Math 1820A Spring 2024 - Homework 6

Instructions: This assignment is worth twenty points. Please complete the following problems assigned below. Submissions with insufficient explanation may lose points due to a lack of reasoning or clarity. If you are handwriting your work, please ensure it is readable and well-formatted for the grader.

Be sure when uploading your work to **assign problems to pages**. Problems with pages not assigned to them **may not be graded**.

Textbook Problems:

Additional Problems: For these problems if you see an S in front of a group, you can assume it means determinant 1, e.g. all elements of $SO(3,\mathbb{R})$ and $SO(2,1)$ have determinant 1.

1. Let $\mathfrak{n}, \mathfrak{h}$, and $\mathfrak{g}$ be Lie-algebras that fit into the short exact sequence $\mathfrak{n} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{h}$. Finish our proof in class that $\mathfrak{g}$ is solvable if and only if both $\mathfrak{n}$ and $\mathfrak{h}$ are solvable by showing if both $\mathfrak{n}$ and $\mathfrak{h}$ are solvable, so is $\mathfrak{g}$.

Suppose both $\mathfrak{n}$ and $\mathfrak{h}$ are solvable. Then we have that

$$1 = \mathfrak{n}_k \triangleleft \mathfrak{n}_{k-1} \triangleleft \cdots \triangleleft \mathfrak{n}_1 \triangleleft \mathfrak{n}$$

and

$$1 = \mathfrak{h}_m \triangleleft \mathfrak{h}_{m-1} \triangleleft \cdots \triangleleft \mathfrak{h}_1 \triangleleft \mathfrak{h}$$

where $\mathfrak{n}_i/\mathfrak{n}_{i+1}$ and $\mathfrak{h}_j/\mathfrak{h}_{j+1}$ are abelian.

Since we have the short exact sequence, we can say

$$\mathfrak{n}/[\mathfrak{n},\mathfrak{n}] \hookrightarrow \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \twoheadrightarrow \mathfrak{h}/[\mathfrak{h},\mathfrak{h}]$$

and since $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ and $\mathfrak{h}/[\mathfrak{h},\mathfrak{h}]$ are abelian, the commutator subgroups fit into the derived sequences above.

Thus denote $\mathfrak{g}_{n+1} = \mathfrak{g}_n/[\mathfrak{g}_n,\mathfrak{g}_n]$. Then we have that

$$\begin{tikzcd}
\mathfrak{n} \arrow[r, hook] \arrow[d, two heads] & \mathfrak{g} \arrow[r, two heads] \arrow[d, two heads] & \mathfrak{h} \arrow[d, two heads] \\
\mathfrak{n}_1 \arrow[r, hook] \arrow[d, dotted] & \mathfrak{g}_1 \arrow[r, two heads] \arrow[d, dotted] & \mathfrak{h}_1 \arrow[d, dotted] \\
\mathfrak{n}_n \arrow[r, hook] & \mathfrak{g}_n \arrow[r, two heads] & \mathfrak{h}_n
\end{tikzcd}$$

However, since $\mathfrak{n}$ and $\mathfrak{h}$ are solvable, we have that $\mathfrak{n}_n = 0 = \mathfrak{h}_n$ for $n \geq \max\{k,m\}$. Then we have that

$$0 \hookrightarrow \mathfrak{g}_n \twoheadrightarrow 0$$

so $\mathfrak{g}_n = 0$ and $\mathfrak{g}$ is solvable. ∎

2. A *derivation* of a Lie-algebra $\mathfrak{n}$ is any linear map $D : \mathfrak{n} \longrightarrow \mathfrak{n}$ satisfying $D[X, Y] = [DX, Y] + [X, DY]$ for all $X, Y \in \mathfrak{n}$. For example, and $\mathrm{ad}_X(Y) := [X, Y]$ is a derivation for any $X \in \mathfrak{n}$. A *representation* of a Lie-algebra $\mathfrak{h}$ into $\mathfrak{n}$ is a Lie-algebra homomorphism $\phi : \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{n})$ where $\mathfrak{gl}(\mathfrak{n})$ denotes the Lie-algebra of all derivations. The Lie bracket on $\mathfrak{gl}(\mathfrak{n})$ is the difference of function composition: $[A, B] := A \circ B - B \circ A$.

Let H and N be groups and $\Phi : H \longrightarrow \mathrm{Aut}(N)$ be a homomorphism. Use this to construct a representation of a Lie-algebra of $\mathfrak{h}$ into $\mathfrak{gl}(\mathfrak{n})$.

