Math 1010 Midterm Review

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1 Class Notes

A **set** is a collection of objects.

De Morgan's Laws:

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

A **function** $f : A \to B$ assigns each $a \in A$ to a unique element $f(a) \in B$. A is the **domain** of f and B is the **codomain** of f. The **range** of f is the set of all possible outputs of f (a subset of B)

Properties of the absolute value:

- $|ab| = |a|\ |b|$
- Triangle inequality: $|a + b| \leq |a| + |b|$

Theorem: $a = b \iff \forall \varepsilon > 0, |a - b| < \varepsilon$

The Real Numbers: $\mathbb{R}$ is a field. $\exists P = \{x \in \mathbb{R} : x > 0\}$ such that the positive numbers are closed under addition and multiplication.

Completeness Axiom: If $A \subseteq \mathbb{R}$ such that $A \neq \emptyset$ and A is bounded above, then $\sup A$ (the least upper bound for A) exists, i.e. $\sup A \geq y$ for all $y \in A$ and if z is an upper bound for A, then $\sup A \leq z$.

For $A \subseteq \mathbb{R}$, $\inf A$ and $\sup A$ exist and are unique. If $\sup A \in A$, it is the **maximum**. If $\inf A \in A$, it is the **minimum**.

Lemma: $\forall \varepsilon > 0$, exists $a \in A$ such that $\sup A - \varepsilon < a$.

Theorem (Nested Interval Property): If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ is a sequence of closed intervals, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Archimedean Property: $\mathbb{N}$ is not bounded above.

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$): For all $a, b \in \mathbb{R}$, exists $r \in \mathbb{Q}$ such that $a < r < b$.

A function is **injective/one-to-one** if $a_1 \neq a_2$ in A implies $f(a_1) \neq f(a_2)$

A function is **surjective/onto** if $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$.

A function is **bijective** if it is both injective and surjective.

Two sets have the same cardinality ($A \sim B$) if there exists a bijection $f : A \to B$.

- $\mathbb{N} \sim \mathbb{Z}$
- $(a, b) \sim \mathbb{R}$

A set is **countable** if it has the same cardinality as $\mathbb{N}$

- $\mathbb{Q}$ is countable
- $\mathbb{R}$ is uncountable

A **sequence** is a function whose domain is $\mathbb{N}$

A sequence (a_n) **converges** if $\exists L \in \mathbb{R}$ such that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - L| < \varepsilon$. Alternatively, $(a_n) \to a$ if, given any ε-neighborhood of a, exists a point in the sequence after which all points are in the neighborhood.

The **ε-neighborhood** of $a \in \mathbb{R}$ (given $\varepsilon > 0$) is the set

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$$

A template for convergence proofs:

1. Let $\varepsilon > 0$
2. Choose $N \in \mathbb{N}$
3. Verify that $n \geq N$ implies $|a_n - L| < \varepsilon$

Theorem (Uniqueness of Limits): The limit of a sequence, if it exists, is unique.

Theorem: Every convergent sequence is **bounded**, i.e. $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Algebraic Limit Theorem: Let $(a_n) \to a$, $(b_n) \to b$, then

1. $\lim(ca_n) = ca$
2. $\lim(a_n + b_n) = a + b$
3. $\lim(a_n b_n) = ab$
4. $\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$ if $b \neq 0$

Order Limit Theorem: If $(a_n) \to a$ and $(b_n) \to b$, then

1. If $a_n \geq 0$, then $a \geq 0$
2. If $a_n \leq b_n$ for all n, then $a \leq b$
3. If $c \leq b_n$ for all n, then $c \leq b$

Monotone Convergence Theorem: If a sequence is bounded and monotone (either increasing or decreasing for all $n \in \mathbb{N}$), then it converges.

A **series** $\sum_{n=1}^{\infty} a_n$ converges if the sequence of partial sums $S_n = \sum_{n=1}^{m} a_n$ converges.

Cauchy Condensation Test: Suppose (b_n) is decreasing and $b_n \geq 0$.

$$\sum_{n=1}^{\infty} b_n = b \iff \sum_{n=1}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + \cdots = b$$

Corollary: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Let (a_n) be a sequence and let $n_1 < n_2 < n_3 < \ldots$ be a sequence of natural numbers. Then the sequence (a_{n_k}) is called a **subsequence** of (a_n). The order of terms in a subsequence is the same as in the original sequence and no repetitions are allowed.

Theorem: A subsequence of a convergent sequence converges to the same limit as the original sequence.

Corollary: If two convergent subsequences of a sequence have different limits, then the sequence does not converge.

