Math 1820A Spring 2024 - Homework 8

Instructions: This assignment is worth twenty points. Please complete the following problems assigned below. Submissions with insufficient explanation may lose points due to a lack of reasoning or clarity. If you are handwriting your work, please ensure it is readable and well-formatted for the grader.

Be sure when uploading your work to **assign problems to pages**. Problems with pages not assigned to them **may not be graded**.

Textbook Problems:

Additional Problems: For these problems let $\mathbb{H}$ denote the algebra of Hamiltonians and $\mathbb{H}^\times$ denote all the non-zero elements as a *group* under multiplication. For the context of this homework, assume a 'rotation' is orientation preserving, and a 'reflection' is preserves a codimension 1 subspace. Also denote L_A by a line and also the line-symmetry through L_A.

1. Let L_A and L_B be non-parallel lines through in $\mathbb{R}^2$ intersecting at p. Prove that $L_B \circ L_A$ is a rotation in $\mathbb{R}^2$ about p by an angle of 2θ where θ is the oriented angle between L_A and L_B.

It suffices to construct a reference frame isomorphic to $\mathbb{R}^2$ with p at the origin. Thus, L_A and L_B in the new frame are two line symmetries through the origin. By the Euler-Rodrigues Theorem (see Problem 7), $L_B \circ L_A$ is a rotation about p by an angle of 2θ in the clockwise direction where θ is the oriented angle between L_A and L_B. ■

2. Let L_A and L_B be two parallel lines in $\mathbb{R}^2$. Let $\Gamma = \langle L_A, L_B \rangle$. Does there exist a point $P \in \mathbb{R}^2$ preserved by Γ, i.e. is there a point $P \in \mathbb{R}^2$ such that $\Gamma P = P$? What about a line?

Notice that $L_A \circ L_A = I$ and $L_B \circ L_B = I$ by the properties of reflection in $\mathbb{R}^2$. Further, we know the only points fixed by a line symmetry L_X are precisely the points on the line generated by X. In this case, this means that the only points fixed by L_A are the points on L_A and the only points fixed by L_B are the points on L_B. Since L_A and L_B are parallel, they do not intersect, so there is no point fixed by both L_A and L_B unless $L_A = L_B$.

Considering the action of line symmetries on lines in $\mathbb{R}^2$, though, we know that the only lines fixed by a line symmetry are the lines perpendicular to the line of reflection or the line of reflection itself. Since L_A and L_B are parallel, the only line fixed by both L_A and L_B is the line perpendicular to L_A and L_B.

3. In the following problems, we construct an embedding of the free group of rank 2, $F_2 \longrightarrow \mathrm{SO}(3, \mathbb{R})$. Let $\Gamma = \langle A, B \rangle$. We prove that A and B have no relations. Denote

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Prove that A generates an infinite cyclic group. (Hint: Some basic number theory may be helpful here)

Let $G = \langle A \rangle$. After reduction of $AA^{-1} = A^{-1}A = I$, we have that all elements $g \in G$ will be of the form A^n for some $n \in \mathbb{Z}$.

It suffices to show that that for all $n \neq 0 \in \mathbb{Z}$, $A^n \neq I$ since if $A^m = A^n$ for some $m \neq n$, then $A^{m-n} = I$.

Suppose $A^n = I$.

We proceed by cases: If n is even then $A^n = A^{2m}$ for some m. Thus,

$$A^m A^m = I \implies A^m = A^{-m} \implies m = -m \implies m = 0$$

This is a contradiction so no even n will satisfy $A^n = I$.

If n is odd, then $A^n = A^{2m+1}$ for some m. Thus,

$$A^{2m} A = I \implies A^{2m} = A^{-1} \implies 2m = -1$$

but this has no solution in the integers, so no odd n will satisfy $A^n = I$.

