Math 1820A Spring 2024 - Homework 4

Instructions: This assignment is worth twenty points. Please complete the following problems assigned below. Submissions with insufficient explanation may lose points due to a lack of reasoning or clarity. If you are handwriting your work, please ensure it is readable and well-formatted for the grader.

Be sure when uploading your work to **assign problems to pages**. Problems with pages not assigned to them **may not be graded**.

Additional Problems: For these problems if you see an S in front of a group, you can assume it means determinant 1, e.g. all elements of $SO(3,\mathbb{R})$ and $SO(2,1)$ have determinant 1.

1. Prove that $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^\times$ are not isomorphic groups. Show however that $\mathrm{GL}(3,\mathbb{R})$ and $\mathrm{SL}(3,\mathbb{R}) \times \mathbb{R}^\times$ are isomorphic.

Notice that the center of $\mathrm{GL}(2,\mathbb{R})$ is

$$Z(\mathrm{GL}(2,\mathbb{R})) = \{tI_2 \mid t \in \mathbb{R}^\times\} \simeq \mathbb{R}^\times$$

and the center of $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^\times$ is

$$Z(\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^\times) = Z(\mathrm{SL}(2,\mathbb{R})) \times Z(\mathbb{R}^\times) = \{(\pm I_2, t) \mid t \in \mathbb{R}^\times\} \simeq \mathbb{Z}_2 \times \mathbb{R}^\times$$

Clearly, these centers are not isomorphic because any map $\phi : Z(\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^\times) \to Z(\mathrm{GL}(2,\mathbb{R}))$ will be two-to-one. Therefore, the groups are not isomorphic. $\blacksquare$

We introduce the homomorphism $\phi : \mathrm{GL}(3,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R}) \times \mathbb{R}^\times$ given by $\phi(A) = (A/\sqrt[n]{\det A}, \sqrt[n]{\det A})$.

We can check that ϕ is a homomorphism:

$$\begin{aligned}
\phi(A)\phi(B) &= (\frac{A}{\sqrt[n]{\det A}}, \sqrt[n]{\det A})(\frac{B}{\sqrt[n]{\det B}}, \sqrt[n]{\det B}) \\
&= (\frac{AB}{\sqrt[n]{\det A \det B}}, \sqrt[n]{\det A \det B}) \\
&= (\frac{AB}{\sqrt[n]{\det AB}}, \sqrt[n]{\det AB}) \\
&= \phi(AB)
\end{aligned}$$

and introduce the map $\psi : \mathrm{SL}(3,\mathbb{R}) \times \mathbb{R}^\times \to \mathrm{GL}(3,\mathbb{R})$ given by $\psi(A,t) = tA$ which is also a homomorphism:

$$\psi(A,t)\psi(B,s) = (tA)(sB) = tsAB = \psi(AB, ts)$$

Then, clearly,

$$\psi(\phi(A)) = \psi(\frac{A}{\sqrt[n]{\det A}}, \sqrt[n]{\det A}) = \sqrt[n]{\det A} \cdot \frac{A}{\sqrt[n]{\det A}} = A$$

$$\phi(\psi(A,t)) = \phi(tA) = (\frac{tA}{\sqrt[n]{\det tA}}, \sqrt[n]{\det tA}) = (\frac{tA}{\sqrt[n]{t^n \det A}}, \sqrt[n]{t^n \det A}) = (\frac{A}{\det A}, t \det A) = (A,t)$$

so ϕ is an isomorphism.

Therefore, $\mathrm{GL}(3,\mathbb{R}) \simeq \mathrm{SL}(3,\mathbb{R}) \times \mathbb{R}^{\times}$. ■

2. Show that $\mathrm{GL}^+(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det(A) > 0\}$, is isomorphic to $\mathrm{SL}(n,\mathbb{R}) \times \mathbb{R}^+$ where $\mathbb{R}^+$ is the group of *positive* real numbers under multiplication. Show that $\mathfrak{gl}(n,\mathbb{R})$ is isomorphic to $\mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R}$ as Lie-algebras.

As above, we introduce the map $\phi : \mathrm{GL}^+(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R}) \times \mathbb{R}^+$ by

$$\phi(A) = \left(\frac{A}{\sqrt[n]{\det A}}, \sqrt[n]{\det A}\right)$$

which we showed to be a homomorphism in Problem 1.

