Math 1820A: Homework 5

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Problem 1

Finish the proof we started in class that every two dimensional non-abelian Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{aff}(1, \mathbb{R})$, the Lie-algebra generated by X and Y defined by $[Y, X] = X$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be non-commutative 2-dimensional algebras. We seek to show that $\mathfrak{g} \simeq \mathfrak{h} \simeq \mathfrak{aff}(1, \mathbb{R})$.

It suffices to show that $\mathfrak{g} \simeq \mathfrak{aff}(1, \mathbb{R}) = \mathbb{R}X \oplus \mathbb{R}Y$ with $[Y, X] = X$. Let $\mathfrak{g} = \mathbb{R}A \oplus \mathbb{R}B$. Suppose $[A, B] = xA + yB$. Since $\mathfrak{g}$ is not commutative, x and y are not both zero.

WLOG, suppose $y = 0$. Then $[A, B] = xA$ for $x \neq 0$ so $[A, B/x] = A$. By symmetry, $[B/x, A] = -A$. We make the substitutions $A' = A, B' = -B/x$ and define $\phi : \begin{matrix} A' \mapsto X \\ B' \mapsto Y \end{matrix}$.

Then

$$[B', A'] = [-\frac{B}{x}, A] = A = A'$$

and we are done.

Now we check the case $y \neq 0$, i.e. $[A, B] = xA + yB$.

We make the substitutions $A' = A + \frac{y}{x}B$, $B' = -\frac{B}{x}$ and define $\phi : \begin{matrix} A' \mapsto X \\ B' \mapsto Y \end{matrix}$ so

$$\begin{aligned}
[X, Y] \sim [A', B'] &= [A + \frac{y}{x}B, -\frac{B}{x}] \\
&= [A, -\frac{B}{x}] + [\frac{y}{x}B, -\frac{B}{x}] \\
&= -\frac{1}{x}[A, B] - \frac{y}{x^2}[B, B] \\
&= -\frac{1}{x}(xA + yB) \\
&= -A - \frac{y}{x}B \\
&= -A' \\
[Y, X] \sim [B', A'] = A'
\end{aligned}$$

and we are done. ■

Problem 2

Let $\mathfrak{g}$ be a Lie-algebra and $\mathfrak{h}$ an ideal. Prove that $\mathfrak{g}/\mathfrak{h}$ is abelian if and only if $\mathfrak{h}$ contains the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$.

We want to show that $\mathfrak{g}/\mathfrak{h} \simeq \mathbb{R}^n$ iff $[\mathfrak{g}, \mathfrak{g}] \in \mathfrak{h}$.

If $\mathfrak{g}/\mathfrak{h}$ is abelian, then $[\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h}] = 0$. Let $X, Y \in \mathfrak{g}$ so we can write

$$\begin{aligned}
[\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h}] &= [X + \mathfrak{h}, Y + \mathfrak{h}] \\
&= [X, Y] + [\mathfrak{h}, Y] + [X, \mathfrak{h}] + [\mathfrak{h}, \mathfrak{h}] \\
&= [X, Y] - [Y, \mathfrak{h}] + [X, \mathfrak{h}] + [\mathfrak{h}, \mathfrak{h}] \\
&= [X, Y] + [\mathfrak{h}, \mathfrak{h}]
\end{aligned}$$

Then if $[\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h}] = 0$, we must have that $[X, Y] = [\mathfrak{g}, \mathfrak{g}] \in \mathfrak{h}$.

Conversely, if $[\mathfrak{g}, \mathfrak{g}] \in \mathfrak{h}$, then for all $X, Y \in \mathfrak{g}$, we have that $[X, Y] \in \mathfrak{h}$. Then

$$\begin{aligned}
[\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h}] &= [X + \mathfrak{h}, Y + \mathfrak{h}] \\
&= [X, Y] + [\mathfrak{h}, Y] + [X, \mathfrak{h}] + [\mathfrak{h}, \mathfrak{h}] \\
&= [X, Y] - [Y, \mathfrak{h}] + [X, \mathfrak{h}] + [\mathfrak{h}, \mathfrak{h}] \\
&= [X, Y] + [\mathfrak{h}, \mathfrak{h}] \\
&= [\mathfrak{h}, \mathfrak{h}]
\end{aligned}$$

But this implies that $X + \mathfrak{h}, Y + \mathfrak{h} \in \mathfrak{h} \implies X, Y \in \mathfrak{h}$ so $\mathfrak{g} = \mathfrak{h}$.

