Math 1010: Homework 7

Problem 1

For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ε challenge

1. $\lim_{x\to 3}(5x-6)=9$, where $\varepsilon = 1$

   We need a $\delta > 0$ such that the system of inequalities

   $$\begin{cases} |x-3| < \delta \\ |5x-6-9| < 1 \end{cases}$$

   is satisfied. From the second inequality, we have

   $$|5x-15| < 1 \implies -1 < 5x-15 < 1 \implies \frac{14}{5} < x < \frac{16}{5}$$

   Therefore,

   $$-\frac{1}{5} < |x-3| < \frac{1}{5} \implies \boxed{\delta = \frac{1}{5}}$$

2. $\lim_{x\to 4}\sqrt{x} = 2$, where $\varepsilon = 1$.

   $$\begin{cases} |x-4| < \delta \\ |\sqrt{x}-2| < 1 \end{cases}$$

   From the second inequality, we have

   $$-1 < \sqrt{x}-2 < 1 \implies 1 < \sqrt{x} < 3 \implies 1 < x < 9$$

   so

   $$(1-4) < x-4 < (9-4) \implies -3 < x-4 < 5 \implies |x-4| < 3$$

   so $\boxed{\delta = 3}$.

Use the definition of functional limits to supply a proper proof for the following limit statements

1. $\lim_{x\to 2}(x^2+x-1) = 5$

   Let $\varepsilon > 0$. We want to show that $\exists \delta > 0$, such that for $|x-2| < \delta$,

   $$\left|(x^2+x-1)-5\right| = \left|x^2+x-6\right| = |x-2|\,|x+3| < \varepsilon$$

   We construct a δ-neighborhood around $c = 2$ with radius no bigger than $\delta = 1$ so

   $$|x+3| \leq 6$$

   Choose $\delta = \min\{1, \frac{\varepsilon}{6}\}$ so $|x-2| < \delta$ implies

   $$\left|(x^2+x-1)-5\right| \leq 6\,|x-2| < 6\delta = \varepsilon \quad \blacksquare$$

2. $\lim_{x\to 0} x^3 = 0$

   Let $\varepsilon > 0$. $|x - 0| < \delta$ implies
$$\left|x^3 - 0\right| < \varepsilon$$
   if $\delta = \sqrt[3]{\varepsilon}$:
$$|x| < \delta \implies \left|x^3\right| = |x|^3 < \delta^3 = \varepsilon \quad \blacksquare$$

3. $\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}$.

   Let $\varepsilon > 0$. We want to show that $\exists\delta > 0$, such that for $|x - 3| < \delta$, $\left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon$.

   We have
$$\left|\frac{1}{x} - \frac{1}{3}\right| = \left|\frac{3 - x}{3x}\right| = \frac{|x - 3|}{|3x|}$$
   Construct a δ-neighborhood of at most 2 around $x = 3$ so $x > 1$ and
$$\frac{|x - 3|}{|3x|} < \frac{|x - 3|}{|3|}$$
   Let $\delta = \min\{2, 3\varepsilon\}$ so $|x - 3| < \delta$ implies
$$\left|\frac{1}{x} - \frac{1}{3}\right| < \frac{|x - 3|}{3} < \frac{\delta}{3} = \varepsilon \quad \blacksquare$$

Problem 2

Are the following claims true or false and give a justification for each conclusion.

1. If a particular δ has been constructed as a suitable response to a particular ε challenge, then any smaller positive δ will also suffice

   True. Suppose $\lim_{x\to c} f(x) = L$, that is $\exists\delta > 0$ such that for $|x-c| < \delta$, $|f(x) - L| < \varepsilon$ for all $\varepsilon > 0$.

   If $\delta' < \delta$, then $|x-c| < \delta'$ implies $|x-c| < \delta$ so $|f(x) - L| < \varepsilon$. Therefore, any smaller δ will also suffice. ■

2. If $\lim_{x\to a} f(x) = L$ and a happens to be in the domain of f, then $L = f(a)$

   False. Consider
   $$f(x) = \begin{cases} 1 & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

   $\lim_{x\to a} f(x) = 1$ but $f(a) = 0$. ■

3. If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f)

   False. For any continuous function g, the limit $\lim_{x\to a} f(x)g(x) = 0$ by the ALT for functional limits. However, if g is not continuous at a, then the limit may not exist. For example, consider $g(x) = \frac{1}{x-a}$. Then $\lim_{x\to 0} f(x)g(x) = \frac{0}{0}$ which is indeterminate.

4. The limit $\lim_{x\to 2} \frac{|x-2|}{x-2}$ exists (compute it if it does, or prove that it doesn't)

   False. Notice that
   $$f(x) = \frac{|x-2|}{x-2} = \begin{cases} 1 & \text{if } x > 2 \\ -1 & \text{if } x < 2 \end{cases}$$

   Consider the sequences $x_n = 2 + \frac{1}{n}$ and $y_n = 2 - \frac{1}{n}$. Clearly, $x_n \to 2$ and $y_n \to 2$. However, $\lim_{x_n\to 2} f(x_n) = 1$ and $\lim_{x_n\to 2} f(y_n) = -1$ so $f(x_n)$ and $f(y_n)$ do not converge to the same value. Therefore, by the Divergence Criterion, the limit does not exist. ■