Since $\Phi : H \to \mathrm{Aut}(N)$, $h \mapsto \{\psi(g) : N \hookrightarrow\!\!\!\!\twoheadrightarrow N\}$ is a lie group homomorphism, we can calculate the lie algebra homomorphism $\phi : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{n})$ by

$$\phi(X) = \frac{d}{dt}\bigg|_{t=0} \Phi(e^{tX})$$

and $\frac{d}{dt}\Big|_{t=0} \Phi(e^{tX})$ is an endomorphism $\mathfrak{n} \to \mathfrak{n}$ so ϕ is a lie algebra homomorphism. ■

3. Let $\mathfrak{h}$ and $\mathfrak{n}$ be Lie-algebras, and $\phi : \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{n})$ be a Lie-algebra representation. Consider the *vector space homomorphisms* $i_{\mathfrak{n}} : \mathfrak{n} \longrightarrow \mathfrak{n} \oplus \mathfrak{h}$ and $i_{\mathfrak{h}} : \mathfrak{h} \longrightarrow \mathfrak{n} \oplus \mathfrak{h}$ defined by the embeddings

$$i_{\mathfrak{n}}(X) = (X, 0) \text{ and } i_{\mathfrak{h}}(Y) = (0, Y)$$

Prove there exists a unique *Lie-algebra* on $\mathfrak{n} \oplus \mathfrak{h}$ for which both $i_{\mathfrak{n}}$ and $i_{\mathfrak{h}}$ are Lie-algebra homomorphisms and $[(0, Y), (X, 0)] := \phi(Y)(X)$ for all $X \in \mathfrak{n}$ and $Y \in \mathfrak{h}$. We denote this Lie-algebra by $\mathfrak{n} \oplus_\phi \mathfrak{h}$ and call it the semi-direct product of $\mathfrak{n}$ and $\mathfrak{h}$ over ϕ. With this Lie-algebra equipped on $\mathfrak{n} \oplus \mathfrak{h}$, prove that the sequence of Lie-algebras in Equation 1 is split-exact.

$$\mathfrak{n} \xrightarrow{i_{\mathfrak{n}}} \mathfrak{n} \oplus_\phi \mathfrak{h} \xrightarrow{p} \mathfrak{h} \tag{1}$$

where $p : \mathfrak{n} \oplus_\phi \mathfrak{h} \longrightarrow \mathfrak{h}$ denotes projection $p(X, Y) = Y$

To be lie algebra homomorphisms, the vector space homomorphisms must satisfy

$$\psi([X, Y]) = [\psi(X), \psi(Y)]$$

for all X and Y.

This introduces the conditions,

$$i_{\mathfrak{n}}([X, Y]) = [i_{\mathfrak{n}}(X), i_{\mathfrak{n}}(Y)] \implies ([X, Y], 0) = [(X, 0), (Y, 0)]$$

and

$$i_{\mathfrak{h}}([X, Y]) = [i_{\mathfrak{h}}(X), i_{\mathfrak{h}}(Y)] \implies (0, [X, Y]) = [(0, X), (0, Y)]$$

Clearly, $\text{im } i_{\mathfrak{n}} = (X, 0) = \ker p$ and further, p admits the section $\sigma = i_{\mathfrak{h}}$:

$$p(i_{\mathfrak{h}}(Y)) = p(0, Y) = Y$$

so the sequence is split exact.

4. Let $\mathfrak{g}$ be a Lie-algebra and assume one can find an *ideal* $\mathfrak{n} \leq \mathfrak{g}$ and a *sub-algebra* $\mathfrak{h} \leq \mathfrak{g}$ for which $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ as *vector spaces*. Prove that $\mathfrak{g}$ is isomorphic to a semi-direct product of $\mathfrak{n}$ and $\mathfrak{h}$. Exhibit such an $\mathfrak{n}$ and $\mathfrak{h}$ in the case where $\mathfrak{g} = \mathfrak{i}$ is the Lie-algebra of $\text{Isom}^+(\mathbb{R}^2)$.

$$\mathfrak{i} = \left\{ \begin{pmatrix} 0 & -z & x \\ z & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

We know that $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ as vector spaces. To show that $\mathfrak{g} \simeq \mathfrak{n} \rtimes_\phi \mathfrak{h}$, it suffices to show that there exists a split exact sequence

$$\mathfrak{n} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{h}$$

Since $\mathfrak{n}$ is an ideal of $\mathfrak{g}$, we have the inclusion map $i_\mathfrak{n} : \mathfrak{n} \hookrightarrow \mathfrak{g}$. For the projection, it suffices to define $\mathfrak{h} = \mathfrak{g}/\mathfrak{n}$ and $p : \mathfrak{g} \to \mathfrak{h}$ as the quotient map. Then, as normal, the sequence is split exact.

In HW 3, we proved that $\text{Isom}^+(\mathbb{R}^2) \simeq \mathbb{R}^2 \rtimes_\phi \mathbb{R}$ for $\phi : \mathbb{R} \to \text{Aut}(\mathbb{R}^2)$ by the map $\phi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$.

