Math 1820A Spring 2024 - Homework 10

Instructions: This assignment is worth twenty points. Please complete the following problems assigned below. Submissions with insufficient explanation may lose points due to a lack of reasoning or clarity. If you are handwriting your work, please ensure it is readable and well-formatted for the grader.

Be sure when uploading your work to **assign problems to pages**. Problems with pages not assigned to them **may not be graded**.

Additional Problems: For these problems let $\mathbb{H}$ denote the algebra of Hamiltonians and $\mathbb{H}^\times$ denote all the non-zero elements as a *group* under multiplication. For the context of this homework, assume a 'rotation' is orientation preserving, and a 'reflection' is preserves a codimension 1 subspace. By a circle on an n-sphere we mean the intersection of an affine plane with S^n. By a circle in $\mathbb{R}^n$ we mean a typical Euclidean circle, or, also a line.

1. Let $H : S^3 \longrightarrow S^2$ be the Hopf-fibration map as in class, namely $H(q) = q\hat{k}q^{-1}$ where here we are identifying $S^3 \subset \mathbb{H}$ with the 3-sphere of unit quaternions. Prove that the fiber over a point $v \neq -\hat{k}$ in S^2 is given by

$$H^{-1}(v) = \frac{1}{\sqrt{2+2c}}\left(1 + c - b\hat{\imath} + a\hat{\jmath}\right) S$$

where $v = (a, b, c) \in S^2$ and $S = \mathrm{Stab}_{\hat{k}} = \left\{\cos(\theta) + \sin(\theta)\hat{k} \,|\, \theta \in S^1\right\}$. (Note in class we technically proved that everything in here is in the fiber, but we didn't prove that was entire fiber)

Together with what we proved in class, it suffices to show that this expression is the entire fibre, i.e. there does not exist $p \in S^3$ for which $H(p) = v = (a, b, c)$ but where

$$p \neq \frac{1}{\sqrt{2+2c}}\left(1 + c - b\hat{\imath} + a\hat{\jmath}\right)\left(\cos\theta + \sin\theta\hat{k}\right)$$

for any $\theta \in S^1$.

Suppose such a p exists. Then it must be the fibre of some $v \in S^2$, i.e. $p = H^{-1}(v)$.

However, if we have any point $p \in S^3$ which is in the fibre of v, then we can write the entire fibre as

$$H^{-1}(v) = p\mathrm{Stab}(\hat{k}) = p(\cos\theta + \sin\theta\hat{k})$$

because the Hopf fibration is a principal S^1-bundle over S^2 so for $s \in S^1$,

$$H(qs) = qs\hat{k}(qs)^{-1} = qs\hat{k}s^{-1}q^{-1} = q\hat{k}q^{-1} = H(q)$$

and for $q_1, q_2 \in S^3$,

$$H(q_1q_2) = q_1q_2\hat{k}q_2^{-1}q_1^{-1} = q_1H(q_2)q_1^{-1} = I_{q_1}(H(q_2))$$

is a rotation of $H(q_2)$ about the vector part of q_1.

In class, we defined q as the quaternion in S^3 which rotated $\hat{k}$ to v for all $v \neq -\hat{k}$ so $H^{-1}(v) = q\hat{k}S = qS$.

Since $p \neq q$, we have

$$pS = H^{-1}(v) = qS$$

but this is a contradiction. ■

2. Let $U \subset S^2$ be defined by $U = S^2 \setminus -\hat{k}$. Construct a map from $\Phi : H^{-1}(U) \longrightarrow U \times S^1$ via

$$\Phi(q) = (v, \theta)$$

where $v = H(q) \in S^2$ and $\theta \in S^1$ is the unique $\theta \in S^1$ for which

$$q = \frac{1}{\sqrt{2+2c}} \left(1 + c - b\hat{i} + a\hat{j}\right) \left(\cos(\theta) + \sin(\theta)\hat{k}\right)$$

Prove this map is bijective, thus $\Phi^{-1} : H^{-1}(U) \longrightarrow U \times S^1$ exists.

We first show that Φ is injective.