Since $\det A > 0$ for $A \in \mathrm{GL}^+(n,\mathbb{R})$, and

$$\det\left(\frac{A}{\sqrt[n]{\det A}}\right) = \det\left(\frac{1}{\sqrt[n]{\det A}} \cdot A\right) = \left(\frac{1}{\sqrt[n]{\det A}}\right)^n \det A = \frac{\det A}{\det A} = 1$$

ϕ is surjective.

To see that it is injective, let $\phi(A) = \phi(B)$ and suppose $A \neq B$. Then

$$\begin{cases} \frac{A}{\sqrt[n]{\det A}} = \frac{B}{\sqrt[n]{\det B}} \\ d := \sqrt[n]{\det A} = \sqrt[n]{\det B} > 0 \end{cases} \quad \Longrightarrow \frac{A}{d} = \frac{B}{d} \Longrightarrow A = B$$

but this is a contradiction. Therefore, ϕ is an isomorphism.

We know that $\mathfrak{gl}(n,\mathbb{R}) = M_n(\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \mathrm{tr}\,(A) = 0\}$, and $\mathbb{R}$ as a Lie algebra is just $\mathbb{R}$.

Therefore, it suffices to show that

$$M_n(\mathbb{R}) \simeq \{A \in M_n(\mathbb{R}) \mid \mathrm{tr}\,(A) = 0\} \oplus \mathbb{R}$$

First, embed $i : \mathfrak{sl}(n,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R})$ by simple inclusion so $\mathrm{im}\, i = \mathfrak{sl}(n,\mathbb{R})$. Then, define the projection $p : \mathfrak{gl}(n,\mathbb{R}) \to \mathbb{R}$ by $p(A) = \mathrm{tr}\,(A)$.

$$\ker p = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid \mathrm{tr}\,(A) = 0\} = \mathfrak{sl}(n,\mathbb{R})$$

Therefore, we have the short exact sequence

$$\mathfrak{sl}(n,\mathbb{R}) \overset{i}{\hookrightarrow} \mathfrak{gl}(2,\mathbb{R}) \overset{\mathrm{tr}}{\twoheadrightarrow} \mathbb{R}$$

We also notice that p admits a section $\sigma : \mathbb{R} \to \mathfrak{gl}(n,\mathbb{R})$ given by $\sigma(t) = \frac{t}{n} I_n$ such that

$$p(\sigma(t)) = p\left(\frac{t}{n} I_n\right) = \mathrm{tr}\,\left(\frac{t}{n} I_n\right) = t$$

therefore, the extension is split. Then by definition of a split exact sequence, we have the commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{sl}(n,\mathbb{R}) \arrow[r, "i"] \arrow[d, "id"] & \mathfrak{gl}(n,\mathbb{R}) \arrow[r, "p"] \arrow[d, "\phi"] & \mathbb{R} \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & \mathfrak{sl}(n,\mathbb{R}) \arrow[r, "i"] & \mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R} \arrow[r, "p"] & \mathbb{R} \arrow[r] & 0 \end{tikzcd}$$

where ϕ is an isomorphism.

Therefore, $\mathfrak{gl}(n,\mathbb{R}) \simeq \mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R}$ ■

3. For $n > 1$ prove that $\mathrm{SL}(n, \mathbb{C}) \times \mathbb{C}^\times$ is not isomorphic to $\mathrm{GL}(n, \mathbb{C})$. Prove that $\mathfrak{gl}(n, \mathbb{C})$ is isomorphic to $\mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}$ as Lie-algebras.

As above, consider the centers of the groups:

$$\begin{aligned} Z(\mathrm{SL}(n, \mathbb{C}) \times \mathbb{C}^\times) &= Z(\mathrm{SL}(n, \mathbb{C})) \times Z(\mathbb{C}^\times) = \{tI_n \mid t^n = 1\} \times \mathbb{C}^\times \\ Z(\mathrm{GL}(n, \mathbb{C})) &= \{tI_n \mid t \in \mathbb{C}^\times\} \simeq \mathbb{C}^\times \end{aligned}$$

Clearly these groups are not isomorphic as any map $\phi : Z(\mathrm{SL}(n, \mathbb{C}) \times \mathbb{C}^\times) \to Z(\mathrm{GL}(n, \mathbb{C}))$ will be n-to-one.

