Math 1820A Spring 2024 - Homework 3

Instructions: This assignment is worth twenty points. Please complete the following problems assigned below. Submissions with insufficient explanation may lose points due to a lack of reasoning or clarity. If you are handwriting your work, please ensure it is readable and well-formatted for the grader.
Be sure when uploading your work to **assign problems to pages**. Problems with pages not assigned to them **may not be graded**.

Textbook Problems:

Additional Problems: For these problems if you see an S in front of a group, you can assume it means determinant 1, e.g. all elements of $SO(3,\mathbb{R})$ and $SO(2,1)$ have determinant 1.

1. Recall in class we defined the Lie-algebra of a group $G \subset \mathrm{GL}(n,\mathbb{C})$ to be

$$\mathfrak{g} = \{B \in M_n(\mathbb{C}) \,|\, B = \gamma'(0) \text{ where } \gamma \text{ is a smooth path in } G \text{ satisfying } \gamma(0) = 1\}$$

Prove there is a one-to-one correspondence between one-parameter subgroups of G and $\mathfrak{g}$. Note this says that every one-parameter subgroup is in fact exponential.

Let $A(t)$ be a one-parameter subgroup of G. By definition, it satisfies

1. A is continuous
2. $A(0) = 1$
3. $A(t+s) = A(t)A(s)$

By Hall Th. 2.13, there exists a unique $n \times n$ complex matrix X such that

$$A(t) = e^{tX}$$

Differentiating with respect to t and evaluating at $t = 0$, we have

$$A'(0) = Xe^{(0)X} = X$$

i.e. $X \in M_n(\mathbb{C})$ and $X = A'(0)$ where A is a smooth path in G satisfying $A(0) = 1$, so $X \in \mathfrak{g}$.
By the uniqueness of X, we have a one-to-one correspondence between one-parameter subgroups of G and $\mathfrak{g}$. ■

2. Explain briefly why if we have a smooth path $\gamma : \mathbb{R} \to G$ so that $\gamma(0) = 1$, that we may approximate γ by a one-parameter subgroup up to first-order. Note this means that we may define

$$\mathfrak{g} = \{B \in M_n(\mathbb{C}) \,|\, \gamma(t) = e^{tB} \text{ is a smooth path in } G\}$$

Since it is smooth and $\gamma(0) = 1$, we may approximate γ by its first order Maclaurin polynomial

$$\gamma(t) \approx \gamma(0) + t\gamma'(0) = 1 + t\gamma'(0)$$

Notice, however, that up to first order, this is exactly $\exp(t\gamma'(0))$, a one parameter subgroup. Therefore, e^{tB} where $B = \gamma'(0)$ is a good approximation for γ.

3. Let $G = \mathrm{SO}(2,1)$ be the group of determinant 1 matrices in $\mathrm{GL}(3,\mathbb{R})$ such that $(Av, Aw) = (v, w)$ where (v, w) is the Lorentzian product with signature $(+,+,-)$. Calculate $\mathfrak{g}$, and find a basis for $\mathfrak{g}$. Write out an interesting exponential to get a neat one-parameter family in G.

From Hall 2.5.6, a matrix A is in $O(2,1)$ if and only if $A^T g A = g$ (or equivalently, $g^{-1}A^T g = A^{-1}$) where

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Then for X a 3×3 matrix, we have $X \in \mathfrak{g}$ if and only if $\exp(tX) \in \mathrm{SO}(2,1)$, i.e.

$$g^{-1}e^{tX^T}g = A^{-1}$$

notice, however, that

$$g^2 = I \implies g = g^{-1}$$

so we have

$$g^{-1}e^{tX^T}g = ge^{tX^T}g = e^{tgX^Tg} = e^{-tX}$$

This condition holds for all t if $gX^Tg = -X$.
Adding the condition that $\det(A) = 1$, we have $\operatorname{tr} X = 0$. Thus,

$$\mathfrak{g} = \{X \in M_3(\mathbb{R}) \mid gX^Tg = -X \text{ and } \operatorname{tr} X = 0\}$$

These are matrices of the form

$$\begin{pmatrix} 0 & -a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}$$

so we can introduce the basis

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and then find that

$$[A, B] = C$$
$$[A, C] = -B$$
$$[B, C] = -A$$

Then notice that

$$\begin{aligned}\exp(tA) &= I + tA + \frac{1}{2}t^2A^2 - \frac{1}{3!}t^3A - \frac{1}{4!}t^4A^2 + \frac{1}{5!}t^5A + \frac{1}{6!}t^6A^2 + \dots \\ &= A\sin(t) - A^2\cos(t) \\ &= \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

Bonus: [**3 pts**] In the context of Problem 3, prove that a vector $v \in T_pH^+$ if and only if $(v, p) = 0$.