Bolzano-Werierstrass Theorem: Every bounded sequence has a convergent subsequence.

A sequence is **Cauchy** if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for $n, m \geq N$,

$$|a_n - a_m| < m$$

Cauchy Criterion: A sequence converges if and only if it is Cauchy.

Corollary: Every Cauchy sequence (every convergent sequence) is bounded.

Algebraic Limit Theorem for Series: If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then

1. $\sum_{k=1}^{\infty} ca_k = cA$
2. $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$

Cauchy Criterion for Series: The series $\sum_{k=1}^{\infty} a_k$ converges iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for $m \geq n \geq N$,

$$\left| \sum_{k=m+1}^{n} a_k \right| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon$$

Theorem: If $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$

Series Comparson Test: If (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$, then

1. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges
2. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges

A series of the form $\sum_{k=0}^{\infty} ar^k$ is called a **geometric series**. It converges if $|r| < 1$ and diverges otherwise. In the case $|r| < 1$,

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

Absolute Convergence Test: If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Alternating Series Test: If (a_n) is a decreasing sequence such that $\lim a_n = 0$, then the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ **converges absolutely**. If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ **converges conditionally**.

Theorem: If $\sum_{n=1}^{\infty} a_n$ converges absolutely, every rearrangement of the series converges to the same sum.

A set $O \subseteq \mathbb{R}$ is **open** if $\forall x \in O$, $\exists \varepsilon > 0$ such that $V_\varepsilon(x) \subseteq O$.

- $\emptyset$ and $\mathbb{R}$ are open
- (c, d) is open
- The union of an arbitrary collection of open sets is open
- The intersection of a finite collection of open sets is open

A point x is a **limit point** of a set A if every ε-neighborhood of x contains a point in A other than x itself. If a point is not a limit point of A, it is an **isolated point** of A. An isolated point is always in the set, the limit point may or may not be in the set.

Theorem: A $x \in A$ is a limit point of $A \iff \exists (a_n) \in A$ such that $(a_n) \to x$ and $a_n \neq x$ for all $n \in \mathbb{N}$.

A set $F \subseteq \mathbb{R}$ is **closed** if it contains its limit points.

- $[c, d]$ is closed

Theorem: A set is closed iff every Cauchy sequence in the set converges to a point in the set.

The **closure** of a set $A \subseteq \mathbb{R}$ is given by $\bar{A} = A \cup L$ where L is the set of all limit points of A.

- $\bar{Q} = \mathbb{R}$
- $A = (a, b) \implies \bar{A} = [a, b]$
- If A is closed, $\bar{A} = A$

Theorem: For any $A \subseteq \mathbb{R}$, $\bar{A}$ is closed and is the smallest closed set containing A.

Theorem: O is open $\iff O^c$ is closed. F is closed $\iff F^c$ is open.

Theorem: The intersection of an arbitrary collection of closed sets is closed. The union of a finite collection of closed sets is closed.

A set $K \subseteq \mathbb{R}$ is **compact** if every sequence in K has a convergent subsequence whose limit is in K.

Characterization of Compactness in $\mathbb{R}$: A set $K \subseteq \mathbb{R}$ is compact $\iff K$ is closed and bounded.

Nested Compact Set Property: If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$ is a sequence of nonempty compact sets, then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

2 Homework Results

Let $A \subset \mathbb{R}$ be nonempty and bounded above. Let $c \in \mathbb{R}$ and define $cA = \{ca : a \in A\}$. If $c \geq 0$, $\sup(cA) = c \sup A$. If $c < 0$, $\sup(cA) = c \inf A$.

If a is an upper bound for A and $a \in A$, then $a = \sup A$.

If $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ then $a + t \in I$ and $at \in I$.

$\mathbb{I}$ is dense in $\mathbb{R}$.

Theorem: If $A_1, A_2, \dots, A_m$ are countable, then $\bigcup_{n=1}^{m} A_n$ is countable.

Lemma: If $(x_n) \to 0$, $\sqrt{x_n} \to 0$. If $(x_n) \to x$, $\sqrt{x_n} \to \sqrt{x}$.

Squeeze Theorem: If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and $\lim x_n = \lim z_n = L$, then $\lim y_n = L$.

Cesaro Means: If $(x_n) \to x$, then $(y_n) \to x$ where

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

The **limit superior** of (a_n), denoted $\limsup a_n$ is given by $\lim\left(\sup\{a_k : k \geq n\}\right)$. The **limit inferior** of (a_n), denoted $\liminf a_n$ is given by $\lim\left(\inf\{a_k : k \geq n\}\right)$.

$$\liminf a_n = \limsup a_n \iff (a_n) \to a$$

Lemma: For two nonempty sets bounded above with $A \subset B$, $\sup A \leq \sup B$.

