseteq B$ and B is countable, then A is countable or finite

Proof: HW

Theorem:

1. If $A_1, A_2, \ldots, A_m$ are countable, then $\bigcup_{n=1}^{m} A_n$ is countable
2. If $A_1, A_2, \ldots$ are countable, then $\bigcup_{n=1}^{\infty} A_n$ is countable

Proof: HW

Chapter 2

Sequences and Series

Lecture 1 - Feb 6 (Continued):

The Limit of a Sequence

Definition: A *sequence* is a function whose domain is $\mathbb{N}$

Examples:

- $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots) = (\frac{1}{n})_{n\in\mathbb{N}}$
- $(\frac{1+n}{n})_{n=1}^{\infty} = (2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots)$
- $x_1 = 2,\ x_{n+1} = \frac{x_n+1}{2}$

Definition (convergence of a sequence): A sequence (a_n) *converges* to a real number a if, for every positive number ε, there exists a $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - a| < \varepsilon$:

$$\begin{aligned} \lim_{n\to\infty} a_n = a &\iff a_n \to a \\ &\iff \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } n \geq N \implies |a_n - a| < \varepsilon \end{aligned}$$

Definition (ε-neighborhood): The *ε-neighborhood* of $a \in \mathbb{R}$ (given $\varepsilon > 0$) is the set $V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$

Here, ε is the radius about the center a.

Definition: A sequence (a_n) converges to a if, given any ε-neighborhood $V_\varepsilon(a)$ if a, there exists a point in the sequence after which all the terms are in $V_\varepsilon(a)$

Lecture 2 - Feb 08:

Convergence

Example: Let $a_n = \frac{1}{\sqrt{n}}$. Show $\lim_{n\to\infty} a_n = 0$.

First we try a few values of epsilon:

- $\varepsilon = \frac{1}{10}$: $(0 - \frac{1}{10}, 0 + \frac{1}{10}) = (-\frac{1}{10}, \frac{1}{10})$

  When $n = 100 \implies a_{100} = \frac{1}{10}$. So the first element in the interval is a_{101}.

- $\varepsilon = \frac{1}{50}$: $(-\frac{1}{50}, \frac{1}{50})$

  Here, the first element in the interval is a_{2501}.

Now for the rigorous version: Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon^2} \implies \frac{1}{\sqrt{N}} < \varepsilon$.

Let $n \geq N$. Then

$$n > \frac{1}{\varepsilon^2} \implies \frac{1}{\sqrt{N}} < \varepsilon \implies \left|\frac{1}{\sqrt{n} - 0}\right| < \varepsilon$$

A template for convergence proofs:

1. Let $\varepsilon > 0$
2. Demonstrate a choice for $N \in \mathbb{N}$
3. Verify N
4. With N well chosen, it should be possible to get $|x_n - x| < \varepsilon$

Example: Prove that $\lim \frac{n+1}{n} = 1$

We want $\left|\frac{n+1}{n} - 1\right| < \varepsilon$. This is equivalent to $\left|\frac{1}{n}\right| < \varepsilon$. So we choose $N \in \mathbb{N} > \frac{1}{\varepsilon}$.

The actual proof then reads: Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ s.t. $N\frac{1}{\varepsilon}$. Let $n \geq N$.

$$n > \frac{1}{\varepsilon} \implies \frac{1}{n} < \varepsilon \implies \left|\frac{n+1}{n} - 1\right| < \varepsilon$$

Theorem (Uniqueness of limits): The limit of a sequence, when it exists, is unique

Proof: HW

The algebraic and order limit theorems

Definition: A sequence (x_n) is bounded if there exists a number $M > 0$, such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem: Every convergent sequence is bounded

Proof: Assume (x_n) converges to l.

Given $\varepsilon > 0$,

$$\exists\, N \in \mathbb{N} \text{ s.t. } x_n \in (l - \varepsilon, l + \varepsilon)\ \forall n \geq N$$

Since we do not know if l is positive or negative, we can only say

$$|x_n| < |l| + \varepsilon$$

From this we know x is bounded for $n \geq N$. Now we check the case $n < N$. Luckily, this is a finite number of cases.

By construction, $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |l| + 1\}$. Then $|x_n| \leq M$ for all $n \in \mathbb{N}$. ■

Theorem (Algebraic Limit Theorems): Let $\lim a_n = a$, $\lim b_n = b$

1. $\lim(ca_n) = ca, \quad \forall c \in \mathbb{R}$
2. $\lim(a_n + b_n) = a + b$
3. $\lim(a_n \cdot b_n) = a \cdot b$
4. $\lim \frac{a_n}{b_n} = \frac{a}{b}$, provided $b \neq 0$

Proof:

1. Let $\varepsilon > 0$. We want to show $|ca_n - ca| < \varepsilon$. Notice

$$|ca_n - ca| = |c| \cdot |a_n - a|$$

Since a_n is convergent, we can make $|a_n - a|$ arbitrarily small.

We choose $N \in \mathbb{N}$ s.t. $|a_n - a| < \frac{\varepsilon}{|c|}$ so $\forall n > N$,

$$|ca_n - ca| < |c| \frac{\varepsilon}{|c|} = \varepsilon \quad \checkmark$$

2. Let $\varepsilon > 0$. We want to show $|a_n + b_n - (a+b)| < \varepsilon$. We can say $|a_n - a + b_n - b| \leq |a_n - a| + |b_n - b|$ by the Triangle inequality. Then since a_n and b_n are convergent, we note that

$$\exists N_1 \in \mathbb{N}, \text{ s.t. } \forall n \geq N_1 : \quad |a_n - a| < \frac{\varepsilon}{2}$$
$$\exists N_2 \in \mathbb{N}, \text{ s.t. } \forall n \geq N_2 : \quad |b_n - b| < \frac{\varepsilon}{2}$$

Choose $N = \max\{N_1, N_2\}$ so

$$\forall n \geq N : \quad |(a_n + b_n) - (a+b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \checkmark$$

3. Let $\varepsilon > 0$. We want to show that $|a_n \cdot b_n - a \cdot b| < \varepsilon$. We can say

$$\begin{aligned} |a_n b_n - ab_n + ab_n - ab| &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n| \cdot |a_n - a| + |a| \cdot |b_n - b| \end{aligned}$$

Since a_n and b_n are convergent, $\exists N_1 \in \mathbb{N}, \text{ s.t. } \forall n \geq N_1 : \quad |b_n - b| < \frac{\varepsilon}{2|a|}$. Note then that b_n is convergent so bounded: $|b_n| \leq M$. Then $\exists N_2, \text{ s.t. } \forall n \geq N_2 : \quad |a_n - a| < \frac{\varepsilon}{2M}$

So with $N = \max N_1, N_2$, $\forall n \geq N$, we have

$$|a_n b_n - ab| \leq M \cdot \frac{\varepsilon}{2M} + |a| \cdot \frac{\varepsilon}{2\,|a|} = \varepsilon$$

4. Let $\varepsilon > 0$. We want to show that $\left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \varepsilon$. This is the same as showing $a_n \cdot \frac{1}{b_n} \to a \cdot \frac{1}{b}$ so it suffices to show that $\frac{1}{b_n} \to \frac{1}{b}$ and apply the multiplicative limit theorem.

Observe:

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_n \cdot b}\right| = \frac{|b - b_n|}{|b_n| \cdot |b|}$$

Intuitively, finding a lower bound for b_n gives an upper bound for $1/b_n$. *Trick:* Choose a large n such that $|b_n - b| > |b_n - 0| \implies |b_n| > \frac{|b|}{2}$. By convergence of (b_n), $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N : |b_n - b| < \frac{|b|}{2}$. Then $|b_n| > \frac{|b|}{2}$.

Now bound $|b_n - b| < \frac{\varepsilon |b|^2}{2}$ by convergence at $N_2 \in \mathbb{N}$.

Finally, let $N = \max\{N_1, N_2\}$ then for $n > N$,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b_n| \cdot |b|} < \frac{\varepsilon \, |b|^2}{2} \cdot \frac{2}{|b|} \cdot \frac{1}{|b|} = \varepsilon \quad \blacksquare$$

Lecture 3 - Feb 15:

Theorem (Order Limit Theorem): Assume $(a_n) \to a$, $(b_n) \to b$.

1. If $a_n \geq 0 \quad \forall n \in \mathbb{N}$, then $a \geq 0$
2. If $a_n \leq b_n \quad \forall n \in \mathbb{N}$, then $a \leq b$
3. If $\exists c \in \mathbb{R}$ s.t. $c \leq b_n \quad \forall n \in \mathbb{N}$, then $c \leq b$

Proof:

1. Suppose $a < 0$. Consider $\varepsilon = |a|$ so $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N : |a_n - a| < |a|$. However, since $a < 0$, this tells us

   $$a < a_n - a < -a \implies a_n < 0$$

   But this contradicts the fact that $a_n \geq 0$.
2. By the Algebraic limit theorem, $(b_n - a_n) \to b - a$. Since $a_n \leq b_n$ for all $n \in \mathbb{N}$, $b_n - a_n \geq 0$, by part 1, $b - a \geq 0 \implies b \geq a$
3. Take $a_n = c \quad \forall n \in \mathbb{N}$. Then $(a_n) \to c$. The result follows from part 2. $\blacksquare$

Monotone Convergence Theorem

Definition: A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. It is *decreasing* if $a_n \geq a_{n+1} \quad \forall n \in \mathbb{N}$.

A sequence is *monotone* if it is either increasing or decreasing for all $n \in \mathbb{N}$.

Theorem (Monotone Convergence Theorem): If a sequence is monotone and bounded, then it is convergent

Proof: Let (a_n) be monotone and bounded. Assume WLOG that (a_n) is increasing. Consider the set $A = \{a_n : n \in \mathbb{N}\}$. SInce (a_n) is bounded, $supA$ exists.

We claim $\lim_{n\to\infty} a_n = \sup A$. Let $\varepsilon > 0$. Since $\sup A$ is the least upper bound, $\sup A - \varepsilon$ is not an upper bound. Thus, $\exists N \in \mathbb{N}$ s.t. $a_N > \sup A - \varepsilon$. Since a_n is monotone, $a_n > \sup A - \varepsilon \quad \forall n \geq N$. Further, $a_n \leq \sup A + \varepsilon$ so

$$|a_n - \sup A| < \varepsilon$$

Series Introduction

Definition (Convergence of Series): Let (b_n) be a sequence. A *infinite series* is an expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \dots$$

The series *converges* to S if the sequence of *partial sums* (S_n) given by

$$S_m = \sum_{n=1}^{m} b_n = b_1 + \cdots + b_m$$

converges to S.

Example: Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$S_m = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{m^2}$$

We seek an upper bound for (S_m). Notice

$$\begin{aligned} S_m &= \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \cdots + \frac{1}{m\cdot m} \\ &< \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(m-1)\cdot m} \\ &= 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{m-1} - \frac{1}{m}) \\ &= 2 - \frac{1}{m} < 2 \end{aligned}$$

Since (S_m) has an upper bound and is increasing, it is convergent to some limit s.

Example (Harmonic Series): Consider $\sum_{n=1}^{\infty} \frac{1}{n}$. Taking partial sums,

$$S_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

Is S_m bounded? No!

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 2$$

But

$$S_8 > 2 + \frac{1}{2}$$

and

$$S_{2^k} > 1 + k(\frac{1}{2})$$

and this is unbounded!

Lecture 4 - Feb 22:

Theorem (Cauchy Condensation Test): Suppose (b_n) is decreasing and $b_n \geq 0 \quad \forall n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} b_n$ converges iff $\sum_{n=1}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + \dots$ converges

Proof: Omitted.

Remark: This is a mostly useless theorem used only for showing the harmonic series diverges.

Corollary: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

Subsequences

Definition: Let (a_n) be a sequence and let $n_1 < n_2 < n_3 < \dots$ be an increasing sequence of natural numbers. Then the sequence $(a_{n_1}, a_{n_2}, a_{n_3}, \dots)$ is a *subsequence* of (a_n) and is denoted by (a_{n_k}) where $k \in \mathbb{N}$ is the index.