Therefore, as a vector space $\mathfrak{i} = \mathbb{R}^2 \oplus \mathbb{R}$ so we can take $\mathfrak{n} = \mathbb{R}^2$ and $\mathfrak{h} = \mathbb{R}$ and from Part 1, we have that $\mathfrak{i} \simeq \mathbb{R}^2 \rtimes \mathbb{R}$. $\blacksquare$

5. Use the Killing form to distinguish $\mathfrak{i}$ and $\mathfrak{e}(1,1)$ as Lie-algebras, where $\mathfrak{e}(1,1)$ is the Lie-algebra

$$\mathfrak{e}(1,1) = \left\{ \begin{pmatrix} 0 & z & x \\ z & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

In class, we calculated the Killing form of $\mathfrak{e}(1,1)$ with lie algebra $[X,Y]=0$, $[Z,X]=Y$, $[Z,Y]=X$ as

$$\begin{pmatrix} \langle X,X\rangle & \langle X,Y\rangle & \langle X,Z\rangle \\ & \langle Y,Y\rangle & \langle Y,Z\rangle \\ & & \langle Z,Z\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 2 \end{pmatrix}$$

By similar process, we can calculate the Killing form of $\mathfrak{i}$.

Let

$$Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that $[X,Y]=0$, $[Z,X]=Y$, $[Z,Y]=-X$.

Then, $\mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}] = \langle X, Y\rangle$, $\mathfrak{g}^2 = [\mathfrak{g}^1,\mathfrak{g}^1] = \langle -X, Y\rangle = 0$ so $\mathfrak{g}$ is solvable.

Thus,

$$\begin{aligned}
B(X,X) &= \operatorname{tr}(\mathrm{ad}_X \circ \mathrm{ad}_X) = \operatorname{tr}([\underbrace{X}_{\in\mathfrak{g}^1}, \underbrace{[X,\cdot]}_{\in\mathfrak{g}^1}]) = \operatorname{tr} 0 = 0 \\
B(X,Y) &= \operatorname{tr}([\underbrace{X}_{\in\mathfrak{g}^1}, \underbrace{[Y,\cdot]}_{\in\mathfrak{g}^1}]) = \operatorname{tr} 0 = 0 \\
B(X,Z) &= \operatorname{tr}([X,[Z,\cdot]]) = \operatorname{tr}([X,\langle -X, Y\rangle]) = 0 \\
B(Y,Y) &= \operatorname{tr}([Y,[Y,\cdot]]) = \operatorname{tr}([Y,\langle X\rangle]) = 0 \\
B(Y,Z) &= \operatorname{tr}([Y,[Z,\cdot]]) = \operatorname{tr}([Y,\langle Y, -X\rangle]) = 0 \\
B(Z,Z) &= \operatorname{tr}([Z,[Z,\cdot]]) = \operatorname{tr}([Z,\langle -X, Y\rangle]) = \operatorname{tr}(\langle -X, -Y\rangle) = -2
\end{aligned}$$

Therefore, the signature of the Killing form of $\mathfrak{i}$ is $(0,0,-)$ but the signature of the Killing form of $\mathfrak{e}(1,1)$ is $(0,0,+)$ so they are not isomorphic. ■

6. Prove that $\mathfrak{g}$ is solvable if and only if there exists a sequence of subalgebras

$$0 = \mathfrak{h}_n < \mathfrak{h}_{n-1} < \ldots < \mathfrak{h}_1 < \mathfrak{h}_0 = \mathfrak{g} \tag{2}$$

such that each $\mathfrak{h}_{i+1}$ is an ideal of $\mathfrak{h}_i$ and each $\mathfrak{h}_i/\mathfrak{h}_{i+1}$ is abelian.

Suppose $\mathfrak{g}$ is solvable. Then, denote

$$\mathfrak{h}_n := [\mathfrak{h}^{n-1}, \mathfrak{h}^{n-1}]$$

so $\mathfrak{h}_0 = \mathfrak{g}$ and $\mathfrak{h}_n = 0$ for some n.

Then, $\mathfrak{h}_{i+1}$ is an ideal of $\mathfrak{h}_i$ since

$$[\mathfrak{h}_{i+1}, \mathfrak{h}_i] = [[\mathfrak{h}_i, \mathfrak{h}_i], \mathfrak{h}_i] \subset [\mathfrak{h}_i, \mathfrak{h}_i] \subset \mathfrak{h}_i$$

Further, $\mathfrak{h}_i/\mathfrak{h}_{i+1}$ is in fact the abelianization of $\mathfrak{h}_i$ since $\mathfrak{h}_{i+1} = [\mathfrak{h}_i, \mathfrak{h}_i]$ so we have a sequence of subalgebras satisfying the conditions.