We know that $\mathfrak{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$, $\mathfrak{sl}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \mathrm{tr}\,(A) = 0\}$, and $\mathbb{C}$ as a Lie algebra is just $\mathbb{C}$.

Therefore, it suffices to show that

$$M_n(\mathbb{C}) \simeq \{A \in M_n(\mathbb{C}) \mid \mathrm{tr}\,(A) = 0\} \oplus \mathbb{C}$$

Consider the map $\phi : \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}$ given by $\phi(A) = (A - \frac{\mathrm{tr}\,A}{n} I_n, \mathrm{tr}\,(A))$ so

$$\phi(A+B) = (A + B - \frac{\mathrm{tr}\,(A+B)}{n} I_n, \mathrm{tr}\,(A+B)) = (A - \frac{\mathrm{tr}\,A}{n}, \mathrm{tr}\,A) + (B - \frac{\mathrm{tr}\,B}{n}, \mathrm{tr}\,B) = \phi(A) + \phi(B)$$

meaning ϕ is a homomorphism of vector spaces. Clearly it is surjective.

To see that it is injective, let $\phi(A) = \phi(B)$ and suppose $A \neq B$. Then

$$\begin{cases} A - \frac{\mathrm{tr}\,A}{n} I_n = B - \frac{\mathrm{tr}\,B}{n} I_n \\ \mathrm{tr}\,A = \mathrm{tr}\,B \end{cases} \implies A = B$$

This is a contradiction, so ϕ is an isomorphism of vector spaces.

Then we just need to show that $\phi([A, B]) = [\phi(A), \phi(B)]$.

$$\begin{aligned} [\phi(A), \phi(B)] &= \phi(A)\phi(B) - \phi(B)\phi(A) \\ &= \left(A - \frac{\mathrm{tr}\,A}{n} I_n, \mathrm{tr}\,A\right)\left(B - \frac{\mathrm{tr}\,B}{n} I_n, \mathrm{tr}\,B\right) - \left(B - \frac{\mathrm{tr}\,B}{n} I_n, \mathrm{tr}\,B\right)\left(A - \frac{\mathrm{tr}\,A}{n} I_n, \mathrm{tr}\,A\right) \\ &= \left(AB - \frac{\mathrm{tr}\,A}{n} B - \frac{\mathrm{tr}\,B}{n} A + \frac{\mathrm{tr}\,A\,\mathrm{tr}\,B}{n} I_n, \mathrm{tr}\,A\,\mathrm{tr}\,B\right) - \left(BA - \frac{\mathrm{tr}\,B}{n} A - \frac{\mathrm{tr}\,A}{n} B + \frac{\mathrm{tr}\,B\,\mathrm{tr}\,A}{n} I_n, \mathrm{tr}\,B\,\mathrm{tr}\,A\right) \\ &= (AB - BA, 0) \\ &= \left(AB - BA - \frac{\mathrm{tr}\,(AB - BA)}{n} I_n, 0\right) \end{aligned}$$

So ϕ is an isomorphism of Lie algebras.

Therefore, $\mathfrak{gl}(n, \mathbb{C}) \simeq \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}$ ■

4. Embed $i : \mathbb{R}^\times \longrightarrow \mathrm{GL}(3, \mathbb{R})$ via $i(t) = tI_3$. Consider the quotient group $\mathrm{PGL}(3, \mathbb{R}) := \mathrm{GL}(3, \mathbb{R})/\mathbb{R}^\times$. Prove that $\mathfrak{pgl}(3, \mathbb{R})$ is isomorphic to $\mathfrak{sl}(3, \mathbb{R})$.

From Problem 1, $\mathrm{GL}(3, \mathbb{R}) \simeq \mathrm{SL}(3, \mathbb{R}) \times \mathbb{R}^\times$.

Consider the map $\phi : \mathrm{SL}(3, \mathbb{R}) \times \mathbb{R}^\times \to \mathbb{R}^\times$ given by $\phi(A, t) = t$. Clearly, $\ker \phi = \mathrm{SL}(3, \mathbb{R}) \times e_{\mathbb{R}^\times}$. Thus, by the first isomorphism theorem, $\mathrm{GL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{R}) \simeq \mathbb{R}^\times$.