Example:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	$\dots$
1	5	-3	10	0	-8	12	$\dots$

If we choose $n_1 = 3, n_2 = 4, n_3 = 6, \dots$ then $(a_{n_k}) = (-3, 10, -8, \dots)$

Note: The order of the terms in the subseq is the same as in the original sequence. Further, no repetitions are allowed.

Examples: $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots)$

- $(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8})$ is a subsequence
- $(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \dots)$ is a subsequence
- $(\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{5}, \dots)$ is *not* a subsequence
- $(1, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \dots)$ is *not* a subsequence

Theorem: A subsequence of a convergent sequence converges to the same limit as the original sequence

Proof: Assume $(a_n) \to a$. Let (a_{n_k}) be a subsequence. Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N : |a_n - a| < \varepsilon$. Since $n_k \geq k \quad \forall k$, the same N will suffice for the subsequence. Then,

$$|a_{n_k} - a| < \varepsilon \quad \forall k \geq N \quad \blacksquare$$

Example: Let $0 < b < 1$. Then $b > b^2 > b^3 > \dots > 0$. Therefore, (b^n) is decreasing and bounded below. By the Monotone Convergence Theorem, $(b^n) \to l$. (b^{2n}) is a subsequence so by the Theorem above, $(b^{2n}) \to l$. However,

$$b^{2n} = b^n \cdot b^n \to l \cdot l \implies l^2 = l \implies l = 0$$

Therefore, $(b_n) \to 0$.

Example: Consider the sequence $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \dots)$. Does it converge?

Consider:

- $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \dots) \to \frac{1}{5}$
- $(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \dots) \to -\frac{1}{5}$

Since the subsequences do not converge to the same limit, the original sequence does not converge.

Theorem (Bolzano-Weierstrass): Every bounded sequence contains a convergent subsequence

Proof: Let (a_n) be a bounded subsequence. $\exists M > 0$ s.t. $|a_n| \leq M \quad \forall n \in \mathbb{N}$.

Split $[-M, M]$ into equal intervals $[-M, 0]$ and $[0, M]$. At least one these intervals must contain infinitely many terms of (a_n). Call this interval I_1. WLOG, suppose $I_1 = [-M, 0]$.

Let (a_{n_1}) to be some term of (a_n) which lies in I_1. Now we repeat: $I_1 = [-M, \frac{M}{2}] \cup [-\frac{M}{2}, 0]$. Label the interval with infinite terms I_2 and pick (a_{n_2}) from I_2 with $n_2 > n_1$.

In general, construct the closed I_k by taking the half of I_{k-1} containing infinitely many terms of (a_n). Select $n_k > n_{k-1} > n_{k-2} > \cdots > n_1$ such that $a_{n_k} \in I_k$.

Notice that the sets $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ are nested and closed. By the Nested Interval Property, $\exists x \in \mathbb{R}$ which lies in every I_k. Intuitively, this is a good limit candidate.

Now we seek to show that $(a_{n_k}) \to x$. Let $\varepsilon > 0$. By construction, each I_k has length $M(\frac{1}{2})^{k-1} \to 0$. $\exists N \in \mathbb{N}$ s.t. $\forall k \geq N$, the length of I_k is less than ε. Since $x \in I_k$ and $a_{n_k} \in I_k$, $|a_{n_k} - x| < \varepsilon$.

Therefore, (a_{n_k}) is a convergent subsequence of the bounded sequence (a_n). $\blacksquare$

Lecture 5 - Feb 27:

Recall:

- A *subsequence* of (a_n) is a sequence (a_{n_k}) where $n_1 < n_2 < n_3 < \dots$
- Any subsequence of a convergent sequence converges to the same limit as the original sequence

- If two convergent subsequences converge to different limits, the original sequence diverges
- *Bolzano-Weierstrass Theorem:* Every bounded sequence contains a convergent subsequence

The Cauchy Criterion

Definition: A sequence (a_n) is called a Cauchy sequence if $\forall \varepsilon > 0$,

$$\exists N \in \mathbb{N} \text{ s.t. } |a_n - a_m| < \varepsilon \quad \forall n, m \geq N$$

Theorem: Every convergent sequence is a Cauchy sequence

Proof: Assume (x_n) converges to x. To prove (x_n) is a Cauchy sequence, we need to find a point in the sequence after which $|x_n - x_m| < \varepsilon$.

Since $(x_n) \to x$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2}$.

$$|x_n - x_m| = |(x_n - x) + (x - x_m)| \leq |x_n - x| + |x_m - x| < \varepsilon$$

Lemma: Cauchy sequences are bounded

Proof: Set $\varepsilon = 1$. Then $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$,

$$|x_n - x_m| < 1 \implies |x_n| < |x_N| + 1 \quad \forall n \geq N$$

Then

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N| + 1\}$$

is a bound for the sequence. ■

Theorem (Cauchy Criternion): A sequence converges iff it is a Cauchy sequence

Proof: The first direction follows from the fact that every convergent sequence is Cauchy.

For the other direction, assume (x_n) is a Cauchy sequence. Then (x_n) is bounded by the Lemma. By the Bolzano-Weierstrass Theorem, (x_n) contains a convergent subsequence $(x_{n_k}) \to x$.

Since (x_n) is Cauchy, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|x_n - x_m| < \frac{\varepsilon}{2} \quad \forall n, m \geq N$.

Since $(x_{n_k}) \to x$, choose x_{n_k} with $n_k \geq N$. Then,

$$|x_{n_k} - x| > \frac{\varepsilon}{2}$$

Now

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \blacksquare$$

Properties of Infinite Series

Recall:

- For a sequence $(a_1, a_2, a_3, \dots)$, the sequence of partial sums is given by

$$(S_m) = (S_1, S_2, S_3, \dots,) = (a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots)$$

- A series $\sum_{n=1}^{\infty} a_n$ converges to A if $\lim(S_m) = A$

Theorem (Algebraic Limit Theory for Series): If $\sum_{k=1}^{\infty} = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

1. $\sum_{k=1}^{\infty} ca_k = cA, \quad \forall c \in \mathbb{R}$
2. $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$

Proof:

1. Since $\sum_{k=1}^{\infty} a_k = A$, $(S_m) = \sum_{k=1}^{m} a_k \to A$. Then $\lim(cS_m) = c \lim S_m = cA$ by the Algebraic Limit Theorem for Sequences (ALT). Then, by definition, $\sum_{k=1}^{\infty} ca_k = cA$
2. Let $S_m = \sum_{k=1}^{m} a_k$ and $T_m = \sum_{k=1}^{m} b_k$. Then $S_m + T_m = \sum_{k=1}^{m} (a_k + b_k)$. Since $(S_m) \to A$ and $(T_m) \to B$, $(S_m + T_m) \to A + B$ by the ALT. Then $\sum_{k=1}^{\infty} (a_k + b_k) = A + B \quad \blacksquare$

Theorem (Cauchy Criterion for Series): The series $\sum_{k=1}^{\infty} a_k$ converges iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall m \geq n \geq N$ we have

$$\left| \sum_{k=m+1}^{n} a_k \right| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon$$

Proof: Define $S_n = a_1 + a_2 + \cdots + a_n$. Observe that

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff (S_n) \text{ converges} \overset{*}{\iff} (S_n) \text{ Cauchy seq}$$

where $\overset{*}{\iff}$ follows from the Cauchy Criterion for sequences.

Further, if and only if (S_n) is Cauchy, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n > m \geq N$,

$$|S_n - S_m| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon \quad \blacksquare$$

Lecture 6 - Feb 29:

Theorem: If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.

Proof: Pick $n = m + 1$ in previous theorem: for $m > N$,

$$|a_{m+1}| < \varepsilon$$

Remark: The converse is *not* true! Consider the harmonic series: $a_n = \frac{1}{n} \to 0$ but $\sum_{n=1}^{\infty} a_n = \infty$

Theorem (Comparison Test): Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. Then

1. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof: Apply Cauchy Criterion for series and observe that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |b_{m+1} + \cdots + b_n|$$

Example (Geometric Series): A series is called a *geometric series* if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots$$

If $r \geq 1$ and $a \neq 0$, then the series diverges. If $r \neq 1$, we use the identity

$$(1-r)(1 + r + r^2 + r^3 + \dots + r^{m-1}) = 1 - r^m$$

Then for partial sums

$$S_m = a + ar + ar^2 + \dots + ar^{m-1} = a(1 + r + r^2 + \dots + r^{m-1}) = a\frac{1-r^m}{1-r}$$

If $|r| < 1$, $a\frac{1-r^m}{1-r} \to \frac{a}{1-r}$. Therefore, for $|r| < 1$,

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

Theorem (Absolute Convergence Test): If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof: Since $\sum_{n=1}^{\infty} |a_n|$ converges, by Cauchy Criterion, given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n > m \geq N$,

$$|a_{m+1}| + |a_{m+1}| + \dots + |a_n| < \varepsilon$$

By triangle inequality,

$$|a_{m+1} + a_{m+2} + \dots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \dots + |a_n| < \varepsilon$$

Remark: The converse is not true! Consider the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges}, \quad \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

Theorem (Alternating Series Test): Let (a_n) be a sequence satisfying

(a) $a_1 \geq a_2 \geq \dots \geq a_n \geq a_{n+1} \geq \dots$ (Decreasing)

(b) $(a_n) \to 0$ (Converges to 0)

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Proof: From conditions (i) and (ii), we have that $a_n \geq 0$. We want to show that the sequence of partial sums (S_n) converges by showing that (S_n) is Cauchy. Let $\varepsilon > 0$ be arbitrary. We need to find an N such that $n > m \geq N$ implies $|S_n - S_m| < \varepsilon$.

$$|S_n - S_m| = |a_{m+1} - a_{m+2} + a_{m+3} - \dots \pm a_n|$$

Since (a_n) is decreasing and all the terms are positive, we can use an induction argument to show $|S_n - S_m| \leq |a_{m+1}|$ for all $n > m$.

Sketch:

$$|a_{m+3}| \leq |a_{m+2}| \leq |a_{m+1}| \implies a_{m+1} - a_{m+2} + a_{m+3} \leq a_{m+1}$$

Since $(a_n) \to 0$, we can choose N such that $m \geq N$ implies $|a_m| < \varepsilon$. Then

$$|S_n - S_m| \leq |a_{m+1}| < \varepsilon$$

Therefore, (S_n) is Cauchy so it converges ■

Definition:

- If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ *converges absolutely*
- If $\sum_{n=1}^{\infty}a_n$ converges but $\sum_{n=1}^{\infty}|a_n|$ diverges, then $\sum_{n=1}^{\infty}a_n$ *converges conditionally*

Definition: Let $\sum_{n=1}^{\infty}a_n$ be a series. A series $\sum_{n=1}^{\infty}b_n$ is called a rearrangement of the original series if there exists $f: \mathbb{N} \hookrightarrow\!\!\!\!\!\rightarrow \mathbb{N}$ such that $b_{f(n)} = a_n$ for all $n \in \mathbb{N}$.

Note: the bijectivity means that every term eventually appears and there are no repetitions.

Theorem: If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then every rearrangement of the series converges to the same limit.

Proof: Omitted

Chapter 3

Basic Topology on $\mathbb{R}$

March 05:

Recall: an ε-*neighborhood* of a point $x \in \mathbb{R}$ is the set

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$$

Definition: A set $O \subseteq \mathbb{R}$ is *open* if for all points $a \in O$, there exists an ε-neighborhood of a such that $V_\varepsilon(a) \subseteq O$.