Let $v_1 = H(q_1)$ and $v_2 = H(q_2)$ associated to $\theta_1, \theta_2 \in S^1$ respectively for $q_1, q_2 \in S^3$.

Suppose $v_1 = v_2$. By definition, the Hopf fibration takes circles in S^3 to points in S^2. Thus, q_1 and q_2 are on the same circle in S^3, i.e. they are the same up to θ.

If $\theta_1 = \theta_2$, then $q_1 = q_2$ and $\Phi(q_1) = \Phi(q_2)$.

If $\theta_1 \neq \theta_2$ then $(v_1, \theta_1) \neq (v_2, \theta_2)$ and $\Phi(q_1) \neq \Phi(q_2)$ by definition. Therefore, $\Phi(q_1) = \Phi(q_2)$ iff $q_1 = q_2$ and $\theta_1 = \theta_2$. Hence, it is injective.

Now we want to show surjectivity. Since it is a fibre bundle, H is surjective. Further, we are certainly free to choose any $\theta \in [0, 2\pi)$ since for a given $v \in U$, all the points in the fibre are the same up to θ. Thus, for any $(v, \theta) \in U \times S^1$, we can find a $q \in S^3$ such that $H(q) = v$ and q is associated to a unique θ.

Since Φ is surjective and injective, it is bijective. $\blacksquare$

3. Show that Φ as defined in Problem 2 satisfies the following commutative diagram.

$$\begin{tikzcd} H^{-1}(U) \arrow[r, "\Phi"] \arrow[dr, "H"'] & U \times S^1 \arrow[d, "p_1"] \\ & U \end{tikzcd}$$

where $p_1 : U \times S^1 \longrightarrow U$ is projection onto the first factor. Moreover, explain why both Φ and Φ^{-1} as defined in Problem 2 are both continuous.

If we define $p_1 : U \times S^1 \to U$ by $p_1(v, \theta) = v$, then

$$p_1 \circ \Phi(q) = p_1(v, \theta) = v$$

for all $q \in H^{-1}(U)$.

Let $v \in U \subset S^2$. Then

$$H(H^{-1}(v)) = H(q) = v$$

Thus, the diagram commutes.

Since the fibres of the Hopf map are circles in S^3 parameterized by θ, we can see that Φ is continuous.

For Φ^{-1} we can see

$$\Phi^{-1}(v, \theta) = \frac{1}{\sqrt{2 + 2c}}(1 + c - b\hat{\imath} + a\hat{\jmath})(\cos\theta + \hat{k}\sin\theta)$$

has a discontinuity only at $c \geq -1$. Since $v = (a, b, c) \in U$,

$$c = -1 \implies v = (0, 0, -1)$$

but $U = S^2 \setminus -\hat{k}$ so this is out of the domain and thus Φ^{-1} is continuous for all points in $U \times S^1$. $\blacksquare$

4. Let $V \subset S^2$ be defined by $V = S^2 \setminus \hat{k}$. Construct a function $\Psi : H^{-1}(V) \longrightarrow V \times S^1$ that enjoys the same properties that Φ does, namely construct a $\Psi : H^{-1}(V) \longrightarrow V \times S^1$ that is a continuous bijection with a continuous inverse which satisfies the diagram below

$$\begin{tikzcd} H^{-1}(V) \arrow[r, "\Psi"] \arrow[dr, "H"'] & V \times S^1 \arrow[d, "p_1"] \\ & V \end{tikzcd}$$

Consider $\Psi(q) = (v, \theta)$ where $v = H(q) \in V$ and $\theta \in S^1$ is the unique θ for which

$$q = \sqrt{\frac{1-c}{2}}\left(1 - \frac{a}{1-c}\hat{\jmath} + \frac{b}{1-c}\hat{k}\right)(\cos\theta + \sin\theta\hat{k})$$