Conversely, suppose we have a sequence of subalgebras satisfying the conditions and let $\mathfrak{h}_0 = \mathfrak{g}$ and $\mathfrak{h}_n = 0$ for some n. Then, since each $\mathfrak{h}_{i+1}$ is an ideal of $\mathfrak{h}_i$ we can take quotients. Now we can induct on n: since $[\mathfrak{h}_n, \mathfrak{h}_n] = 0$ we know that h_n is solvable.

Suppose $\mathfrak{h}_i$ is solvable. Then $\mathfrak{h}_i \subset \mathfrak{h}_{i-1}$ and

$$\mathfrak{h}_i \hookrightarrow \mathfrak{h}_{i-1} \twoheadrightarrow \mathfrak{h}_{i-1}/\mathfrak{h}_i$$

since $\mathfrak{h}_{i-1}/\mathfrak{h}_i$ is abelian, it is nilpotent. Since it is nilpotent, it is solvable. Then since $\mathfrak{h}_i$ and $\mathfrak{h}_{i-1}/\mathfrak{h}_i$ are solvable, $\mathfrak{h}_{i-1}$ is solvable. Therefore, $\mathfrak{h}_0 = \mathfrak{g}$ is solvable. ■

7. Prove that the sum of solvable ideals is solvable. (Hint: Use the 2nd isomorphism theorem for Lie-algebras which states that for two ideals $\mathfrak{a}, \mathfrak{b} \leq \mathfrak{g}$ one has that $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a} \simeq \mathfrak{b}/\mathfrak{a} \cap \mathfrak{b})$. Use this to prove that there exists a *unique* maximal solvable ideal inside a finite dimensional Lie-algebra. We call this maximal ideal the *radical* of $\mathfrak{g}$, and is frequently denoted rad $\mathfrak{g}$. Prove that $\mathfrak{g}/\text{rad } \mathfrak{g}$ is semi-simple.

Let $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{g}$ be solvable ideals.

We can create the short exact sequence by the standard inclusion and quotient maps:

$$\mathfrak{a} \hookrightarrow \mathfrak{a}+\mathfrak{b} \twoheadrightarrow (\mathfrak{a}+\mathfrak{b})/\mathfrak{a}$$

Then, $\mathfrak{a}+\mathfrak{b}$ is solvable if and only if $\mathfrak{a}$ and $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a}$ are solvable. We already have that $\mathfrak{a}$ is solvable so it suffices to show that $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a}$ is solvable.

By the 2nd isomorphism theorem, $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a} \simeq \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$ so we just need to show that $\mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$ is solvable. Trivially, $\mathfrak{b} \cap \mathfrak{a}$ is a subgroup of $\mathfrak{b}$ so it is solvable since $\mathfrak{b}$ is solvable.

Therefore, $\mathfrak{a}+\mathfrak{b}$ is solvable. $\blacksquare$

Now consider the sum of all solvable ideals M in $\mathfrak{g}$. By the above, this is a solvable ideal. Further, this sum is maximal since for any solvable ideal $\mathfrak{a} \subset \mathfrak{g}$, $\mathfrak{a} \subset M$. Finally, this maximal ideal is unique since if M and N are maximal solvable ideals, then $M+N$ is a solvable ideal containing both M and N so $M+N=M=N$.

Let rad $\mathfrak{g}$ be the unique maximal solvable ideal in $\mathfrak{g}$. We seek to show that $\mathfrak{g}/\text{rad } \mathfrak{g}$ is semi-simple.

By definition, a lie algebra is semi-simple if it has no non-zero solvable ideals. Assume $\mathfrak{g}/\text{rad } \mathfrak{g}$ contains a non-zero solvable ideal A. Then consider the pre-image of A under the quotient map $\pi : \mathfrak{g} \to \mathfrak{g}/\text{rad } \mathfrak{g}$. $pi^{-1}(A)$ is a solvable ideal of $\mathfrak{g}$ since

$$A = \pi^{-1}(A)/\text{rad } \mathfrak{g}$$

and we have that A and rad $\mathfrak{g}$ are solvable. But then by uniqueness and maximality of rad $\mathfrak{g}$, $\pi^{-1}(A) \subseteq \text{rad } \mathfrak{g}$ so $\pi^{-1}(A) = \text{rad } \mathfrak{g} \implies A = \text{rad } \mathfrak{g}/\text{rad } \mathfrak{g} = 0$. This is a contradiction so $\mathfrak{g}/\text{rad } \mathfrak{g}$ is semi-simple. $\blacksquare$

Bonus: **[4 pts]** Let B_4 denote the 4×4-upper triangular matrices with 1's along the main diagonal. Express $G = B_4/Z(B_4)$ as the semi-direct product familiar groups and obtain a faithful representation of $G \longrightarrow \mathrm{Aut}(\mathbb{R}^n)$ for some sufficiently large n.