Let $g \in \mathfrak{g}$ and $h \in \mathfrak{h}$. Since $g \in \mathfrak{h}$ too,

$$ghg^{-1}h^{-1} \in \mathfrak{h}$$

so $\mathfrak{g}/\mathfrak{h}$ is abelian. $\blacksquare$

Problem 3

Let $\mathfrak{g}$ be the Lie-algebra of a connected Lie-group G. If $\mathfrak{g}$ is abelian, prove that G is abelian. (You may use the fact that $\exp(\mathfrak{g}) \subset G$ generates G as a group.)

Let $\mathfrak{g}$ be the abelian lie algebra of a connected lie group G. Let $A, B \in G$.

Since $\exp(\mathfrak{g}) \subset G$ generates G, there must be some $X_1, X_2, \ldots X_n, Y_1, Y_2, \ldots, Y_m \in \mathfrak{g}$ such that $\prod_{i=1}^{n} \exp(X_i)^{k_i} = A$ and $\prod_{i=1}^{m} \exp(Y_i)^{j_i} = B$ with $k_1, \ldots, k_n, j_1, \ldots, j_n \in \mathbb{N}$.

Since $\mathfrak{g}$ is abelian, we have

$$\prod_{i=1}^{n} \exp(X_i)^{k_i} = \exp\left(\sum_{i=1}^{n} k_i X_i\right) = A$$
$$\prod_{i=1}^{n} \exp(Y_i)^{j_i} = \exp\left(\sum_{i=1}^{m} j_i Y_i\right) = B$$

Since $\mathfrak{g}$ is closed under addition and scalar multiplication, we may make the substitutions,

$$X' = \sum_{i=1}^{n} k_i X_i$$
$$Y' = \sum_{i=1}^{m} j_i Y_i$$

So

$$AB = \exp(X') \exp(Y')$$

and since $\mathfrak{g}$ is abelian, $[X', Y'] = 0 \implies \exp(X') \exp(Y') = \exp(X' + Y')$ so

$$AB = \exp(X' + Y') = \exp(Y') \exp(X') = BA$$

Therefore, G is abelian. $\blacksquare$

Problem 4

Prove that $\exp : \mathfrak{g} \longrightarrow G$ is surjective if $\mathfrak{g}$ is abelian. (Note these exercises effectively prove that if G is connected, compact, and abelian, then it is an n-torus)

Exactly as in Problem 3, let $\mathfrak{g}$ be the abelian lie algebra of a connected lie group G. Let $A \in G$.

Since $\exp(\mathfrak{g}) \subset G$ generates G, there must be some $X_1, X_2, \ldots X_n \in \mathfrak{g}$ such that

$$\prod_{i=1}^{n} \exp(X_i)^{k_i} = A$$

Since $\mathfrak{g}$ is abelian, we have

$$\prod_{i=1}^{n} \exp(X_i)^{k_i} = \exp\left(\sum_{i=1}^{n} k_i X_i\right) = A$$

Since this argument holds for all $A \in G$, every element of G is the image of some element in $\mathfrak{g}$ under the exponential map. Hence, exp is surjective if $\mathfrak{g}$ is abelian and G is connected. ■

Problem 5

Let G be the lie-group

$$G = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c, d, e \in \mathbb{R} \right\}$$

Calculate the lie-algebra $\mathfrak{g}$. Prove that the exponential map $\exp : \mathfrak{g} \longrightarrow G$ is a bijection. Is it a group-homomorphism?

$$\mathfrak{g} = \mathbb{R}A \oplus \mathbb{R}B \oplus \mathbb{R}C \oplus \mathbb{R}D \oplus \mathbb{R}E$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

then calculating brackets, we get

$$\begin{aligned} [A, B] &= 0 \\ [A, C] &= 0 \\ [A, D] &= C \\ [A, E] &= 0 \\ [B, C] &= 0 \\ [B, D] &= 0 \\ [B, E] &= C \\ [C, D] &= 0 \\ [C, E] &= 0 \\ [D, E] &= 0 \end{aligned}$$

Let $X \in \mathfrak{g}$. Then

$$X = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X^2 = \begin{pmatrix} 0 & 0 & 0 & ad + be \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so

$$\exp(X) = I + X + \frac{1}{2}X^2 = \begin{pmatrix} 1 & a & b & c + \frac{ad+be}{2} \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Trivially, the map

$$\exp : \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & b & c + \frac{ad+be}{2} \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a bijection.

