Math 1820A: Introduction to Lie Algebras

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Lecture 1 - Jan 24:

Historically arose from study of DEs.

Example 1: Consider the system

$$\frac{dx_1}{dt} = 2x_1 \tag{1}$$

$$\frac{dx_2}{dt} = \frac{1}{2}x_2 \tag{2}$$

$$x_1(0) = x_2(0) = 1 \tag{3}$$

Solution:

$$\frac{dx_1}{dt} = 2x_1 \implies \int \frac{dx_1}{x_1} = \int 2\ dt \implies x_1(t) = C_1 e^{2t}$$

Now just plug in the initial condition:

$$x_1(0) = C_1 e^0 = 1 \implies C_1 = 1$$

So the solution is

$$x_1(t) = e^{2t}$$

and similarly,

$$x_2(t) = e^{\frac{1}{2}t}$$

so we can write

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{t/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Example 2: Notice that the problem becomes harder with interaction between the variables.

$$\begin{aligned}\frac{dx_1}{dt} &= 3x_1 - 4x_2 \\ \frac{dx_2}{dt} &= 2x_1 - 3x_2 \\ x_1(0) &= x_2(0) = 1\end{aligned}$$

Solution: We can write this system as a matrix equation:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

If we can diagonalize the matrix (a change variables), then we can remove dependence between the variables and solve as before.

First look at the characteristic equation:

$$(3-\lambda)(-3-\lambda) + 8 = \lambda^2 - 1 \implies \lambda = \pm 1$$

These are distinct, so we have a change of coordinates that look like

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

(and in fact this is just a linear change of coordinates).

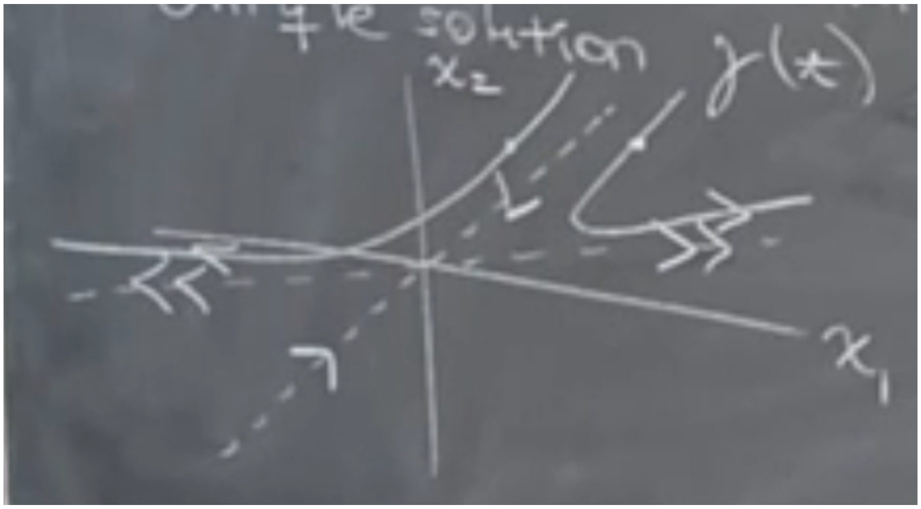
For any choice of $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} \in \mathbb{R}^2$, you get a solution to

$$\frac{d}{dx}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where the initial condition is satisfied. We call this unique solution $\gamma(t)$ a *trajectory.*

Returning to the example above we have

$$\lambda_1 = -1,\ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = 1,\ v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



We can plot the solutions simply from the eigenvalues and eigenvectors. (Positive eigenvalues correspond to expanding solutions, negative eigenvalues correspond to contracting solutions).

More strongly, we observe that the dynamics of $\frac{dx}{dt} = Ax$ are defined by the eigenvectors and eigenvalues of A.

For a diagonalizable matrix in general, we will have (up to conjugation), n different axes which are positively or negatively oriented and then trajectories will fall along linear combinations of these.

A more interesting question is **what conditions can fail for a matrix to be diagonalizable?**

In the case where A is diagonalizable, we know what to do: Change the basis so that $A = PDP^{-1}$. Focus on $y' = Dy$

There are two other cases to consider:

- The eigenvalues of A are not real (e.g. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$). This is annoying but still solvable.
- The eigenvalues of A are *deficient* (the geometric multiplicity is less than the algebraic multiplicity). (e.g. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$)

Example 3:

$$\frac{dx_1}{dt} = x_1 + x_2$$
$$\frac{dx_2}{dt} = x_2$$

First notice, that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has an eigenvector corresponding to the solution

$$s_1(t) = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

But dim $N(A - I) = 1, \quad p(\lambda) = (1 - \lambda)^2$

There is a trick: *Jordan Canonical Form.*

We will let $s_1(t) = e^t u$ and introduce a new solution:

$$s_2(t) = te^t u + e^t v$$

where v is unspecified (but we will find a condition to make this a real solution.)

Of course,

$$s_2(t) = ts_1(t) = e^t v$$

By the product rule,

$$s_2'(t) = s_1(t) + ts_1'(t) + e^t v$$

For this to be a solution, we want

$$\frac{d}{dt} s_2(t) - As_2(t) = 0$$

Thus

$$s_2'(t) = As_2(t) = A(ts_1(t) + e^t v) = tAs_1(t) + e^t Av$$

Putting these together,

$$s_1(t) + ts_1'(t) + e^t v = tAs_1(t) + e^t v$$

But notice that $ts_1'(t) = tAs_1(t)$ so we have

$$s_1(t) + e^t v - e^t Av = e^t u + e^t v - e^t Av = 0$$

Factoring out an e^t, we have

$$u + v - Av = 0 \implies (A - I)v = u$$

which is just a generalized eigenvalue.

Thus, if we can find a v such that $(A - I)v = u$, then we have a solution.

How do we know such a v exists? Because it is Jordan Canonical Form.

Exercise: Find the solutions corresponding to

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Lecture 2 - Jan 26:

Jordan Canonical Form

Let V be a finite dimensional vector space over $\mathbb{C}$. (Algebraic reasons: we want the characteristic polynomial to split). Let A be a linear transformation of V to itself.

Ideally, find a basis for V such that

$$(A)_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_d \end{pmatrix}$$

(i.e. A is diagonalizable) where λ_i's are the eigenvalues of A and

$$p(\lambda) = \det(A - \lambda I) \qquad (\text{ or } \det(\lambda I - A))$$

is the n-th degree polynomial with coefficients in $\mathbb{C}$.

By the FTOA,

$$p(\lambda) = (-1)^n(\lambda - \lambda_1)\dots(\lambda - \lambda_n)$$

where the lambdas could be repeated. But this is also equal to

$$(-1)^n(\lambda - \lambda_1)^{d_1}(\lambda - \lambda_2)^{d_2}\dots(\lambda - \lambda_k)^{d_k}$$

where the lambdas are distinct and d_i is the algebraic multiplicity of λ_i.

We denote $E_i = N(A - \lambda_i I),\ g_i = \dim E_i =$ geometric multiplicity of λ_i.

Theorem: $g_i \leq d_i$

Theorem: The map A is diagonalizable iff $g_i = d_i$ for each λ_i. (the geometric multiplicity is equal to the algebraic multiplicity)

Example: if there are n-distinct eigenvalues

$$p(\lambda) = (-1)^n(\lambda - \lambda_1)^1(\lambda - \lambda_2)^1 \dots (\lambda - \lambda_k)^1$$

then $g_i = d_i = 1$, so diagonalizes.

Example: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ p(\lambda) = (1-\lambda)^2,\ \lambda_1$ has algebraic multiplicity of 2 but $\dim N(A - I) = \dim N \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1$. Since $1 < 2$, A is not diagonalizable.

Jordan Canonical Form: Let V be a finite dimensional vector space over $\mathbb{C}$, let A be a linear map from V to itself. Then B decomposes a direct sum of A-invariant subspaces $V = \bigoplus_{i=1}^k W_i$ such that for each i, there exists a basis $\mathcal{B}_i$ for which

$$(A \mid w_i)_{\mathcal{B}} = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \ddots & \\ & & & & \lambda \end{pmatrix}$$

where λ is an eigenvalue of A.

Observation: For any map A from V to itself, there exists a basis $\mathcal{B}$ of V for which

$$(A)_{\mathcal{B}} = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}$$

where each $J_i = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \ddots & \\ & & & & \lambda \end{pmatrix}$ is called a *Jordan Block.*

For any linear map A, A preserves a complete *flag*

$$0 \subsetneq z_1 \subsetneq z_1 \subsetneq \cdots \subsetneq z_n = V$$

with $\dim(Z_{i+1}, Z_i) = 1$, every linear map up to conjugation is upper triangular.

Example:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \subsetneq \begin{pmatrix} z_1 \\ 0 \\ 0 \end{pmatrix} \subsetneq \begin{pmatrix} z_1 \\ z_2 \\ 0 \end{pmatrix} \subsetneq \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \mathbb{C}^3$$

Jodan Chevalley Decomposition: More strongly, this tells us that *each linear map decomposes as a sum of a diagonalizable part and a nilpotent part.* i.e. $A = D + N$ where D is diagonalizable and N is nilpotent. Further, the pieces commute.

Example:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{N}$$

and $DN = ND$.

Put differently, all of these pieces are cyclic in the sense that they are generated by a single vector and its iterates under the map A.

Claim: $\mathbb{C}^3 = \mathbb{C}\{e_3, Ae_3, A^2e_3\}$

Proof:

$$e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad Ae_3 = 2e_3 + e_2, \quad A^2e_3 = \dots$$

Example:

$$A = \begin{pmatrix} 3 & 2 & -3 \\ 1 & 3 & -2 \\ 1 & 1 & 0 \end{pmatrix}$$

has char. poly. $p(\lambda) = (2 - \lambda)^3$ which only has eigenvalue $\lambda = 2$. Thus, this is not diagonalizable ($\dim N(A - 2I) \neq 3$)

But what we can do is look at powers of $A - 2I$:

$$\begin{aligned} A - 2I &= \begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{pmatrix} \\ (A-2I)^2 &= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \\ (A-2I)^3 &= 0 \end{aligned}$$

This makes sense because of nilpotency and the Cayley Hamilton Theorem.

Cayley Hamilton Theorem: If there is a linear map A, then $p(A) = 0$ where $p(\lambda)$ is characteristic polynomial.

Thus, we just choose a vector that is not in the nullspace of $N((A - 2I)^2)$ (the previous non-zero term), say $u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Then, $\{(A - 2I)^2 u, (A - 2I)u, u\}$ will be a Jordan basis.

Proof: denote $v_1 = (A - 2I)^2 u, v_2 = (A - 2I)u, v_3 = u$. Then

$$\begin{aligned} (A-2I)v_1 = (A-2I)^3 v_1 = 0 &\implies Av_1 - 2v_1 = 0 \implies v_1 \text{ eigenvector of } A \\ Av_2 &= (A-2I)v_2 + 2Iv_2 = v_1 + 2v_2 \\ Av_3 &= (A-2I)v_3 + 2Iv_3 = v_2 + 2v_3 \end{aligned}$$

so in regard to this basis,

$$(A)_{\mathcal{B}} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Lecture 3 - Jan 29:

Recall

Jordan Canonical Form says that, up to similarity, every matrix in $\mathbb{C}^{n\times m}$ looks like

$$\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}$$

where each J_i is a *Jordan Block* of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & \\ & & & \lambda_i \end{pmatrix}$$

which means that any operator can be written as a diagonal and nilpotent part:

$$A = D + N$$

where D and N commute.

This discussion was originally motivated by the problem

$$\frac{d}{dx} = Ax$$

with constraint $x(0) = x_0$, $x(t)$ a function into $\mathbb{R}^n$, and A some real $n \times n$ matrix

Theorem: For any choice of $x_0 \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, there exists a unique solution $x(t)$ to the above problem.

In some sense, this gives us a map. Fix $A \in \mathbb{R}^{n\times n}$, choose $x(0) = x_0$, $x_0 \in \mathbb{R}^n$, and $t \in \mathbb{R}$. Then

$$\theta(t, x_0) = x(t)$$

where $x(t)$ is the solution to $\frac{d}{dt}x = Ax$ with $x(0) = x_0$.

(i.e. There is some trajectory $x(t)$ which passes through the point $x_0 = x(0)$ representing a flow)

Further, this flow gives us an $\mathbb{R}$-action:

$$\theta(t, x_0) = x(t)$$

Recall: Let G be a group. Let X be a set. We say a function $\alpha : G \times X \to X$ is a *group action* if

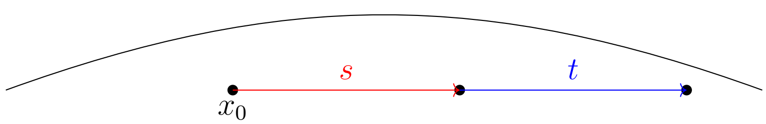
1. $\alpha(1, x) = x$, for all $x \in X$
2. $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for all $g, h \in G$ and $x \in X$

Also note that the definition of a group action is equivalent to saying there is a homomorphism from $G \to \mathrm{Sym}(X)$

In this case, our group is $\mathbb{R}$. So the flow gives us a homomorphism from $\mathbb{R} \to \mathrm{Diff}(\mathbb{R}^n)$ (the *diffeomorphism groups*, a collection of infinitely differentiable functions)

We can imagine the trajectory-flow situation as the map $(t, x_0) \mapsto \theta(t, x_0)$ where t refers to the movement along the trajectory. Thus, $\theta(0, x_0)$ refers to *not* moving along the trajectory so $\theta(0, x_0) = x_0$ – exactly in line with our first group action condition.

Now consider $\theta(t + s, x_0)$.



We notice that moving $t + s$ corresponds to moving s then moving t But, the final point is the same as if we had simply moved a distance of t from the second point:

$$\theta(t + s, x_0) = \theta(s, \theta(t, x_0))$$

Proof: Show that it is the unique solution corresponding to the initial value problem $\frac{d}{dt}x = Ax$ with $x(0) = x_0$. Define $y(t) = x(t + s)$ for some fixed s. Claim $\frac{dy}{dt} = Ay$ and $y(0) = x(s)$. (This is called the *translation lemma*)

Notice too, this is exactly the second group action condition. Thus, we have an $\mathbb{R}$-action. More generally, for any $A \in \mathbb{R}^{n\times n}$, we get an induced $\mathbb{R}$-action on $\mathbb{R}^n$, given by flowing along trajectories:

$$\mathbb{R} \to \mathrm{Diff}(\mathbb{R}^n)$$

but in fact, we can do better and heavily restrict the target group by looking at the solutions to $\frac{d}{dt}x = Ax$.

Let's fix a time $t \in \mathbb{R}$, then

$$\theta_t : \mathbb{R}^n \rightarrowtail \mathbb{R}^n$$

(i.e. there is a bijection from $\mathbb{R}^n$ to $\mathbb{R}^n$)

And

$$\theta_t(x_0 + y_0) = \theta_t(x_0) + \theta_t(y_0)$$

So each θ_t is an (invertible) linear map.

Thus, we can restrict the homomorphism to

$$\theta : \mathbb{R} \to \mathrm{GL}\left(\mathbb{R}^n\right)$$

(the general linear group on $\mathbb{R}^n$)

Definition: $F : \mathbb{R}^n \rightarrowtail \mathbb{R}^n$ is a *diffeomorphism* iff it has infinitely differential partial derivatives and is invertible with same property.

More precisely, the Jacobian is non-zero. Heuristically, it is a smooth, bijective, invertible map.

Definition: Let V be a finite dimensional vector space, $\mathbb{R}^n$, $T \in \mathrm{GL}\left(\mathbb{R}^n\right)$ iff T is linear and has a linear inverse iff $\det(T) \neq 0$.

Definition: A one parameter subgroup is a homomorphism from $\mathbb{R} \xrightarrow{\alpha} \mathrm{GL}\left(n, \mathbb{R}\right)$

Example: Every matrix A gives us a a one parameter subgroup.