If (a_n) and (b_n) are Cauchy, then $c_n = |a_n - b_n|$ is Cauchy. Similarly, $(a_n + b_n)$ and $(a_n \cdot b_n)$ are Cauchy.

Limit Ratio Test: Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1$$

then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Summation by Parts: Let $s_n = x_1 + \cdots + x_n$. Then

$$\sum_{j=m+1}^{n} x_j y_j = s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j (y_j - y_{j+1})$$

Dirichlet's Test: If the partial sums of $\sum_{n=1}^{\infty} x_n$ are bounded and if $y_n \geq 0$ and decreasing with $\lim y_n = 0$, then $\sum_{n=1}^{\infty} x_n y_n$ converges.

Practice Problems

1.3.6: Given sets A and B, define $A+B = \{a+b : a \in A, b \in B\}$. Follow these steps to prove that if A, B nonempty and bounded above, then $\sup(A+B) = \sup A + \sup B$.

1. Let $s = \sup A$ and $t = \sup B$. Show $s+t$ is an upper bound for $A+B$

   Notice that for any $a \in A$ and $b \in B$, $a+b \leq s+b$. Similarly, $a+b \leq a+t$. Thus,

   $$2a + 2b \leq s + b + a + t \implies a + b \leq s + t \implies s + t \text{ is upper bound}$$

2. Let u be an arbitrary upper bound for $A+B$ and fix $a \in A$. Show $t \leq u - a$

   From (1), $a+b \leq s+t$. First suppose $s+t \leq u$. Since $a \leq s$, $a+t \leq s+t \leq u \implies t \leq u-a$.

   Now consider the case $s+t > u$. Let $\varepsilon > 0$. By definition of supremum, $s+t-\varepsilon \leq a+b$ for some $a, b \in A, B$. Since u is an upper bound for $A+B$, $s+t-\varepsilon \leq u \implies s+t \leq u+\varepsilon$. This is a contradiction so the $s+t \leq u$ for all upper bounds u. The first case shows $t \leq u-a$

3. Show $\sup(A+B) = s+t$

   From (1), $s+t$ is an upper bound for $A+B$. From part (2), $s+t$ is the least upper bound for $A+B$. Thus, $\sup(A+B) = s+t$

4. Construct another proof of this same fact using Lemma 1.3.8

1.3.8: Compute, without proof, the sup and inf (if they exist) of the following sets:

1. $\{m/n : m, n \in \mathbb{N}, m < n\}$

   Since $m < n$, $m/n < 1$. Thus, $\sup = 1$. The inf is 0.

2. $\{(-1)^m/n : m, n \in \mathbb{N}\}$

   The set is bounded above by 1 and below by -1. Thus, $\sup = 1$ and $\inf = -1$.

3. $\{n/(3n+1) : n \in \mathbb{N}\}$

   The set is bounded above by $1/3$ and below by 0. Thus, $\sup = 1/3$ and $\inf = 0$.

4. $\{m/(m+n) : m, n \in \mathbb{N}\}$

   $\inf = 0$. $\sup = 1$

1.4.2: Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Let $s \in \mathbb{R}$ have the property $\forall n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound of A and $s - \frac{1}{n}$ is not an upper bound for A. Prove that $s = \sup A$.

Let $\varepsilon > 0$ and $N = \frac{1}{\varepsilon} \implies \varepsilon = \frac{1}{N}$. Choose $n > N$ so

$$s - \frac{1}{n} < \sup A < s + \frac{1}{n} \implies s - \varepsilon < \sup A < s + \varepsilon \implies |\sup A - s| < \varepsilon \implies s = \sup A$$

2.3.6 Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1 ($(x_n) \to x \implies (\sqrt{x_n}) \to \sqrt{x}$), show $\lim b_n$ exists and find its value.

Consider $\frac{1}{b_n}$:

$$\begin{aligned}
\lim \frac{1}{b_n} &= \lim \frac{1}{n + \sqrt{n^2 + n}} \\
&= \lim \frac{1}{n} + \lim \frac{1}{\sqrt{n^2 + n}} && \text{(ALT)} \\
&= \lim \sqrt{\frac{1}{n^2 + n}} && ((1/n) \to 0) \\
&= \sqrt{\lim \frac{1}{n^2 + n}} && \text{(Ex 2.3.1)} \\
&= \sqrt{\lim \frac{1}{n(n+1)}} \\
&= \sqrt{\lim \frac{1}{n} \cdot \frac{1}{n+1}} \\
&= \sqrt{\lim \frac{1}{n} \cdot \lim \frac{1}{n+1}} && \text{(ALT)} \\
&= \sqrt{0 \cdot 0} && ((1/n) \to 0) \\
&= 0
\end{aligned}$$

Thus, $\lim \frac{1}{b_n} = 0$.