Thus,

$$\mathrm{PGL}(3, \mathbb{R}) \simeq \mathrm{SL}(3, \mathbb{R}) \implies \mathfrak{pgl}(3, \mathbb{R}) \simeq \mathfrak{sl}(3, \mathbb{R}) \quad \blacksquare$$

5. Define the *center* of a Lie algebra to be $\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = 0\}$. Let

$$G = \left\{ \begin{pmatrix} 1 & w & w+w^2 & x \\ 0 & 1 & 2w & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| x, y, z, w \in \mathbb{R} \right\}$$

Calculate the center of $\mathfrak{g}$.

$$\mathfrak{g} = \mathbb{R} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbb{R}W \oplus \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z$$

We can explicitly calculate brackets using Matlab

$$\begin{aligned} [W, X] &= 0 \\ [W, Y] &= X \\ [W, Z] &= X + 2Y \\ [X, Y] &= 0 \\ [X, Z] &= 0 \\ [Y, Z] &= 0 \end{aligned}$$

Therefore,

$$\boxed{Z(\mathfrak{g}) = \left\{ \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\}}$$

6. Inductively define $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_{n+1} = [\mathfrak{g}, \mathfrak{g}_n]$. This is called the *lower central series*. Call a Lie-algebra *nilpotent* if and only if the lower central series stabilizes at the zero subspace. Prove the Lie-algebra in Problem 5 is nilpotent.

From Problem 5,

$$\mathfrak{g}_0 = \mathfrak{g} = \begin{pmatrix} 0 & w & w & x \\ 0 & 0 & 2w & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then we can start explicitly calculating elements of the lower central series:

$$\begin{aligned}
\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] &= \begin{pmatrix} 0 & w & w & x \\ 0 & 0 & 2w & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & a & b \\ 0 & 0 & 2a & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & a & a & b \\ 0 & 0 & 2a & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & w & w & x \\ 0 & 0 & 2w & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & cw - ay - az + dw \\ 0 & 0 & 0 & 2dw - 2az \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & r \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}_1] &= \begin{pmatrix} 0 & w & w & x \\ 0 & 0 & 2w & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & r \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & r \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & w & w & x \\ 0 & 0 & 2w & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & sw \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathfrak{g}_3 = [\mathfrak{g}, \mathfrak{g}_2] &= \begin{pmatrix} 0 & w & w & x \\ 0 & 0 & 2w & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & sw \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & sw \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & w & w & x \\ 0 & 0 & 2w & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Therefore, $\mathfrak{g}$ is nilpotent. ■

7. Prove the Lie algebra of

$$G = \left\{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z, w \in \mathbb{R} \right\}$$

is *not* nilpotent.

Let

$$\mathfrak{g} = \left\{ \begin{pmatrix} -z & 0 & x \\ 0 & z & y \\ 0 & 0 & 0 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

Then let $X = \begin{pmatrix} -z & 0 & x \\ 0 & z & y \\ 0 & 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} -c & 0 & a \\ 0 & c & b \\ 0 & 0 & 0 \end{pmatrix}$. Then,

$$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] = [X, Y] = \begin{pmatrix} 0 & 0 & cx - az \\ 0 & 0 & bz - cy \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix}$$

Now we can take

$$\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}_1] = \begin{pmatrix} -z & 0 & x \\ 0 & z & y \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -z & 0 & x \\ 0 & z & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -rz \\ 0 & 0 & sz \\ 0 & 0 & 0 \end{pmatrix}$$

This is precisely the same form as $\mathfrak{g}_1$ so $\mathfrak{g}_{n \geq 1}$ is of the form $\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$ with $x, y \in \mathbb{R}^\times$. Therefore, $\mathfrak{g}$ is not nilpotent. $\blacksquare$

8. Consider the Lie-group

$$G = \left\{ \begin{pmatrix} e^s & te^s & x \\ 0 & e^s & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, s,t,x,y \in \mathbb{R} \right\}$$

Prove that G possesses a normal two-dimensional abelian subgroup H, whose quotient, G/H, is isomorphic to $\mathbb{R}^2$. On the level of Lie-algebras, show that there exists short-exact sequence of Lie-algebras

$$\mathbb{R}^2 \longrightarrow \mathfrak{g} \longrightarrow \mathbb{R}^2$$

Inductively define $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$. This is called the *derived series*. Call a Lie-algebra *solvable* if and only if its derives series stabilizes at the zero subspace. Is $\mathfrak{g}$ solvable?