Example: With

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and $x(0) = x_0 = \begin{pmatrix} a \\ b \end{pmatrix}$, solve the system.

The general solution is just

$$x(t) = ae^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + be^{\frac{1}{2}t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{\frac{1}{2}t} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

Note that $\theta(t) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{\frac{1}{2}t} \end{pmatrix}$ is a group homomorphism $\mathbb{R} \to \mathrm{GL}\left(2, \mathbb{R}\right)$!

$$\theta(t + s) = \theta(t)\theta(s)$$

Then if

$$x(t) = M(t)x_0,\ M(0) = I$$
$$\frac{dx}{dt} = M'(t)x_0 = AM(t)x_0$$
$$\implies M'(t) = AM(t)$$

which tells us that for a one parameter subgroup, $\theta(s+t) = \theta(s)\theta(t)$, θ takes addition into multiplication.

Matrix Exponential:

Let $A \in \mathbb{C}^{n\times n}$. Define

$$e^A = I_n + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}A^n$$

Justification of well-defined:

$$\left|\left|e^A\right|\right| \leq \sum_{n=0}^{\infty} \frac{1}{n!}\left|\left|A\right|\right|^n < \infty$$

So e^A always makes sense and we have

$$\mathbb{R} \to GL(n, r)$$
$$t \mapsto e^{tA}$$

gives a one parameter subgroup.

Lecture 4 - Jan 31:

Recall: The matrix exponential arises from an $n \times n$ (say real) matrix which gives a homomorphism $\phi : \mathbb{R} \to \mathrm{GL}\left(n, \mathbb{R}\right)$. $\phi(\cdot)$ is the flow of the ODE $\frac{dx}{dt} = Ax$ when

$$x(t) = \phi(t)x_0,\ x_0 \in \mathbb{R}^n$$

with

$$\phi(t+s) = \phi(t)\phi(s)$$
$$\frac{d\phi}{dt}(t) = A\phi(t)$$
$$\phi(0) = I$$

Further, the existence of ϕ is guaranteed by existence and uniqueness of ODEs.

What exactly is ϕ? Precisely the matrix exponential:

$$\phi(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{1}{n!}(tA)^n$$

Claim: $x(t) = e^{tA}x_0$ solves $\frac{dx}{dt} = Ax$, $x(0) = x_0$

Proof:

First we verify the initial condition:

$$x(0) = e^{0A}x_0 = (I + 0 + 0 + \dots)x_0 = x_0$$

(this is a little bit of an abuse of notation saying $I = 1$)

Now taking the derivative:

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dx}(1 + tA + \frac{1}{2}t^2A^2 + \frac{1}{3!}t^3A^3 + \dots)x_0 \\ &= (0 + A + tA^2 + \frac{1}{2}t^2A^3 + \dots)x_0 \\ &= A(1 + tA + \frac{1}{2!}t^2A^2 + \dots)x_0 & = Ae^{tA}x_0 \\ &= Ax(t) \end{aligned}$$

So it is a solution to the ODE.

Consequently, we know

$$e^{tA}e^{sA} = e^{(t+s)A}$$

because of the homomorphism conditions. But we can also say $\phi(0) = I$ and $\phi'(0) = A$.

Example:

$$\frac{dx}{dt} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} x \implies e^{tA} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{t/2} \end{pmatrix}$$

Example:

$$\frac{dx}{dt} = Ax = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix} x$$

Note:

$$\begin{aligned} e^{PBP^{-1}} &= 1 + PBP^{-1} + \frac{1}{2!}(PBP^{-1})^2 + \frac{1}{3!}(PBP^{-1}) + \dots \\ &= 1 + PBP^{-1} + \frac{1}{2}PB^2P^{-1} + \frac{1}{3!}PB^3P^{-1} + \dots \\ &= Pe^BP^{-1} \end{aligned}$$

This leads to the observation that calculating e^{tA} with A diagonalizable, we can just write

$$e^{tA} = e^{tPDP^{-1}} = Pe^{tD}P^{-1}$$

Thus for $A = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}$, we have $\lambda_1 = -1,\ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \lambda_2 = 1,\ v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Currently, we have $(A)_{\mathcal{B}} = A$ where $\mathcal{B} = \{e_1, e_2\}$. We introduce a new basis $\gamma = \{v_1, v_2\}$ and write $A = DPP^{-1}$ by

$$(A)_{\mathcal{B}} = (I)^{\gamma}_{\mathcal{B}}(A)_{\gamma}[(I)^{\gamma}_{\mathcal{B}}]^{-1}$$

so

$$\begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1}$$

Thus,

$$e^{tA} = Pe^{tD}P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} e^{-t} + 2e^t & 2e^{-t} + 2e^t \\ e^{-t} + e^t & 2e^{-t} + e^t \end{pmatrix}$$

Then if we want to solve the initial value problem (say $x(0) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$) then we just have

$$x(t) = e^{tA}x_0 = \begin{pmatrix} e^{-t} + 2e^t & 2e^{-t} + 2e^t \\ e^{-t} + e^t & 2e^{-t} + e^t \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 10e^{-t} + 14e^t \\ 10e^{-t} + 7e^t \end{pmatrix}$$

We would like to generalize more exponential properties (such as saying $e^{A+B} = e^Ae^B = e^Be^A$). But in general, this almost *never* holds. Except under a very special condition!

Theorem: Let A, B be matrices that commute, i.e. $AB = BA$. Then $e^A e^B = e^B e^A = e^{A+B}$

Calculation:

$$\begin{aligned}
e^A e^B &= (I + A + \frac{1}{2}A^2 + \dots)(I + B + \frac{1}{2}B^2 + \dots) \\
&= I + (A+B) + (\frac{1}{2!}A^2 + AB + \frac{1}{2!}B^2) + (\frac{1}{3!}A^3 + \frac{1}{2!\,1!}A^2B + \frac{1}{2!\,1!}AB^2 + \frac{1}{3!}B^3) + \dots \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!(n-k)!} A^{n-k} B^k \right) \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^{n-k} B^k \right) \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (A+B)^n \qquad \text{(note that this needs commutativity!!)}
\end{aligned}$$

Example: $\frac{dx}{dt} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} x$

$$\begin{aligned}
e^{tA} &= e^{t(I+N)} \qquad \text{(Jordan Canonical!)} \\
&= e^{tI} e^{tN} \\
&= \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} 1 & ta \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Notice that the Nilpotent part will always be a finite sum. In the example above $N = I + tN$ in the Taylor Expansion (because $N^2 = 0$).

And indeed, we can generalize this. Up to conjugation,

$$e^{tA} \sim e^{tD} e^{tN}$$

for some diagonal and nilpotent matrices D and N.

Lecture 5 - Feb 2:

Recall:

Example: With $A = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$, $\lambda \in \mathbb{R}$, then

$$e^{tA} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix}$$

Its determinant is 1 so $e^{tA} \in \mathrm{SL}(2, \mathbb{R})$. This has some cool properties: because it is traceless, it preserves volume and up to conjugation, these matrices preserve the Lorentzian product (Physics)

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1 y_1 - x_2 y_2$$

so are of the form $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$

Example: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\lambda = \pm i$ so we cannot diagonalize over the reals.

$$e^{tA} = \sum_{n=0}^{\infty} \frac{1}{n!} (tA)^n$$

but

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \implies A^3 = A,\ A^4 = I$$

so

$$\begin{aligned} e^{tA} &= \frac{1}{0!} I + \frac{1}{1!} tA - \frac{1}{2!} t^2 I - \frac{1}{3!} t^3 A + \frac{1}{4!} t^4 I \pm \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n!)} t^{2n} I + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1} A \\ &= \cos(t) I + \sin(t) A \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2, \mathbb{R}) \end{aligned}$$

(Remark: You have absolute convergence on compact subspaces so the summation splitting is well-defined)

Definition: $SO(2,\mathbb{R})$ is the group of 2×2 matrices that preserve the standard euclidean product $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1 y_1 + x_2 y_2$

Further, $\det(e^{tA}) = \cos^2 t + \sin^2 t = 1$

This leads to a generalization: Given a matrix, you get a whole family of matrices in a group, on which you can impose conditions on the input matrix to land in certain groups.

A matrix group H then looks at one parameter families about $1 \in G$.

$$\gamma : \mathbb{R} \to G$$

where $\gamma'(0)$ is in the Tangent space of G so exp maps the tangent space at $1 \in G$ to the open ball about $1 \in G$.

Example:

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} \oplus \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \xrightarrow{\exp} \mathrm{SL}\,(2,\mathbb{R})$$

Example: $A = \begin{pmatrix} -1/2 & 1 \\ 1 & -1/2 \end{pmatrix}$ so you can just decompose it into

$$A = \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since the two blocks (R, S) commute,

$$e^{tA} = e^{tR} e^{tS} = e^{-t/2} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Claim: We end up having e^{tA} is invertible and lands in $\mathrm{GL}\,(n,\mathbb{R})$ or $\mathrm{GL}\,(n,\mathbb{C})$ (note: only one of those is surjective)

Theorem: $\det(e^A) = e^{\operatorname{tr} A}$ for any matrix $A \in M_n(\mathbb{C})$

Proof: (Sketch - diagonalize and invoke density of diagonalizable matrices in $\mathrm{GL}\,(n,\mathbb{R})$)

Using Jordan Canonical Form: note that det is conjugation invariant so

$$\begin{aligned}
\det(e^A) &= \det(Pe^AP^{-1}) \\
&= \det(e^{PAP^{-1}}) \\
&= \det(e^{D+N}) \\
&= \det(e^D)\det(e^N) \\
&= e^{\lambda_1+\lambda_2+\dots+\lambda_n} \cdot 1 \\
&= e^{\operatorname{tr} A}
\end{aligned}$$

(because $D = \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix}$ and $e^N = 1 + N + \frac{1}{2}N^2 + \dots$ is an upper triangular matrix with 1's on the diagonal. Also, $\operatorname{tr} A = \sum_{i=1}^n \lambda_i$ and $\det A = \prod_{i=1}^n \lambda_i$)

Remark: this is one of the first hints at a fundamental relation between Lie groups and Lie algebras

An immediate consequence is that $\exp : M_n(\mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ lies in the identity component of $\mathrm{GL}(n, \mathbb{R})$ (i.e. $\det(e^A) > 0$)

Lecture 6 - Feb 05:

Theorem: Let $A(t)$ be a smooth family of matrices in $\mathrm{GL}(2, \mathbb{R})$ and $A(0) = I$. Then

$$\left.\frac{d}{dt}\right|_{t=0} \det A(t) = \operatorname{tr} A'(0)$$

Proof:

Let $A(t)$ be smooth with $A(0) = I$ and $A'(0) = B$. Consider the curve $\gamma(t) = e^{tB}$.

Notice that up to first order, $A(t)$ and $\gamma(t)$ agree:

$$A(0) = \gamma(0) = I$$
$$A'(0) = B = \gamma'(0)$$

put differently,

$$A(t) = I + tB + O(t^2)$$
$$\gamma(t) = I + tB + O(t^2)$$

Taking the derivative linearizes the curves, so we can safely ignore higher order terms.

Then taking the derivatives,

$$\frac{d}{dt}\bigg|_{t=0} \det A(t) = \frac{d}{dt}\bigg|_{t=0} \det \gamma(t)$$

(Note that this is actually a chain rule property assuming $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is differentiable at I).

Because det is differentiable at $A(0) = \gamma(0)$ (det is polynomial in entries of input matrix), we can use the chain rule.

$$\frac{d}{dt}\bigg|_{t=0} A(t) = D(\det)_I \circ \frac{d}{dt}\bigg|_{t=0} A(t) = D(\det)_I \frac{d}{dt}\bigg|_{t=0} \gamma(t) = \frac{d}{dt}\bigg|_{t=0} \det \gamma(t)$$

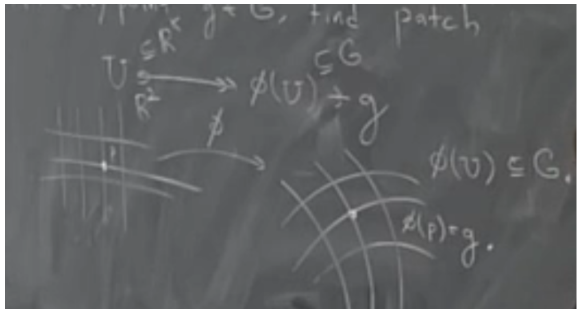
Since this is a 1-parameter subgroup,

$$\frac{d}{dt}\bigg|_{t=0} \det \gamma(t) = \frac{d}{dt}\bigg|_{t=0} \det e^{tB} = \frac{d}{dt}\bigg|_{t=0} e^{\operatorname{tr}(tB)}$$

by the earlier theorem. Then,

$$\frac{d}{dt}\bigg|_{t=0} e^{\operatorname{tr}(tB)} = (e^{\operatorname{tr}(tB)}\bigg|_{t=0}) \cdot \frac{d}{dt}\bigg|_{t=0} \operatorname{tr}(tB) = \operatorname{tr} B$$

(Notice that $\operatorname{tr}(tB) = \sum_{i=1}^n tB_{ii}$ so $\frac{d}{dt}\Big|_{t=0} \operatorname{tr}(tB) = \sum_{i=1}^n B_{ii} = \operatorname{tr} B$)



We can represent these relationships by a set of commutative diagrams:

$$\begin{tikzcd} G \arrow[r, "\phi"] & H \\ g \arrow[u, hook, "\exp"] \arrow[r, "d\phi"] & h \arrow[u, hook, "\exp"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{GL}\,(n,\mathbb{R}) \arrow[r, "\det"] & \mathbb{R}^* \\ M_n(\mathbb{R}) \arrow[u, hook, "\exp"] \arrow[r, "\mathrm{tr}"] & \mathbb{R} \arrow[u, hook, "\exp"'] \end{tikzcd}$$

This hints at deep concept of lie homomorphisms on a group level and algebra level

Definition: Let $G \subseteq \mathrm{GL}\,(n,\mathbb{C})$ be a group of matrices. We say G is a *matrix group* if it is a smooth submanifold of $\mathrm{GL}\,(n,\mathbb{C})$ (i.e. at every point $\mathfrak{g} \in G$, we can find a "patch" $U \subseteq \mathbb{R}^k \hookrightarrow\!\!\!\!\twoheadrightarrow \phi(I) \subseteq G$)

We want to do this in such a way that if we have two charts

$$\phi : U \hookrightarrow\!\!\!\!\twoheadrightarrow \phi(U) \subseteq G$$
$$\psi : V \hookrightarrow\!\!\!\!\twoheadrightarrow \psi(V) \subseteq G$$

and $U \cap V$,

$$\begin{tikzcd} & (U \cap V) \arrow[dl, hook, two heads, "\phi"'] \arrow[dr, hook, two heads, "\psi"] & \\ \phi(U \cap V) \subseteq G \arrow[rr, hook, two heads, "\psi \circ \phi^{-1}"] & & \psi(U \cap V) \subseteq G \end{tikzcd}$$

where $(\psi \circ \phi^{-1})$ is smooth and differentiable infinitely many times.

Examples:

- With
$$\mathrm{SL}\left(n,\mathbb{C}\right) \subseteq \mathrm{GL}\left(2,\mathbb{R}\right) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}\left(2,\mathbb{R}\right) \middle| \det -1 \right\}$$
$\mathrm{SL}\left(n,\mathbb{C}\right)$ is a Lie group
-
$$\mathrm{SO}\left(2,\mathbb{R}\right) \subseteq \mathrm{GL}\left(2,\mathbb{R}\right) = \left\{ A \in \mathrm{GL}\left(2,\mathbb{R}\right) \middle| \langle Av, Aw \rangle = \langle v, w \rangle \right\}$$
$\forall v, x \in \mathbb{R}^2$ where $\langle \cdot, \cdot \rangle$ is euclidean inner product is a 1-dim Lie group

These charts allow us to do calculus on our space G.

