Math 1010 - Homework 11

Problem 1

Supply the details for the proof of the Weirerstrass M-Test. (Corollary 6.4.5)

Theorem: For each $n \in \mathbb{N}$, let f_n be a function defined on $A \subseteq \mathbb{R}$ and $M_n > 0$ a real number satisfying $|f_n(x)| \leq M_n$ for all $x \in A$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A.

Proof: Suppose $\sum_{n=1}^{\infty} M_n$ converges.

By the Cauchy Criterion for Uniform Convergence of Series, $\sum_{n}^{\infty} f_n(x)$ converges uniformly if and only if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n > m \geq N$ and $x \in A$, we have

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \varepsilon$$

However, by boundedness of $f_n(x)$,

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| \leq M_{m+1} + M_{m+2} + \cdots + M_n$$

Since $\sum_{n=1}^{\infty} M_n$ converges, by the Cauchy Criterion for Series,

$$|M_{m+1} + M_{a+2} + \cdots + M_n| < \varepsilon$$

Since $M_{m+1} + M_{m+2} + \cdots + M_n \leq |M_{m+1} + M_{m+2} + \cdots + M_n|$, we have

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \varepsilon$$

so $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A. $\blacksquare$

Problem 2

Let
$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}.$$

1. Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.

   Let $g_k(x) = \frac{\sin(kx)}{k^3}$. We know that $\sin(kx)$ and $\frac{1}{k^3}$ are differentiable for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$. Since g_k is a composition of differentiable functions, g_k is differentiable for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$.

   Consider
   $$\sum_{k=1}^{\infty} g_k'(x) = \sum_{k=1}^{\infty} \frac{k\cos(kx)}{k^3} = \frac{\cos(kx)}{k^2}$$
   We know $|\cos(kx)| \leq 1$ so $|g_k'(x)| \leq \frac{1}{k^2}$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$. Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, by the Weierstrass M-Test, $\sum_{k=1}^{\infty} g_k'(x)$ converges uniformly on $\mathbb{R}$.

   Clearly, for $x = 0$,
   $$f(0) = \sum_{k=1}^{\infty} \frac{\sin(0)}{k^3} = 0$$
   so there exists $x_0 \in \mathbb{R}$ for which $\sum_{k=1}^{\infty} g(x_0)$ converges.

   All together, by the Term-by-Term differentiability theorem, $f(x)$ converges uniformly and
   $$f'(x) = \sum_{k=1}^{\infty} g_k'(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$
   so f is differentiable.

   Clearly, $\left(\frac{\cos(kx)}{k^2}\right)$ is a sequence of functions which are continuous for all $x \in \mathbb{R}$ and $k \neq 0$.

   Above, we showed that $\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$ converges uniformly on $\mathbb{R}$. By the term-by-term continuity theorem, $f'(x)$ is continuous on $\mathbb{R}$. ■

2. Can we determine if f is twice-differentiable?

   We can repeat the same argument:
   $$g_k(x) = \frac{\sin(kx)}{k^3} \implies g_k'(x) = \frac{\cos(kx)}{k^2} \implies g_k''(x) = \frac{-\sin(kx)}{k}$$
   However, $|\sin(kx)| \leq 1$ so $|g_k''(x)| \leq \frac{1}{k}$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$. Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, we are not able to use the Weierstrass M-test to show that $\sum_{k=1}^{\infty} g_k''(x)$ converges uniformly on $\mathbb{R}$.

   Therefore, we cannot determine that f is twice-differentiable. ■

Problem 3

1. Recall the Ratio Test from PSET 5. Use this to show that if s satisfies $0 < s < 1$, show ns^{n-1} is bounded for all $n \geq 1$.

   The Ratio Test states that, given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, if (a_n) satisfies

   $$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1$$

   then the series converges absolutely.

   Suppose $0 < s < 1$. Consider the sequence $a_n = ns^{n-1}$.