Examples:

- $\mathbb{R}$ is open
- $\emptyset$ is open
- $(c, d) = \{x \in \mathbb{R} : c < x < d\}$ is open (*Proof:* Let $x \in (c, d)$. Then $V_{\min\{x-c,d-x\}}(x) \subseteq (c, d)$)

Theorem:

1. The union of an arbitrary collection of open sets is open
2. The intersection of a finite collection of open sets is open

Proof:

1. Let $\{O_\lambda : \lambda \in \Lambda\}$ be a collection of open sets.

Let $O = \bigcup_{\lambda \in \Lambda} O_\lambda$. We need an ε-neighborhood of an arbitrary $a \in O$ to be completely contained in O.

Notice that $a \in O \implies a \in O_{\lambda'}$ for some $\lambda' \in \Lambda$. Since $O_{\lambda'}$ is open, $\exists \varepsilon > 0$ such that $V_\varepsilon(a) \subseteq O_{\lambda'} \subseteq O$.

2. Let $\{O_1, O_2, \dots, O_n\}$ be a finite collection of open sets. Denote $O = \bigcap_{k=1}^n O_k$. We need to show that O is open.

   Let $a \in O$. Then $a \in O_k$ for all $k = 1, 2, \dots, n$. Since O_k is open, $\exists \varepsilon_k > 0$ such that $V_{\varepsilon_k}(a) \subseteq O_k$ for all k.

   Now, we have different ε-neighborhoods in each O_k. We want an ε-neighborhood which is contained in *every* O_k.

   Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$. Then $V_\varepsilon(a) \subseteq O_k$ for all $k = 1, 2, \dots, n$. Therefore, $V_\varepsilon(a) \subseteq \bigcap_{k=1}^n O_k$. ■

Definition: A point x is a *limit point* (cluster point/accumulation point) of a set A if every ε-neighborhood of x intersects A at some point other than x.

Theorem: A point x is a limit point of a set A iff there exists a sequence (a_n) in A such that $(a_n) \to x$ and $a_n \neq x$ for all $n \in \mathbb{N}$

Proof:

Assume x is a limit point of A. We need a sequence (a_n) in A such that $(a_n) \to x$. By definition, every ε-neighborhood of x intersects A at some point other than x. Pick $\varepsilon = \frac{1}{n}$. Then for all $n \in \mathbb{N}$, pick

$$a_n \in V_{1/n}(x) \cap A, \quad a_n \neq x$$

Now we want $(a_n) \to x$. Given $\varepsilon > 0$ choose N such that $\frac{1}{N} < \varepsilon$ so $|a_n - x| < \varepsilon$ for all $n \in N$

Now, suppose there exists a sequence (a_n) in A such that $(a_n) \to x$ and $a_n \neq x$ for all $n \in \mathbb{N}$. We need to show that x is a limit point of A.

Let $V_\varepsilon(x)$ be an arbitrary ε-neighborhood. By definition of convergence, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - x| < \varepsilon$. Then $a_n \in V_\varepsilon(x)$ for all $n \geq N$. ■

Definition: A point $a \in A$ is an isolated point of A if it is *not* a limit point of A

Note: An isolated point is *always* a point in the set. A limit point does not necessarily belong to the set.

Definition: a set $F \subseteq \mathbb{R}$ is closed if it contains its limit points.

Theorem: A set $F \subseteq \mathbb{R}$ is closed iff every Cauchy sequence contained in F has a limit in F

Proof: HW

Example: Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Show each point in A is isolated.

Given $\frac{1}{n} \in A$, choose $\varepsilon = \frac{1}{n} - \frac{1}{n+1}$. Therefore, $V_\varepsilon(\frac{1}{n}) \cap A = \{\frac{1}{n}\}$ so $\frac{1}{n}$ is an isolated point and not a limit point.

Further, the limit of A is 0. Therefore, $\forall \varepsilon > 0$, $V_\varepsilon(0)$ contains points in A. Since $0 \notin A$, A is not closed.

However, we can create a closed set $F = A \cup \{0\}$. This is the *closure* of A.

Example: Show $[c, d] = \{x \in \mathbb{R} : c \leq x \leq d\}$ is closed.

If x is a limit point, then $\exists (x_n) \in [c, d]$ with $(x_n) \to x$. We want to show that $x \in [c, d]$. Since $c \leq x_n \leq d$, by the Order Limit Theorem,

$$c \leq \lim x_n \leq d \implies \lim x_n \in [c, d] \implies x \in [c, d]$$

so the set is closed.

Example: $\mathbb{Q} \subseteq \mathbb{R}$. The set of all limit point in $\mathbb{Q}$ is $\mathbb{R}$.

Proof: Let $y \in \mathbb{R}$. Consider any neighborhood $V_\varepsilon(y) = (y - \varepsilon, y + \varepsilon)$. From the density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists r \neq y$ such that $y - \varepsilon < r < y + \varepsilon$. Therefore, $r \in V_\varepsilon(y)$ so y is a limit point of $\mathbb{Q}$.

Lecture 1 - March 7:

Definition: given a set $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. The *closure* of A is the set $\overline{A} = A \cup L$.

Example:

- $\overline{\mathbb{Q}} = \mathbb{R}$
- $A = (a, b) \implies \overline{A} = [a, b]$

- If A is closed, $\overline{A} = A$

Theorem: For any $A \subseteq \mathbb{R}$, the closure $\overline{A}$ is a closed set and it is the smallest closed set containing A

Proof: Let L be the set of limit points of A. Then $\overline{A} = A \cup L$ is closed (it contains all its limit points, obviously). Any closed set containing A must contain L. Therefore $\overline{A}$ is the smallest closed set containing A. ■

Complement: Recall that $A^c = \{x \in \mathbb{R} : x \notin A\}$

Theorem:

1. A set O is open $\iff$ O^c is closed
2. A set F is closed $\iff$ F^c is open

Proof:

1. Let $O \subseteq \mathbb{R}$ be open. We want to show O^c is closed. By definition, if x is a limit point of O^c, then every ε-neighborhood of x contains some point of O^c. Thus, any ε-neighborhood of x cannot be a subset of O so $x \notin O$. Since $x \in O^c$, O^c is closed.

   Now assume O^c is closed. We want to show that O is open, i.e. for any $x \in O$, $\exists V_\varepsilon(x) \subseteq O$. By definition, O^c is closed so x is not a limit point of O^c. Therefore, $\exists V_\varepsilon(x)$ which does not intersect O^c. Then $V_\varepsilon(x) \subseteq O$.
2. $(E^c)^c = E$. The rest of the proof follows from 1).

Theorem:

1. The union of a finite collection of closed sets is closed
2. The intersection of an arbitary collection of closed sets is closed

Proof: Follows from previous theorem and de Morgan's laws:

$$\left(\bigcup_{\lambda \in \Lambda} E_\lambda\right)^c = \bigcap_{\lambda \in \Lambda} E_\lambda^c, \quad \left(\bigcap_{\lambda \in \Lambda} E_\lambda\right)^c = \bigcup_{\lambda \in \Lambda} E_\lambda^c$$

Compact Sets

Motivation: Bring "finite" quality to infinite arguments.

Definition: A set $K \subseteq \mathbb{R}$ is *compact* if every sequence in K has a convergent subsequence whose limit is in K.

Example: $[c, d]$ is compact. *Proof:* if $(a_n) \in [c, d]$, then it is bounded so by Bolzano-Weierstrass, $\exists (a_{n_k})$ which converges to a. Further $a \in [c, d]$ since $[c, d]$ is closed.

Definition: A set $A \subseteq \mathbb{R}$ is bounded if $\exists M > 0$ such that $|a| < M$ for all $a \in A$.

Theorem (Characterization of compactness in $\mathbb{R}$): A set $K \subseteq \mathbb{R}$ is compact iff it is closed and bounded

Proof: Assume K is compact. Suppose K is not bounded. Since K is not bounded:

$$\forall n \in \mathbb{N}: \quad \exists x_n \in K, \text{ s.t. } |x_n| > n$$

Since K is compact, (x_n) should have a convergent subsequence. However, (x_n) is unbounded so (x_{n_k}) is unbounded. Therefore, there is no convergent subsequence in (x_n). This is a contradiction of compactness so K is bounded.

Now we want to show K is closed. Let $x = \lim x_n$ with $(x_n) \in K$. It suffices to show $x \in K$. By definition, K is compact so (x_n) has a convergent subsequence (x_{n_k}) which converges to x and lies in K. $(x_{n_k}) \to x \implies x \in K \implies$ K is closed.

It remains to prove that K is compact if it is closed and bounded. This is left for HW.

Theorem (Nested Compact Set Property): If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ is a nested sequence of nonempty compact sets, then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$

Proof: Use compactness of K_n to produce a sequence that belongs to each set. $\forall n \in \mathbb{N}$, pick $x_n \in K_n$. Therefore, $(x_n) \in K_1 \implies \exists (x_{n_k}) \in K_1$ with $\lim x_{n_k} = x \in K_1$.

Given an $n_0 \in \mathbb{N}$, the terms of (x_n) are contained in K_{n_0} as long as $n > n_0$. We now ignore the finite number of terms for which $n_k < n_0$. Therefore,

$(x_{n_k}) \in K_{n_0}$ so $\lim x_{n_k} = x \in K_{n_0}$. Since n_0 was arbitrary,

$$x \in \bigcap_{n=1}^{\infty} K_n$$

March 12:

Definition: Let $A \subseteq \mathbb{R}$. An *open cover* of A is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ such that

$$A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$$

Given an open cover for A, a *finite subcover* is a finite collection of open sets from the original open cover, whose union still contains A

Example: Find an open cover for $(0, 1)$.

$\forall x \in (0, 1)$, let O_x be the open interval $(\frac{x}{2}, 1)$ so we have the infinite collection

$$\{O_x : x \in (0, 1) \text{ covering } (0, 1)\}$$

However, it is impossible to find a finite subcover for $(0, 1)$ using this open cover: Construct $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$ and set $x' = \min\{x_1, \ldots, x_n\}$. But then any $y \in \mathbb{R}$ with $0 < y \leq \frac{x'}{2}$ is not in $\bigcup_{i=1}^{n} O_{x_i}$

Example: Find an open cover for $[0, 1]$.

Naturally, we can use the same open cover as $(0, 1)$. However, this does not include the endpoints. Now let $\varepsilon > 0$ and define $O_0 = \{-\varepsilon, \varepsilon\}, O_1 = (1 - \varepsilon, 1 + \varepsilon)$. Then

$$\{O_0, O_1, O_x : x \in (0, 1)\}$$

is an open cover if $[0, 1]$.

To find a finite subcover, choose x' such that $\frac{x'}{2} < \varepsilon$:

$$\{O_0, O_1, O_{x'}\}$$

Theorem (Heine-Borel): For $K \subseteq \mathbb{R}$, then the following are equivalent:

(i) K is compact

(ii) K is closed and bounded

(iii) Every open cover of K has a finite subcover

Proof: (i) $\iff$ (ii) follows from the Characterization of compactness in $\mathbb{R}$.

It suffices to show (ii) $\iff$ (iii):

Assume that every open cover of K has a finite subcover. We want to show that K is closed and bounded. Let $O_x = \{|x-a| < 1 : a \in \mathbb{R}\} = V_1(x)$. Since $\{O_x : x \in K\}$ must have finite subcover, $\exists x_1, x_2, \ldots, x_n \in K$ such that $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$ is a finite subcover of K.

Since K is contained in a finite collection of sets, it is bounded.

To show K is closed, let (y_n) be a Cauchy sequence is K with $(y_n) \to y$. Suppose $y \notin K$, i.e. $\forall x \in K$, x lies some positive distance away from y.

Construct an open cover by taking O_x to be the interval of radius $\frac{|x-y|}{2}$ around $x \in K$. By (iii), we have a finite subcover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$.

Let $\varepsilon_0 = \min\left\{\frac{|x_i-y|}{2} : 1 \leq i \leq n\right\}$. Since $(y_n) \to y$, $\exists y_N$ such that $|y_N - y| < \varepsilon_0$.

This means that y_N must be excluded from each O_x so certainly, $y \notin \bigcup_{i=1}^{n} O_{x_i}$. Therefore, this finite collection cannot be a subcover since it does not contain all of K. This is a contradiction so K contains every limit point, and therefore K is closed.