Example: $G = \mathrm{GL}\left(n,\mathbb{R}\right)$ is an n^2-dimensional lie group.

$$\mathrm{GL}\left(n,\mathbb{R}\right) = \left\{ A \in M_n(\mathbb{R}) \middle| \det A \neq 0 \right\}$$

is a group with group law multiplication. We have closure because with $A, B \in \mathrm{GL}\left(n,\mathbb{R}\right)$,
$$\det(AB) = \det(A)\det(B) \neq 0$$

Why is this n^2-dimensional? Because $\det : \mathrm{GL}\left(n,\mathbb{R}\right) \to \mathbb{R}^*$ is a smooth map (continuous) so the neighborhood around a point is mapped to a neighborhood around $\det(A)$ in $\mathbb{R}^*$.

Example:
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : |a-1| < \varepsilon, |d-1| < \varepsilon, |b-1| < \varepsilon, |c-1| < \varepsilon \right\}$$

Example: What is the definition of
$$G = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \middle| \lambda_1, \lambda_2, \lambda_3 > 0 \right\} \subseteq \mathrm{GL}\left(3,\mathbb{R}\right)?$$

We can find a chart
$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \in \left\{ |\lambda_1 - 1| < \varepsilon, |\lambda_2 - 1| < \varepsilon, |\lambda_3 - 1| < \varepsilon \right\}$$

Example:

$$\mathrm{SL}\left(3,\mathbb{R}\right)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\middle| ad-bc=1\right\}\subseteq \mathrm{GL}\left(2,\mathbb{R}\right)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\middle| ad-bc\neq 0\right\}$$

Coming up with an explicit chart is hard! But we can say the dimensionality is 3.

Intuition: a,b,d can be chosen freely but c is determined by $\frac{ad-1}{b}=c$ (note this breaks if $b=0!5$)

Lecture 7 - Feb 07:

Recall: Last time we looked at dimension of groups and the idea of a *chart.* The point of a chart is to do calculus on G.

Definition: G is a *Lie group* if it is a smooth manifold and a group whose operation of multiplication and inversion are smooth maps.

In our context, smoothness is automatically satisfied by "nice" subgroups of $\mathrm{GL}\left(n,\mathbb{C}\right)$

Our ultimate goal is to construct a tangent space at $g=1\in G$:

$$g=\{B\in M_n(\mathbb{C})\Big|\quad B=\frac{d}{dt}\Big|_{t=0}\gamma(t),\ \gamma:\mathbb{R}\to G,\ \gamma(0)=1\}$$

(γ is the *lie algebra*)

Note that this definition does not include a notion of one parameter subgroups. However, you can in fact show that it is the same: *Up to first order, every smooth path in G is going to look exponential or like a one parameter subgroup.*

Example:

$$G=\mathrm{diag}^{+}(3,\mathbb{R})=\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}\text{ where }\lambda_i>0$$

Last time, we found that this is a 3-dimensional group. This notion should be reflected in the lie-algebra (the tangent space of the group)

We construct a family:

$$\gamma(t) = \begin{pmatrix} e^t & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \gamma(0) = 1$$

$$\gamma'(0) = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \in M_3(\mathbb{R})$$

$$\alpha(t) = \begin{pmatrix} 1 & & \\ & e^t & \\ & & 1 \end{pmatrix} \implies \alpha'(0) = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$

$$\beta(t) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & e^t \end{pmatrix} \implies \beta'(0) = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

So the general structure is:

$$g = \left\{ \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\} = \begin{pmatrix} a & & \\ & 0 & \\ & & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & & \\ & b & \\ & & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & & \\ & 0 & \\ & & c \end{pmatrix}$$

Example: $G = \mathrm{SL}\,(2, \mathbb{R})$ is 3-dimensional so we expect g to be 3-dimensional.

$$\mathrm{SL}\,(2, \mathbb{R}) = \{A \in \mathrm{GL}\,(2, \mathbb{R}) \big| \det A = 1\}$$

has $\gamma(t) \in \mathrm{SL}\,(2, \mathbb{R}),\ \gamma(0) = 1$. Differentiating, $\frac{d}{dt}\Big|_{t=0} \mathrm{tr}\, \gamma'(0) = 0$ so

$$\mathfrak{sl}(2, \mathbb{R}) = \{B \in M_2(\mathbb{R}) \big| \mathrm{tr}\,(B) = 0\}$$

Thus for any matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathrm{tr}\, B = a + d = 0$,

$$\mathfrak{sl}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$$

But this is not limited to $n = 2$! In general,

$$\dim \mathfrak{sl}(n, \mathbb{R}) = \dim \mathrm{SL}\,(n, \mathbb{R}) = n^2 - 1$$

(because only the last diagonal entry is fixed)

> *Claim:* $\dim \mathrm{GL}\,(n, \mathbb{R}) = n^2$
>
> *Proof:* $\mathrm{GL}\,(n, \mathbb{R})$ is an open subset of $M_n(\mathbb{R})$ which is n^2-dimensional.

Natural question: Why is the dimension of the lie group equal to the dimension of the lie algebra?

Answer: The exponential map is a local diffeomorphism.

$$F : \mathbb{R}^n \ to \mathbb{R}^n$$

look at a fixed point p and consider the Jacobian matrix at p,

$$dF \sim \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & & \\ \vdots & & \\ \frac{\partial F_n}{\partial x_n} & \dots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}$$

if $\det(\mathrm{Jac})_p \neq 0$, then F preserves dimension.

$$F \approx F(p) + \mathrm{Jac}(p)(x - p) + (\text{higher order stuff})$$

So it suffices to calculate the derivative of $\exp : g \to G$ at $B = 0$ since we are interested in $\exp(0) = 1$.

Calculate $d(\exp)_0(B) = \frac{d}{dt}\Big|_{t=0} \exp(tB)$ where $\gamma(t) = tB$ is a particular path in G with $\gamma(0) = 1$ and $\gamma'(0) = B$.

In fact, we know how to take this derivative.

$$\frac{d}{dt}\Big|_{t=0} e^{tB} = B$$

so technically,

$$d(\exp)_0 : T_0\mathfrak{g} \to T_1 G$$

but
$$\begin{tikzcd} T_0 g \arrow[r] \arrow[d, "\simeq"] & T_1 G \arrow[d, "="] \\ g \arrow[r, tail, two heads, "I"] & g \end{tikzcd}$$

Say you have a point $\mathfrak{g} \in G$ and we want a chart about it.

A lie algebra has a local neighborhood around 0. By exponentiation, we get to a local neighborhood around $1 \in G$, the lie group. If we want to get to $\mathfrak{g} \in G$, all it takes is a left multiplication because G *acts on itself by diffeomorphisms (smooth maps)*

In general, you can always move from one point g to another point h by left multiplication,

Most importantly, this tells us that any structure we want to induce on the Lie group can be induced on the Lie algebra (the tangent space at the identity)

Example: Look at the KAN-decomposition chart of $\mathrm{SL}(2,\mathbb{R})$:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

This happens to be the exponential map locally into $\mathrm{SL}(2,\mathbb{R})$ (done carefully, this is actually a bijection!)

Examples of Lie groups:

- $G = \mathrm{SO}(3,\mathbb{R})$ (the group of 3-dimensional rotations acting on S^2). What is the Lie algebra $\mathfrak{so}(3,\mathbb{R})$? We have two conditions: $\det A = 1$ and $\langle Av, Aw \rangle = \langle v, w \rangle$ where $\langle \cdot, \cdot \rangle$ is the euclidean inner product ($A^T A = 1$)

  Linearizing equation 1, we have $\mathrm{tr}(B) = 0$. To linearize equation 2, notice

  $$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} ([A(t)]^T A(t)) = 0 &\implies (\frac{d}{dt}\bigg|_{t=0} A(t)^T) A(0) + A(0)^T (\frac{d}{dt}\bigg|_{t=0}) = 0 \\ &\implies B^T + B = 0 \end{aligned}$$

  Taking these two conditions together, we have a traceless, skew-symmetric matrix. (**Remark:** skew-symmetric implies traceless but the first condition is still necessary because reflections (for example) preserve inner product but not determinant). So we have a matrix of the form

  $$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

  so $\dim(\mathfrak{so}(3,\mathbb{R})) = 3$

Finally, notice that we have not actually defined *algebra.* Introduce the bracket:

$$[A,B] := \frac{d}{dt}\bigg|_{t=0}\frac{d}{ds}\bigg|_{s=0} e^{tA}e^{sB}e^{-tA}$$

and notice that $e^{tA}e^{sB}e^{-tA} \in G$ for all s,t

Lecture 8 - Feb 09:

Let's go back to the notion of the Lie Algebra. We have a vector space structure associated to the lie group. $g = T_1G$, the tangent space at $1 \in G$.

This vector space also has an algebra structure: we can pair tangent vectors together. Let $A, B \in \mathfrak{g}$. Define

$$[A,B] = \frac{d}{dt}\bigg|_{t=0}\frac{d}{ds}\bigg|_{s=0} e^{tA}e^{sB}e^{-tA}$$

There are two (conflicting) interpretations:

1. failure to commute on the level of flows induced by e^{tB}, e^{sA}.
2. infinitesimal derivative of the conjugation action

Looking at the calculation, we see that we are really doing conjugation by e^{tB} applied to e^{sA}.

Taking derivatives,

$$[A,B] = \frac{d}{dt}\bigg|_{t=0} e^{tA}Be^{-tA}$$

but this is weird! $B \in \mathfrak{g}$ but $e^{tA}, e^{-tA} \in G$. This is the magic of the matrix group property. This is an ambient manifold allowing us to identify tangent vectors with elements of the group.

Regardless,

$$\frac{d}{dt}\bigg|_{t=0} e^{tA}Be^{-tA} = (ABe^{-tA} + e^{tA}(-BA)e^{-tA})\bigg|_{t=0} = AB - BA$$

This gives us a way of taking tangent vectors to other tangent vectors ($g \times g \to g$). Further, this pairing is motivated by conjugation.

Note: $[A,A] = AA - AA = 0$. (This is not surprising! Two flows generated by the same matrix will commute.)

The bracket is also bilinear:

$$\begin{aligned} [A+cB, D] &= (A+cB)D - D(A+cB) \\ &= AD + cBD - DA - cDB \\ &= AD - DA + c(BD - DB) \\ &= [A,D] + c[B,D] \end{aligned}$$

Finally,

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$. This is called the *Jacobi identity*.

Definition: A vector space with a pairing $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denoted by $(\mathfrak{g}, [\cdot,\cdot])$ is called a *Lie Algebra* iff it is:

1. Skew-symmetric ($[B,A] = -[A,B]$)
2. Bilinear in both variables
3. Jacobi identity holds ($[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$)

So we can think of this as

$$\begin{aligned} G &\to \mathrm{Aut}(G) \\ g &\mapsto ghg^{-1} \end{aligned}$$

Now we have calculated the vector space structure, but not the algebraic structure.

Examples:

- $(\mathbb{R}^n, +)$. Clearly, this is a group. We can look at $T_0\mathbb{R}^n = \mathbb{R}^n$ (this is the lie-algebra of $\mathbb{R}^n$ and the vector space). We want addition in $\mathbb{R}^n$ to commute so $[X,Y] = 0$.

  Say we want to embed $\mathbb{R}^n \hookrightarrow \mathrm{GL}(\mathbb{R}^m)$? Just consider the exponential map:

  $$(\lambda_1, \lambda_2, \ldots, \lambda_n) \mapsto \begin{pmatrix} e^{\lambda_1} & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix}$$

(This is exactly analogous to the mapping $\mathbb{R} \overset{\exp}{\to} \mathrm{GL}(1) \simeq \mathbb{R}^*$)

If we specify to $\mathbb{R}^3$,

$$g = \left\{ \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$$

and pair any two elements,

$$\left[\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}, \begin{pmatrix} d & & \\ & e & \\ & & f \end{pmatrix} \right] = 0$$

because this is simply isomorphic to $\mathrm{Diag}^+(3, \mathbb{R})$. So the tangent space is just isomorphic to the vector space $\mathbb{R}^3$. Moreover, it is the $\mathbb{R}^3$ lie algebra (this is the vector space with the bracket $[X, Y] = 0$)

- $G = \mathrm{Aff}^+(1, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a > 0,\ b \in \mathbb{R} \right\}$ acts on $\mathbb{R}^2$. In a sense, it maps $x \mapsto ax + b$ and preserves a parallel hyperplane.

  What is the Lie Algebra? Taking derivatives,

  $$g = \left\{ \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \middle| c, d \in \mathbb{R} \right\}$$

  We will choose a basis $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Checking the pairings,

  $$[X, Y] = XY - YX = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = Y$$

  Why? The subgroup of vectors formed by $\begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$ is a normal subgroup. Obviously, normal subgroups should be preserved in the Lie algebra because they are preserved under conjugation.

- $G = \mathrm{SL}(2, \mathbb{R})$. $g = \mathfrak{sl}(2, \mathbb{R})$. We choose the (trace-less) basis

  $$X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

to check the lie algebra structure, we check the pairings.

$$
\begin{aligned}
[X, Y] &= Z \\
[X, Z] &= 2X \\
[Z, Y] &= -2Y
\end{aligned}
$$

(up to signs)

Lecture 9 - Feb 12:

Exercise: The identity component of a lie group is normal in G.

Identity Component: G^0, is the set of matrices $g \in G$ such that there is a path $\gamma : [0,1] \to G$ from $1 \in G$ to $g \in G$.

Example: $G = \mathrm{GL}(2,\mathbb{R})$, $G^0 =$ matrices in $\mathrm{GL}(2,\mathbb{R})$ with $\det A > 0$. So

$$G/G_0 \simeq Z_2 \implies G_0 \hookrightarrow G \hookrightarrow Z_2$$

Notation: S (as in SO and SL) stands for the special linear groups ($\det = 1$) so $\mathrm{SO}(p,q)$ preserve bilinear forms of signature

Recall: The Lie-algebra of G is

$$[A, B] = AB - BA = \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} e^{tA} e^{tB} e^{-tA}$$

and it is

1. Bilinear in each variable
2. Skew-symmetric
3. Satisfies the Jacobi identity

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

Example: $G = \mathrm{SL}(2,\mathbb{R})$ with basis $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ generated by basis A, B, C:

$$\mathfrak{sl}(2,\mathbb{R}) = \left\{ \begin{pmatrix} c & a \\ b & -c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$$

Pairing brackets,

$$
\begin{aligned}
[A, B] &= C \\
[A, C] &= 2A \\
[B, C] &= -2B
\end{aligned}
$$

(you only need three brackets because order is invariant up to a sign)

Note that for each $x \in \mathfrak{g}$, you get a natural map $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ defined by $\mathrm{ad}(x)(y) = [x, y]$.

Notice:

$$
\begin{aligned}
\mathrm{ad}(C)(A) &= 2A \\
\mathrm{ad}(C)(B) &= -2B \\
\mathrm{ad}(C)(C) &= 0
\end{aligned}
$$

Written more evocatively with respect to $\beta = \{A, B, C\}$,

$$(\mathrm{ad}(c))_\beta \sim \begin{pmatrix} 2 & & \\ & -2 & \\ & & 0 \end{pmatrix}$$

Example: $G = \mathrm{SO}\,(3, \mathbb{R})$.

$$\mathfrak{g} = \mathfrak{so}(3, \mathbb{R}) = \{B \in M_3(\mathbb{R}) \big| B + B^T = 0\}$$

these are of the form $\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$ so we can choose the basis

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

calculating the pairings,

$$
\begin{aligned}
[A, B] &= C \\
[B, C] &= A \\
[C, A] &= B
\end{aligned}
$$

This is particularly cool because it is exactly the cross-product map $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

So now we have an algebra on $\mathfrak{g}$. This is remarkable! From a group and calculus, we get an enormous amount of structure.

Definition: $\mathfrak{s} \subseteq \mathfrak{g}$ is a *subalgebra* if it is a vector subspace and closed under lie-brackets. $[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{s}$.