4. Prove that the linear action of G on $\mathbb{R}^3$ in Problem 3 preserves the set $S = \{p \in \mathbb{R}^3 \,|\, (p,p) = -1\}$. Let $H = \mathrm{Stab}(p)$ where $p = (0,0,1)$. Calculate a basis for its Lie-algebra $\mathfrak{h}$.

Let $p = (x,y,z) \in \mathbb{R}^3$. Then the set of points S is given by the set of points for whom

$$(p,p) = x^2 + y^2 - z^2 = -1$$

From calculus, this is a hyperboloid of two sheets. Problem 3 shows that the action of G is rotations in 3-space. Clearly, the linear action of G preserves S since the hyperboloid is symmetric about the z-axis.
In fact, H is given by precisely the one-parameter family from Problem 3:

$$\begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

In problem 3, we showed that $\exp(tA) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since H is a one parameter family, a basis for $\mathfrak{h}$ is given simply by $\{A\}$.

5. Recall the construction of a semi-direct product. Let N and H be groups, and let $\phi : H \longrightarrow \mathrm{Aut}(N)$ be a group homomorphism. Form the semi-direct product $G = N \rtimes_\phi H$ to the following. As a *set*, G is simply $N \times H$. As a *group*, the group law is

$$(n, h)(a, b) := (n\phi_h(a), hb)$$

Prove that N is a normal subgroup of G, and that $G/N \simeq H$. In fact, show that there exists a *section* q in the short exact sequence

$$N \xrightarrow{i} G \xrightarrow{q} H$$

That is, show there exists a $\sigma : H \longrightarrow G$ such that $q\sigma = 1_H$.

N is normal in G iff for all $n \in N$ and $g \in G$, $gng^{-1} \in N$. Let $n \in N$ and $g = (n', h) \in G$.
First, we need to embed $n \mapsto (n, 1) \in G$. Then notice that the inverse of an element $(a, b) \in G$ is given by $(\phi_b^{-1}(a^{-1}), b^{-1})$:

$$(a, b)(\phi_b^{-1}(a^{-1}), b^{-1}) = (a\phi_b(\phi_b^{-1}(a^{-1})), bb^{-1}) = (aa^{-1}, bb^{-1}) = (a, b)$$

Now taking conjugates, let $n \in N$ and $(a, b) \in G$ so

$$\begin{aligned}
(a, b)(n, 1)(a, b)^{-1} &= (a, b)(n, 1)(\phi_b^{-1}(a^{-1}), b^{-1}) \\
&= (a\phi_b(n), b)(a\phi_b^{-1}(a^{-1}), b^{-1}) \\
&= (a\phi_b(n) \cdot \phi_b(\phi_b^{-1}(a^{-1})), bb^{-1}) \\
&= (a\phi_b(n)a^{-1}, 1)
\end{aligned}$$

Now we just take the inverse $(n, h) \mapsto n$ of the initial embedding to get $a\phi_b(n)a^{-1} \in N$.
Since $a \in N$, clearly conjugation is in N and thus N is normal in G. $\blacksquare$

6. Let $\phi : \mathbb{R} \longrightarrow \mathrm{Aut}(\mathbb{R}^2)$ be a homomorphism, and form $G_\phi := \mathbb{R}^2 \rtimes_\phi \mathbb{R}$. For each $v \in \mathbb{R}^2$, and $(n, h) \in G$, define $(n, h)v := \phi(h)v + n$. Prove this defines a group action of G_ϕ on $\mathbb{R}^2$.