Thus, A generates an infinite cyclic group. $\blacksquare$

4. Convince yourself that any word ending in A in Γ may be expressed as $C^{k_m} \dots A^{k_3} B^{k_2} A^{k_1}$ where $k_i \neq 0$ and $n \geq 1$ and where C is either A and B depending on the parity of m. Denote such words by $\mathcal{R}_A$. Prove that for each such $g \in \mathcal{R}_A$ one has that there exists $i, j, k \in \mathbb{Z}$ such that

$$ge_3 = \left(\frac{i}{5^N}, \frac{j}{5^N}, \frac{k}{5^N} \right) \text{ where } N = |k_m| + |k_{m-1}| + \cdots + |k_1|$$

We can write

$$A = \frac{1}{5} A_0 = \frac{1}{5} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{pmatrix}, \quad B = \frac{1}{5} B_0 = \frac{1}{5} \begin{pmatrix} 3 & -4 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Therefore,

$$\begin{aligned} g &= C^{k_m} \dots A^{k_3} B^{k_2} A^{k_1} \\ &= \frac{C_0^{k_m}}{5^{|k_m|}} \cdots \frac{A_0^{k_3}}{5^{|k_3|}} \cdot \frac{B_0^{k_2}}{5^{|k_2|}} \cdot \frac{A_0^{k_1}}{5^{|k_1|}} \\ &= \frac{1}{5^{|k_m|} \dots 5^{|k_2|} \cdot 5^{|k|} \cdot 5^{|k_1|}} \cdot C_0^{k_m} \cdots A_0^{k_3} \cdot B_0^{k_2} \cdot A_0^{k_1} \\ &= \frac{1}{5^{|k_m| + \cdots + |k_2| + |k_1|}} \cdot C_0^{k_m} \cdots A_0^{k_3} \cdot B_0^{k_2} \cdot A_0^{k_1} \\ &= \frac{1}{5^N} C_0^{k_m} \cdots A_0^{k_3} \cdot B_0^{k_2} \cdot A_0^{k_1} \\ &= \frac{1}{5^N} \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \end{aligned}$$

where $N = |k_m| + \cdots + |k_2| + |k_1|$.

We can thus write

$$ge_3 = \frac{1}{5^N} \begin{pmatrix} * \end{pmatrix}$$

However, since all entries of A_0 and B_0 are integers, we know the product $C_0^{k_m} \cdots A_0^{k_3} \cdot B_0^{k_2} \cdot A_0^{k_1}$ will have integer entries. Thus, we can more strongly say

$$ge_3 = \left(\frac{i}{5^N}, \frac{j}{5^N}, \frac{k}{5^N} \right)$$

for $i, j, k \in \mathbb{Z}$ and $N = |k_m| + \cdots + |k_2| + |k_1|$. ■

5. Calculate the following quantities.

$$Ae_3, \quad A^2e_3 \quad A^3e_3 \quad BAe_3 \quad BA^2e_3 \quad BA^3e_3 \quad B^2Ae_3 \quad B^2A^2e_3 \quad B^2A^3e_3$$

Rationalize each by powers of 5^N where N is as it was defined in Problem 4. Consider the second coordinate of each of the above post rationalization, and evaluate it mod 4. You should notice a pattern.

Observe that for each such choice of element $g \in \mathcal{R}_A$, the second coordinate of ge_3 is expressible as $(4j)/5^m$ where $j \in \mathbb{Z}$ *and* $j \neq 0 \mod 5$ and $m \geq 1$.

$$Ae_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4/5 \\ 3/5 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix}$$

$$A^2e_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -7/25 & -24/25 \\ 0 & 24/25 & -7/25 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -24/25 \\ 7/25 \end{pmatrix} = \frac{1}{5^2} \begin{pmatrix} 0 \\ -24 \\ 7 \end{pmatrix}$$

$$A^3e_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -117/125 & -44/125 \\ 0 & 44/125 & -117/125 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -44/125 \\ 117/125 \end{pmatrix} = \frac{1}{5^3} \begin{pmatrix} 0 \\ -44 \\ 117 \end{pmatrix}$$

$$BAe_3 = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/5 & -12/25 & 16/25 \\ 4/5 & 9/25 & -12/25 \\ 0 & 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 16/25 \\ -12/25 \\ 3/5 \end{pmatrix} = \frac{1}{5^2} \begin{pmatrix} 16 \\ -12 \\ 15 \end{pmatrix}$$

$$BA^2e_3 = \begin{pmatrix} 3/5 & 28/125 & 96/125 \\ 4/5 & -21/125 & -72/125 \\ 0 & 24/25 & -7/25 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 96/125 \\ -72/125 \\ -7/25 \end{pmatrix} = \frac{1}{5^3} \begin{pmatrix} 96 \\ -72 \\ -35 \end{pmatrix}$$