The other direction, (ii) $\implies$ (iii), is left for homework. ■

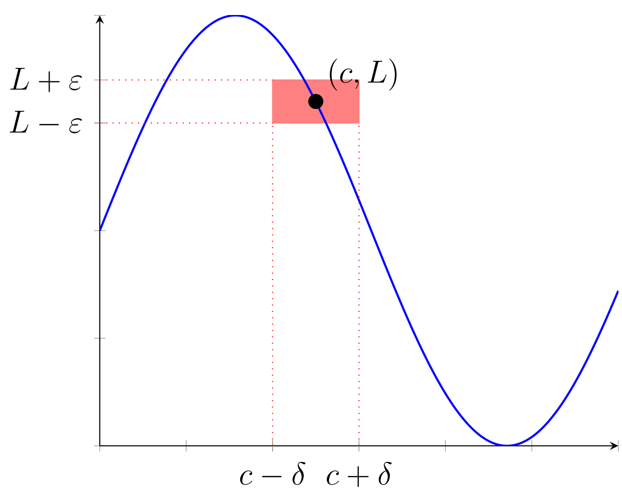
Chapter 4

Functional Limits and Continuity

March 12 (Continued)

Definition (Functional limit): Let $f : A \to \mathbb{R}$ be a function and let c be a limit point of the domain A. We say $\lim f(x) = L$ if $x \to c$.

Then $\forall \varepsilon > 0$, $\exists \delta > 0$, such that whenever $0 < |x - c| < \delta$ and $x \in A$, we have $|f(x) - L| < \varepsilon$.



Topological Definition: Let c be a limit point in A of $f : A \to \mathbb{R}$. We say that

$$\lim_{x \to c} f(x) = L$$

if $\forall V_\varepsilon(L)$, there exists $V_\delta(c)$ such that $\forall x \in V_\delta(c),\ f(x) \in V_\varepsilon(L)$

Example: Show $\lim_{x\to 2} f(x) = 7$ with $f(x) = 3x + 1$.

Let $\varepsilon > 0$. We need to produce a $\delta > 0$ such that $0 < |x-2| < \delta$ implies $|f(x) - 7| < \varepsilon$.

$$|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3\,|x - 2|$$

Choose $\delta = \frac{\varepsilon}{3}$ so $0 < |x - 2| < \delta \implies |f(x) - 7| < 3\delta = \varepsilon$.

Example: Show $\lim_{x\to 2} g(x) = 4,\ g(x) = x^2$.

Let $\varepsilon > 0$. We want $|g(x) - 4| < \varepsilon$ by restricting $|x - 2| < \delta$.

Notice

$$|g(x) - 4| = \left|x^2 - 4\right| = |x + 2|\,|x - 2|$$

So we construct a δ-neighborhood around $c = 2$ with radius no bigger than $\delta = 1$:

$$|x + 2| \le |3 + 2| = 5$$

Choose $\delta = \min\{1, \frac{\varepsilon}{5}\}$. Then when $0 < |x - 2| < \delta$, we have

$$|g(x) - 4| < \varepsilon$$

Lecture 1 - March 19:

Theorem (Sequential Criterion for Functional Limits): Given $f : A \to \mathbb{R}$ and c is a limit point of A, then the following are equivalent:

1. $\lim_{x\to c} f(x) = L$
2. For every sequence (x_n) in A with $(x_n) \to c$ and $x_n \neq c$, we have $f(x_n) \to L$

Proof:

Assume $\lim_{x\to c} f(x) = L$. Let (x_n) be a sequence in A with $(x_n) \to c$ and $x_n \neq c$. We want to show that $\forall \varepsilon > 0$, $\exists V_\delta(c)$ such that $\forall x \in V_\delta(c),\ f(x) \in V_\varepsilon(L)$.

We assume that $(x_n) \to c \implies \exists N \in \mathbb{N}$ such that $x_n \in V_{delta}(c)$ for all $n \ge N$. Then for all $n \ge N$, $f(x_n) \in V_\varepsilon(L)$.

For the other direction, we argue the contrapositive statement:

$$\lim_{x \to c} f(x) \neq L \implies \exists \varepsilon_0 \text{ s.t. } \forall \delta > 0, \exists x \in V_\delta(c) \text{ s.t. } f(x) \notin V_{\varepsilon_0}(L)$$

Consider $\delta_n = \frac{1}{n}$. Then $\exists x_n \in V_{\delta_n}(c)$ such that $f(x_n) \notin V_{\varepsilon_0}(L)$. Then $(x_n) \to c$ but $f(x_n) \nrightarrow L$. ■

Corollary (ALT for Functional Limits): Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$ and assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then

1. $\lim_{x\to c} kf(x) = kL$ for any $k \in \mathbb{R}$
2. $\lim_{x\to c}(f(x) + g(x)) = L + M$
3. $\lim_{x\to c}(f(x)g(x)) = LM$
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, provided $M \neq 0$

Proof: Direct consequence of ALT for sequences and Sequential Criterion for Functional Limits.

Corollary (Divergence Criterion): If $f : A \to \mathbb{R}$ with c a limit point of f, if $\exists (x_n) \to c$ and $(y_n) \to c \in A$ but $\lim_{x_n\to c} f(x_n) \neq \lim_{y_n\to c} f(y_n)$, then $\lim_{x\to c} f(x)$ does not exist.

Proof: Omitted

Example: To show $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist, consider the sequences $(x_n) = \frac{1}{2n\pi}$ and $(y_n) = \frac{1}{2n\pi + \frac{\pi}{2}}$.

Clearly, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. However, $\sin(\frac{1}{x_n}) = 0$ and $\sin(\frac{1}{y_n}) = 1$ so $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.

Definition: A function $f : A \to \mathbb{R}$ is *continuous* at a point $c \in A$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for $x \in V_\delta(c)$ (and $x \in A$), it follows that

$$|f(x) - f(c)| < \varepsilon$$

If f is continuous at every point in A, then we say f is *continuous* on A.

Theorem (Characterization of continuity): Let $f : A \to \mathbb{R}$ and $c \in A$. The following definition of continuity of f at c are equivalent:

1. $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$
2. $\forall V_\varepsilon(f(c)), \exists \delta > 0$ such that $x \in V_\delta(c) \implies f(x) \in V_\varepsilon(f(c))$
3. If for $x_n \in A$ we have $(x_n) \to c$, then $f(x_n) \to f(c)$

Proof:

Equivalently, if c is a limit point of A, then f is continuous at c if $\lim_{x \to c} f(x) = f(c)$.

Corollary (Criterion for discontinuity): $f : A \to \mathbb{R}$, $c \in A$ be a limit point of A. If $\exists (x_n) \in A$ with $(x_n) \to c$ but with $f(x_n) \nrightarrow f(c)$, then f is not continuous at c.

Proof: Direct from Characterization of continuity

Algebraic Continuity Theorem: Assume $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$ are continuous at $c \in A$. Then

1. $kf(x)$ is continuous at c for any $k \in \mathbb{R}$
2. $f(x) + g(x)$ is continuous at c for any continuous functions f and g
3. $f(x)g(x)$ is continuous at c
4. $\frac{f(x)}{g(x)}$ is continuous at c provided $g(c) \neq 0$

Proof: Direct from sequential criterion and sequences

Example: All polynomials (and in fact all rational functions) are continuous. Consider

$$g(x) = x \implies |g(x) - g(c)| = |x - c|$$

so $\forall \varepsilon > 0$, pick $\delta = \varepsilon$ and then $|x - c| < \delta \implies |g(x) - g(c)| < \varepsilon$ so $g(x) = x$ is continuous.

Now consider $f(x) = k$. Clearly, with $\varepsilon > 0$ and $\delta = 1$, k is continuous.

Notice the general form of a polynomial:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Each term is continuous by the Algebraic Continuity Theorem so the sum is continuous.

Example: Consider

$$g(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then we can estimate

$$|g(x) - g(0)| = \left| x\sin(\frac{1}{x}) - 0 \right| \leq x$$

since $|\sin x| \leq 1$.

Then, $\forall \varepsilon > 0$, set $\delta = \varepsilon$ so whenever $|x - 0| = |x| < \delta$,

$$|g(x) - g(0)| < \varepsilon \implies g \text{ is continuous at } 0$$

Lecture 2 - March 21:

Example: $f(x) = \sqrt{x}$ on $A = \{x \in \mathbb{R} : x \geq 0\}$. Show that $f(x)$ is continuous in A.

Let $\varepsilon > 0$. We need to show that

$$|f(x) - f(c)| < \varepsilon$$

for all x in some $V_\delta(c)$.

Case 1 ($c = 0$):

$$\left| f(x) - \sqrt{0} \right| = \sqrt{x} < \varepsilon \implies x < \varepsilon^2$$

Choosing $\delta = \varepsilon^2$, we have $|x - 0| < \delta \implies |f(x) - f(0)| < \varepsilon$.

Case 2 ($c > 0$):

$$|f(x) - f(c)| = \left|\sqrt{x} - \sqrt{c}\right| = \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \leq \frac{|x - c|}{\sqrt{c}}$$

Choose $\delta = \sqrt{c} \cdot \varepsilon$. Then

$$|x - c| < \delta \implies \left|\sqrt{x} - \sqrt{c}\right| < \frac{\varepsilon\sqrt{c}}{\sqrt{c}} = \varepsilon$$

Theorem (Composition of Continuous Functions): Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ with $f(A) \subseteq B$, if f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

Proof: HW

Continuous Functions on Compact Sets

Let $f : A \to \mathbb{R}$ with $B \subseteq A$. We say $f(B) = \{f(x) : x \in B\}$ is the *image* of B under f.

Question: does a continuous function map open sets to open sets?

No! $f(x) = x^2$ maps $(-1, 1)$ to $[0, 1)$ which is not open. In fact, this property is also not true for closed sets: $g(x)\frac{1}{1+x^2}$ maps $[0, \infty)$ to $(0, 1)$ which is not closed.

This leads to another natural question: is there any property which is preserved under continuous maps?

Theorem (Preservation of Compact Sets): Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact.

Proof: Let $(y_n) \in f(K)$. It suffices to find a subsequence (y_{n_k}) which converges to a limit contained in $f(K)$.

Note that $(y_n) \in f(K) \implies \forall n \in \mathbb{N}, \exists x_n \in K$ such that $f(x_n) = y_n$.

Since K is compact, $\exists (x_{n_k})$ such that $x_{n_k} \to x \in K$.

Since f is continuous on A, clearly f is continuous at x. Therefore,

$$(x_{n_k}) \to x \implies y_{n_k} = f(x_{n_k}) \to f(x) \in f(K)$$

Extreme Value Theorem: If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attain a maximum and minimum value, i.e. $\exists x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.

Proof: Since $f(K)$ is compact, we can set $\alpha = \sup f(K)$ and we know $\alpha \in f(K)$. Therefore, $\exists x_1 \in K$ such that $f(x_1) = \alpha$ and we call it the maximum.

The minimum follows by similar argument

Uniform Continuity

Example:

1. $f(x) = 3x + 1$ is continuous at $c \in \mathbb{R}$ so

$$|f(x) - f(c)| = |3x + 1 - 3c - 1| = 3|x - c|$$

Then, given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{3}$ so $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$.

2. $g(x) = x^2$. Given $c \in \mathbb{R}$,

$$|g(x) - g(c)| = \left|x^2 - c^2\right| = |x - c|\,|x + c|$$

we need and upper bound so choose $\delta \leq 1$ which bounds $x \in (c - 1, c1)$.

Then,

$$|x + c| \leq |x| + |c| \leq (|c| + 1) + |c| = 2|c| + 1$$

Let $\varepsilon > 0$, choose $\delta = \min\{1, \frac{\varepsilon}{2|c|+1}\}$. Finally,

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$$

Conclusion: In example 1, we chose an arbitrary c. In example 2, our value of δ depended on c.