To be a group action, $(n, h) \cdot v$ must satisfy

1. $(e_{\mathbb{R}^2}, e_{\mathbb{R}}) \cdot v = v$
2. $(n, h) \cdot ((m, k) \cdot v) = ((n, h)(m, k)) \cdot v \quad$ for $(m, k) \in G$

For the first condition,

$$(e_{\mathbb{R}^2}, e_{\mathbb{R}}) \cdot v = \phi(e)v + e_{\mathbb{R}^2}$$

since homomorphisms preserve identity, $\phi_{\mathbb{R}}(e) = e$, so

$$(e_{\mathbb{R}^2}, e_{\mathbb{R}}) \cdot v = v + (0, 0) = v$$

For the second condition,

$$\begin{aligned} (n, h) \cdot ((m, k) \cdot v) &= (n, h) \cdot (\phi(k)v + m) \\ &= \phi(h)(\phi(k)v + m) + n \\ &= \phi(hk)v + \phi(h)m + n \end{aligned}$$

and

$$\begin{aligned} ((n, h)(m, k)) \cdot v &= (n + \phi(h)m, h + k) \cdot v \\ &= \phi(h + k)v + n + \phi(h)m \\ &= \phi(hk)v + \phi(h)m + n \quad \blacksquare \end{aligned}$$

7. Let $\phi : \mathbb{R} \longrightarrow \mathrm{Aut}(\mathbb{R}^2)$ be defined by

$$\phi(h) = \begin{pmatrix} \cos(h) & -\sin(h) \\ \sin(h) & \cos(h) \end{pmatrix}$$

Prove the group G_ϕ is isomorphic to

$$G = \left\{ \begin{pmatrix} \cos(h) & -\sin(h) & a \\ \sin(h) & \cos(h) & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \, h, a, b \in \mathbb{R} \right\} = \mathrm{Isom}^+(\mathbb{R}^2)$$

Elements of G_ϕ are of the form $(\begin{pmatrix} a \\ b \end{pmatrix}, c)$ where $a, b, c \in \mathbb{R}$. A natural choice of homomorphism $\psi : G_\phi \to G$ is

$$\psi((\begin{pmatrix} a \\ b \end{pmatrix}, c)) = \begin{pmatrix} \cos(c) & -\sin(c) & a \\ \sin(c) & \cos(c) & b \\ 0 & 0 & 1 \end{pmatrix}$$

We need to show that ψ is a homomorphism and that it is bijective.

$$\begin{aligned}
\psi((\begin{pmatrix} a \\ b \end{pmatrix}, c)(\begin{pmatrix} d \\ e \end{pmatrix}), f) &= \psi((\begin{pmatrix} a \\ b \end{pmatrix} + \phi_c(\begin{pmatrix} d \\ e \end{pmatrix}), c + f)) \\
&= \psi((\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} \cos(c)d - \sin(c)e \\ \sin(c)d + \cos(c)e \end{pmatrix}, c + f)) \\
&= \psi((\begin{pmatrix} a + \cos(c)d - \sin(c)e \\ b + \sin(c)d + \cos(c)e \end{pmatrix}, c + f)) \\
&= \begin{pmatrix} \cos(c+f) & -\sin(c+f) & a + \cos(c)d - \sin(c)e \\ \sin(c+f) & \cos(c+f) & b + \sin(c)d + \cos(c)e \\ 0 & 0 & 1 \end{pmatrix} \\
\psi((\begin{pmatrix} a \\ b \end{pmatrix}), c)\psi((\begin{pmatrix} d \\ e \end{pmatrix}, f)) &= \begin{pmatrix} \cos(c) & -\sin(c) & a \\ \sin(c) & \cos(c) & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(f) & -\sin(f) & d \\ \sin(f) & \cos(f) & e \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(c)\cos(f) - \sin(c)\sin(f) & -\cos(c)\sin(f) - \sin(c)\cos(f) & a + d\cos(c) - e\sin(c) \\ \cos(c)\sin(f) + \cos(f)\sin(c) & -\sin(c)\sin(f) + \cos(c)\cos(f) & b + e\cos(c) + d\sin(c) \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(c+f) & -\sin(c+f) & a + d\cos(c) - e\sin(c) \\ \sin(c+f) & \cos(c+f) & b + e\cos(c) + d\sin(c) \end{pmatrix}
\end{aligned}$$