   Immediately, we note $a_n > 0$ for all $n \in \mathbb{N}$. Further,

   $$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{(n+1)s^n}{ns^{n-1}} \right| = \lim \left| \frac{n+1}{n} \cdot s \right| = \lim \left| s + \frac{s}{n} \right| = |s| < 1$$

   so $\sum_{n=1}^{\infty} ns^{n-1}$ converges absolutely. This tells us $(a_n) \to 0$. Since (a_n) is convergent, it is Cauchy and hence bounded. ■

2. Given an arbitrary $x \in (-R, R)$, pick t to satisfy $|x| < t < R$. Use the observation

   $$|na_n x^{n-1}| = \frac{1}{t}\left(n \left| \frac{x^{n-1}}{t^{n-1}} \right| \right) |a_n t^n|$$

   to construct a proof for Theorem 6.5.6 from the book.

   Theorem 6.5.6: If $\sum_{n=0}^{\infty} a_n x^n$ converges for $x \in (-R, R)$, then the series differentiated series $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges at each $x \in (-R, R)$ as well. Consequently, the convergence is uniform on compact sets contained in $(-R, R)$.

   Let $x \in (-R, R)$ and pick t such that $|x| < t < R$. Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges.

   Notice

   $$\begin{aligned} \left| na_n x^{n-1} \right| &= \frac{n}{t} \left| \frac{x^{n-1}}{t^{n-1}} \right| |a_n t^n| \\ &= \frac{1}{t} \left(n \left| \frac{x}{t} \right|^{n-1} \right) |a_n t^n| \end{aligned}$$

   But $|x| < t$ so $\left| \frac{x}{t} \right| < 1$. By Part 1, $n \left| \frac{x}{t} \right|^{n-1}$ is bounded for all $n \in \mathbb{N}$ so we can write

   $$\frac{1}{t} \left(n \left| \frac{x}{t} \right|^{n-1} \right) |a_n t^n| \leq M \frac{|a_n t^n|}{t}$$

   Therefore,

   $$\sum_{n=1}^{\infty} \left| na_n x^{n-1} \right| = \sum_{n=1}^{\infty} \frac{M}{t} |a_n t^n| = \frac{M}{t} \sum_{n=1}^{\infty} |a_n t^n|$$

   but $t \in (-R, R)$ so $\sum_{n=1}^{\infty} |a_n t^n|$ converges (converges absolutely because it is a power series which converges on $(-R, R)$). Therefore, $\sum_{n=1}^{\infty} |na_n x^{n-1}|$ converges so $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges absolutely for all $x \in (-R, R)$.

   Since the series converges absolutely for all $x_0 \in (-R, R)$, it converges uniformly on all compact sets $[x_0, x_0] \subset (-R, R)$. ■

Problem 4

1. Show that the power series representations are unique. If we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in the interval $(-R, R)$, prove that $a_n = b_n$ for all $n = 0, 1, 2, \cdots$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

Expanding terms,

$$a_0 + a_1 x^1 + a_2 x^2 + \cdots = b_0 + b_1 x^1 + b_2 x^2 + \ldots$$

Clearly, at $x = 0 \in (-R, R)$, $a_0 = b_0$.

Since the power series converge, we have uniform convergence on $(-R, R)$ so we may take term-by-term derivatives:

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

by identical argument as above, with $x = 0$, we have

$$a_1 x^0 + 2a_2 x^1 + 3a_3 x^2 + \cdots = b_1 x^0 + 2b_2 x^1 + 3b_3 x^2 + \cdots \implies a_1 = b_1$$

Now suppose $a_n = b_n$ for all $n < k$. Then

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \implies \sum_{n=k}^{\infty} a_n x^n = \sum_{n=k}^{\infty} b_n x^n$$

Taking the k-th termwise derivative,

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k) a_n x^{n-k} = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k) b_n x^{n-k}$$

so at $x = 0$,

$$n(n-1)\cdots(n-k) a_k = n(n-1)\cdots(n-k) b_k \implies a_k = b_k$$

Therefore, by induction, $a_n = b_n$ for all $n \in \mathbb{N}$. $\blacksquare$

2. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge on $(-R, R)$, and assume $f'(x) = f(x)$ for all $x \in (-R, R)$ and $f(0) = 1$. Deduce the values of a_n.