Definition: A function $f : A \to \mathbb{R}$ is *uniformly continuous* on A if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x, y \in A$,

$$|x - y| < \delta \implies |f(x) - f(c)| < \varepsilon$$

Remark: Continuity requires only that for each c there exists at least one $\delta > 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$.

Uniform continuity requires that a single $\delta > 0$ works for all $c \in A$.

Seqential Criterion for Absence of Uniform Continuity: $f : A \to \mathbb{R}$ fails to be uniformly continuous iff $\exists \varepsilon_0 > 0$ and $(x_n), (y_n) \in A$ such that $\forall \delta > 0$,

$$|x - y| \to 0 \text{ but } |f(x) - f(y)| \geq \varepsilon_0$$

Proof: Suppose that f fails to be uniformly continuous. Then, by definition, $\exists \varepsilon_0 > 0$ such that $\forall \delta > 0$, $\exists x, y \in A$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$.

Then we can construct (x_n) and (y_n) by choosing $\delta_n = \frac{1}{n}$ $(n \in \mathbb{N})$ so that $\exists x_n, y_n$ with

$$|x_n - y_n| < \frac{1}{n} \text{ but } |f(x_n) - f(y_n)| \geq \varepsilon_0$$

For the other direction, suppose that $\exists \varepsilon_0 > 0$ and $(x_n), (y_n) \in A$ such that $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

Obviously, as $|f(x_n) - f(y_n)| \geq \varepsilon_0$, f fails to be uniformly continuous. ■

Lecture 3 - April 2:

Example of uniform continuity: We showed that $h(x) = \sin(\frac{1}{x})$ is continuous on $(0, 1)$. However, it is not uniformly continuous: Take $\varepsilon_0 = 2$ so

$$x_n = \frac{1}{\frac{\pi}{2} + 2n\pi}, \quad y_n = \frac{1}{\frac{3\pi}{2} + 2n\pi}$$

As $n \to \infty$, $(x_n) \to 0$ and $(y_n) \to 0$ so $|x_n - y_n| = 0$. However,

$$|h(x_n) - h(y_n)| = 2$$

so it is not uniformly continuous

Theorem (Uniform Continuity on Compact Sets): A function that is continuous on a compact set K is uniformly continuous on K.

Proof (by Contradiction): Assume $f : K \to \mathbb{R}$ is continuous on K. Suppose that f is not uniformly continuous on K.

Then by the Criterion for Absence, $\exists \varepsilon_0 > 0$ and $(x_n), (y_n) \in K$ such that $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

Since K is compact, (x_n) has a convergent subsequence (x_{n_k}) which converges to $x \in K$. We can then consider (y_{n_k}) consisting of the terms in y_n that correspond to the terms in (x_{n_k}). We know these exist since $|x_n - y_n| \to 0$.

By the ALT,

$$\lim(y_{n_k}) = \lim((y_{n_k} - x_{n_k}) + x_{n_k}) = 0 + x \implies y_{n_k} \to x$$

Since f is continuous at x, we have $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$. However, this implies

$$\lim(f(x_{n_k}) - f(y_{n_k})) = 0$$

but this contradicts the Criterion for Absence. Therefore, f is uniformly continuous on K. ■

Intermediate Value Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then $\exists c \in (a, b)$ such that $f(c) = L$.

Proof: HW

Chapter 5

Derivatives

April 2 (Continued):

Definition: Let $g : A \to \mathbb{R}$. Given $c \in A$, the *derivative* of g at c is given

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

provided the limit exists. In this case, we say g is *differentiable* at c.

If g is differentiable at every point in A, then we say g is differentiable on A.

Example:

- $f(x) = x^n$, $n \in \mathbb{N}$.

  For $c \in \mathbb{R}$, we have

  $$x^n - c^n = (x - c)(x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1})$$

  so

  $$\begin{aligned} f'(c) &= \lim_{x \to c} \frac{x^n - c^n}{x - c} \\ &= \lim_{x \to c} (x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1}) \\ &= c^{n-1} + c^{n-1} + \cdots + c^{n-1} \\ &= nc^{n-1} \end{aligned}$$

- $g(x) = |x|$.

  Attempting to calculate the derivative at $c = 0$, we have

  $$g'(0) = \lim_{x \to 0} \frac{|x|}{x} = \pm 1$$

  so the limit does not exists.

Theorem: If $g : A \to \mathbb{R}$ is differentiable at $c \in A$, then g is continuous at c.

Proof: Let g be differentiable at c. Then

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

exists.

It suffices to show $\lim_{x \to c} g(x) = g(c)$. By ALT,

$$\lim_{x \to c} g(x) - g(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \cdot (x - c) = g'(c) \cdot 0 = 0$$

so g is continuous at c. ■

Algebraic Differentiability Theorem: Let f, g be defined on an interval A and assume both are differentiable at some point $c \in A$. Then:

1. $(f + g)'(c) = f'(c) + g'(c)$
2. $(kf)'(c) = kf'(c) \quad \forall k \in \mathbb{R}$
3. $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$
4. $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g^2(c)}$ provided $g(c) \neq 0$

Proof: (1) and (2) come directly from the definition of the derivative and the ALT.

(3):

$$\begin{aligned}\frac{(fg)(x)-(fg)(c)}{x-c} &= \frac{f(x)g(c)-f(x)g(c)+f(x)g(c)-f(c)g(c)}{x-c} \\ &= f(x)\frac{g(x)-g(c)}{x-c}+g(c)\frac{f(x)-f(c)}{x-c}\end{aligned}$$

Since f is differentiable at c, it is continuous, i.e. $\lim_{x\to c} f(x) = f(c)$. Therefore, by the ALT,

$$\lim_{x\to c}\frac{(fg)(x)-(fg)(c)}{x-c} = f(c)g'(c)+f'(c)g(c)$$

(4): Similar

Theorem (Chain Rule): Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ with $f(A) \subseteq B$. If f is differentiable at $c \in A$ and g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c and

$$(g\circ f)'(c) = g'(f(c)) \cdot f'(c)$$

Proof: Since g is differentiable at $f(c)$,

$$g'(f(c)) = \lim_{y\to f(c)}\frac{g(y)-g(f(c))}{y-f(c)}$$

We can write

$$\begin{aligned} d(y) &= \frac{g(y)-g(f(c))}{y-f(c)} \\ &\implies \lim_{y\to f(c)} d(y) = g'(f(c)) \\ &\implies g(y)-g(f(c)) = d(y)(y-f(c))\end{aligned}$$

This new equation is defined $\forall y \in B$, including $g(c)$. Make the substitution $y = f(t)$ for some $t \in A$.

If $t \neq c$,

$$\frac{g(f(t))-g(f(c))}{t-c} = d(f(t))\frac{f(t)-f(c)}{t-c}$$

Taking the limit as $t \to c$ and applying the ALT gives the result. ■

Interior Extremum Theorem: Let f be differentiable on an open interval (a, b). If f attains a max value at some point $c \in (a, b)$ (i.e. $f(c) \geq f(x)$ forall $x \in (a, b)$), then $f'(c) = 0$.

Proof: Since $c \in (a, b)$, construct $(x_n), (y_n)$ such that $(x_n) \to c$, $(y_n) \to c$, and $x_n < c < y_n$ forall $n \in \mathbb{N}$.

Since $f(c)$ is a max, $f(y_n) - f(c) \leq 0$ $\forall n \in \mathbb{N}$. So (by the Order Limit Theorem),

$$f;(c) = \lim_{n\to\infty} \frac{f(y_n) - f(c)}{y_n - c} \leq 0$$

Similarly,

$$\frac{f(x_n) - f(c)}{x_n - c} \geq 0 \implies f'(c) = \lim_{n\to\infty} \frac{f(x) - f(c)}{x - c} \geq 0$$

Therefore, $f'(c) = 0$. ■

Darboux's Theorem: If f is differentiable on an interval $[a, b]$ and if α satisfies $f'(a) < \alpha < f'(b)$ then $\exists c \in (a, b)$ where $f'(c) = \alpha$.

Proof: HW

The Mean Value Theorem

Rolle's Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then $\exists c \in (a, b)$ such that $f'(c) = 0$.

Proof: Since f is continuous on a compact set, f attains extrema on that set.

If both the max and min occur at the endpoints, then f is constant since $f(a) = f(b)$. Trivially, $f'(c) = 0$ for all $c \in [a, b]$.

If either the max or the min occur some point $c \in (a, b)$, then by the Interior Extremum Theorem, $f'(c) = 0$.

The Mean Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof: Notice that Rolle's Theorem is a special case of the MVT when $f(a) = f(b)$. We seek to reduce the general case to the special case.

The equation of the line through the points $(a, f(a))$ and $(b, f(b))$ is given by:

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

Consider the difference between this line and the function $f(x)$:

$$d(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right]$$

Clearly, $d(x)$ is continuous on $[a, b]$. Further, it is differentiable on (a, b) and $d(a) = d(b) = 0$. Therefore, we can apply Rolle's Theorem.

This gives that $\exists c \in (a, b)$ where $d'(c) = 0$. Since

$$d'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \implies 0 = f'(c) - \frac{f(b) - f(a)}{b - a} \quad \blacksquare$$

Corollary: If $g : A \to \mathbb{R}$ is differentiable on an interval A and satisfies $g'(x) = 0$ for all $x \in A$, then $g(x) = k$ with $k \in \mathbb{R}$.

Proof: Take $x, y \in A$ and assume $x < y$. By MVT (applied to g on $[x, y]$), $\exists c \in (x, y)$ such that

$$g'(c) = \frac{g(y) - g(x)}{y - x} = 0 \implies g(y) = g(x) = k$$

Since x, y were arbitrary, $g(x) = k \quad \forall x \in A$. $\blacksquare$

Corollary: If f and g are differentiable functions on an interval A and satisfy $f'(x) = g'(x)\ \forall x \in A$, then $f(x) = g(x) + k$ for $k \in \mathbb{R}$

Proof: Let $h(x) = f(x) - g(x)$. $h'(x) = 0$ so by the previous corollary, $h(x) = k$. ■

Theorem (Generalized MVT): If f and g are continuous on the closed interval $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ where

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

If $g \neq 0$ on (a, b),

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof: Follows from applying MVT to $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$

Lecture 1 - April 9:

Theorem (L'Hopital's Rule - 0 case): Let f, g be continuous on an interval containing a and assume f, g are differentiable on this interval. If $f(a) = g(a) = 0$, and $g'(x) \neq 0$ for all $x \neq a$ then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to a} \frac{f(x)}{g(x)} = L$$

Proof: Direct application of Generalized Mean Value Theorem

Definition: Give $g : A \to \mathbb{R}$ and a limit point $c \in A$, we say that

$$\lim_{x \to c} g(x) = \infty$$

if $\forall M > 0$, $\exists \delta > 0$ such that

$$0 < |x - c| < \delta \implies g(x) \geq M$$

Theorem (L'Hopital's Rule - ∞ case): Assume f and g are differentiable on (a, b) and that $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x\to a} \frac{f(x)}{g(x)} = L$$

Proof: Omitted (again largely by Generalized MVT)

Chapter 6

Sequences and Series of Functions

April 9 (Continued):

Pointwise Convergence

Definition: For each $n \in \mathbb{N}$, let f_n be defined on a set $A \subseteq \mathbb{R}$. The seuqnece (f_n) of functions *converges pointwise of* A to a function f if $\forall x \in A$, the sequence of real numbers $f_n(x)$ converges to $f(x)$

Notation: to designate pointwise convergence, we can write

- $f_n \to f$
- $\lim f_n = f$
- $\lim_{n\to\infty} f_n(x) = f(x)$

Example: Consider $f_n(x) = \frac{x^2+nx}{n}$ on $\mathbb{R}$.

We compute

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2+nx}{n} = \lim_{n\to\infty} \frac{x^2}{n} + x = x$$

Example: Let $g_n(x) = x^n$ on $[0, 1]$

If $0 \leq x < 1$, then $x^n \to 0$ as $n \to \infty$. For $x = 1$, $x^n \to 1$ as $n \to \infty$.