Theorem: There is a correspondence between sub-algebras of $\mathfrak{g}$ and connected lie-subgroups of G.

$$S \subseteq G \to T_1 S \subseteq T_1 G \implies \mathfrak{s} \subseteq \mathfrak{g}$$

Proof: This is closed under the bracket because if $A, B \in \mathfrak{s}$,

$$[A, B] = \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} e^{tA} e^{tB} e^{-tA}$$

and $e^{tA}, e^{sB}, e^{-tA} \in S$ and certainly conjugation within a group is closed. So the derivative should certainly lie in $\mathfrak{s}$:

$$\mathfrak{s} = \{B \in \mathfrak{g} \big| e^{tB} \in S\}$$

Example: $\mathrm{SO}(2, \mathbb{R}) \subseteq \mathrm{SL}(2, \mathbb{R})$ because

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \subseteq \mathrm{SL}(2, \mathbb{R}) \implies \mathfrak{so}(2, \mathbb{R}) = \left\{ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \in \mathfrak{g} \right\}$$

but the lie-algebra is trivial.

Just as we have subgroups, we have the stronger notion of normal subgroups: If $gHg^{-1} = H$ for all $g \in G$, take any $A \in \mathfrak{g}$ and $B \in \mathfrak{h}$, then

$$e^{tA} e^{sB} e^{-tA} \in h$$

so

$$[A, B] \in \mathfrak{h}$$

and this gives the notion of an ideal!

Definition: We say $\mathfrak{h}$, a sub-algebra of $\mathfrak{g}$ is an *ideal* iff $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$. (This is the linearized version of $gHg^{-1} = H$!!)

Natural question: If we have an ideal and a subalgebra, can we take quotients?

Proposition: If $\mathfrak{h}$ is an ideal of g, then we may form a quotient lie-algebra $\mathfrak{g}/\mathfrak{h}$. Further, $\mathfrak{g}/\mathfrak{h}$ is the lie-algebra of the quotient group G/H. In fact, we have an exact sequence on the level of lie-algebras,

$$\begin{tikzcd} \mathfrak{h} \arrow[r, hook] & \mathfrak{g} \arrow[r, two heads] \arrow[d, leftrightarrow] & \mathfrak{g}/\mathfrak{h} \\ H \arrow[r, hook] & G \arrow[r, two heads] & G/H \end{tikzcd}$$

Lecture 10 - Feb 14:

Recall: We have a correspondence between sub-algebras $\mathfrak{s} \subseteq \mathfrak{g}$ and connected lie subgroups of G ($S \subseteq G$).

We also have a correspondence between ideals of $\mathfrak{g}$ (sub-algebras such that $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$) and connected normal lie subgroups of G ($gHg^{-1} = H$ differentiated in t along a path $\gamma : \mathbb{R} \to H$ gives $g\gamma(t)g^{-1} \in H$. Differentiating again, $[A, B] \in H$ where A corresponds to g and B corresponds to $\gamma(t)$).

With ideals we can form quotient algebras. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be an ideal. As a vector space $\mathfrak{g}/\mathfrak{h}$ is the vector space quotient of $\mathfrak{g}$ b6 $\mathfrak{h}$ (the cosets).

Denote $[\overline{X}, \overline{Y}] := \overline{[X, Y]}$ where $\overline{z} := z + \mathfrak{h}$:

$$[X + \mathfrak{h}, Y + \mathfrak{h}] \in [X, Y] + [\mathfrak{h}, Y] + [X, \mathfrak{h}] + [\mathfrak{h}, \mathfrak{h}] \in \mathfrak{h}$$

so the operation is well-defined.

Under what contexts do normal subgroups arise?

- $H \subseteq G$ is normal
- $\phi : G \to Q$, yields a normal subgroup because $\ker \phi$ is normal in G

Definition:

1. A *short exact sequence* of groups is a sequence of groups and group homomorphisms
$$N \overset{i}{\hookrightarrow} G \overset{p}{\twoheadrightarrow} Q$$
where $\ker p = \operatorname{im} i$ and i is injective and p is surjective.

2. A long exact sequence is given by

$$1 \xrightarrow{\phi_0} A_1 \overset{\phi_1}{\hookrightarrow} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} A_4 \xrightarrow{\phi_4} \dots$$

where $\ker \phi_{n+1} = \operatorname{im} \phi_n$

Example: $\mathrm{GL}\,(2, \mathbb{R}) \xrightarrow{\det} \mathbb{R}^*$ has

$$\ker \det = \{A \in \mathrm{GL}\,(2, \mathbb{R}) \big| \det A = 1\} = \mathrm{SL}\,(2, \mathbb{R})$$

What happens at the level of Lie Algebras?

$$\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R}), \quad \mathfrak{h} = \mathfrak{sl}(2, \mathbb{R})$$

Claim: $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. $\mathfrak{h}$ is co-dimension 1 in $\mathfrak{g}$.

$$[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$$

$\mathfrak{h}$ is a sub-algebra so choosing $\mathfrak{g} = h + cX, \quad X \notin \mathfrak{h}$ and $z = Y + cX, \quad Y \in \mathfrak{h},\ X \notin \mathfrak{h}$, we claim $[z, \mathfrak{h}] \subseteq \mathfrak{h}$

$$[Z, \mathfrak{h}] = [Y, \mathfrak{h}] + c[X, \mathfrak{h}]$$

so all we need to show is that $[X, \mathfrak{h}] \subseteq \mathfrak{h}$ for some $X \notin \mathfrak{h}$.

We choose $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{g}$. Certainly $x \notin \mathfrak{h}$ because $\operatorname{tr} X \neq 0$. Is $[X, H] \in \mathfrak{h}$?

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right] = 1 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} - \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = 0 \in \mathfrak{h}$$

So $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

What does $\mathfrak{g}/\mathfrak{h}$ look like as an algebra? $\mathfrak{g}/\mathfrak{h} \simeq \mathbb{R}$ by dimension counting.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d + a \end{pmatrix}$$

so in $\mathfrak{g}/\mathfrak{h}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & d + a \end{pmatrix}$$

and we can just bracket those two.

Theorem: Let $\phi : G \to H$ be a group homomorphism, then the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\phi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "d\phi_0"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

(We will see that $d\phi_0$ is actually a lie-algebra homomorphism)

Example:

$$\begin{tikzcd} \mathrm{GL}\,(n, \mathbb{R}) \arrow[r, "\det"] & \mathbb{R}^* \\ M_2(\mathbb{R}) \arrow[u, "\exp"] \arrow[r, "d\det_1 = \mathrm{tr}"] & \mathbb{R} \arrow[u, "\exp"'] \end{tikzcd}$$

this alone tells us $e^{\mathrm{tr}\, A} = \det(e^A)$ which we spent a long time proving!

Proof: Let $x \in \mathfrak{g}$, consider the path $tX \in \mathfrak{g}$.

$$\gamma(t) := \phi(\exp(tX))$$

$$\gamma(t+s) = \phi(\exp(tX)\exp(sX)) = \phi(\exp(tX))\phi(\exp(sX))$$

so $\gamma : \mathbb{R} \to H$ is a one parameter-subgroup of H. By Homework 3, γ is in one-to-one correspondence with a vector in $\mathfrak{h}$ so $\gamma'(0) \in \mathfrak{h}$ exponentiates to the one-parameter subgroup:

$$\gamma'(0) = \frac{d}{dt}\bigg|_{t=0} \phi(\exp(tX)) = d\phi_1(x) \implies \exp(t\, d\phi_1(x)) = \gamma(t) = \phi(\exp(tX))$$

Corollary: In the same context, let $X, Y \in \mathfrak{g}$. Then

$$d\phi_1([X,Y]) = [d\phi_1(X), d\phi_1(Y)]$$

(i.e. $d\phi_1$ preserves lie-algebra structure)

Proof:

$$\begin{aligned}
[d\phi_1(X), d\phi_1(Y)] &= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} e^{t\, d\phi_1(X)} e^{s\, d\phi_1(Y)} e^{-t\, d\phi_1(X)} \\
&= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} \phi(\exp(tX))\phi(\exp(sY))\phi(\exp(-tX)) \\
&= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} \phi(\exp(tX)\exp(sY)\exp(-tX)) \\
&= \frac{d}{dt}\bigg|_{t=0} d\phi_1(\exp(tX) Y \exp(tX)) \\
&= d\phi_1([X,Y])
\end{aligned}$$

Definition: A linear map $\psi : \mathfrak{g} \to \mathfrak{h}$ is a *Lie Algebra Homomorphism* if it preserves brackets ($\psi([X,Y]) = [\psi(X), \psi(Y)]$ for all $X, Y \in \mathfrak{g}$).

Importantly, there is a correspondence between lie group homomorphisms and lie algebra homomorphisms

Exercise: If H is normal in G, then you have the short exact sequence

$$H \hookrightarrow G \twoheadrightarrow G/H$$

and there is an induced lie-algebra short exact sequence

$$\mathfrak{h} \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{g}/\mathfrak{h}$$

Show this is short exact ($\ker p = \operatorname{im} i$). Consider

$$\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{gl}(2,\mathbb{R}) \twoheadrightarrow \mathfrak{gl}(2,\mathbb{R})/\mathfrak{sl}(2,\mathbb{R})$$

Lecture 11 - Feb 16:

Recall: We have a group homomorphism $\phi : G \to Q$ with an induced lie-algebra homomorphism $d\phi_1 : \mathfrak{g} \to \mathfrak{q}$ which preserves lie-algebra structure:

$$d\phi_1([X,Y]) = [d\phi_1(X), d\phi_1(Y)]$$

In general, we introduce a lie-algebra homomorphism as a linear map $\psi : \mathfrak{g} \to \mathfrak{h}$ such that

$$\psi([X,Y]) = [\psi(X), \psi(Y)]$$

We are particularly interested in the case where $H \trianglelefteq G$ (normal) and we have the short exact sequence of lie groups

$$H \overset{i}{\hookrightarrow} H \overset{p}{\twoheadrightarrow} G/H$$

which descends to a short exact sequence of lie-algebras

$$\mathfrak{h} \overset{di_1}{\hookrightarrow} \mathfrak{g} \overset{dp_1}{\twoheadrightarrow} \mathfrak{g}/\mathfrak{h}$$

Thus the Lie-algebra of G/H identifies with the lie-algebra of $\mathfrak{g}/\mathfrak{h}$.

Example:

$$\mathrm{SL}\,(2,\mathbb{R}) \hookrightarrow \mathrm{GL}\,(2,\mathbb{R}) \twoheadrightarrow \mathrm{GL}\,(2,\mathbb{R})/\mathrm{SL}\,(2,\mathbb{R}) \simeq \mathbb{R}^*$$

$$\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{gl}(2,\mathbb{R}) \twoheadrightarrow \mathbb{R}$$

where $\mathbb{R}^k$ as a lie algebra is the $\mathbb{R}^k$ vector space with trivial brackets.

In fact, there exists a lie-algebra homomorphism from $\mathbb{R} \xrightarrow{\sigma} \mathfrak{gl}(2,\mathbb{R})$ so that $p\sigma = 1_{\mathbb{R}}$ (*section* of $\mathfrak{gl}(2,\mathbb{R}) \to \mathbb{R}$)

Let's inspect on the lie-group level:

$$\mathrm{SL}\,(2,\mathbb{R}) \hookrightarrow \mathrm{GL}\,(2,\mathbb{R}) \overset{p}{\twoheadrightarrow} \mathbb{R}^*$$

where

$$t \overset{\sigma}{\mapsto} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$$

is the inverse of p and

$$p\sigma(t) = p\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = t$$

Split exact sequences

Definition: A short exact sequence of groups so that there exists $\sigma : H \to G$ homomorphism satisfying

$$\begin{tikzcd} N \arrow[r, hook, "i"] & G \arrow[r, two heads, "p"] & H \arrow[l, bend right, dashed, "\sigma"'] \end{tikzcd}$$

With $p\sigma = 1_H$ are called *split extensions.*

Example:

$$\mathrm{SL}\,(2,\mathbb{R}) \overset{i}{\hookrightarrow} \mathrm{SL}\,(2,\mathbb{R}) \times \mathrm{SO}\,(3,\mathbb{R}) \overset{p}{\twoheadrightarrow} \mathrm{SO}\,(3,\mathbb{R})$$

where $i : g \mapsto (g,1)$ and $p : (g,h) \mapsto h$

We have

$$\ker p = \{(g,h) \in \mathrm{SL}\,(2,\mathbb{R}) \times \mathrm{SO}\,(3,\mathbb{R}) \big| h = 1\}$$

and

$$\mathrm{im}\, i = \{(g,1) \in \mathrm{SL}\,(2,\mathbb{R}) \times \mathrm{SO}\,(3,\mathbb{R})\}$$

Certainly, this is a short exact sequence. Is it split?

Define $\sigma : \mathrm{SO}\,(3,\mathbb{R}) \to \mathrm{SL}\,(2,\mathbb{R}) \times \mathrm{SO}\,(3,\mathbb{R})$ by $\sigma(h) = (1,h)$ and we have a homomorphism for which $p\sigma = 1_{\mathrm{SO}\,(3,\mathbb{R})}$.

Remarks:

- In general, for any groups G, H, we may define a split exact sequence by

$$G \overset{i}{\hookrightarrow} G \times H \overset{p}{\twoheadrightarrow} H$$

- We call this "split" because $S = G \times H$ is split over G and H but in a particular way such that $S = GH$

Exercise: If G is split over N and H, then there is an embedding of $H \hookrightarrow G$

Example:

$$\mathbb{R}^2 \overset{i}{\hookrightarrow} \begin{pmatrix} 1 & c & a \\ & 1 & b \\ & & 1 \end{pmatrix} \overset{p}{\twoheadrightarrow} \mathbb{R}$$

given by

$$i \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ & 1 & b \\ & & 1 \end{pmatrix}$$

$$p(\begin{pmatrix} 1 & c & a \\ & 1 & b \\ & & 1 \end{pmatrix}) = c$$

This bears some investigation.

1. *Claim:* $p : G \to \mathbb{R}$ is a group homomorphism

$$\begin{pmatrix} 1 & c & a \\ & 1 & b \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & c' & a' \\ & 1 & b' \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & c + c' & a + a' + bc' \\ & 1 & b + b' \\ & & 1 \end{pmatrix}$$

   so $p(g)p(g') = p(gg') \quad \checkmark$

2. *Claim:* $\operatorname{im} i$ is a group homomorphism

$$\begin{pmatrix} 1 & c & a \\ & 1 & b \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a' \\ & 1 & b' \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & a + a' \\ & 1 & b + b' \\ & & 1 \end{pmatrix}$$

3. *Claim:* $\ker p = \operatorname{im} i$ (true by inspection)

4. *Claim:* It is split: $\sigma : \mathbb{R} \to G$ given by $\sigma(c) = \begin{pmatrix} 1 & c & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$

Points 1-3 show that it is a short exact sequence, Point 4 shows it is split.

Continuing, we create a homomorphism $G \twoheadrightarrow \mathbb{R}^2$ by

$$\begin{pmatrix} 1 & c & a \\ & 1 & b \\ & & 1 \end{pmatrix} \overset{p}{\mapsto} \begin{pmatrix} c \\ b \end{pmatrix}$$

with

$$\ker p = \begin{pmatrix} 1 & 0 & a \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

so

$$\mathbb{R} \overset{i}{\hookrightarrow} G \overset{p}{\twoheadrightarrow} \mathbb{R}^2$$

Exercise: Show that this is a split exact sequence.

Examples of non-split extensions:

- $\{\pm 1\} \overset{i}{\hookrightarrow} Q_8 \overset{p}{\twoheadrightarrow} \mathbb{Z}_2 \oplus \mathbb{Z}_2$ where Q_8 is the quaternion group of 8 elements ($Q_8 = \{\langle i, j, k, 1\rangle \,|\, i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j\}$) and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is the Klein-4 group

If this were split, then we could consider $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as a subgroup of Q_8 but then we would have two elements of order 2 in Q_8 which is not true.