Clearly, Ψ is a map $H^{-1}(V) \to V \times S^1$ since $q \in S^3$:

$$\begin{aligned}
||q|| &= \left|\left|\sqrt{\frac{1-c}{2}}\left(1 - \frac{a}{1-c}\hat{\jmath} + \frac{b}{1-c}\hat{k}\right)(\cos\theta + \sin\theta\hat{k})\right|\right| \\
&= \left|\left|\sqrt{\frac{1-c}{2}}\left(\cos\theta + \sin\theta\hat{k} - \frac{a}{1-c}\cos\theta\hat{\jmath} - \frac{a}{1-c}\sin\theta\hat{k}\hat{\jmath} + \frac{b}{1-c}\cos\theta\hat{k} + \frac{b}{1-c}\sin\theta\hat{k}\hat{k}\right)\right|\right| \\
&= \left|\left|\sqrt{\frac{1-c}{2}}\left((\cos\theta - \frac{b}{1-c}\sin\theta) + \frac{a}{1-c}\sin\theta\hat{\imath} - \frac{a}{1-c}\cos\theta\hat{k} + (\sin\theta + \frac{b}{1-c}\cos\theta)\hat{k}\right)\right|\right| \\
&= \sqrt{-\frac{a^2+b^2+c^2-2c+1}{2c-2}} \\
&= \sqrt{-\frac{-2c+2}{2c-2}} = 1
\end{aligned}$$

for all $(a, b, c) \in V = S^2 \setminus \hat{k}$

Further, for $v \in V$, $H(H^{-1}(v)) = H(q) = v$ and

$$p_1(\Psi(H^{-1}(v))) = p_1(\Psi(q)) = p_1(v, \theta) = v$$

so the diagram commutes.

It is bijective and continuous for the same reasons as Φ.

Its inverse $\Psi^{-1} : V \times S^1 \to H^{-1}(V)$ is given by

$$\Psi^{-1}(v, \theta) = \sqrt{\frac{1-c}{2}}\left(1 - \frac{a}{1-c}\hat{\jmath} + \frac{b}{1-c}\hat{k}\right)(\cos\theta + \sin\theta\hat{k})$$

which has discontinuities only at $v = (0, 0, 1)$ which is not in V.

5. Let $w \in U \cap V$. Consider the composition $(U \cap V) \times S^1 \longrightarrow (U \cap V) \times S^1$ given by $\Psi \circ \Phi^{-1} : (U \cap V) \times S^1 \longrightarrow (U \cap V) \times S^1$. Prove that the composition $\Psi \circ \Phi^{-1}$ is of the form,

$$(\Psi \circ \Phi^{-1})(w, \theta) = (w, g_v(\theta))$$

where $g_w : S^1 \longrightarrow S^1$ is a symmetry of S^1 for each choice of $w \in U \cap V$. Explicitly calculate this g_w for each choice of $w \in U \cap V$.

Suppose $\theta \in S^1$ is the unique angle corresponding to each $w \in U \cap V = S^2 \setminus \{\hat{k}, -\hat{k}\}$.

So

$$(\Psi \circ \Phi^{-1})(w, \theta) = \Psi(\Phi^{-1}(w, \theta)) = \Psi(H^{-1}(w)) = (H(H^{-1}(w)), \phi) = (w, \phi)$$

where $\phi = g_w(\theta)$ is some symmetry of S^1 because the quaternion is determined only up to phase in the fibre.

Explicitly, we want to find the rotation between $q = \Phi^{-1}(w, \theta)$ and $p = \Psi^{-1}(w, \phi)$. Since they are in the same fibre, they are rotations within a single circle.

We know

$$qwq^{-1} = pwp^{-1} \implies p^{-1}qwq^{-1}p = (p^{-1}q)w(p^{-1}q)^{-1} = w$$

Thus $(p^{-1}q)$ is a rotation fixing w, i.e.

$$p^{-1}q = \cos\frac{\theta}{2} + w \sin\frac{\theta}{2}$$

But indeed, this is precisely what we want! This θ is the angle between p and q clockwise in the plane of the fibre.