Therefore, $g_n \to g$ pointwise on $[0, 1]$ where

$$g(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

Remark: This is a problem! A continuous sequence might not have a continuous limit. We will see more of this later.

Example: $h_n(x) = x^{1\frac{1}{2n-1}}$ on $[-1, 1]$.

For a fixed $x \in [-1, 1]$, we have

$$\lim_{n\to\infty} h_n(x) = x \cdot x^{\frac{1}{2n-1}} = x \lim_{n\to\infty} x^{\frac{1}{2n-1}} = |x|$$

since $x^{\frac{1}{2n-1}} \to 1$ if $x > 0$ and $\to -1$ if $x < 0$.

Lemma (Failure Continuity of the Limit Function):

Proof: For f to be continuous: fix $c \in A$ with $\varepsilon > 0$. We need to find $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \delta$$

Notice

$$|f(x) - f(c)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|$$

Choose $N \in \mathbb{N}$ such that $|f_N(c) - f(c)| < \frac{\varepsilon}{3}$. By continuity of f_N,

$$|f_N(x) - f_N(c)| < \frac{\varepsilon}{3}$$

However, it might be true that N is not large enough to ensure $|f(x) - f_N(x)|$ converges.

Example: Let $g_n(x) = x^n$ on $[0, 1]$ as above. Notice

$$\left| g_n(\frac{1}{2}) - g(\frac{1}{2}) \right| < \frac{1}{3} \implies n \geq 2$$

but

$$\left| g_n(\frac{9}{10}) - g(\frac{9}{10}) \right| < \frac{1}{3} \implies n \geq 11$$

For any chosen n, there are values of x for which $|g_n(x) - g(x)|$ might not be small enough.

Uniform Convergence

Definition: Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$. Then f_n *converges uniformly* on A to a limit function f defined on A, if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ whenever $n \geq N$ and $x \in A$.

Example: $g_n = \frac{1}{n(1+x^2)}$

For any fixed $x \in \mathbb{R}$, $\lim_{n\to\infty} g_n(x) = 0 \implies g(x) = 0$ is the pointwise limit of (g_n).

Do we have uniform convergence?

Notice:

$$\frac{1}{1+x^2} \leq 1 \quad \forall x \in \mathbb{R} \implies |g_n(x) - g(x)| = \left|\frac{1}{n(1+x^2)} - 0\right| \leq \frac{1}{n}$$

Given $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon}$ so $n \geq N \implies |g_n(x) - g(x)| < \varepsilon$ for all $x \in \mathbb{R}$.

Since N *does not depend on* x, $g_n \to 0$ uniformly on $\mathbb{R}$.

Example: $f_n(x) = \frac{x^2+nx}{n} \xrightarrow{p.w.} f(x) = x$ but not uniformly!

$$abs f_n(x) f(x) = \left|\frac{x^2+nx}{n} - x\right| = \frac{x^2}{n}$$

so $|f_n(x) - f(x)| < \varepsilon$ requires $N > \frac{x^2}{\varepsilon}$ which depends on x.

Theorem (Cauchy Criterion for Uniform Convergence): A sequence of functions (f_n) defined on a set $A \subseteq \mathbb{R}$ converges uniformly on A iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \varepsilon$ whenever $n, m \geq N$ and $x \in A$.

Proof: HW

Theorem (Continuous Limit Theorem): Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f. If each f_n is continuous at $c \in A$, then f is continuous at c.

Proof: Fix $c \in A$ and let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$|f_N(x) - f(x)| < \frac{\varepsilon}{3}$$

Since f_N is continuous at c, $\exists \delta > 0$ for which

$$|f_N(x) - f_N(c)| < \frac{\varepsilon}{3}$$

wheneber $|x - c| < \delta$.

This implies:

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

Note: we only have the convergence of the first term because of uniform continuity

Lecture 1 - April 11:

Differentiable Limit Theorem: Let $f_n \to f$ pointwise on the closed interval $[a, b]$ and assume that each f_n is differentiable if (f_n') converges uniformly on $[a, b]$ to a function g. Then f is differentiable on $[a, b]$ and $f' = g$.

Proof: We want to show that $f'(c)$ exists and equals $g(c)$.

Since

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

it suffices to show that $\exists \delta > 0$ such that whenever $0 < |x - c| < \delta$.

$$\left|\frac{f(x) - f(c)}{x - c} - g(c)\right| < \varepsilon$$

By the triangle inequality, $\forall x \neq c$,

$$\begin{aligned}\left|\frac{f(x) - f(c)}{x - c} - g(c)\right| \leq \left|\frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| \\ + \left|\frac{f_n(x) - f_n'(c)}{x - c} - f_n'(c)\right| \\ + |f_n'(c) - g(c)|\end{aligned}$$

Start by choosing $N \in \mathbb{N}$ such that $|f_m'(c) - g(c)| < \frac{\varepsilon}{3}$ for all $m \geq N$.

By uniform convergence of (f_n'), $\exists N_2$ such that $\forall m, n \geq N_2$,

$$|f_m'(x) - f_n'(x)| < \frac{\varepsilon}{3} \quad \forall x \in [a, b]$$

Pick $N = \max\{N, N_2\}$. Since f_n is differentiable at c, $\exists \delta > 0$ such that

$$0 < |x - c| < \delta \implies \left|\frac{f_N(x) - f_N(c)}{x - c} - f_N'(c)\right| < \frac{\varepsilon}{3}$$

Now all that remains is to bound the first term. Fix x satisfying $0 < |x - c| < \delta$ and let $m \geq n$. We can apply the MVT to $f_m - f_n$ on $[c, x]$

By the MVT, $\exists \alpha \in [c, x]$ such that

$$\begin{aligned}f_m'(\alpha) - f_N'(\alpha) &= \frac{(f_m(x) - f_N(x)) - (f_m(c) - f_N(c))}{x - c} \\ &\implies \left|\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c}\right| < \frac{\varepsilon}{3}\end{aligned}$$

Since $f_m \to f$, take the limit as $m \to \infty$. By OLT,

$$\left|\frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c}\right| < \frac{\varepsilon}{3}$$

Therefore,

$$\begin{aligned}\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| &\leq \left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right| \\ &\quad + \left|\frac{f_n(x)-f_n'(c)}{x-c} - f_n'(c)\right| \\ &\quad + |f_n'(c) - g(c)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

Theorem: Let (f_n) be a sequence of differentiable functions defined on $[a, b]$ and assume that (f_n') converges uniformly on $[a, b]$. If $\exists x_0 \in [a, b]$ where $f_n(x)$ is convergent, then (f_n) converges uniformly on $[a, b]$.

Proof: HW

Theorem: Let (f_n) be a sequence of differentiable functions defined on $[a, b]$ and assume (f_n') converges uniformly to g on $[a, b]$. If $\exists x_0 \in [a, b]$ for which $(f_n(x_0))$ converges, then (f_n) converges uniformly and $f = \lim f_n$ is differentiable with $f' = g$.

Proof: Follows from previous two theorems

Series of Functions

Definition: For each $n \in \mathbb{N}$, let f_n and f be defined on a set $A \subseteq \mathbb{R}$. Then infinite series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + \dots$$

converges pointwise on A to $f(x)$ if the sequence $(S_k(x))$ of partial sums

$$S_k(x) = f_1(x) + \dots + f_k(x)$$

converges pointwise to $f(x)$.

Similarly, the series *converges uniformly* on A to $f(x)$ if the sequence of partial sums converges uniformly to $f(x)$.

Term-by-term Continuity Theorem: Let f_n be continuous functions defined on $A \subseteq \mathbb{R}$ and assume $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to f. Then f is continuous on A.

Proof: Apply Continuous Limit Theorem to (S_k)

Term-by-term Differentiability Theorem: Let f_n be a sequence of differentiable functions defined on $A \subseteq \mathbb{R}$ and assume that $\sum_{n=1}^{\infty} f_n'$ converges uniformly on A to g. If $\exists x_0 \in A$ where $\sum_{n=1}^{\infty} f_n(x)$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A to a function f that is differentiable on A with $f' = g$.

Proof: Apply Differentiable Limit Theorem to (S_k)

Theorem (Cauchy Criterion for Uniform Convergence of Series): The series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $A \subseteq \mathbb{R}$ iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \varepsilon$ when $n > m \geq N$ and $x \in A$.

Proof: Follows immediately

Corollary (Weierstrass M-test): For each $n \in \mathbb{N}$, let f_n be a function defined on $A \subseteq \mathbb{R}$ and $M_n > 0$ a real number satisfying

$$|f_n(x)| \leq M_n \quad \forall x \in A$$

If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A

Proof: HW

Lecture 2 - April 16:

Power Series

Definition: Functions expressed in the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

are called *power series*.

Theorem: If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \in \mathbb{R}$, then it converges absolutely for all x with $|x| < |x_0|$

Proof: If $\sum_{n=0}^{\infty} a_n x_0^n$ converges, then $(a_n x_0^n)$ converges to 0 and is bounded.

Since it is bounded, $\exists M > 0$ such that $|a_n x_0^n| < M$ for all $n \in \mathbb{N}$. If $|x| < |x_0|$, then

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M \left|\frac{x}{x_0}\right|^n$$

Since $\sum_{n=0}^{\infty} M \left|\frac{x}{x_0}\right|^n$ is a geometric series with $\left|\frac{x}{x_0}\right| < 1$, it converges.

By the Comparison Test, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

The set of points for which the power series converges is $\{0\}, \mathbb{R}$, or some interval around 0: $(-R, R), [-R, R), (-R, R], [-R, R]$ where R is the *radius of convergence*

Theorem: If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point x_0, then it converges uniformly on the closed interval $[-c, c]$ where $c = |x_0|$.

Proof: Weierstrass M-theorem.

Notice! If $g(x) = \sum_{n=0}^{\infty} a_n x^n$ converges conditionally at $x = R$, then it *is* possible for it to diverge when $x = -R$

Example: With $R = 1$

$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n}} = \sum_{n=1}^{\infty} \frac{1}{n}$$

Abel's Lemma: Let (b_n) satisfy $b_1 \geq b_2 \geq \cdots \geq 0$ and let $\sum_{n=0}^{\infty} a_n$ be a series for which the partial sums are bounded. Then, $\exists A$ such that

$$|a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \leq A \cdot b_1$$

Proof: Let $s_n = a_1 + a_2 + \cdots + a_n$. Recall Summation by Parts: with $s_0 = 0$,

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1})$$

Therefore,

$$\begin{aligned}
\left|\sum_{k=1}^{n} a_k b_k\right| &= \left|s_n b_{n+1} + \sum_{k=1}^{n} s_k(b_k - b_{k+1})\right| \\
&\leq \left|A b_{n+1} + \sum_{k=1}^{n} A(b_k - b_{k+1})\right| \\
&\leq A b_{n+1} + A\sum_{k=1}^{n}(b_k - b_{k+1}) \\
&= A b_{n+1} + A\left[(b_1 - b_2) + (b_2 - b_3) + \cdots + (b_n - b_{n+1})\right] \\
&= A b_{n+1} + A(b_1 - b_{n+1}) \\
&= A b_1
\end{aligned}$$

Abel's Theorem: Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series that converges at the point $x = R > 0$. Then $g(x)$ converges uniformly on $[0, R]$. A similar result holds if the series converges at $x = -R$.

Proof: To be able to apply Abel's Lemma, we write

$$g(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (a_n R^n)\left(\frac{x}{R}\right)^n$$

Let $\varepsilon > 0$. By Cauchy Criterion for uniform convergence of a series, if we can produce N such that for $n > m \geq N$,

$$\left|(a_{m+1}R^{m+1})\left(\frac{x}{R}\right)^{m+1} + \cdots + (a_n R^n)\left(\frac{x}{R}\right)^n\right| < \varepsilon$$

then we are done.