$$BA^3e_3 = \begin{pmatrix} 3/5 & 468/625 & 176/625 \\ 4/5 & -351/625 & -132/625 \\ 0 & 44/125 & -117/125 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 176/625 \\ -132/625 \\ -117/125 \end{pmatrix} = \frac{1}{5^4} \begin{pmatrix} 176 \\ -132 \\ -585 \end{pmatrix}$$

$$B^2Ae_3 = \begin{pmatrix} -7/25 & -72/125 & 96/125 \\ 24/25 & -21/25 & 28/25 \\ 0 & 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 96/125 \\ 28/25 \\ 3/5 \end{pmatrix} = \frac{1}{5^3} \begin{pmatrix} 96 \\ 140 \\ 75 \end{pmatrix}$$

$$B^2A^2e_3 = \begin{pmatrix} -7/25 & 168/625 & 576/625 \\ 24/25 & 49/625 & 168/625 \\ 0 & 24/25 & -7/25 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 576/625 \\ 168/625 \\ -7/25 \end{pmatrix} = \frac{1}{5^4} \begin{pmatrix} 576 \\ 168 \\ -175 \end{pmatrix}$$

$$B^2A^3e_3 = \begin{pmatrix} -7/25 & 2808/3125 & 1056/3125 \\ 24/25 & 819/3125 & 308/3125 \\ 0 & 44/125 & -117/125 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1056/3125 \\ 308/3125 \\ -117/125 \end{pmatrix} = \frac{1}{5^5} \begin{pmatrix} 1056 \\ 308 \\ -2925 \end{pmatrix}$$

Looking at the second coordinates of each of these vector and reducing mod 4, we have:

g	$ge_3 \mod 4$
-4	0
-24	0
-44	0
-12	0
-72	0
-132	0
140	0
168	0
308	0

so we conclude the second coordinate of each ge_3 is expressible $(4j)/^m$ where $j \in \mathbb{Z}$ with $j \neq 0 \mod 5$ and $m \geq 1$.

6. Assuming the above observation, prove that for each $g \in \mathcal{R}_A$, that g is non-trivial. (By entirely analogous arguments this shows that each element of $g \in \mathcal{R}_B$ is non-trivial, so Γ is free)

From (4), for each $g \in \mathcal{R}_A$,

$$ge_3 = \left(\frac{i}{5^N}, \frac{j}{5^N}, \frac{k}{5^N} \right)$$

From (5), we have the stronger condition that the second coordinate is expressible as $(4j)/5^m$ with $j \in \mathbb{Z}$ and $j \neq 0 \mod 5$ and $m \geq 1$.

To show that g is non-trivial, it suffices to show that the second coordinate of ge_3 is non-zero.

Suppose $(4j)/5^m = 0$. Clearly, this suggests

$$4j = 0 \implies j = 0 \implies j \mod 5 = 0 \mod 5 = 0$$

but this is a contradiction, so the second coordinate of ge_3 is non-zero. Therefore, g is non-trivial. $\blacksquare$

7. Finish the proof of the Euler-Rodrigues theorem we mostly finished in class. Namely that for each $q \in S^3$, if we express $q = \cos(\theta/2) + u\sin(\theta/2)$ for $\theta \in [0, 2\pi]$ and $u \in P \cap S^3$, where P is the pure quaternions, then $I_q : P \longrightarrow P$ is rotation about u by an angle of θ in the *clockwise* direction. Finish the proof by showing the rotation is indeed clockwise as claimed.

Let $q = v^{-1}w$ where v and w are pure unit (based on a lemma from class). Since $v, w \in P \cap S^3$, $q = (-v)w$.

Further, $I_q = I_{-v} \circ I_w$ (where I_{-v} and I_w are line symmetries generated by $-v$ and w respectively) is a rotation by $\theta = 2\phi$ where ϕ is the angle from w to $-v$. So

$$\begin{aligned} q &= (-v)w \\ &= -(-v \cdot w) + (-v) \times w \\ &= v \cdot w - v \times w \end{aligned}$$

Since $v \cdot w = |v|\ |w| \cos\phi = \cos\phi = \cos\frac{\theta}{2}$, we can say

$$q = \cos(\theta/2) - v \times w$$

Since $q \in S^3$, $||q|| = 1$ so $\cos^2(\theta/2) + ||-v \times w||^2 = 1$. Thus, $\sin^2(\theta/2) = ||-v \times w||^2$ so $\sin(\theta/2) = ||-v \times w||$.