- $\mathbb{Z} \hookrightarrow \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ is not split because $\mathbb{Q}/\mathbb{Z}$ is a torsion group (all elements are finite order) but $\mathbb{Q}$ only has one element of finite order
- $\mathbb{Z}_2 \overset{\times 2}{\hookrightarrow} \mathbb{Z}_4 \twoheadrightarrow \mathbb{Z}_4/\{(\mathbb{Z}_2)\}$

Lie algebra exact sequences

Definition: We say a sequence $\mathfrak{h} \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{k}$ is a *short exact* iff $\ker p = \operatorname{im} i$.

We say a sequence is *split exact* iff p admits a section, $\sigma : \mathfrak{k} \to \mathfrak{g}$ so that $p\sigma = 1_{\mathfrak{k}}$

Example: $\mathfrak{sl}(2,\mathbb{R}) \overset{i}{\hookrightarrow} \mathfrak{gl}(2,\mathbb{R}) \overset{\mathrm{tr}}{\twoheadrightarrow} \mathbb{R}$ is a split exact sequence with section $t \mapsto \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$

Example:

$$N = \begin{pmatrix} 1 & 0 & a \\ & 1 & b \\ & & 1 \end{pmatrix} \trianglelefteq G = \begin{pmatrix} 1 & c & a \\ & 1 & b \\ & & 1 \end{pmatrix} \twoheadrightarrow c$$

with lie algebra sequence given by

$$\mathfrak{n} = \begin{pmatrix} 0 & 0 & a \\ & 0 & b \\ & & 0 \end{pmatrix} \hookrightarrow \mathfrak{g} = \begin{pmatrix} 0 & c & a \\ & 0 & b \\ & & 0 \end{pmatrix} \twoheadrightarrow \mathfrak{g}/\mathfrak{n} = c$$

with section $\sigma(c) = \begin{pmatrix} 0 & c & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$

Remark: $\mathfrak{n} \simeq \mathbb{R}^2$ as a lie algebra because all brackets are zeros. $\mathfrak{g} \simeq \mathbb{R}^3$ as a vector space but *not* as a lie algebra because there are non-trivial brackets in $\mathbb{R}^3$ because

$$\begin{gathered} \mathfrak{g} = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z \\ [X,Y] = Z \\ [Z,X] = 0 \\ [Z,Y] = 0 \end{gathered}$$

(G is the *Heisenberg group*)

Notation: $\mathbb{R}^3 = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$ where $\oplus$ is the direct sum, meaning $\{e_1, e_2, e_3\}$ form a basis for $\mathbb{R}^3$

Lecture 12 - Feb 21:

Motivation: By the classification of Semisimple Lie Groups, not all lie groups admit ideals. This is frustrating. Semidirect products give us a natural way to solve this problem.

Definition: a semi-direct product is equivalent to the data of $\phi : H \to \mathrm{Aut}(N)$ where H, N are Lie groups. We form $G_\phi = N \rtimes_\phi H$ (where N is normal because the triangle points to it) with multiplication

$$(n, h)(a, b) = (n\phi_h(a), hb)$$

From the homework, we showed this is equivalent to

$$\begin{tikzcd} N \arrow[r, hook] & G \arrow[r, two heads] & H \arrow[l, bend right, "\sigma"'] \end{tikzcd}$$

Example: $H = \mathbb{R}, N = \mathbb{R}$ with $\phi : H \to \mathrm{Aut}(\mathbb{R})$ and $\phi = 1 \in \mathrm{Aut}(\mathbb{R})$. Then

$$N \rtimes_\phi \mathbb{R} \simeq \mathbb{R}^2$$

(more generally, the identity map yields the direct product $N \times H$)

Alternatively, we could say $\phi : \mathbb{R} \to \mathrm{Aut}(\mathbb{R})$ squeezes or expands (arbitrarily) by $t \mapsto \{x \to e^t x\}$. Now we can form the semi-direct product $N \rtimes_\phi H$.

Exercise:

$$G_\phi \simeq \left\{ \begin{pmatrix} e^t & b \\ 0 & 1 \end{pmatrix} \middle| t, b \in \mathbb{R} \right\}$$

Which we can see by considering G_ϕ's action on $\mathbb{R}$.

Put differently,

$$\begin{tikzcd} \mathbb{R} \arrow[r, hook] & G_\phi \arrow[r, two heads] & \mathbb{R} \arrow[l, bend right, "\sigma"'] \end{tikzcd}$$

$$b \hookrightarrow \begin{pmatrix} e^t & b \\ 0 & 1 \end{pmatrix} \twoheadrightarrow t$$

What is the lie algebra of G_ϕ? We know $\mathbb{R} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathbb{R}$ so

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$$

Notice there is a nontrivial ideal of $\mathfrak{g}$!

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| b \in \mathbb{R} \right\}$$

which corresponds to the normal subgroup

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{R} \right\} \trianglelefteq G_\phi$$

But notice that $N = \mathbb{R}X$ with

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

so

$$\mathfrak{g} = \operatorname{Span}\{X, Y\}$$

(a decomposition into vector spaces) which allows us to explicitly calculate the brackets:

$$[Y, X] = X$$
$$[X, X] = 0$$

so

$$[\mathfrak{g}, \mathfrak{n}] \subseteq \mathfrak{n}$$

which confirms that this is an ideal!

Note: This shows that $\mathbb{R}^2$ and $\mathfrak{aff}(1, \mathbb{R}) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| b \in \mathbb{R} \right\}$ are the *only* 2-d nontrivial lie algebras because $\mathfrak{aff}(1, \mathbb{R})$ is the only non-commutative 2-d lie algebra.

Remark: For all vector spaces, you can create a split exact sequence $v \hookrightarrow w \twoheadrightarrow w/v$. But this is *not* true in general for lie algebras.

Further Examples:

- $\phi : \mathbb{R} \to \mathrm{SL}\left(\mathbb{R}^2\right)$. You can find a ϕ easily by matrix exponentials! For example,

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \implies \phi(t) = e^{tB} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Thus we can form the semidirect product

$$G_\phi = N \rtimes_\phi H = \mathbb{R}^2 \rtimes_\phi \mathbb{R} \sim \begin{pmatrix} 1 & t & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

which is the simple result of the group action $(n, h)v = \phi_h(v) + n$ which $G_\phi \mapsto \mathrm{Diff}(\mathbb{R}^2)$. Then

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, hook] & G_\phi \arrow[r, two heads] & \mathbb{R} \arrow[l, bend right, "\sigma"'] \end{tikzcd}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & t & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow t$$

We have previously shown that $\mathbb{R}^2 \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathbb{R}$ and

$$\mathfrak{g} = \begin{pmatrix} 0 & c & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

We can pick a basis

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so without even doing calculations, we know that $[X, Y]$ corresponds to $\mathbb{R}^2$ so has trivial bracket. Similarly, $\sigma(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ so $[Z, X] = 0$ because $\exp(tX) \in Z(G)$ (center) so $[\mathfrak{g}, X] = 0$ for any element of $\mathfrak{g}$. Finally, $[Z, Y] = X$. Again, this makes sense because we are *close* to abelian but not quite.

Note generally,

$$\begin{aligned} \mathrm{ad}(Z) &: \mathfrak{g} \to \mathfrak{g} \\ \mathrm{ad}(Z)(v) &: [z, v] \end{aligned}$$

so with $\beta = \{X, Y, Z\}$

$$\begin{aligned} \mathrm{ad}(Z)(X) &= 0 \\ \mathrm{ad}(Z)(Y) &= [Z, Y] = X \\ \mathrm{ad}(Z)(Z) &= 0 \\ (\mathrm{ad}(Z))_\beta &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{d}{dt}\bigg|_{t=0} \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline 0 \quad 0 & & 1 \end{array}\right) \end{aligned}$$

Lecture 13 - Feb 23:

Recall: The Heisenberg algebra

$$G = \begin{pmatrix} 1 & c & a \\ & 1 & b \\ & & 1 \end{pmatrix}$$

has lie algebra $\mathfrak{g} = \mathrm{Span}\{X, Y, Z\}$ with

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

determined by three brackets:

$$[X, Y] = 0, \quad [X, Z] = 0, \quad [Y, Z] = X$$

We note that X is special because it is in the center of $\mathfrak{g}$:

$$X \in Z(\mathfrak{g}) = \{A \in \mathfrak{g} \mid [A, \mathfrak{g}] = 0\}$$

This is not surprising because

$$\exp(aX) = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in Z(G)$$

Natural Question: Is $Z(\mathfrak{g})$ the lie-algebra of the center of G? Yes!

Proof Sketch:

$$e^{sB} \in Z(G), \quad e^{tA} \in G \implies \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} e^{tA}e^{sB}e^{-tA} = \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} e^{sB} = 0 = [A, B]$$

Try to write a homomorphism $\mathrm{SO}\left(3, \mathbb{R}\right) \twoheadrightarrow H$. You cannot! $\mathrm{SO}\left(3, \mathbb{R}\right)$ is a simple lie group – it has no non-trivial ideals.

In fact, this is also true for $\mathrm{SL}\left(2, \mathbb{R}\right)$.

Fact: The only ideals of $\mathrm{GL}\left(n, \mathbb{R}\right)$ are

$$\begin{aligned} \mathrm{SL}\left(n, \mathbb{R}\right) &\hookrightarrow \mathrm{GL}\left(n, \mathbb{R}\right) \\ \mathbb{R}^{\times} 000c &\hookrightarrow \mathrm{GL}\left(n, \mathbb{R}\right) \end{aligned}$$

and that is it!

In fact,

$$\mathrm{SL}\left(n, \mathbb{R}\right) \hookrightarrow \mathrm{GL}\left(n, \mathbb{R}\right) \twoheadrightarrow \mathbb{R}^{\times}$$

is a split exact sequence.

But, lie group homomorphisms are easily creatable from $N \trianglelefteq_{\phi} H$: to map $H \to \mathrm{Aut}(N)$, we just need to use one-parameter subgroups!

$$\mathbb{R} \xrightarrow{\phi} \mathrm{Aut}(\mathbb{R}^n) = \mathrm{GL}\left(n, \mathbb{R}\right)$$

$$t \mapsto \exp(tB) \quad B \in M_n(\mathbb{R})$$

Zooming out, our project in this course so far has been classifying some low-dimensional lie algebras (and thus lie groups!):

1. Dimension 1: $\mathfrak{g} \simeq \mathbb{R}$

2. Dimension 2: $\mathfrak{g} \simeq \mathbb{R}^2$ or $\mathfrak{g} \simeq \mathfrak{aff}(1, \mathbb{R})$

Definition: Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a lie-algebra homomorphism. We say its an isomorphism iff it's bijective. We say $\mathfrak{g} \simeq \mathfrak{h}$ iff there exists a lie-algebra isomorphism between $\mathfrak{g}$ and $\mathfrak{h}$.

Claim: Let $\mathfrak{g}$ and $\mathfrak{h}$ be non-commutative, 2-dimensional lie-algebras. Then $\mathfrak{g} \simeq \mathfrak{h}$.

Proof: It suffices to show $\mathfrak{g} \simeq \mathfrak{aff}(1, \mathbb{R}) = \mathbb{R}X \oplus \mathbb{R}Y$ with $[Y, X] = X$. Note that this decomposition is on the level of vector spaces, not lie algebras!

Let $\mathfrak{g} = \mathbb{R}A \oplus \mathbb{R}B$. Its structure is completely determined by $[A, B]$.

Suppose $[A, B] = xA + yB$. Since $\mathfrak{g}$ is not commutative, x and y are not both zero.

First suppose $y = 0$. Then $[A, B] = xA \quad x \neq 0$ and we can just consider $[A, B/x] = A$. Make the substitutions $A' = A$, $B' = \frac{B}{x}$, and define $\phi : \begin{matrix} A' \mapsto X \\ B' \mapsto Y \end{matrix}$. Then

$$[B', A'] = [-\frac{B}{x}, A] = A$$

and we are done.

Now we check the case $y \neq 0$. We can divide by y and get $[A, B] = \frac{x}{y}A + B$. Then

$$\begin{aligned} [A - \frac{B}{y}, \frac{B}{y}] &= [A, \frac{B}{y}] - [\frac{B}{y}, \frac{B}{y}] \\ &= xA + B \end{aligned}$$

$$\vdots$$

Consider the map $\phi : \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ by

$$t \mapsto \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} = \exp(t \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{B \in \mathfrak{sl}(2, \mathbb{R})})$$

with

$$
\begin{aligned}
\cosh(t) &= \frac{e^t + e^{-t}}{2} \\
\sinh(t) &= \frac{e^t - e^{-t}}{2} \\
\cosh'(t) &= \sinh(t) \\
\sinh'(t) &= \cosh(t)
\end{aligned}
$$

Then we can write

$$
\mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \underbrace{\mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{Sol}} \oplus \underbrace{\mathbb{R}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\text{Heisenberg}}
$$

so

$$
\mathrm{Isom}(\mathbb{R}^{1,1}) \simeq G = \left(\begin{array}{cc|c} \cosh(t) & \sinh(t) & a \\ \sinh(t) & \cosh(t) & b \\ \hline 0 & 0 & 1 \end{array}\right), \quad \mathfrak{g} = \left(\begin{array}{cc|c} 0 & t & a \\ t & 0 & b \\ \hline 0 & 0 & 0 \end{array}\right)
$$

since

$$
\left.\frac{d}{dt}\right|_{t=0} \cosh(t) = \sinh(0) = 0
$$

We have the sequence

$$
\mathbb{R}^2 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \hookrightarrow \mathfrak{g} \underset{p}{\twoheadrightarrow} \mathbb{R}
$$

What is p?

$$
X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

and

$$
\begin{aligned}
[X,Y] &= 0 \\
[Z,X] &= Y \\
[Z,Y] &= X
\end{aligned}
$$

Is the Heisenberg algebra isomorphic to this lie algebra? They both fit as split extensions of $\mathbb{R}^2$ and $\mathbb{R}$ so it would make sense. But in fact, if you look at the centers of these lie algebras, you will see that they are not. In the Heisenberg case, $X \in Z(\mathfrak{g})$. But in this case, there is no central element of $\mathfrak{sol} = E(1,1)$

Lecture 14 - Feb 26:

Some General Notes:

1. Why the Jacobi Identity? Why is it true that $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] = 0$ for all $x,y,z \in \mathfrak{g}$?

   For any groups there is a natural representation $\Phi : G \to \mathrm{Aut}(G)$ by $g \mapsto \phi_g : G \hookrightarrow\!\!\!\!\!\twoheadrightarrow G$ where $\phi_g(h) = ghg^{-1}$. This gives us a short exact sequence $Z(G) \hookrightarrow G \twoheadrightarrow \mathrm{Im}(\Phi)$.

   This means that for any $g \in G$, we have an automorphism $\phi_g : G \hookrightarrow\!\!\!\!\!\twoheadrightarrow G$ by $\phi(g)(h) = ghg^{-1}$. On the level of algebras, we form the commutative diagram

$$\begin{tikzcd} G \arrow[r, hook, two heads, "\phi_g"] & G \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, hook, two heads, "d\phi_g"] & \mathfrak{g} \arrow[u, "\exp"'] \end{tikzcd}$$

   So for a smooth path $H(g)$ in G, we get another smooth path $gH(g)g^{-1} \in G$.

   Note that we get a representation of $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{GL}(\mathfrak{g})$ by $g \mapsto d\phi_g$ (the derivative of conjugation by a fixed element $g \in G$.)

   Further, $d\phi_g : \mathfrak{g} \to \mathfrak{g}$ is a lie-algebra isomorphism. By definition it preserves

$$d\phi_g[X,Y] = [d\phi_g(X), d\phi_g(Y)]$$

   Then we take derivatives along paths (say $g(t) \sim \exp(tZ)$) such that $g'(0) = Z$. The principal question here is what is $d\phi_g'(0)$?