Since $f'(x) = f(x)$ for all $x \in (-R, R)$, we have

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

From $f(0) = 1$,

$$f(0) = \sum_{n=0}^{\infty} a_n 0^n = a_0 = 1$$

But this also means $f'(0) = 1$ so

$$(1) a_1 = 1 \implies a_1 = 1$$

We may take further derivatives:

$$f'(x) = f(x) \implies f''(x) = f'(x) \implies f''(x) = f(x) \implies f''(0) = 1$$

so

$$f''(x) = \sum_{n=2}^{\infty} (n)(n-1)a_n x^{n-2}, \quad f''(0) = 2(1)a_2 = 1 \implies a_2 = \frac{1}{2}$$

Now we will induct on the derivatives of $f(x)$. Suppose $f^{(n)} = f$ for all $n < k$. Then

$$\frac{d}{dx} f^{(k)} = \frac{d}{dx} f = f \implies f^{(k+1)}(x) = f(x) \implies f^{(k+1)}(0) = f(0) = 1$$

Therefore,

$$f^{(k+1)}(x) = \sum_{n=k+1}^{\infty} (n)(n-1)\cdots(n-k)a_n x^{n-k-1}$$

and

$$f^{(k+1)}(0) = (k+1)(k)(k-1)\cdots(2)(1)a_{k+1} + 0 + 0 + \cdots = 1 \implies (k+1)!a_{k+1} = 1 \implies a_{k+1} = \frac{1}{(k+1)!}$$

Therefore, $a_n = \frac{1}{n!}$ for all $n \in \mathbb{N}$. $\blacksquare$

Problem 5

1. Generate the Taylor coefficients for the exponential function $f(x) = e^x$, and then prove that the corresponding Taylor series converges uniformly to e^x on any interval of the form $[-R, R]$.

$$a_1 = \frac{\frac{d}{dx}e^x\Big|_{x=0}}{1!} = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{6}, a_4 = \frac{1}{24}, \cdots, a_n = \frac{1}{n!}$$

   Define $f(x) = e^x$ and $S_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$.

   Let $\varepsilon > 0$. To show that the Taylor series converges uniformly on $[-R, R]$, we want to show that for all $x \in [-R, R]$,

$$|E_n(x)| = |f(x) - S_n(x)| < \varepsilon$$

   By Lagrange's Remainder Theorem, if $x \neq 0$ in $(-R, R)$, exists c such that $|c| < |x|$ such that

$$|E_N(x)| = \frac{f^{(N+1)}(c)}{(N+1)!}x^{n+1}$$

   Since $f(x) = e^x$, $f^{(N+1)}(c) = e^c$ for all $N \in \mathbb{N}$. Further, since $c \in [-R, R]$ and e^x is monotone increasing,

$$\left|f^{(N+1)}(c)\right| = |e^c| \leq \left|e^R\right|$$

   Similarly, $|x^{n+1}| \leq |R^{n+1}|$ for all $x \in [-R, R]$.

   Therefore,

$$|E_N(x)| \leq \left|\frac{e^R}{(N+1)!}R^{N+1}\right| \to 0$$

   so the Taylor series converges uniformly on $[-R, R]$. ■

2. Verify the formula $f'(x) = e^x$.

   Assume $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. By a theorem in class,

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

   But $a_n = \frac{1}{n!}$ so

$$f'(x) = \sum_{n=1}^{\infty} \frac{n}{n!}x^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}x^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = e^x \quad ■$$

3. Use a substitution to generate the series for e^{-x}, and then informally calculate $e^x \cdot e^{-x}$ by multiplying together the two series and collecting common powers of x.

   Let $u = -x$. Then from above,

$$e^{-x} = e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n$$

So

$$\begin{aligned}
e^x \cdot e^{-x} &= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n\right) \\
&= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots\right)\left(1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \dots\right) \\
&= \left(1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \dots\right) + \left(x - x^2 + \frac{x^3}{2} - \frac{x^4}{3!} + \dots\right) + \left(\frac{x^2}{2} - \frac{x^3}{2} + \frac{x^4}{4} - \frac{x^5}{2 \cdot 3!}\right) + \dots \\
&= 1 + (-x + x) + \left(\frac{x^2}{2} - x^2 + \frac{x^2}{2}\right) + \left(-\frac{x^3}{3!} + \frac{x^3}{2} - \frac{x^3}{2} + \frac{x^3}{3!}\right) + \dots \\
&= 1 + 0 + 0 + 0 + \dots = 1
\end{aligned}$$

which matches what we would expect!