Therefore, we can write $q = \cos(\theta/2) + u\sin(\theta/2)$ where u is a unit vector in the direction of $v \times w$ corresponding to a clockwise rotation by θ (since the frame is right-handed, $-v \times w$ corresponds to a counterclockwise rotation so the rotation would be clockwise relative to $v \times w$). $\blacksquare$

8. Consider the embedding of $\mathbb{H}^\times \longrightarrow \mathrm{GL}(4,\mathbb{R})$ as described in Homework 7. Let $\mathfrak{h}$ denote the Lie-algebra of $\mathbb{H}^\times \subset \mathrm{GL}(4,\mathbb{R})$. Prove that $\mathfrak{q} = \mathfrak{h}/\mathfrak{z}(\mathfrak{h})$ is isomorphic to $\mathfrak{so}(3,\mathbb{R})$.

Elements of $\mathfrak{h}$ are of the form

$$\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$$

Elements of $\mathfrak{so}(3,\mathbb{R})$ are of the form

$$\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

Consider the map $\phi : \mathfrak{h} \to \mathfrak{so}(3,\mathbb{R})$ given by

$$\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \mapsto \begin{pmatrix} 0 & -b & d \\ b & 0 & -c \\ -d & c & 0 \end{pmatrix}$$

We can check that this is a homomorphism:

$$\begin{aligned}
\phi\left(\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} + \begin{pmatrix} w & -x & -y & -z \\ x & w & -z & y \\ y & z & w & -x \\ z & -y & x & w \end{pmatrix}\right) &= \phi\left(\begin{pmatrix} a+w & -(b+x) & -(c+y) & -(d+z) \\ b+x & a+w & -(d+z) & c+y \\ c+y & d+z & a+w & -(b+x) \\ d+z & -(c+y) & b+x & a+w \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 & -(b+x) & d+z \\ b+x & 0 & -(c+y) \\ -(d+z) & c+y & 0 \end{pmatrix} \\
\phi\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} + \phi\begin{pmatrix} w & -x & -y & -z \\ x & w & -z & y \\ y & z & w & -x \\ z & -y & x & w \end{pmatrix} &= \begin{pmatrix} 0 & -b & d \\ b & 0 & -c \\ -d & c & 0 \end{pmatrix} + \begin{pmatrix} 0 & -x & z \\ x & 0 & -y \\ -z & y & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -b-x & d+z \\ b+x & 0 & -c-y \\ -d-z & c+y & 0 \end{pmatrix}
\end{aligned}$$

And clearly it is surjective.

Notice that $\ker\phi = \mathbb{R}I_4$. This is precisely $\mathfrak{z}(\mathfrak{h})$ since in HW 7, we found that with basis

$$A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, B = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, C = \begin{pmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}, D = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$$

the lie algebra of $\mathfrak{h}$ is defined by brackets

$$[A,B] = 0, [A,C] = 0, [A,D] = 0, [B,C] = 2A, [B,D] = -2C, [C,D] = 2B$$

so $\mathfrak{z}(\mathfrak{h}) = \langle A \rangle = \mathbb{R}I_4$.

Since ϕ is a surjective homomorphism $\mathfrak{h} \to \mathfrak{so}(3, \mathbb{R})$ with kernel $(z)(\mathfrak{h})$, by the first isomorphism theorem,

$$\mathfrak{h}/\mathfrak{z}(\mathfrak{h}) \simeq \mathfrak{so}(3, \mathbb{R}) \quad \blacksquare$$

Bonus: [**3 pts**] In the context of the proof above where we construct a free group of rank 2 inside $\mathrm{SO}(3, \mathbb{R})$, prove that for any $\epsilon > 0$, there exists an $1 \neq g \in \Gamma$ such that $||g - I_3|| < \epsilon$.

Bonus: [**4 pts**] Construct an element $1 \neq g \in \Gamma$ such that $||g - I_3|| < 10^{-4}$. Prove your bound is sharp.

Bonus: [**5 pts**] Provide a rigorous proof that each element $g \in \mathcal{R}_B$ is not-trivial.