Since $\sum_{n=0}^{\infty} a_n R^n$ converges, by Cauchy Criterion for series,

$$\left|a_{m+1}R^{m+1} + \cdots + a_n R^n\right| < \frac{\varepsilon}{2}$$

when $n > m \geq N$.

Notice that $\left(\frac{x}{R}\right)^{m+j}$ is monotone decreasing so we can apply Abel's Lemma to get

$$\left|(a_{m+1}R^{m+1})\left(\frac{x}{R}\right)^{m+1} + \cdots + (a_n R^n)\left(\frac{x}{R}\right)^n\right| \leq \frac{\varepsilon}{2}\left(\frac{x}{R}\right)^{m+1} < \varepsilon$$

Theorem: If a power series converges pointwise on a set $A \subseteq \mathbb{R}$, then it converges uniformly on any compact set $K \subseteq A$.

Proof: A compact set contains a max x_1 and a min x_0. By Abel's theorem, the series converges uniformly on $[x_0, x_1]$ and thus also on k.

Corollary: A power series is continuous at every point at which it converges

Theorem: If $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall x \in (-R, R)$ then the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges a $x \in (-R, R)$.

Proof: HW

Corollary: The convergence is uniform on compact sets contained in $(-R, R)$

Lecture 3 - April 18:

Theorem: Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $A \subseteq \mathbb{R}$. The function f is continuous on A and differentiable on any open interval $(-R, R) \subseteq A$. The derivative is given by

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Moreover, f is infinitely differentiable on $(-R, R)$ with successive derivatives obtained term-by-term differentiation.

Proof: Continuity follows from uniform convergence on compact sets (by pointwise convergence of f). Differentiability follows from term-by-term differentiation theorem.

The radius of convergence is constant because $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is also a power series with the same radius of convergence.

Infinite derivatives follow immediately by induction.

Taylor Series

Consider:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

for $|x| < 1$.

We can equally replace x with $-x^2$ and have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \dots$$

We recall from calculus that

$$\int \frac{1}{1+x^2}\, dx = \arctan(x)$$

so we can integrate term-by-term to get

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{6} + \dots$$

for $|x| < 1$.

Taylor's Formula: Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

be defined on some nontrivial interval centered at 0. Then

$$a_n = \frac{f^n(0)}{n!}$$

Proof: HW

Example: Suppose $\sin(x)$ has a Taylor series. Then it could be written using Taylor's formula as

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Converse: Assume f is infinitely times differentiable around 0. If $a_n = \frac{f^(n)(0)}{n!}$, does $\sum_{n=0}^{\infty} a_n x^n \to f(x)$?

Potentially no!

Let $S_n = \sum_{k=0}^{n} a_k x^k$. Does $S_n \to f$? Consider $E_N(x) = f(x) - S_N(x)$.

Lagrange's Remainder Theorem: Let f be $N+1$ times differentiable on $(-R, R)$. Define

$$a_n = \frac{f^{(n)}(0)}{n!} \quad n = 0, 1, 2, \ldots$$

and let $S_N(x) = a_0 + a_1 x + \cdots + a_N x^N$. Given $x \neq 0$ in $(-R, R)$, $\exists c$ such that $|c| < |x|$ where the error function $E_N(x)$ satisfies

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{n+1}$$

Proof: By assumption $f^{(n)} = S_n^{(n)}(0)$ for all n such that $0 \leq n \leq N$.

Clearly, $E_N(x) = f(x) - S_N(x)$ satisfies $E_N^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots, N$.

Assume WLOG $x > 0$. Apply the generalized MVT to $E_N(x)$ and x^{N+1} on $[0, x]$ so

$$\exists x_1 \in (0, x), \quad \frac{E_N(x)}{x^{N+1}} = \frac{E_N'(x_1)}{(N+1)x_1^N}$$

We can apply the MVT to $E'N(x)$ and $(N+1)x^N$ on $[0, x_1]$. Then, $\exists x_2 \in (0, x_2)$ such that

$$\frac{E_N(x)}{x^{N+1}} = \frac{E_N'(x_1)}{(N+1)x_1^N} = \frac{E_N''(x_2)}{(N+1)(N)x_2^{N-1}}$$

We may repeat this until we have $x_{N+1} \in (0, x_n) \subset \cdots \subset (0, x)$ with

$$\frac{E_N(x)}{x^{N+1}} = \frac{E_N^{(N+1)}(x_{N+1})}{(N+1)!}$$

Set $c = x_{N+1}$. Since $S_N^{(N+1)}(x) = 0$, we have

$$E_N^{(N+1)}(x) = f^{(N+1)}(x) \implies E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$$

Example: How well does $S_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ approximate $\sin(x)$ for $x \in [-2, 2]$?

By Lagrange's Remainder Theorem,

$$E_5(x) = \sin(x) - S_5(x) = -\frac{\sin(c)}{6!}x^6$$

for some $c \in (-|x|, |x|)$.

$|\sin(c)| \leq 1$ so for $x \in [-2, 2]$, $|E_5| \leq \frac{2^6}{6!} \approx 0.089$

To prove $S_N \to \sin(x)$ uniformly on $[-2, 2]$, using $\left|f^{(N+1)}(c)\right| \leq 1$, we get

$$|E_N(x)| = \left|\frac{f^{(N+1)(c)}}{(N+1)!}x^{N+1}\right| \leq \frac{2^{N+1}}{(N+1)!} \to 0$$

But! We can replace $[-2, 2]$ by $[-R, R]$ with R arbitrary. The Taylor Series thus converges to $\sin(x)$ on every $[-R, R]$.

Theorem: If f is defined in some neighborhood of $a \in \mathbb{R}$ and infinitely differentiable at a, then

$$\sum_{n=0}^{\infty} c_n(x-a), \quad c_n = \frac{f^{(n)}(a)}{n!}$$

Proof: By Lagrange's Remainder Theorem, $\exists c \in (a, x)$ such that

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

Chapter 7

Riemann Integral

Lecture 1 - April 23:

Throughout this chapter, we assume f is bounded on $[a, b]$: that is, there exists $M > 0$ such that $|f(x)| < M$ for all $x \in [a, b]$

Definition: A *partition* P of $[a, b]$ is a finite set of points from $[a, b]$ that includes a nad b:

$$P = \{x_0, x_1, \ldots, x_n\} \quad \text{such that} \quad a = x_0 < x_1 < \cdots < x_n = b$$

For each subinterval $[x_{k-1}, x_k]$ of P:

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$$
$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$$

The lower sum of f with respect to P is

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

(sum of areas of rectangles with height m_k and width $x_k - x_{k-1}$ which *underestimates* the value)

The upper sum of f with respect to P is

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

(sum of areas of rectangles which *overestimate* the value)

Definition: A partition Q is *refinement* of P if $P \subseteq Q$ (Q contains all points of P)

Lemma: If $P \subseteq Q$, then $L(f, P) \leq L(f, Q)$ and $u(f, P) \geq u(f, Q)$

Proof: Consider the refinement of P by considering $\{z\} \cup [x_{k-1}, x_k]$.

Now, our lower sum is

$$\begin{aligned} m_k(x_k - x_{k-1}) &= m_k(x_k - z) + m_k(z - x_{k-1}) \\ &\leq m_k'(x_k - z) + m_k''(z - x_{k-1}) \end{aligned}$$

where

$$\begin{aligned} m_k' &= \inf\{f(x) : x \in [z, x_k]\} \\ m_k'' &= \inf\{f(x) : x \in [x_{k-1}, z]\} \end{aligned}$$

By induction on k, we have $L(f, P) \leq L(f, Q)$

Similar argument shows the upper sum case.

Lemma: If P_1 and P_2e are any two partitions of $[a, b]$, then $L(f, P_1) \leq u(f, P_2)$

Proof: Let $Q = P_1 \cup P_2$ (the common refinement of P_1 and P_2)

Since $P_1 \subseteq Q$ and $P_2 \subseteq Q$,

$$L(f, P_1) \leq L(f, Q) \leq u(f, Q) \leq u(f, P_2)$$

by the previous lemma

Integrability

A function is integrable if the upper and lower sums "meet" as partitions get more refined.

Idea: instead of limits, use AoC and lim/sup

Definition: Let $\mathcal{P}$ be the collection of all possible partitions of $[a, b]$. The *upper* *integral* of f is

$$U(f) = \inf\{U(f,P) : P \in \mathcal{P}\}$$

the *lower integral* of f is

$$L(f) = \sup\{L(f,P) : P \in \mathcal{P}\}$$

Lemma: For any bounded function f on $[a,b]$ we always have $U(f) \geq L(f)$

Proof: HW

Definition (Riemann Integrability): A bounded function f defined on $[a,b]$ is Riemann-integrable if $U(f) = L(f)$. Then we write

$$\int_a^b f = U(f) = L(f)$$

Criterion for Integrability

To review:

$$\begin{aligned} \sup\{L(f,P), P \in \mathcal{P}\} &= L(f) \leq U(f) \\ &= \inf\{U(f,P), P \in \mathcal{P}\} \end{aligned}$$

and f is integrable if $L(f) = U(f)$

Theorem (Integrability Criterion): A bounded function f is interable on $[a,b]$ iff $\forall \varepsilon > 0$, $\exists P_\varepsilon$, a partition of a, b such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

Proof: ($\Longleftarrow$) Let $\varepsilon > 0$. If such a partition P_ε exists, then $U(f) - L(f) \leq U(f, P_\varepsilon) L(f, P_\varepsilon) < \varepsilon$

Since ε is arbitrary, $U(f) = L(f)$ so f is integrable.

($\Longrightarrow$) Since $U(f)$ is the greatest lower bound of the upper sums, given $\varepsilon > 0$, $\exists P_1$ such that

$$U(f, P_1) < U(f) + \frac{\varepsilon}{2}$$

and $\exists P_2$ such that

$$L(f, P_2) > L(f) - \frac{\varepsilon}{2}$$

Let $P_\varepsilon = P_1 \cup P_2$ be the common refinement of P_1 and P_2. Then...

Theorem: If f is continuous on $[a, b]$ then it is integrable.

Proof: Since f is continuous on a compact set, it is bounded, and uniformly continuous.

Therefore, given $\varepsilon > 0$, $\exists\delta$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b-a}$

Let P be a partition on $[a, b]$ where

$$\Delta x_k = x_k - x_{k-1} < \delta$$

Given a particular $[x_{k-1}, x_k]$, by the Extreme Value Theorem

$$\begin{aligned} \sup &\implies \exists M_k = f(z_k) \quad \text{for some} \quad z_k \in [x_{k-1}, x_k] \\ \inf &\implies \exists m_k = f(y_k) \quad \text{for some} \quad y_k \in [x_{k-1}, x_k] \end{aligned}$$

Therefore, $|z_k - y_k| < \delta$ so $M_k - m_k = f(z_k) - f(y_k) < \frac{\varepsilon}{b-a}$

Therefore,

$$\begin{aligned} U(f, P) - L(f, P) &:= \sum_{k=1}^{n} (M_k - m_k)\delta x_k \\ &< \frac{\varepsilon}{b-a} \sum_{k=1}^{n} \delta x_k \\ &= \frac{\varepsilon}{b-a}(b-a) = \varepsilon \end{aligned}$$

Therefore, f is integrable.

Lecture 2 - April 25:

Recall

- $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\},\ M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$
- $L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}),\ U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$
- Upper integral: $U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}$, Lower integral: $L(f) = \sup\{L(f, P) : P \in \mathcal{P}\}$
- Integrable: $U(f) = L(f) \iff \forall\varepsilon > 0, \exists P_\varepsilon$ such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$
- f continuous $\implies f$ integrable

Integrating functions with discontinuities

Example: $f(x) = \begin{cases} 1 & x \neq 1 \\ 0 & x = 1 \end{cases}$ on $[0, 2]$.