   We have another commutative diagram

$$\begin{tikzcd} G \arrow[r, hook, two heads, "\mathrm{Ad}"] & \mathrm{Aut}(G) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, hook, two heads, "d(\mathrm{Ad})_1"] & \mathfrak{gl}(\mathfrak{g}) \arrow[u, "\exp"'] \end{tikzcd}$$

so

$$d(\mathrm{Ad}_1(z) \in \mathfrak{gl}(\mathfrak{g}))$$

and

$$d(\mathrm{Ad})_1(z)(x) \in \mathfrak{g} = \cdots = [Z, X]$$

(proof by linearization of conjugates) so

$$d\phi_g[X,Y] = [Z,[X,Y]] = [d\phi_g(X),Y] + [X, d\phi_g(Y)] = [[Z,X],Y] + [X,[Z,Y]]$$

2. Is there a more geometric interpretation of the Lie bracket?

   In general, if you hae two vector fields X, Y, you can define the lie bracket. To the vector field X, you can assign a flow $\theta : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which is an $\mathbb{R}$-action and has property that $\frac{d}{dt}\Big|_{t=0} \theta(t,p) = X_p$. Then you can think of θ as an integral curve

3. Lie's Theorems:

   - For any Lie Algebra $\mathfrak{g}$, you can find a group G so that the lie algebra of $G = \mathfrak{g}$ (further this can be done such that G is simply connected)

     We can make this slightly nicer by relaxing the topological constraint: any lie algebra $\mathfrak{g}$ can be realized as the lie algebra of a linear group (though not necessarily simply connected)

   - For any lie algebra homomorphism of $\psi : \mathfrak{g} \to \mathfrak{h}$ where $\mathfrak{g}$ and $\mathfrak{h}$ are the lie algebras of G, H (and where G is simply connected – loops can be deformed to points) there exists a group homomorphism $\phi : G \to H$ such that $d\phi = \psi$

   - The lie sub algebras of $\mathfrak{g}$ correspond (bijectively) to connected lie subgroups of G

     $$\left\{ \begin{array}{c} \text{connected lie} \\ \text{subgroup } H \subseteq G \end{array} \right\} \to \{\text{lie subalgebra}\}$$

     $$H \mapsto \mathfrak{h}$$

     $$\exp(\mathfrak{h}) \underset{\exp}{\leftarrow\!\shortmid} \mathfrak{h}$$

     (note that $\exp(\mathfrak{h})$ is not always a subgroup! So we really mean a connected "lie "subgroup" which is the image of an exponential map)

A consequence of the first two theorems is that the lie algebra uniquely determines the group if the lie group is simply connected (loops can be contracted to points).

This is a really powerful result: a lie algebra homomorphism is just as good as a group homomorphism

Lecture 15 - Feb 28:

Recall: The adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is given by $g \mapsto \{x \mapsto dI_g(X)\}$ where $I_g : G \hookrightarrow\!\!\!\!\rightarrow G$ by $h \to ghg^{-1}$.

Then letting $X \sim e^{tX}$

$$\mathrm{Ad}_g(X) = \frac{d}{dt}\bigg|_{t=0} g e^{tX} g^{-1} = gXg^{-1}, \quad g \in G$$

so

$$\mathrm{Ad}_g : \mathfrak{g} \hookrightarrow\!\!\!\!\rightarrow \mathfrak{g}, \quad X \mapsto gXg^{-1}$$

(one might wonder why we are allowed to compose lie algebras elements and lie group elements – this is weird if you think about tangent spaces. In fact, this is just a nice consequence of matric groups)

Further, each Ad_g is a lie-algebra automorphism ($\mathrm{Ad}_g([X,Y]) = [\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)]$) because I_g is a group automorphism.

Example: Calculate the adjoint representation of $G = \mathrm{SL}\,(2, \mathbb{R})$.

First, we pick the basis for our lie-algebra:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then

$$[X,Y] = 2Y, \quad [X,Z] = 2Z, \quad [Y,Z] = X$$

Now consider the group element $g(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$:

$$\begin{aligned} \mathrm{Ad}_{g(t)}(X) &= g(t)Xg(t)^{-1} = X \\ \mathrm{Ad}_{g(t)}(Y) &= g(t)Yg(t)^{-1} = e^{2t}Y \\ \mathrm{Ad}_{g(t)}(Z) &= g(t)Zg(t)^{-1} = e^{-2t}Z \end{aligned}$$

We know these results either by brute calculation or by looking at the adjoint representation in the basis $\beta = \{X, Y, Z\}$

$$(\mathrm{Ad}_{g(t)})_\beta = \begin{pmatrix} 1 & & \\ & e^{2t} & \\ & & e^{-2t} \end{pmatrix}$$

We also notice that the bracket equations are precisely the derivatives of the adjoint representation! And again this makes sense because the derivative of the adjoint is the lie bracket!

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{g(t)} = \mathrm{ad}(g'(0)) = \mathrm{ad}(X)$$

since

$$\mathrm{Ad}_{g(t)} \in \mathrm{Aut}(\mathfrak{g}), \quad \mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$$

which creates the commutative diagram

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}"] & \mathrm{Aut}(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{ad}"] & \mathfrak{gl}(\mathfrak{g}) \arrow[u, "\exp"'] \end{tikzcd}$$

Now consider the same group $\mathrm{SL}(2, \mathbb{R})$ but let $h(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \sim Y$. As before

$$\begin{aligned} \mathrm{Ad}_{h(t)}(X) &= h(t)Xh(t)^{-1} = -2tY \\ \mathrm{Ad}_{h(t)}(Y) &= h(t)Yh(t)^{-1} = Y \\ \mathrm{Ad}_{h(t)}(Z) &= h(t)Zh(t)^{-1} = tX - t^2Y + Z \end{aligned}$$

these largely make sense because we want the derivative at 0 to still equal the brackets. Y is of course invariant, but we get some higher order terms for Z. Indeed, this in a sense measures the failure of commutativity of the exponential map.

In general, the lie algebra $\mathfrak{g}$ has a natural bilinear form that is Ad-invariant.

Definition: A *bilinear form* is a map $\mathfrak{g} \times \mathfrak{g}\overset{B}{\mathbb{R}}$ or $\mathbb{C}$ which is

(a) Symmetric: $B(X, Y) = B(Y, X)$

(b) Bilinear in each variable: $B(\cdot, X)$ with X fixed is a linear map $\mathfrak{g} \to \mathbb{R}$ or $\mathbb{C}$ (and same for $B(X, \cdot)$)

If in addition, we have that for some $x \in \mathfrak{g}$ if $B(X,Y) = 0$ for all $Y \in \mathfrak{g}$, then $X = 0$, we call this pairing *non-degenerate*

Example 1: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ takes $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \big(x_2, y_2\big) \mapsto x_1x_2 + y_1y_2$ is a non-degenerate bilinear form.

Example 2: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ takes $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \big(x_2, y_2\big) \mapsto x_1x_2 - y_1y_2$ is also bilinear and non-degenerate but has different signature.

To each bilinear form, you may assign a symmetric matrix A by $B(X,Y) = x^T Ay$. Since it is symmetric, the eigenvalues are all real. Further, the signs of these eigenvalues is na invariant of the choice of A.

In the examples above, example 1 has signature $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow (+,+)$. Example 2 is given by $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \leftrightarrow (+,-)$.

Proposition: A bilinear is non-degenerate iff its signature has no zeros.

Example: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \mapsto x_1x_2$ is a degenerate bilinear form with signature $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \leftrightarrow (+,0)$

Definition: We say a bilinear form on V is *G-invariant* if we have a G-action on V that preserves the bilinear form. i.e. for $g : V \hookrightarrow\!\!\!\!\rightarrow V$, we have that $B(gv, gw) = B(v,w)$.

Lemma: There exists a G-invariant non-degenerate bilinear form on finite-dimensional V iff $V \simeq V^*$ where V^* is the dual representation of G

$$\begin{tikzcd}
V \arrow[r, hook, two heads, "g"] \arrow[d, hook, two heads, "\phi"] & V \arrow[d, hook, two heads, "\phi"] \\
V^* \arrow[r, hook, two heads, "(g^*)^{-1}"] & V^*
\end{tikzcd}$$

Example: $G \to \mathrm{GL}(\mathbb{R}^2)$ with $V = \mathbb{R}^2$ given by $t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ has no G-invariant non-degenerate metric.

Definition: Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). Then define the *dual space* $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ where elements $f \in V^*$ are linear maps $f : V \to \mathbb{R}$.

Definition: For each $X, Y \in \mathfrak{g}$, define

$$B(X,Y) = \mathrm{tr}\,(\mathrm{ad}_X \circ \mathrm{ad}_Y)$$

by where $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}_x(Y) := [X,Y]$ and $B(X,Y)$ is called the *Killing Form*

Lecture 16 - March 1:

Recall: The *Killing form* of a lie algebra $\mathfrak{g}$ is a bilinear symmetric pairing $B\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ or $\mathbb{C}$ given by

$$B(X,Y) = \mathrm{tr}\,(\mathrm{ad}_x \circ \mathrm{ad}_y)$$

where $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$, $Y \mapsto [X,Y]$.

This is a bilinear pairing and Ad-invariant:

$$B(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = B(X,Y)$$

(like an isometry)

In a sense, this is analogous to $C \in \mathrm{SO}\,(3,\mathbb{R})$ preserving $x^2 + y^2 + z^2 - \langle Cv, Cw \rangle = \langle v, w \rangle$.

Note: in general, a symmetric bilinear pairing $B(X,Y)$ is entirely determined along diagonal, $B(X,X)$.

Polarization identity:

$$B(x+y, x+y) = B(x,x) + 2B(x,y) + B(y,y)$$

This makes sense because (in a basis) the Killing form preserves $B(X,Y) = \mathrm{tr}\,(AB) = \mathrm{tr}\,(BA) = B(Y,X)$

Example: Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z$ with $[X,Y] = 2Y$, $[X,Z] = -2Z$, $[Y,Z] = X$.

Then $\mathrm{ad}_X \circ \mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ so we need to calculate the basis representation with respect ot $\beta = \{X, Y, Z\}$:

$$\begin{aligned}
(\mathrm{ad}_X \circ \mathrm{ad}_X)(X) &= [X, [X, X]] = 0X + 0Y + 0Z \\
(\mathrm{ad}_X \circ \mathrm{ad}_X)(Y) &= [X, [X, Y]] = [X, 2Y] = 0X + 4Y + 0Z \\
(\mathrm{ad}_X \circ \mathrm{ad}_X)(Z) &= [X, [X, Z]] = [X, -2Z] = 0X + 0Y + 4Z
\end{aligned}$$

so $\mathrm{tr}\,(\mathrm{ad}_X \circ \mathrm{ad}_X) = 0 + 4 + 4 = 8$ so $B(X, X) = 8$.

Then to know everything, we need to calculate the matrix

$$\begin{pmatrix} \langle X, X\rangle & \langle X, Y\rangle & \langle X, Z\rangle \\ & \langle Y, Y\rangle & \langle Y, Z\rangle \\ & & \langle Z, Z\rangle \end{pmatrix}$$

Symmetry gives us the rest and the eigenvalues of this matrix give the signature of the Killing form.

Example (Heisenberg Algebra): $\mathfrak{h} = \{X, Y, Z\} = \begin{pmatrix} 1 & c & a \\ & 1 & b \\ & & 1 \end{pmatrix}$ subject to $[X, Y] = 0$, $[Z, X] = 0$, $[Z, Y] = X$.

If we want the Killing form, we note that $[X, \cdot] = 0$ because X is central so $\mathrm{ad}_X = 0$

Further,

$$\begin{aligned}
(\mathrm{ad}_Y \circ \mathrm{ad}_Y)(X) &= 0 \\
(\mathrm{ad}_Y \circ \mathrm{ad}_Y)(Y) &= 0 \\
(\mathrm{ad}_Y \circ \mathrm{ad}_Y)(Z) &= 0
\end{aligned}$$

Which gives us the matrix

$$\begin{pmatrix} B(X, X) & B(X, Y) & B(X, Z) \\ & B(Y, Y) & B(Y, Z) \\ & & B(Z, Z) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \end{pmatrix}$$

Recall the notion of nilpotency: $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_n = [\mathfrak{g}, \mathfrak{g}_{n-1}]$ is the *lower central series*. We say a lie-algebra is nilpotent iff $g_N = 0$ for sufficiently large $N \in \mathbb{N}$.

In the case of the Heisenberg algebra,

$$\begin{aligned}
&\mathfrak{h}_0 = \mathfrak{h} \\
&\mathfrak{h}_1 = [\mathfrak{h}, \mathfrak{h}] = \langle [X,Y], [Z,X], [Z,Y] \rangle = \mathbb{R}X \\
&\mathfrak{h}_2 = [\mathfrak{h}, \mathbb{R}X] = 0
\end{aligned}$$

Then we can notice that for any $U, V, W \in \mathfrak{h}$, $[U,[V,W]] = 0$ so $\mathrm{ad}_U \circ \mathrm{ad}_V = 0$ and for any U, V in the Heisenberg algebra, $B(U,V) = 0$ so the Killing form on $\mathfrak{h}$ vanishes.

Lemma: Let A be a linear map on V to V finite dimensional so that A is nilpotent. Then $\operatorname{tr}(A) = 0$

Proof:

$$A \sim \begin{pmatrix} \lambda_1 & & \text{stuff} \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \quad A^N \sim \begin{pmatrix} \lambda_1^N & & \text{stuff} \\ & \ddots & \\ & & \lambda_n^N \end{pmatrix}$$

By Cayley-Hamilton, $A^N = 0$ so the eigenvalues of A^N are all zero. Then $\operatorname{tr}(A^N) = 0$ so $\operatorname{tr}(A) = 0$

Theorem: Let $\mathfrak{g}$ be a nilpotent lie-algebra. Then $K = 0$ on $\mathfrak{g}$.

Idea: For each $X \in \mathfrak{g}$, we have $\mathrm{ad}_{X^N} = 0$ for sufficiently large N.

Note: the converse is not true: there are lie-algebras where $K = 0$, yet $\mathfrak{g}$ is not nilpotent.

Theorem: The Killing Form is Ad-invariant.

Proof: Suffices to calculate $\mathrm{Ad}_g \circ \mathrm{ad}_X$ in order to show

$$\begin{aligned}
B(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) &= B(X,Y) \\
&= \operatorname{tr}((\mathrm{Ad}_g \circ \mathrm{ad}_X) \circ (\mathrm{Ad}_g \circ \mathrm{ad}_Y)) \\
&= \operatorname{tr}\,[\mathrm{Ad}_g \mathrm{ad}_X, [\mathrm{Ad}_g \mathrm{ad}_Y, \cdot]] \\
&= \operatorname{tr}\,[\mathrm{Ad}_g \mathrm{ad}_X, \mathrm{Ad}_g \circ \mathrm{Ad}_{g^{-1}}[\mathrm{Ad}_g \mathrm{ad}_Y, \cdot]] \\
&= \operatorname{tr}\left(\mathrm{Ad}_g[\mathrm{ad}_X, \mathrm{Ad} g^{-1}[\mathrm{Ad}_g \circ \mathrm{ad}_Y, \cdot]]\right) \\
&\vdots
\end{aligned}$$

The rest is annoying algebra and notation. Just show

$$\mathrm{ad}_{\mathrm{Ad}_g} = \mathrm{Ad}_g \circ \mathrm{ad} \circ \mathrm{Ad}_{g^{-1}}$$

If you believe that B is Ad-invariant, then it follows that B is an invariant of the lie algebra: If $\phi : \mathfrak{g} \hookrightarrow\!\!\!\!\rightarrow \mathfrak{h}$ where ϕ is a lie-algebra isomorphism, then the Killing form is preserved, $(\mathfrak{g}, B) \hookrightarrow\!\!\!\!\rightarrow (\mathfrak{h}, K)$ so the pull-back metric is preserved

$$\phi^+ K = B$$

Pull-back metric: for $X, Y \in \mathfrak{g}$

$$(\phi^+ K)(X, Y) = K(\phi(X), \phi(Y)) = B(X, Y)$$

Conclusion: for two isomorphic lie algebras, the killing forms are isometric.