Thus, for $w = (a, b, c) \in U \cap V$,

$$g_w(\theta) = (2\text{acos}(\theta), 0, 0, 0) \cdot p^{-1}q$$

where

$$p = \sqrt{\frac{1-c}{2}} \left(1 - \frac{a}{1-c}\hat{\jmath} + \frac{b}{1-c}\hat{k}\right)$$
$$q = \frac{1}{\sqrt{2+2c}}(1 + c - b\hat{\imath} + a\hat{\jmath})$$

6. Let $C \subset S^2 \subset \mathbb{H}$ be defined as the set of all points in $\mathbb{H}$ of unit norm spanned by $\{\hat{\imath}, \hat{\jmath}\}$. Consider $H^{-1}(C) \subset S^3$ under stereographic projection $F : S^3 \setminus \hat{k} \longrightarrow \mathbb{R}^3$. Prove the image of $H^{-1}(C)$ under F is a torus in $\mathbb{R}^3$.

By definition,

$$C = \{q \in \mathbb{H} \mid \; ||q|| = 1,\; q = \mathbb{R}\hat{\imath} + \mathbb{R}\hat{\jmath}\} = \{a\hat{\imath} + b\hat{\jmath} \mid a, b \in \mathbb{R},\; a^2 + b^2 = 1\}$$

By the norm constraint, $a, b \in [0, 1]$ so we can write

$$C = \{\cos\theta\hat{\imath} + \sin\theta\hat{\jmath} \mid \theta \in S^1\}$$

which is clearly a circle in S^2.

H is a fibre bundle so

$$H^{-1}(C) = C \times S^1 = S^1 \times S^1 \simeq T^2 \subset S^3$$

Since $H^{-1}(C)$ is a torus contained in S^3, it is equivalent to the Clifford Torus which stereographically projects onto the torus of rotation in $\mathbb{R}^3$ $\blacksquare$

7. Using Problem 6, explain why S^3 is one solid torus glued to another.

Consider S^3 as $\mathbb{R}^3$ plus a point at infinity (as under the stereographic projection).

Let T be a solid torus in $\mathbb{R}^3$ centered on the z-axis and consider $\mathbb{R}^3 \setminus T$.

We want to show that what remains is another solid torus.

Consider the hole left by T. Projected onto the plane, this hole is a disk. From problem 6, we know that the image of the fibre over a circle is a torus so the image of the fibre over a disk under stereographic projection will be a solid torus.

Locally, we can consider each the fibre of each point on the disk as a vertical line in $\mathbb{R}^3$ which is a circle in S^3. Since we want these to be disjoint, the circles must have larger and larger radius as we approach $(0, 0, 0)$ until at the origin, we have a circle of radius ∞ which is the point at infinity.

Thus, all of $\mathbb{R}^3$ plus the point at infinity is covered by the two solid tori so S^3 is a solid torus glued to another. $\blacksquare$

8. Prove that $S^3 \setminus S^1$ is an S^1-bundle over $\mathbb{R}^2$. Draw (to the best of your abilities) a cartoon of this; specifically if $g : S^3 \setminus S^1 \longrightarrow \mathbb{R}^2$ is your bundle map, what does $g^{-1}(U)$ look like for a small set U about a point $v \in S^2$?