For *any* partition P of $[0, 2]$ we have $U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) < \sum_{k=1}^{n} 1(2) = 2$. Therefore, the lower sum is less than 2

We will proceed by constructing a partition that embeds $x = 1$ into a very small subinterval. Let $\varepsilon > 0$, consider

$$P_\varepsilon = \{0, 1 - \frac{\varepsilon}{3}, 1 + \frac{\varepsilon}{3}, 2\}$$

(we are tightly constraining our discontinuity)

Then

$$L(f, P_\varepsilon) = 1 \cdot (1 - \frac{\varepsilon}{3}) + 0 + 1 \cdot (1 - \frac{\varepsilon}{3}) = 2 - \frac{2\varepsilon}{3} < 2$$

Since $U(f, P_\varepsilon) = 2$,

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) = \frac{2}{3}\varepsilon < \varepsilon$$

Theorem: If $f : [a, b] \to \mathbb{R}$ is bounded and f is integrable on $[c, b]$ for all $c \in (a, b)$, then f is integrable on $[a, b]$

Proof: Let $\varepsilon > 0$. It sufficesd to produce a P such that $U(f, P) - L(f, P) < \varepsilon$.

For any partition, we have

$$
\begin{aligned}
U(f,P) - L(f,P) &= \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} (M_k - m_k)\Delta x_k \\
&= (M_1 - m_1)(x_1 - a) + \sum_{k=2}^{n} (M_k - m_k)\Delta x_k
\end{aligned}
$$

We want to choose x_1 close enough to a such that

$$(M_1 - m_1)(x_1 - a) < \frac{\varepsilon}{2}$$

Since f is bounded, $\exists M > 0$ such that $|f(x)| < M$ for all $x \in [a,b]$. Now $M_1 - m_1 \leq 2M$, so we can pick x_1 such that so

$$x_1 - a < \frac{\varepsilon}{4M}$$

By assumption, f is integrable on $[x_1, b]$ so $\exists P_1$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2}$$

Let $P = \{a\} \cup P$. Then

$$U(f,P) - L(f,P) \leq 2M(x_1 - a) + U(f,P_1)L(f,P_1) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

Properties of the Integral

Theorem: Assume $f : [a,b] \to \mathbb{R}$ is bounded and let $c \in (a,b)$. Then, f is integrable on $[a,b]$ iff f is integrable on $[a,c]$ and $[c,b]$. In that case, we may write

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof: ($\Longrightarrow$) If f is integrable on $[a,b]$, then $\forall \varepsilon > 0$, $\exists P$ such that $U(f,P) - L(f,P) < \varepsilon$.

Let $P_1 = P \cap [a,c]$ (a partition of $[a,c]$) and $P_2 = P \cap [c,b]$ (a partition of $[c,b]$).

Then

$$U(f,P_1) - L(f,P_1) < \varepsilon \quad \text{and} \quad U(f,P_2) - L(f,P_2) < \varepsilon$$

Therefore, f is integrable on $[a,c]$ and $[c,b]$

($\Longleftarrow$) Assume f is integrable on $[a,c]$ and $[a,b]$. Then given $\varepsilon > 0$, $\exists P_1 \subset [a,c]$ and $P_2 \subset [c,b]$ such that

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2} \quad \text{and} \quad U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$. Then

$$U(f,P) - L(f,P) < \varepsilon$$

so f is integrable on $[a,b]$

Finally, we want to show that $\int_a^b f = \int_a^c f + \int_c^b f$.

We know

$$\begin{aligned}
\int_a^b \leq U(f,P) &< L(f,P) + \varepsilon \\
&= L(f,P_1) + L(f,P_2) + \varepsilon \\
&\leq \int_a^c f + \int_c^b f + \varepsilon \\
&\leq \int_a^c f + \int_c^b f
\end{aligned}$$

Now we show the other direction:

$$\begin{aligned}\int_a^c f + \int_c^b f &\le U(f,P_1) + U(f,P_2) \\ &< L(f,P_1) + L(f,P_2) + \varepsilon \\ &= L(f,P) + \varepsilon \\ &\le \int_a^c f + \varepsilon \\ &\le \int_a^b f\end{aligned}$$

Therefore, $\int_a^b f = \int_a^c f + \int_c^b f$

Theorem: Assume f, g are integrable on $[a,b]$. Then

(i) $f+g$ is integrable on $[a,b]$ with $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

(ii) kf is integrable on $[a,b]$ for any $k \in \mathbb{R}$ with $\int_a^b kf = k\int_a^b f$

(iii) If $m \le f(x) \le M$ on $[a,b]$, then $m(b-a) \le \int_a^b f \le M(b-a)$

(iv) $f(x) \le g(x)$ for all $x \in [a,b]$ implies $\int_a^b f \le \int_a^b g$

(v) $|f|$ is integrable and $\left|\int_a^b f\right| \le \int_a^b |f|$

Proof: f is integrable $\iff \exists (P_n)$ such that

$$\lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0 \qquad (\textit{Proof:}\ \text{HW})$$

Therefore

$$\int_a^b f = \lim U(f,P_n) = \lim L(f,P_n)$$

The rest of the results follow:

(i) HW

(ii) For $k \geq 0$, from properties of sup and inf

$$U(kf, P) = kU(f, P), \quad L(kf, P) = kL(f, P)$$

since f is integrable, $\exists (P_n)$ such that

$$\lim_{n\to\infty} [U(kf, P_n) - L(kf, P_n)] = \lim_{n\to\infty} k(U(f, P_n) - L(f, P_n)) = 0$$

For $k < 0$, similar except $U(kf, P) = kL(f, P)$ and $L(kf, P) = kU(f, P)$

(iii) Since $U(f, P) \geq L(f, P)$ for all P, pick $P = [a, b]$ so

$$M(b-a) \geq \int_a^b f \geq m(b-a)$$

(iv) Set $h = g - f$ and apply (i), (ii), (iii)

(v) HW

Defintion: If f is integrable on $[a, b]$, define

$$\int_a^b = -\int_b^a f$$

Moreover $\forall c \in [a, b]$: $\int_c^c f = 0$

Question: If $f_n \to f$, does $\int f_n = \int f$? No! Unless the convergence is uniform as we will see.

Interlude: Arzela-Ascoli Theorem

Recall the Bolzano-Weierstrass Theorem for sequences: a_n bounded implies $\exists a_{n_k}$ convergent.

Does this hold for functions, i.e. does $f_n(x)$ bounded imply $\exists f_{n_k}$ convergent?

Not quite! We can show $f_n(x)$ bounded on a countable set A implies $\exists f_{n_k}$ pointwise convergent on A (Exercise 6.2.13)

Definition: (f_n) is *equicontinuous* on $E \subset \mathbb{R}$ if $\varepsilon > 0$, $\exists \delta > 0$ such that $|x - y| < \delta \implies |f_n(x) - f_n(y)| < \varepsilon$ for all $n \in \mathbb{N}$

Theorem: If $f_n(x)$ is bounded and equicontinuous then $\exists f_{n_k}$ uniformly convergent on A.

Proof: HW

Remark: This is a much stronger result!

Lecture 3 - April 30:

Question: If $(f_n(x)) \to f(x)$ of integrable functions on $[a, b]$, do we have that $\int f_n \to \int f$?

Answer: When $(f_n(x)) \to f(x)$ uniformly, then yes!

Integrable Limit Theorem: Assume that $f_n \to f$ uniformly on $[a, b]$ and that each f_n is integrable on $[a, b]$. Then f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

Proof: Proof that f is integrable is HW.

From the properties of integrals,

$$\left| \int_a^b f_n - \int_a^b f \right| = \left| \int_a^b f_n - f \right| \leq \int_a^b |f_n - f|$$

But $f_n \to f$ uniformly so for all $\varepsilon > 0$, $\exists N$ such that $|f_n - f| < \frac{\varepsilon}{b-a}$ for all $n \geq N$ so

$$\left| \int_a^b f_n - \int_a^b f \right| \leq \int_a^b |f_n(x) - f(x)| < \int_a^b \frac{\varepsilon}{b-a} = \varepsilon \quad \blacksquare$$

Fundamental Theorem of Calculus:

(i) If $f : [a,b] \to \mathbb{R}$ is integrable and $F : [a,b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a,b]$, then

$$\int_a^b f = F(b) - F(a)$$

(ii) Let $g : [a,b] \to \mathbb{R}$ be integrable and define $G(x) = \int_a^x f$ for all $x \in [a,b]$. Then: G is continuous on $[a,b]$. If g is continuous at some point $c \in [a,b]$, then G is differentiable at c and $G'(c) = g(c)$

Proof:

(i) Let P be a partition of $[a,b]$ and apply the MVT to F on a subinterval $[x_{k-1}, x_k]$ of P to get $\exists t \in (x_{k-1}, x_k)$ such that

$$\begin{aligned} F(x_k) - F(x_{k-1}) &= F'(t_k)(x_k - x_{k-1}) \\ &= f(t_k)(x_k - x_{k-1}) \end{aligned}$$

Now, consider the upper and lower sums $U(f,P)$ and $L(f,P)$. Since

$$m_k = \inf_{t_k \in (x_{k-1}, x_k)} f(t_k) \leq f(t_k) \leq M_k = \sup_{t_k \in (x_{k-1}, x_k)} f(t_k)$$

we have

$$\begin{gathered} m_k \leq f(t_k) \leq M_k \\ \sum m_k(x_k - x_{k-1}) \leq \sum f(t_k)(x_k - x_{k-1}) \leq \sum M_k(x_k - x_{k-1}) \\ L(f,P) \leq \sum F(x_k) - F(x_{k-1}) \leq U(f,P) \\ L(f,P) \leq \sum_{k=1}^{n} [F(x_k) - F(x_{k-1})] \leq U(f,P) \end{gathered}$$

but this is a telescoping sum! so

$$L(f,P) \leq F(b) - F(a) \leq U(f,P)$$

but this is independent of P which gives

$$L(f) \leq F(b) - F(a) \leq U(f)$$

Since f is integrable, $U(f) = L(f)$ so $F(b) - F(a) = \int_a^b f$

(ii) Take $x > y$ in $[a, b]$. Observe:

$$|G(x) - G(y)| = \left|\int_a^x g - \int_a^y g\right| = \left|-\int_x^a g - \int_a^y g\right| = \left|\int_y^x g\right| \leq \int_y^x |g| \leq M(x-y)$$

where $M > 0$ is a bound on $|g|$.

Therefore, G is Lipschitz on $[a, b]$ so G is uniformly continuous on $[a, b]$.

Assume g is continuous at $c \in [a, b]$.

By definition,

$$G'(c) = \lim_{x \to c} \frac{G(x) - G(c)}{x - c} = \lim_{x \to c} \frac{1}{x - c}\left(\int_a^x g(t)\ dt - \int_a^c g(t)\ dt\right) = \lim_{x \to c} \frac{1}{x - c} \int_c^x g(t)\ dt$$

It suffices to show this limit equals $g(c)$.

Given $\varepsilon > 0$, we want a $\delta > 0$ such that $|x - c| < \delta$ implies

$$\left|\frac{1}{x - c} \int_c^x g(t)\ dt - g(c)\right| < \varepsilon$$

By continuity of g, $\exists \delta > 0$ such that $|x - c| < \delta$ implies $|g(x) - g(c)| < \varepsilon$.

We write

$$g(c) = \frac{1}{x - c} \int_c^x g(c)\ dt$$

Since $|x - c| \geq |t - c|$, we have $\forall\, |x - c| < \delta$,

$$\begin{aligned}
\left|\frac{1}{x - c} \int_c^x g(t)\ dt - g(c)\right| &= \left|\frac{1}{x - c} \int_c^x g(t)\ dt - \frac{1}{x - c} \int_c^x g(c)\ dt\right| \\
&= \left|\frac{1}{x - c} \int_c^x [g(t) - g(c)]\ dt\right| \\
&\leq \frac{1}{|x - c|} \int_c^x |g(t) - g(c)|\ dt \\
&< \frac{1}{x - c} \int_c^x \varepsilon\ dt = \varepsilon \quad \blacksquare
\end{aligned}$$