Example: $\mathfrak{h}$ and $\mathfrak{sl}(2, \mathbb{R})$ are *not* isomorphic! The Killing form of $\mathfrak{h}$ vanishes identically but the Killing form of $\mathfrak{sl}(2, \mathbb{R})$ is non-zero

Lecture 17 - March 4:

Recall: The *Killing form* $B(X, Y) = \mathrm{tr}\,(\mathrm{ad}_X \circ \mathrm{ad}_Y)$ is a symmetric bilinear form on $\mathfrak{g}$ which is Ad-invariant for all $g \in G$, $X, Y \in \mathfrak{g}$:

$$B(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = B(X, Y)$$

where

$$\mathrm{Ad}_g X = d(I_g)_e(X) = gXg^{-1}$$

Theorem: If G is nilpotent, then the killing form of G is identically zero.

Theorem (Cartan's Criterion for Semi-simplicity): A lie-algebra $\mathfrak{g}$ is semi-simple (it has no non-zero solvable ideals) iff the Killing form is non-degenerate.

Proof:

Definition: A bilinear symmetric pairing $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ (or $\mathbb{C}$) is *non-degenerate* iff for some $X \in \mathfrak{g}$, we have $B(X, Y) = 0$ for all $Y \in \mathfrak{g}$, then $X = 0$.

Equivalently, $B : V \to V^*$ given by $v \mapsto \{w \mapsto B(v, \cdot)\}$ is an isomorphism. Equivalently, the signature has no zeroes.

Note: because the Killing form of $\mathfrak{sl}(2, \mathbb{R})$ is non-degenerate (by explicit calculation), we have $\mathrm{SL}\,(2, \mathbb{R}) \to \mathrm{SO}\,(2, 1)$ because the Killing form on $\mathfrak{sl}(2, \mathbb{R})$ has signature $(+, +, -)$. Further, for $g \in \mathrm{SL}\,(2, \mathbb{R})$, that map is precisely the adjoint representation:

$$\mathrm{Ad}_g : \mathfrak{g} \hookrightarrow\!\!\!\!\rightarrow \mathfrak{g}, \quad K(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = K(X, Y)$$

Put differently, we know

$$\mathrm{SL}\,(2, \mathbb{R}) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g})$$

but more strongly.

$$\mathrm{SL}\,(2, \mathbb{R}) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g} \mid \text{ which preserves K}) \subset \mathrm{Aut}(\mathfrak{g})$$

The automorphisms of $\mathfrak{g}$ which preserve K are precisely those which preserve $x^2 + y^2 - z^2$ which is the isometry group of $\mathrm{SO}\,(2, 1)$.

Since Ad is trivial along the center of G,

$$\mathbb{Z}_2 \hookrightarrow \mathrm{SL}\,(2, \mathbb{R}) \twoheadrightarrow \mathrm{SO}^{\,+}(2, 1) \simeq \mathrm{PSL}(2, \mathbb{R})$$

Example: Calculate the Killing form of $E(1, 1)$.

$$\phi : \mathbb{R} \to \mathrm{SL}\,(2, \mathbb{R}), \quad c \mapsto \begin{pmatrix} \cosh c & \sinh c \\ \sinh c & sinhx \end{pmatrix}$$

$$E(1, 1) = \mathbb{R}^2 \rtimes_\phi \mathbb{R} = \left\{ \begin{pmatrix} \cosh z & \sinh z & x \\ \sinh z & \cosh z & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

so

$$\mathfrak{e}(1, 1) = \begin{pmatrix} 0 & z & x \\ z & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z$$

Then

$$[X, Y] = 0, \quad [Z, X] = Y, \quad [Z, Y] = X$$

Thus we can write $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \langle X, Y \rangle$ so $\mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}^1] = 0$ so $\mathfrak{g}$ is solvable.

This helps us calculate the Killing form because

$$\langle X, X\rangle = \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_X = \operatorname{tr}([X, \underbrace{[X, \cdot]}_{\in \mathfrak{g}^1}])) = \operatorname{tr} 0 = 0$$

Then

$$\begin{pmatrix} \langle X, X\rangle & \langle X, Y\rangle & \langle X, Z\rangle \\ & \langle Y, Y\rangle & \langle Y, Z\rangle \\ & & \langle Z, Z\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 2 \end{pmatrix}$$

since

$$\begin{aligned} (\operatorname{ad}_z \circ \operatorname{ad}_z)(X) &= [Z, [Z, X]] = [Z, Y] = X \implies \text{ eigenvector so tr} = 1 \\ (\operatorname{ad}_z \circ \operatorname{ad}_z)(Y) &= [Z, [Z, Y]] = [Z, X] = Y \implies \text{ eigenvector so tr} = 1 \\ (\operatorname{ad}_z \circ \operatorname{ad}_z)(Z) &= 0 \\ \operatorname{tr}(\operatorname{ad}_z \circ \operatorname{ad}_z) &= 1 + 1 = 2 \end{aligned}$$

Because the Killing form is an invariant of the lie algebra, we can conclude that $\mathfrak{h} \not\cong \mathfrak{e}(1,1)$ because the Killing form of $\mathfrak{h}$ is identically zero.

Another way to see that they are different is that

$$[\mathfrak{h}, \mathfrak{h}] = \mathbb{R}X$$
$$[\mathfrak{e}(1,1), \mathfrak{e}(1,1)] = \langle X, Y\rangle$$

so $\mathbb{Z}(\mathfrak{h})$ is non-trivial but $Z(\mathfrak{e}(1,1))$ is not.

How do you distinguish $\mathfrak{e}(1,1)$ and $\mathfrak{i}$ where $\mathfrak{i}$ is the lie-algebra of

$$\operatorname{Isom}^+(\mathbb{R}^2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \theta, x, y \in \mathbb{R} \right\}$$

We note that both $\mathfrak{i}$ and $\mathfrak{e}(1,1)$ fit into

$$\mathbb{R}^2 \hookrightarrow \mathfrak{i} \twoheadrightarrow \mathbb{R}$$
$$\mathbb{R}^2 \hookrightarrow \mathfrak{e}(1,1) \twoheadrightarrow \mathbb{R}$$

Further, with $\mathfrak{i} = \langle X', Y', Z'\rangle$ we have

$$[X', Y'] = 0, \quad [Z', X'] = Y', \quad [Z', Y'] = -X' \implies [\mathfrak{i}, \mathfrak{i}] = \langle x', y'\rangle$$

therefore, it is not particularly easy to see that these are different. However! The signature of K on $\mathfrak{i}$ is $(0, 0, -)$ while the signature of K on $\mathfrak{e}(1,1)$ is $(0, 0, +)$ so they are not isomorphic!

Lecture 18 - March 6:

Solvability

Last time we concluded that we are able to distinguish lie algebras by their Killing forms (and in particular, using the signature of the Killing form)

Theorem (Cartan's Criternion for Semisimplicity): A lie algebra is semi-simple iff its Killing form is non-degenerate.

Definition: a lie algebra is *semi-simple* if it has no non-zero solvable ideals.

Example: $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}), \mathfrak{so}(3, \mathbb{R})$.

Motivation: to each semi-simple lie group, you can associate a geometry. For example, $\mathrm{SO}\,(3, \mathbb{R})$ admits a transitive action on S^2 whose point-stabilizer is $\mathrm{SO}\,(2, \mathbb{R}) \subseteq \mathrm{SO}\,(3, \mathbb{R})$ stabilizes the north pole (rotation of a ball around the axis.)

From the orbit stabilizer theorem with geometry,

$$\mathrm{SO}\,(3, \mathbb{R})/\mathrm{SO}\,(2, \mathbb{R}) \simeq S^2$$

where $\mathrm{SO}\,(2, \mathbb{R})$ is the maximal compact subgroup (which is not equal to $\mathrm{SO}\,(3, \mathbb{R})$)

From a topological view, we can create a fiber bundle (note this is a relation of spaces, not groups)

$$\mathrm{SO}\,(2, \mathbb{R}) \hookrightarrow \mathrm{SO}\,(3, \mathbb{R}) \twoheadrightarrow S^2$$

and then use that the long-exact sequences is a homotopy to show $\pi_1(\mathrm{SO}\,(n, \mathbb{R})) \simeq \mathbb{Z}_2$ for $n > 1$

Recall: A lie algebra is solvable iff the derived series vanishes in finitely many steps:

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^{n+1} = [\mathfrak{g}^n, \mathfrak{g}^n]$$

Example: is $\mathfrak{e}(1, 1)$ solvable? Yes:

$$[X, Y] = 0, [Z, X] = Y, [Z, Y] = X \implies \mathfrak{g}^1 = \langle X, Y \rangle . \mathfrak{g}^2 = 0$$

Therefore, $\mathfrak{e}(1, 1)$ is solvable.

However, $\mathfrak{sl}(2, \mathbb{R})$ is *not* solvable. (*Proof:* exercise)

Theorem (Levi decomposition): Every finite dimensional lie algebra fits into a short exact sequence

$$\mathfrak{h} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{h}$$

where $\mathfrak{h}$ is the unique maximal solvable ideal and $\mathfrak{g}/\mathfrak{h}$ is semi-simple. Further, this sequence is split. (Though this is much harder to show)

Proof: Omitted

Definition: A maximal solvable ideal, denoted $\mathrm{Rad}(\mathfrak{g})$ is defined by inclusion using the FACT: if $\mathfrak{a}$ and $\mathfrak{b}$ are solvable, $\mathfrak{a} + \mathfrak{b}$

Comparison to group theory: A group is called solvable iff the derived series $G^1 = [G, G] = g^{-1}h{-}1gh, G^{n+1} = [G^n, G^n]$ terminates in finite time

Exercise: A group is solvable iff there exists a composition series

$$1 = G_n \triangleleft G_{n-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$$

so that G_i/G_{i+1} is abelian.

Observation: Any abelian group is solvable

Example:

$$U_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \middle| a, d, g \neq 0, b, c, e \in \mathbb{R} \right\}$$

is a solvable group since

$$[U_3, U_3] = H = \begin{pmatrix} 1 & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix}, [H, H] = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Z(H), [Z(H), Z(H)] = 1$$

In fact, any solvable group (over an algebraically closed field) can be embedded in the upper triangular matrices

Example: $G = \mathrm{SO}(2, \mathbb{R})$ is not upper triangular in $\mathbb{R}$ but it is in $\mathbb{C}$. $\mathrm{SO}(2, \mathbb{R})$ is abelian (hence solvable) but cannot conjugate to anything upper triangular in $\mathbb{R}$ (but you can in $\mathbb{C}$!)

Lemma: If $H \subseteq G$ and G is solvable, then H is solvable

Proof: if $H \subseteq G$, then $H^n \subseteq G^n$. At $n = 0$, $a \in H^{n+1}$ looks like $a = h_n h'_n h_n^{-1} (h'_n)^{-1} \in G^{n+1}$ and $h_n \in H^n \subseteq G^n$. Proceed by induction.

Because $H^n \subseteq G^n$ and $G^n \to 1$, $H^n \to 1$

Lemma: If $\phi : G \twoheadrightarrow H$ and G is solvable then H is solvable

Proof: (Same as above)

Lemma: If $N \hookrightarrow G \twoheadrightarrow H$, G is solvable iff both N and H are solvable.

Conclusion: group extensions are closed under solvability

Proof: Exercise.

Nilpotency

In a sense, Abelian $\subset$ Nilpotent $\subset$ Solvable.

Definition: the commutator of x, y is given by

$$[x, y] = xyx^{-1}y^{-1} \in G$$

Definition: the commutator subgroup $[G, G]$ is the group generated by the set $\{[x, y] \mid x, y \in G\}$

Exercise: $G^{\text{ab}} = G/[G, G]$ (the abelianization of G) is abelian. In fact, it is the largest abelian quotient of G.

Definition: Let G be a group and define the *lower central series* by

$$G_0 = G, G_{n+1} = [G, G_n]$$

A group is *nilpotent* iff the lower central series converges to 1 in finite time.

Observation: $G^{n+1} = [G^n, G^n] \subseteq [G, G_n] = G_{n+1}$ thus nilpotent implies solvable (but solvable does not imply nilpotent)

Example: $G = \text{Aff}^+(1, \mathbb{R})$ and $G = E(1, 1)$ are both solvable but not nilpotent.

Claim: each G_n is normal in G. This is obvious if $n = 1$ since $G = [G, G]$ and the commutator subgroup is normal. Assume it is true for n. Then with $z, x \in G$ and $y \in G_n$

$$z[x,y]z^{-1} = zxyx^{-1}y^{-1}z^{-1} = (zxz^{-1})(zyz^{-1})(zx^{-1}z^{-1})(zy^{-1}z^{-1}) = [zxz^{-1}, zyz^{-1}]$$

By normality $zxz^{-1} \in G$ and $zyz^{-1} \in G_n$ so $z[x,y]z^{-1}$ gives a composition series

$$1 = G_n \triangleleft G_{n-1} \triangleleft \ldots G_1 \triangleleft G_0 = G$$

and each G_i is further normal in G (this is strictly stronger than a normal composition series!)

Thus, we can form the quotient G/G_i.

Example:

$$H = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, H_1 = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so

$$1 = H_2 \triangleleft H_1 \triangleleft H$$

which gives us short exact sequences

$$\begin{tikzcd}
 & & H_1 \arrow[r, hook] \arrow[d, dash] & H \arrow[r, two heads] & H/H_1 \\
 & H_2 \arrow[r, hook] \arrow[d, dash] & H_1 \arrow[r, two heads] & H/H_1 & \\
H_2 \arrow[r, hook] & H_2 \arrow[r, two heads] & 1 & &
\end{tikzcd}$$

Specifically,

$$\begin{tikzcd}
 & & \mathbb{R} \arrow[r, hook] \arrow[d, dash] & H \arrow[r, two heads] & \mathbb{R}^2 \\
 & 1 \arrow[r, hook] \arrow[d, dash] & \mathbb{R} \arrow[r, two heads] & \mathbb{R} & \\
1 \arrow[r, hook] & 1 \arrow[r, two heads] & 1 & &
\end{tikzcd}$$

Exercise:

$$\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} = P_4$$

what is $P_4 / \begin{pmatrix} 1 & & & c \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$?

Lecture 19 - March 8:

Definition: A *composition series* of a group is a finite sequence of normal subgroups

$$1 = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_m = G$$

This is useful because we can build G from the smaller pieces.

Jordan-Holder Theorem: For finite groups, a composition series always exists with "nice quotients"

Two types:

- *Ascending sequence:* $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G$
- *Descending sequence:* $1 \triangleleft \cdots \triangleleft G_2 \triangleleft G_1 \triangleleft G$

We say that these sequences *terminate* if either sequence reaches G (or 1) in finitely many steps.

Example: The lower central series of G,

$$G_0 = G, G_1 = [G, G], G_2 = [G, [G, G]], G_{n+1} = [G, G_n]$$

is a descending sequence. If it terminates in finite time, we say G is *nilpotent*

Example:

$$G_0 = H = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

so

$$G_1 = [H,H] = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$G_2 = [H, G_1] = \left[\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] = 1$$

(because G_1 is central) so the lower central series terminates and H is nilpotent.