However, it is not a group homomorphism: Let $Y = \begin{pmatrix} 0 & x & y & z \\ 0 & 0 & 0 & v \\ 0 & 0 & 0 & w \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then

$$\begin{aligned}
\exp(X)\exp(Y) &= \begin{pmatrix} 1 & a & b & c + \frac{ad+be}{2} \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y & cz + \frac{xw+yv}{2} \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & a+x & b & c + z + \frac{ad+be}{2} + av + bw + \frac{vx+wy}{2} \\ 0 & 1 & 0 & d + w \\ 0 & 0 & 1 & e + v \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\exp(X+Y) &= \begin{pmatrix} 1 & a+x & b+y & c + z + \frac{ad+be}{2} + \frac{av+bw}{2} + \frac{dx+ey}{2} + \frac{vx+wy}{2} \\ 0 & 1 & 0 & d + v \\ 0 & 0 & 1 & e + w \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Problem 6

Write G in Problem 5 as a non-split central extension of $\mathbb{R}^4$ by $\mathbb{R}$ and a split non-central extension of $\mathbb{R}^2$ by $\mathbb{R}^3$. That is, show there exists short exact sequences

$$\mathbb{R} \xrightarrow{i_1} G \xrightarrow{p_1} \mathbb{R}^4 \text{ and } \mathbb{R}^3 \xrightarrow{i_2} G \xrightarrow{p_2} \mathbb{R}^2$$

where the map $p_1 : G \longrightarrow \mathbb{R}^4$ admits no section and the map $i_1 : \mathbb{R} \longrightarrow G$ takes its image in the center of G, and, for the second sequence, the map $p_2 : G \longrightarrow \mathbb{R}^2$ admits a section, and the map $i_2 : \mathbb{R}^3 \longrightarrow G$ takes it image outside the center of G.

From problem 5, we have

$$G = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

First, we will construct the sequence

$$\mathbb{R} \hookrightarrow G \twoheadrightarrow \mathbb{R}^4$$

The center of G is the set of matrices of the form

$$Z(G) = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus for $\operatorname{im} i = Z(G)$, we can take $i : \mathbb{R} \to G$ to be

$$i(x) = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and clearly,

$$i(x+y) = \begin{pmatrix} 1 & 0 & 0 & x+y \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & y \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = i(x)i(y)$$

Then for $\ker p = \operatorname{im} i$, the obvious choice for $p : G \to \mathbb{R}^4$ is

$$p \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} = (a, b, d, e)$$

and again,

$$p\begin{pmatrix}1 & a & b & c\\ 0 & 1 & 0 & d\\ 0 & 0 & 1 & e\\ 0 & 0 & 0 & 1\end{pmatrix} p\begin{pmatrix}1 & x & y & z\\ 0 & 1 & 0 & r\\ 0 & 0 & 1 & s\\ 0 & 0 & 0 & 1\end{pmatrix} = (a,b,d,e)+(x,y,r,s) = (a+x, b+y, d+r, e+s)$$

$$p\left(\begin{pmatrix}1 & a & b & c\\ 0 & 1 & 0 & d\\ 0 & 0 & 1 & e\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x & y & z\\ 0 & 1 & 0 & r\\ 0 & 0 & 1 & s\\ 0 & 0 & 0 & 1\end{pmatrix}\right) = p\begin{pmatrix}1 & a+x & b+y & c+z+ar+bs\\ 0 & 1 & 0 & d+r\\ 0 & 0 & 1 & e+s\\ 0 & 0 & 0 & 1\end{pmatrix} = (a+x, b+y, d+r, e+s)$$

so we do have a short exact sequence. However, p admits no section: Let

$$\sigma(a,b,d,e) = \begin{pmatrix}1 & a & b & f(a,b,d,e)\\ 0 & 1 & 0 & d\\ 0 & 0 & 1 & e\\ 0 & 0 & 0 & 1\end{pmatrix}$$

for some function $f:\mathbb{R}^4\to\mathbb{R}$. Then,

$$p\sigma(a,b,d,e) = p\begin{pmatrix}1 & a & b & f(a,b,d,e)\\ 0 & 1 & 0 & d\\ 0 & 0 & 1 & e\\ 0 & 0 & 0 & 1\end{pmatrix} = (a,b,d,e) \quad \checkmark$$