Let $H = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x,y \in \mathbb{R} \right\}$. Clearly $H \subseteq G$. Further, it is closed under multiplication because

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a+c \\ 0 & 1 & b+d \\ 0 & 0 & 1 \end{pmatrix}$$

and abelian because

$$\begin{pmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & c+a \\ 0 & 1 & d+b \\ 0 & 0 & 1 \end{pmatrix}$$

Finally, to see that it is normal in G, we note that

$$\begin{pmatrix} e^s & te^s & x \\ 0 & e^s & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^s & te^s & x \\ 0 & e^s & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} e^s te^s & x + ae^s + bte^s & \\ 0 & e^s & y + be^s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-s} & -te^{-s} & -xe^{-s} + yte^{-s} \\ 0 & e^{-s} & -ye^{-s} \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & ae^s + bte^s \\ 0 & 1 & be^s \\ 0 & 0 & 1 \end{pmatrix} \in H$$

By the first isomorphism theorem, it suffices to find a surjective homomorphism ϕ for which $\ker \phi = H$.

A natural choice is

$$\phi : G \to \mathbb{R}^2 \quad \begin{pmatrix} e^s & te^s & x \\ 0 & e^s & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (s,t)$$

We can calculate

$$\phi\left(\begin{pmatrix} e^a & be^a & x \\ 0 & e^a & y \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^c & de^c & w \\ 0 & e^c & z \\ 0 & 0 & 0 \end{pmatrix} \right) = \phi\left(\begin{pmatrix} e^{a+c} & (b+d)e^{a+c} & (w+bx)e^a \\ 0 & e^{a+c} & ze^a \\ 0 & 0 & 0 \end{pmatrix} \right) = (a+c, b+d)$$
$$\phi \begin{pmatrix} e^a & be^a & 0 \\ 0 & e^a & 0 \\ 0 & 0 & 0 \end{pmatrix} \phi \begin{pmatrix} e^c & de^c & 0 \\ 0 & e^c & 0 \\ 0 & 0 & 0 \end{pmatrix} = (a,b)(c,d) = (a+c, b+d)$$

Therefore, ϕ is a homomorphism and clearly it is surjective.

Further,

$$\phi\begin{pmatrix}1 & 0 & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} = (0,0) \implies \ker\phi = \begin{pmatrix}1 & 0 & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} = H$$

Therefore, $G/H \simeq \mathbb{R}^2$.

Further, since $H \trianglelefteq G$, we have the commutative diagram

$$\begin{tikzcd}
H \arrow[r, hook] & G \arrow[r, two heads] \arrow[d, leftrightarrow] & G/H \\
\mathfrak{h} \arrow[r, hook] & \mathfrak{g} \arrow[r, two heads] & \mathfrak{g}/\mathfrak{h}
\end{tikzcd}$$

Since $G/H \simeq \mathbb{R}^2$, we have $\mathfrak{g}/\mathfrak{h} \simeq \mathbb{R}^2$ as well.

Similarly, $H \simeq \mathbb{R}^2$ because $\mathfrak{h} = \mathbb{R}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \oplus \mathbb{R}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$ and

$$\left[\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\right] = 0$$

so all the brackets are trivial.

Together, these give us the short exact sequence of Lie algebras

$$\mathbb{R}^2 \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathbb{R}^2$$

Finally, we can calculate the derived series of $\mathfrak{g}$.

$$\mathfrak{g} = \left\{ \begin{pmatrix}s & t & x\\ 0 & s & y\\ 0 & 0 & 0\end{pmatrix} \middle| s,t,x,y \in \mathbb{R} \right\}$$

Now let $X = \begin{pmatrix}a & b & c\\ 0 & a & d\\ 0 & 0 & 0\end{pmatrix}$ and $Y = \begin{pmatrix}e & f & g\\ 0 & e & h\\ 0 & 0 & 0\end{pmatrix}$. Then,

$$\mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}] = [X,Y] = \begin{pmatrix}0 & 0 & ag-ce+bh-df\\ 0 & 0 & ah-de\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & x\\ 0 & 0 & y\\ 0 & 0 & 0\end{pmatrix}$$

Iterating again,

$$\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}] = \begin{pmatrix}0 & 0 & a\\ 0 & 0 & b\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & c\\ 0 & 0 & d\\ 0 & 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0 & c\\ 0 & 0 & d\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & a\\ 0 & 0 & b\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Therefore, $\mathfrak{g}$ is solvable. $\blacksquare$