In general, for a lower central series,

$$1 = G_n \lhd \cdots \lhd G_2 \lhd G_1 \lhd G$$

we can form quotients which will be central:

$$G_k/G_{k+1} \subseteq Z(G/G_{k+1})$$

Claim: With $\bar{g} = gG_{k+1}$,

$$\bar{g}\,\bar{g_k} = \bar{g_k}\,g \simeq \bar{g}\,\bar{g_k}\,\bar{g}^{-1}\,\bar{g_k}^{-1} = \bar{1}$$

iff $gg_kg^{-1}g_k^{-1} \in G_{k+1}$ since

$$[g, g_k] \in G_{k+1}, [G, G_k] \subseteq G_{k+1}$$

Continuing Example: From above, we have

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \lhd \begin{pmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{pmatrix} \lhd \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}$$

Observation: if $G_{n+1} = 1$ and $G_n \neq 1$, where does G_n live? We have

$$G_{n+1} = [G, G_n] = 1$$

so G_n commutes with everything so $G_n \subseteq Z(G)$. We can construct the diagram

$$\begin{tikzcd}
 & G_1 \arrow[r, hook] \arrow[d, no head] & G_2 \arrow[r, two heads] & G_0/G_1 \subseteq Z(G/G_1) \\
G_2 \arrow[r, hook] & G_1 \arrow[r, two heads] & G_1/G_2 \subseteq Z(G/G_2) &
\end{tikzcd}$$

which is isomorphic to

$$
\begin{tikzcd}
 & \mathbb{R} \arrow[r, hook] \arrow[d, dash] & H \arrow[r, two heads] & \mathbb{R}^2 \subseteq Z(G/G_1) \\
1 \arrow[r, hook] & \mathbb{R} \arrow[r, two heads] & \mathbb{R}/1 = \mathbb{R} \subseteq Z(H) &
\end{tikzcd}
$$

Example:

$$G = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so

$$G_1 = \begin{pmatrix} 1 & 0 & b & c \\ 0 & 1 & 0 & e \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$G_2 = [G, G_1] = \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$G_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now building up the group again,

$$
\begin{tikzcd}
 & & G_1 \arrow[r, hook] \arrow[d, dash] & G_0 \arrow[r, two heads] & G_0/G_1 \\
 & G_2 \arrow[r, hook] \arrow[d, dash] & G_1 \arrow[r, two heads] & G_1/G_2 & \\
G_3 \arrow[r, hook] & G_2 \arrow[r, two heads] & G_2/G_3 & &
\end{tikzcd}
$$

Looking at the isomorphisms,

$$\begin{tikzcd}
 & & \mathbb{R}^3 \arrow[r, hook] \arrow[d, dash] & G \arrow[r, two heads] & \mathbb{R}^3 \\
 & \mathbb{R} \arrow[r, hook] \arrow[d, dash] & \mathbb{R}^3 \arrow[r, two heads] & \mathbb{R}^2 & \\
1 \arrow[r, hook] & \mathbb{R} \arrow[r, two heads] & \mathbb{R}_1 & &
\end{tikzcd}$$

Observation: all the quotients are abelian but they are also central!

$$\begin{aligned}
G_2/G_3 &\simeq Z(G) \subseteq Z(G/G_3) \\
G/G_2 &= \begin{pmatrix} 1 & a & b & 0 \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \subseteq Z(G/G_2) \\
G/G_1 &\subseteq Z(G/G_1) \qquad \text{(because } G/G_1 \text{ is abelian)}
\end{aligned}$$

This is the difference between solvable and nilpotent! Nilpotency says that the quotients are central, not just abelian. Solvability just says the quotients are abelian.

Further, up to diffeomorphism, each of the quotients is of the form $\mathbb{R}^k$

Theorem: Let G be a connected lie group. Then G is nilpotent as a group iff $\mathfrak{g}$ is a nilpotent lie algebra. Further, $\mathfrak{g}$ is solvable iff G is solvable.

Proof: Surjectivity of exponential

Lecture 20 - March 11:

Recall: When G is connected, G is solvable/nilpotent iff $\mathfrak{g}$ is solvable/nilpotent.

Last time, we focused on the lower central series

$$1 = G_n \triangleleft G_{n-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$$

where $G_0 = G$ and $G_{n+1} = [G, G_n]$.

This gave rise to a way of writing our group as a series of extensions with central quotients:

$$G_n/G_{n+1} \subseteq Z(G/G_{n+1})$$

Example: $H_0 = H$, $H_1 = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $H_2 = 1$ so

$$\begin{tikzcd}
 & H_1 \arrow[r, hook] \arrow[d, dash] & H \arrow[r, two heads] & H/H_1 \\
H_2 \arrow[r, hook] & H_1 \arrow[r, two heads] & H/H_1 &
\end{tikzcd}$$

where H/H_1 and H_1/H_2 are both central.

Writing a nilpotent group G as a series of central extensions comes from the upper central series

$$1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_n = G$$

(note this is ascending!)

Intuitively, in the lower central series we wanted the largest central quotient, while in the upper central series we want the largest central subgroup (we are moving in opposite directions!) Of course, the largest central subgroup is simply the center: $G_1 = Z(G)$

We have a series of natural projections

$$G \overset{p_1}{\twoheadrightarrow} G/Z(G) = H_1 \overset{p_2}{\twoheadrightarrow} H_2 = H_1/Z(H_1) \overset{p_3}{\twoheadrightarrow} \dots \overset{p_{n+1}}{\twoheadrightarrow} H_{n+1} = H_n/Z(H_n)$$

so we have maps defined by iterated compositions and we can define $G_1 = \ker p_1, G_2 = \ker p_2 \circ p_1, \dots, G_n = \ker p_n \circ \cdots \circ p_1$. These *generalized centers of* G are all normal subgroups of G and are increasing.

Definition: we say G is *nilpotent* if the sequence $G_i \to G$ in finitely many steps. The sequence of G_i's is called the *upper central series*

Definition: We say a nilpotent group is k-steps if we need k extensions of G, i.e. the length of the upper/lower central series is k.

Example: A 1-step nilpotent group is abelian. A 2-step nilpotent group looks like the Heisenberg group,

$$\begin{pmatrix} 1 & v & c \\ 0 & I_n & u \\ 0 & 0 & 1 \end{pmatrix}, \quad v, u \in \mathbb{R}^n, c \in \mathbb{R}$$

Using the upper central series, one can write the nilpotent group G as a sequence of central extensions. (As the generalized centers converge to G, the quotient subgroups converge to 0)

Example:

$$G = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

so

$$G_0 = 1, G_1 = Z(G) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad p_1 : G \twoheadrightarrow H_1 = G/Z(G) \simeq \mathbb{R}^2$$

Then

$$p_2 : H_1 = G/Z(G) = \mathbb{R}^2 \twoheadrightarrow H_1/Z(H_1) = 1$$

So

$$G_2 = G$$

and hence G is 2-step nilpotent.

We can draw the series of extensions

$$1 \triangleleft G_0 \triangleleft G_1 \triangleleft G_2 = G$$

with $G_{n+1}/G_n = Z(G/G_{n+1})$ by

$$\begin{tikzcd} & G_2/G_1 \arrow[r, hook, "z"] & G/G_1 \arrow[r, two heads] \arrow[d, dash] & G/G_2 \\ G_1/G_0 \arrow[r, hook, "z"] & G/G_0 \arrow[r, two heads] & G/G_1 & \end{tikzcd}$$

which is isomorphic to

$$\mathbb{R}^2 \xhookrightarrow{z} \mathbb{R}^2 \twoheadrightarrow 1$$

$$\mathbb{R} \xhookrightarrow{z} G \twoheadrightarrow G/\mathbb{R}$$

Theorem (of Lie): Let $\mathfrak{g}$ be a solvable lie algebra and let $\pi : \mathfrak{g} \to \mathfrak{gl}(v)$ (as a lie algebra $\mathfrak{gl}(v)$ is given by $[A, B] = AB - BA$ though V does not have a lie algebra structure). Further, assume for $x \in \mathfrak{g}$, the eigenvalues of $\pi(x)$ are real (alternatively, extend everything to $\mathbb{C}$ so we can solve characteristic polynomials). Then, there exists a $v \neq 0$ so that v is an eigenvector of $\pi(\mathfrak{g})$

Consequence: You can find a complete, invariant flag of $\pi(\mathfrak{g})$, i.e. there exists a basis of v so that $\pi(\mathfrak{g})$ is upper triangular. Further, if $\mathfrak{g}$ is nilpotent, then the main diagonal will be all 1's.

Example: $G = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then

$$G_0 = 1,\ G_1 = Z(G) = \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ G_2 = \begin{pmatrix} 1 & 0 & b & c \\ 0 & 1 & 0 & e \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ G_3 = G$$

Since $G_{n+1}/G_n \subseteq Z(G/G_n)$, we have that for $n = 0$, $G_1/G_0 \subseteq Z(G/G_0)$. For $n = 1$, $G_2/G_1 \subseteq Z(G/G_1)$ and for $n = 2$, $G_3/G_2 \subseteq Z(G/G_2)$.

Then

$$\begin{tikzcd}
& & G_3/G_2 \arrow[r, hook] & G/G_2 \arrow[r, two heads] \arrow[d, no head] & G/G_3 \\
& G_2/G_1 \arrow[r, hook] & G/G_1 \arrow[r, two heads] \arrow[d, no head] & G/G_2 & \\
G_1/G_0 \arrow[r, hook] & G/G_0 \arrow[r, two heads] & G/G_1 & &
\end{tikzcd}$$

which is isomorphic to

$$\begin{tikzcd}
 & & \mathbb{R}^3 \arrow[r, hook, "z"] & \mathbb{R}^3 \arrow[r, two heads] \arrow[d, dash] & 1 \\
 & \mathbb{R}^2 \arrow[r, hook, "z"] & G/\mathbb{R} \arrow[r, two heads] \arrow[d, dash] & \mathbb{R}^3 & \\
\mathbb{R} \arrow[r, hook, "z"] & G \arrow[r, two heads] & G/\mathbb{R} & &
\end{tikzcd}$$

The only one which is not obviously central is $R^2 \hookrightarrow G/\mathbb{R}$ but we can just multiply it out.

Conclusion: we can think of a nilpotent group as a finite series of central extensions or by taking iterative centers and eventually killing off the group. Thus, nilpotent groups are morally abelian.

Lecture 21 - March 13:

Definition: *Solvable groups* are those groups G for which the derived series $G^0 = G$, $G^{n+1} = [G^n, G^n]$ converges in finite time.

Theorem: If a group G is connected, G is solvable iff $\mathfrak{g}$ is solvable.

Remark: Solvability on the lie algebra level is defined in exactly the same way; $\mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^{n=1} = [\mathfrak{g}^n, \mathfrak{g}^n]$ *Proof:* HW

Remark: Solvable is much weaker than nilpotent!

Example: $G = \mathrm{Aff}^+(1,\mathbb{R}) \simeq \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a > 0, b \in \mathbb{R} \right\}$ is solvable but not nilpotent:

We can calculate $\mathfrak{g} = \langle X, Y \rangle$ with $[Y, X] = X$ so $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \langle X \rangle$, $\mathfrak{g}^2 = 0$ so $\mathfrak{g}$ is solvable.

However, $\mathfrak{g}$ is not nilpotent:

$$\mathfrak{g}_2 = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = [\mathfrak{g}, \langle X \rangle] = \langle X \rangle$$

so the lower central series does not terminate.

Recall that solvability is "flexible" in that it is inherited by quotients/subgroups and is closed under extensions. So if $\mathfrak{n} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{h}$, then $\mathfrak{g}$ is solvable iff $\mathfrak{n}$ and $\mathfrak{h}$ are solvable. But this fails for nilpotency!

Example: $\mathbb{R} \hookrightarrow \mathfrak{aff}(1,\mathbb{R}) \twoheadrightarrow \mathbb{R}$ gives an example

Exercise: If $\widehat{G} \overset{p}{\twoheadrightarrow} G$ is a covering, then $\widehat{\mathfrak{g}} \simeq \mathfrak{g}$ as lie algebras. (Hint: $p(t) = (\cos t, \sin t)$ is a homomorphism $\mathbb{R} \to S^1 \subseteq \mathbb{C}^\times$)

Equivalent notions of solvability: A group G is solvable iff there exists a sequence of subgroups

$$1 = A_n \lhd A_{n-1} \lhd \cdots \lhd A_1 \lhd A_0 = G$$

so that each $A_{k+1} \lhd A_k$ and the quotients A_k/A_{k+1} are abelian.

Note that in context of connected lie groups G, $A_k/A_{k+1} \simeq \mathbb{R}^k \oplus Tj$ where $k, j \geq 0$ and $Tj := S^1 \times \cdots \times S^1$ is a j-torus.

Example: (Orientation preserving isometry group of Minkowski-plane) $G = E(1,1) = \mathbb{R}^2 \rtimes_\phi \mathbb{R}$ with $\phi : \mathbb{R} \to (\mathbb{R}^2)$ and $\phi(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ then

$$1 \lhd \mathbb{R}^2 \lhd G$$

so $\mathbb{R}^2/1 \simeq \mathbb{R}^2$ is abelian and $G/\mathbb{R}^2 \simeq \mathbb{R}$ is abelian.

Example: $G = \mathrm{Isom}^+(\mathbb{R}^2) \simeq \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}$ has

$$1 \lhd \mathbb{R}^2 \lhd G$$

so again $\mathbb{R}^2/1 \simeq \mathbb{R}^2$ but now $G/\mathbb{R}^2 = S^1$ because the group has a linear part and a translation part so really $G/\mathbb{R}^2 \simeq \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \simeq \mathrm{SO}\,(2,\mathbb{R})$ but this is "parameterized" by an angle which is unique up to 2π so we just have a circle.

Example: Is the following group solvable?

$$\left\{ \begin{pmatrix} \cos a & -\sin a & x & y \\ \sin a & \cos a & z & w \\ 0 & 0 & \cos b & -\sin b \\ 0 & 0 & \sin b & \cos b \end{pmatrix} \middle| a, b, x, y, z, w \in \mathbb{R} \right\}$$

Yes! Notice that each of the 2×2 blocks on the diagonal give an angle up to 2π so there is a natural quotient $\phi : G \to T^2 = S^1 \times S^1$ given by $g \mapsto (a, b)$. The kernel is naturally

$$K = \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

T^2 is abelian and so is K so we just take $K \oplus \mathbb{R}^2$ and we have an abelian quotient so G is solvable.

Claim: The upper triangular matrices (U_n) are solvable

Proof: Let B_n be of the form

$$B_n = \begin{pmatrix} 1 & & & \text{stuff} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

so

$$B_n \hookrightarrow U_n \overset{\phi}{\twoheadrightarrow} (\mathbb{R}^\times)^n$$

where

$$\phi \begin{pmatrix} u_{11} & & & \text{stuff} \\ & u_{22} & & \\ & & \ddots & \\ & & & u_{nn} \end{pmatrix} = \begin{pmatrix} u_{11} \\ u_{22} \\ \vdots \\ u_{nn} \end{pmatrix} \in (\mathbb{R}^\times)^n$$

We note that U_n is an extension of $(\mathbb{R}^\times)^n$ by B_n. $(\mathbb{R}^\times)^n$ is clearly solvable (it is abelian), so it suffices to show B_n is solvable.

One proof is to just look at the lie-algebras and proceed by induction. Another way is to look at the derived series:

$$[B_n, B_n] = \begin{pmatrix} 1 & 0 & & \text{stuff} \\ & 1 & 0 & \\ & & \ddots & 0 \\ & & & 1 \end{pmatrix}$$

this is normal in B_n. We can keep taking quotients and we will eventually get to the identity so it is solvable.

Alternatively, observe B_{n+1} contains a normal $\mathbb{R}^n$. For example,

$$\left(\begin{array}{cc|c} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} & & \begin{matrix} a \\ b \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & & 1 \end{array}\right)$$

Then, $\mathbb{R}^2 \hookrightarrow B_{n+1} \twoheadrightarrow B_n$. $\mathbb{R}^2$ and B_n are solvable so B_{n+1} is solvable and we can build by induction.

Example: $\mathrm{GL}(2, \mathbb{R}) \supseteq \mathrm{SL}(2, \mathbb{R})$ is not solvable. Why is $\mathrm{SL}(2, \mathbb{R})$ not solvable?

$$\mathfrak{sl}(2, \mathbb{R}) = \langle X, Y, Z \rangle \text{ with } [Z, X] = 2X,\ [Z, Y] = -2Y,\ [X, Y] = Z$$

so $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and the derived series does not terminate.