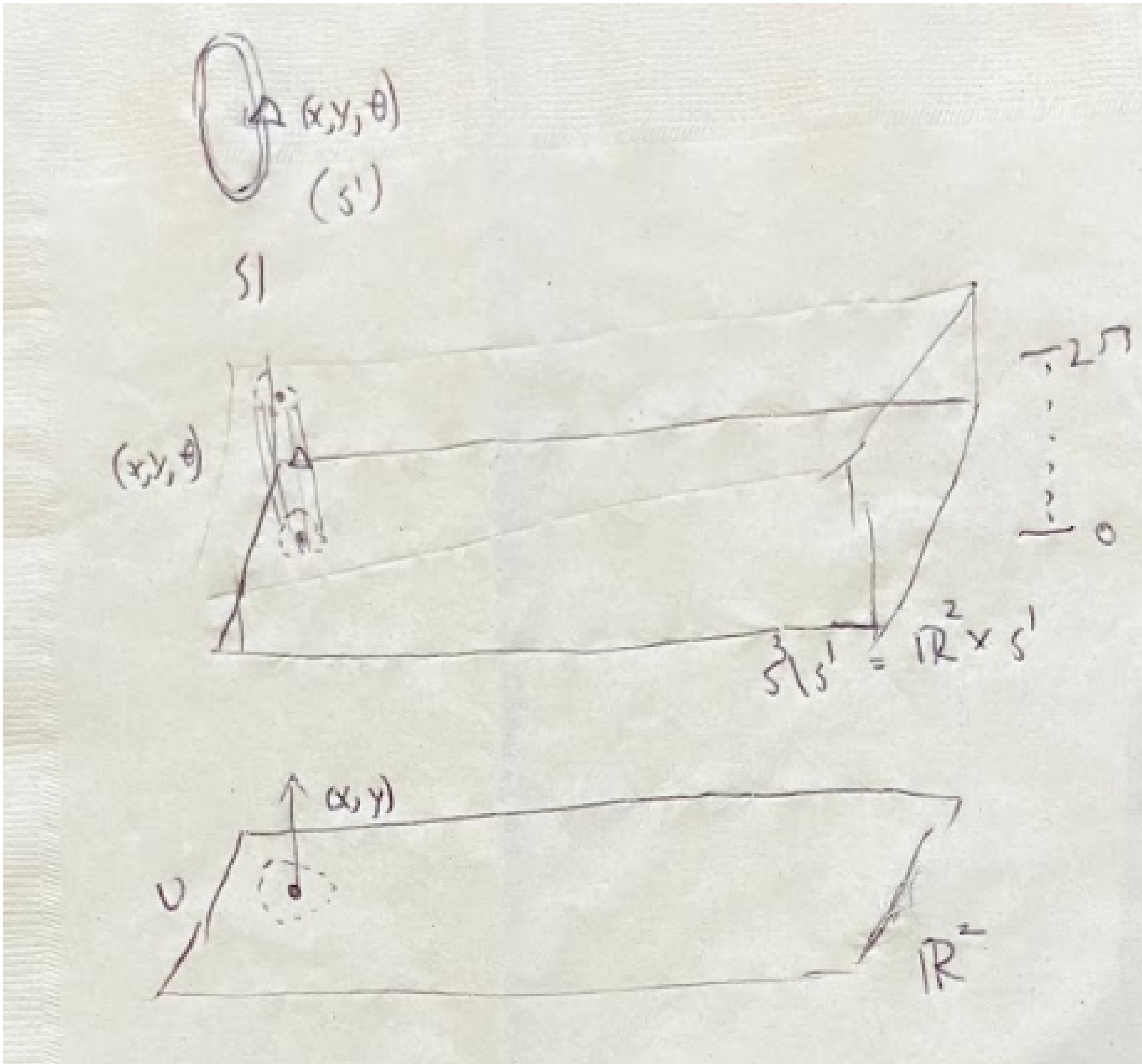
Let $g : S^3/S^1 \to \mathbb{R}^2$. Since $S^3 \setminus S^1 \simeq S^1 \times \mathbb{R}^2$,

$$g^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 \times S^1$$

Therefore, a small set U about a point $v \in S^2$ will be

$$g^{-1}(U) \simeq U \times S^1$$

Hence, $S^3 \setminus S^1$ is an S^1-bundle over $\mathbb{R}^2$.



Here, the bottom picture is the image of $\mathbb{R}^2$ with a small neighborhood U around a point. This pullsback into a cylinder in $\mathbb{R}^2 \times S^1$. Since the fibre of each point is a circle, $g^{-1}(U)$ looks like a solid torus.

Bonus: [3 pts]. Prove that S^2 is a not an S^1-bundle over S^1.

Bonus: [3 pts]. Let X be S^2. Glue a copy of D^4 to S^2 via taking $\partial D^4 = S^3$ to S^2 through the Hopf-fibration. Prove the resulting manifold is $\mathbb{C}P^2$, the complex projective plane.

Bonus: [2 pts]. Draw a cartoon illustrating that $S^3 \setminus S^1$ is the same thing as $S^1 \times \mathbb{R}^2$.