2.3.8: Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

1. Show $p(x_n) \to p(x)$

   Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$. Then

   $$p(x_n) = a_0 + a_1 x_n + \cdots + a_n x_n^n$$

   $x_n \to x$ so $p(x_n) \to a_0 + a_1 x + \cdots + a_n x^n = p(x)$ by the ALT.

2. Find an example of a function $f(x)$ and a convergent series $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$

   Let $(x_n) = 1/n \to 0$ and $f(x) = \sin x$. Then $f(x_n) = \sin(1/n) \to 0$ but $f(x)$ does not converge.

2.4.2:

1. $y_1 = 1$, $y_{n+1} = 3 - y_n$, $\lim y_n = y$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we find $y = 3/2$. What is wrong with this argument?

2. This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the argument in (1) be used to compute this limit?

2.4.8: For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.

1. $\sum_{n=1}^{\infty} \frac{1}{2^n}$

   $$\begin{aligned} S_1 &= \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\ S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\ S_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} \\ S_n &= 1 - \frac{1}{2^n} \end{aligned}$$

   $(\frac{1}{2^n}) \to 0$ so $S_n \to 1$.

2. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$\begin{aligned}
S_1 &= \frac{1}{1(2)} = \frac{1}{2} \\
S_2 &= \frac{1}{1(2)} + \frac{1}{2(3)} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \\
S_3 &= \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4} \\
S_4 &= \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \frac{1}{4(5)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5} \\
S_n &= \frac{n}{n+1} = 1 - \frac{1}{n+1}
\end{aligned}$$

$(\frac{1}{n+1}) \to 0$ so S_n converges.

3. $\sum_{n=1}^{\infty} \log(\frac{n+1}{n})$

$$\begin{aligned}
S_1 &= \log(2) \\
S_2 &= \log(2) + \log(3/2) = \log(\frac{3}{2} \cdot 2) = \log(3) \\
S_3 &= \log(2) + \log(3/2) + \log(4/3) = \log(4) \\
S_4 &= \log(2) + \log(3/2) + \log(4/3) + \log(5/4) = \log(5) \\
S_n &= \log(n+1)
\end{aligned}$$

$\log(n+1)$ is unbounded so S_n diverges.

2.6.2: Give an example of each of the following (or show they are impossible):

1. A Cauchy sequence that is not monotone

$$a_n = \frac{(-1)^n}{n^2}$$

2. A Cauchy sequence with an unbounded subsequence

A Cauchy sequence is convergent. Every subsequence of a convergent sequence converges. Every convergent sequence is bounded. Thus, a Cauchy sequence cannot have an unbounded subsequence.

3. A divergent monotone sequence with a Cauchy subsequence

   A divergent monotone sequence is unbounded so it cannot have a bounded infinite subsequence.

4. An unbounded sequence containing a Cauchy subsequence

   As above, a Cauchy sequence must be bounded so it cannot be a subsequence of an unbounded sequence.

2.7.2: Decide whether the following series converge or diverge:

1. $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$
2. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$
3. $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \dots$
4. $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \dots$

   $(1/n) \to 0$ and is decreasing so by the Alternating Series Test the series converges.

5. $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \dots$

2.7.4: Give an example of each or explain why it is impossible:

1. Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges

   Let $(x_n) = (y_n) = \frac{1}{n}$. Then $\sum x_n y_n = \sum \frac{1}{n^2} = \frac{\pi^2}{6}$ which converges.

2. A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges

3. Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum(x_n + y_n)$ both converge but $\sum y_n$ diverges

   Let $\sum x_n = A$ and $\sum(x_n + y_n) = B$. Then by the Algebraic Limit Theorem,

   $$\sum y_n = \sum(x_n + y_n) - x_n = B - A$$

   so $\sum y_n$ converges.

4. A sequence (x_n) with $0 \leq x_n \leq 1/n$ where $\sum(-1)^n x_n$ diverges

   $(1/n) \to 0$ so by comparison test, $(x_n) \to 0$. Since $(1/n)$ is decreasing, (x_n) must also be decreasing so $\sum(-1)^n x_n$ converges by Alternating Series Test.