Thus, ψ is a homomorphism. To see that it is a bijection, consider the mapping $\psi^{-1} : G \to G_\phi$ given by

$$\begin{pmatrix} \cos(c) & -\sin(c) & a \\ \sin(c) & \cos(c) & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto (\begin{pmatrix} a \\ b \end{pmatrix}, c)$$

So

$$\psi^{-1}(\psi((\begin{pmatrix} a \\ b \end{pmatrix}, c))) = \psi^{-1}\begin{pmatrix} \cos(c) & -\sin(c) & a \\ \sin(c) & \cos(c) & b \\ 0 & 0 & 1 \end{pmatrix} = (\begin{pmatrix} a \\ b \end{pmatrix}, c)$$

$$\psi(\psi^{-1}\begin{pmatrix} \cos(c) & -\sin(c) & a \\ \sin(c) & \cos(c) & b \\ 0 & 0 & 1 \end{pmatrix}) = \psi((\begin{pmatrix} a \\ b \end{pmatrix}, c)) = \begin{pmatrix} \cos(c) & -\sin(c) & a \\ \sin(c) & \cos(c) & b \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, ψ is a bijective homomorphism and thus an isomorphism. ■

8. Calculate the Lie-algebra of the group G as defined in Problem 7. Show that there is a split-exact sequence of Lie-algebras

$$\mathbb{R}^2 \longrightarrow \mathfrak{g} \longrightarrow \mathbb{R}$$

Let $X \in \mathfrak{g}$. Then e^{tX} is of the form

$$\begin{pmatrix} \cos(t) & -\sin(t) & a \\ \sin(t) & \cos(t) & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \begin{matrix} & R & \end{matrix} & \begin{matrix} a \\ b \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$$

with $R \in \mathrm{SO}(2, \mathbb{R})$.
So $X = \frac{d}{dt}\Big|_{t=0} e^{tX}$ must have the form

$$\begin{pmatrix} Y & \begin{matrix} y_1 \\ y_2 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}$$

Notice that for $n \geq 1$,

$$\begin{pmatrix} Y & \begin{matrix} y_1 \\ y_2 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}^n = \begin{pmatrix} Y^n & \text{---} & Y^{n-1}y \\ | & & | \\ 0 & 0 & 0 \end{pmatrix}$$

where $y = (y_1, y_2)^T$
Thus, e^{tX} is of the form

$$\begin{pmatrix} e^{tY} & \begin{matrix} * \\ * \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$$

Since we need $e^{tY} \in O(2, \mathbb{R})$, we need $Y^T = -Y$. All together,

$$\mathfrak{g} = \left\{ \begin{pmatrix} Y & \begin{matrix} a \\ b \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix} \in M_3(\mathbb{R}) \,\middle|\, Y^T = -Y, \ a, b \in \mathbb{R} \right\}$$

From Problem 5, $\mathbb{R}^2 \trianglelefteq G \simeq \mathbb{R}^2 \rtimes_\phi \mathbb{R}$. Therefore, we have a short exact sequence of groups

$$\mathbb{R}^2 \hookrightarrow G \twoheadrightarrow G/\mathbb{R}^2 = \mathbb{R}$$

Then on the level of lie-algebras, we have a short exact sequence

$$\mathbb{R}^2 \underset{i}{\hookrightarrow} \mathfrak{g} \underset{p}{\twoheadrightarrow} \mathbb{R}$$

(note that this is a slight abuse of notation as the first equation refers to $\mathbb{R}^2, \mathbb{R}$ as groups and the second refers to them as lie-algebras with trivial brackets).
To show the sequence is split exact, we must show that it admits a a section $\sigma : \mathbb{R} \to \mathfrak{g}$ such that $p\sigma = 1$.

Arbitrarily, we can define

$$p\begin{pmatrix} \;\;Y & & a \\ & & b \\ 0 & 0 & 0 \end{pmatrix} = \det Y$$

and

$$\sigma(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then for $t \in \mathbb{R}$,

$$p(\sigma(t)) = p\begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} = \det\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} = 1 \quad \blacksquare$$