5. The limit $\lim_{x\to 7/4} \frac{|x-2|}{x-2}$ exists (compute it if it does, or prove that it doesn't)

   True. Let $f(x) = \frac{|x-2|}{x-2}$.
   $$f(\frac{7}{4}) = \frac{|\frac{7}{4} - 2|}{\frac{7}{4} - 2} = \frac{|-1/4|}{-1/4} = -1$$

   Let $\varepsilon > 0$. Let $\delta = \frac{1}{4} - \varepsilon$ so for $x \in V_\delta(\frac{7}{4})$, $x < 2$. Since $f(x) = -1$ for all $x < 2$, $f(\frac{7}{4}) \in V_\varepsilon(-1)$. Therefore, $\lim_{x\to 7/4} f(x) = -1$. ■

Problem 3 (Squeeze Theorem for functions)

Let f, g and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$ at some limit point c, show that $\lim_{x \to c} g(x) = L$ as well.

By the Sequential Criterion for Functional Limits, there exist a sequence $(x_n) \to c$ such that $f(x_n) \to L$ and $h(x_n) \to L$. Since $f(x) \leq g(x) \leq h(x)$, we have

$$f(x_n) \leq g(x_n) \leq h(x_n)$$

for all n. By the Squeeze Theorem for sequences, we have $g(x_n) \to L$ as well. Therefore, by the Sequential Criterion for Functional Limits, $\lim_{x \to c} g(x) = L$. ■

Problem 4

Let $g(x) = \sqrt[3]{x}$.

1. Prove that g is continuous at $c = 0$.

   Let $\varepsilon > 0$. We want to show that $\exists \delta > 0$, such that for $|x| < \delta$,

   $$|g(x) - 0| = \left|\sqrt[3]{x}\right| < \varepsilon$$

   A natural choice is $\delta = \varepsilon^3$ so $|x| < \delta$ implies

   $$|g(x) - 0| = \left|\sqrt[3]{x}\right| = \sqrt[3]{|x|} < \sqrt[3]{\delta} = \sqrt[3]{\varepsilon^3} = \varepsilon \implies |g(x) - 0| < \varepsilon \quad \blacksquare$$

2. Prove that g is continuous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ will be helpful).

   As above, let $\varepsilon > 0$. We want to show that $|x - c| < \delta$ (for $\delta > 0$) implies

   $$\left|\sqrt[3]{x} - \sqrt[3]{c}\right| < \varepsilon$$

   Notice that

   $$\left|\sqrt[3]{x} - \sqrt[3]{c}\right| = \left|\frac{x - c}{\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}}\right|$$

   If x and c have the same sign, then $\left|\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}\right| = \sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2} > \sqrt[3]{c^2}$ so

   $$\left|\sqrt[3]{x} - \sqrt[3]{c}\right| = \left|\frac{x - c}{\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}}\right| < \frac{|x - c|}{\sqrt[3]{c^2}}$$

   A natural approach is to construct a δ-neighborhood around c such that x and c have the same sign. Let $\delta = \min\{\frac{|c|}{2}, \varepsilon\,|c|^{2/3}\}$ so $|x - c| < \delta$ implies

   $$\left|\sqrt[3]{x} - \sqrt[3]{c}\right| < \frac{|x - c|}{\sqrt[3]{c^2}} < \frac{\delta}{\sqrt[3]{c^2}} = \frac{\varepsilon\sqrt[3]{c^2}}{\sqrt[3]{c^2}} = \varepsilon \quad \blacksquare$$

Problem 5

1. Supply a proof for Theorem 4.3.9 using the $\varepsilon - \delta$ characterization of continuity

   Theorem 4.3.9 states that given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ with $f(A) \subseteq B$, if the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so $g \circ f(x) = g(f(x))$ is defined on A and if f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

   Since f is continuous at c, $\forall \varepsilon > 0$, $\exists \delta_1 > 0$ such that for $|x - c| < \delta_1$, $|f(x) - f(c)| < \varepsilon_1$. Similarly, since g is continuous at $f(c)$, $\forall \varepsilon > 0$, $\exists \delta_2 > 0$ such that for $|y - f(c)| < \delta_2$, $|g(y) - g(f(c))| < \varepsilon$.

   We want to show that there exists a δ such that $|x - c| < \delta$ implies that

   $$|g(f(x)) - g(f(c))| < \varepsilon$$

   By continuity of f,

   $$|x - c| < \delta_1 \implies |f(x) - f(c)| < \varepsilon_1$$

   Let $\delta_2 = \varepsilon_1$. Since $f(x)$ and $f(c)$ are in the domain of g, the continuity of g gives that

   $$|f(x) - f(c)| < \delta_2 \implies |g(f(x)) - g(f(c))| < \varepsilon$$

   Therefore, for $\delta = \delta_1$,

   $$|x - c| < \delta \implies |g(f(x)) - g(f(c))| < \varepsilon \quad \blacksquare$$

2. Give another proof of this theorem using the sequential characterization of continuity.

   Since f is continuous at c, $\forall (x_n) \in A$ such that $(x_n) \to c$, $f(x_n) \to f(c)$. Similarly, by the continuity of g, $\forall (y_n) \to f(c)$, $g(y_n) \to g(f(c))$.

   Therefore, $\forall (x_n) \to c$, $g(f(x_n)) \to g(f(c))$ so by the Sequential Criterion for Functional Limits, $g \circ f$ is continuous at c. $\blacksquare$