But σ is not a homomorphism:

$$\sigma(a+x, b+y, d+z, e+w) = \begin{pmatrix}1 & a+x & b+y & f(*)\\ 0 & 1 & 0 & d+z\\ 0 & 0 & 1 & e+w\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$\sigma(a,b,d,e)\sigma(x,y,z,w) = \begin{pmatrix}1 & a & b & f(a,b,d,e)\\ 0 & 1 & 0 & d\\ 0 & 0 & 1 & e\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x & y & f(x,y,z,w)\\ 0 & 1 & 0 & z\\ 0 & 0 & 1 & w\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & a_x & b+y & f(a,b,d,e)+f(x,y,z,w)+bw+az\\ 0 & 1 & 0 & d+z\\ 0 & 0 & 1 & e+w\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$\sigma(a+x, b+y, d+z, e+w) \neq \sigma(a,b,d,e)\sigma(x,y,z,w)$$

Now, we will construct the sequence

$$\mathbb{R}^3 \hookrightarrow G \twoheadrightarrow \mathbb{R}^2$$

We want $i:\mathbb{R}^3\to G$ to take its image outside the center of G. Let

$$i(x,y,z) = \begin{pmatrix}1 & x & 0 & y\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & z\\ 0 & 0 & 0 & 1\end{pmatrix} \notin Z(G)$$

Then let

$$p\begin{pmatrix}1 & a & b & c\\ 0 & 1 & 0 & d\\ 0 & 0 & 1 & e\\ 0 & 0 & 0 & 1\end{pmatrix} = (b,d)$$

so

$$\ker p = \begin{pmatrix}1 & a & 0 & c\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & e\\ 0 & 0 & 0 & 1\end{pmatrix} = \operatorname{im} i$$

Then both i and p are homomorphisms:

$$i(x+a,y+b,z+c) = \begin{pmatrix}1 & x+a & 0 & y+b\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & z+c\\ 0 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & x & 0 & y\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & z\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & a & 0 & b\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & c\\ 0 & 0 & 0 & 1\end{pmatrix} = i(x,y,z)i(a,b,c)$$

$$p\left(\begin{pmatrix}1 & a & b & c\\ 0 & 1 & 0 & d\\ 0 & 0 & 1 & e\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x & y & z\\ 0 & 1 & 0 & q\\ 0 & 0 & 1 & r\\ 0 & 0 & 0 & 1\end{pmatrix}\right) = p\begin{pmatrix}1 & a+x & b+y & *\\ 0 & 1 & 0 & d+q\\ 0 & 0 & 1 & e+r\\ 0 & 0 & 0 & 1\end{pmatrix} = (a+x,e+r)$$

$$p\begin{pmatrix}1 & a & b & c\\ 0 & 1 & 0 & d\\ 0 & 0 & 1 & e\\ 0 & 0 & 0 & 1\end{pmatrix} + p\begin{pmatrix}1 & x & y & z\\ 0 & 1 & 0 & q\\ 0 & 0 & 1 & r\\ 0 & 0 & 0 & 1\end{pmatrix} = (a,e)+(x,r) = (a+x,e+r)$$

Finally, p admits a section. Let

$$\sigma(a,b) = \begin{pmatrix}1 & a & 0 & a+b\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & b\\ 0 & 0 & 0 & 1\end{pmatrix}$$

Then

$$p\sigma(a,b) = p\begin{pmatrix}1 & a & 0 & a+b\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & b\\ 0 & 0 & 0 & 1\end{pmatrix} = (a,b) \quad \checkmark$$

and σ is a homomorphism:

$$\sigma(a+x,b+y) = \begin{pmatrix}1 & a+x & 0 & a+x+b+y\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & b+y\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$\sigma(a,b)\sigma(x,y) = \begin{pmatrix}1 & a & 0 & a+b\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & b\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x & 0 & x+y\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & y\\ 0 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & a+x & 0 & a+x+b+y\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & b+y\\ 0 & 0 & 0 & 1\end{pmatrix}$$

Problem 7

Let $U_3 \subset \mathrm{GL}(3, \mathbb{R})$ denote the invertible upper triangular matrices. Prove that H, the Heisenberg group, sits normally inside U_3 and has an abelian quotient. Is this short exact sequence split?

H is the group of matrices of the form

$$A = \begin{pmatrix} 1 & c & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

Let $B = \begin{pmatrix} x & y & z \\ 0 & w & v \\ 0 & 0 & u \end{pmatrix} \in U_3$. Then

$$BAB^{-1} = \begin{pmatrix} x & y & z \\ 0 & w & v \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} 1 & c & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{x} & -\frac{y}{xw} & \frac{yv-wz}{xwu} \\ 0 & \frac{1}{w} & -\frac{v}{wu} \\ 0 & 0 & \frac{1}{u} \end{pmatrix} = \begin{pmatrix} 1 & \frac{cx}{w} & \frac{awx-cvx+bwy}{uw} \\ 0 & 1 & \frac{bw}{u} \\ 0 & 0 & 1 \end{pmatrix} \in H$$

Therefore, $H \trianglelefteq U_3$.

U_3/H is abelian if for any $g \in U_3$ and $h \in H$,

$$ghg^{-1}h^{-1} \in H$$

Using A and B above (and the normality calculation) we may calculate this product explicitly:

$$\begin{aligned} BAB^{-1}A^{-1} &= \begin{pmatrix} 1 & \frac{cx}{w} & \frac{awx-cvx+bwy}{uw} \\ 0 & 1 & \frac{bw}{u} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -c & bc-a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\frac{cw-cx}{w} & -\frac{auw-awx+cvx-bwy-bcuw+bcux}{uw} \\ 0 & 1 & -\frac{b(u-w)}{u} \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Clearly, this is in H so the quotient is abelian.

This gives us the short exact sequence

$$H \overset{i}{\hookrightarrow} U_3 \overset{p}{\twoheadrightarrow} U_3/H$$

Then we need $\ker p = \operatorname{im} i$. Since the natural i is simple inclusion, we need a homomorphism $p : U_3 \to U_3/H$ whose kernel is

$$\begin{pmatrix} 1 & c & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

Let $p : U_3 \to \mathbb{R}^3$ by

$$p \begin{pmatrix} x & y & z \\ 0 & w & v \\ 0 & 0 & u \end{pmatrix} \mapsto (x, w, u)$$

Then $\ker p = H$ and the map is surjective and a homomorphism:

$$p\left(\begin{pmatrix} x_1 & y_1 & z_1 \\ 0 & w_1 & v_2 \\ 0 & 0 & u_2 \end{pmatrix}\begin{pmatrix} x_2 & y_2 & z_2 \\ 0 & w_2 & v_2 \\ 0 & 0 & u_2 \end{pmatrix}\right) = p\begin{pmatrix} x_2x_1 & * & * \\ 0 & w_1w_2 & * \\ 0 & 0 & u_2u_1 \end{pmatrix} = (x_2x_1, w_1w_2, u_2u_1)$$

$$p\begin{pmatrix} x_1 & y_1 & z_1 \\ 0 & w_1 & v_2 \\ 0 & 0 & u_2 \end{pmatrix} p\begin{pmatrix} x_2 & y_2 & z_2 \\ 0 & w_2 & v_2 \\ 0 & 0 & u_2 \end{pmatrix} = (x_1, w_1, u_1)(x_2, w_2, u_2) = (x_1x_2, w_1w_2, u_1u_2)$$

so $U_3/\ker p = U_3/H \simeq \mathbb{R}^3$.

So $p : U_3 \to U_3/H$ and the sequence is short exact as desired. To show it is split, we need to find a homomorphism $\sigma : U_3/H \to U_3$ such that $p\sigma = 1_{\mathbb{R}^3}$.

Let

$$\sigma(x, y, z) = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$$

so

$$p\sigma(x, y, z) = p\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} = (x, y, z)$$

Further, this is a homomorphism:

$$\sigma(xa, yb, zc) = \begin{pmatrix} xa & 0 & 0 \\ 0 & yb & 0 \\ 0 & 0 & zc \end{pmatrix}$$

$$\sigma(x, y, z)\sigma(a, b, c) = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} xa & 0 & 0 \\ 0 & yb & 0 \\ 0 & 0 & zc \end{pmatrix}$$

So we are done. $\blacksquare$