Math 1820A: Lie Algebras - Final Exam Review

1 Lecture Notes

Killing form: $B(X,Y) = \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_Y)$ is a symmetric bilinear form on $\mathfrak{g}$ which is Ad-invariant, i.e.

$$B(\operatorname{Ad}_g X, \operatorname{Ad}_g Y) = B(gXg^{-1}, gYg^{-1}) = B(X,Y)$$

Cartan's Criterion for Semisimplicity: a lie algebra is semi-simple (has no non-zero solvable ideals) iff its Killing form is non-degenerate (i.e. $B(X,Y) = 0$ for all $Y \in \mathfrak{g}$ or the signature has no zeros)

Solvability

A lie algebra is **solvable** iff the derived series terminates:

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^{n+1} = [\mathfrak{g}^n, \mathfrak{g}^n]$$

- $\mathfrak{e}(1,1)$ is solvable
- $\mathfrak{sl}(2,\mathbb{R})$ is *not* solvable

A group is solvable iff there exists a composition series

$$1 = G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 \lhd G_0 = G$$

where G_i/G_{i+1} is abelian.

Lemmas:

- Every abelian group is solvable
- If $H \subseteq G$ and G is solvable, then H is solvable
- If G is solvable and $\phi : G \twoheadrightarrow H$ then H is solvable
- If $N \hookrightarrow G \twoheadrightarrow H$, G is solvable iff N and H are solvable

Levi Decomposition: Every finite-dimensional lie algebra fits into a *split* exact sequence

$$\mathfrak{h} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{h}$$

where $\mathfrak{h}$ is the unique maximal solvable ideal and $\mathfrak{g}/\mathfrak{h}$ is semi-simple.

Maximal Solvable Ideal: $\operatorname{Rad}(\mathfrak{g})$, the *maximal solvable ideal* is defined as expected by inclusion using the fact that $\mathfrak{a}, \mathfrak{b}$ solvable implies $\mathfrak{a} + \mathfrak{b}$ is solvable.

Nilpotency

A group is **nilpotent** iff the *lower central series* terminates:

$$G_0 = G, \quad G_{n+1} = [G, G_n]$$

equivalently,

$$1 = G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 \lhd G_0 = G$$

where $G_i \triangleleft G$ and the quotients G_k/G_{k+1} are central:

$$G_k/G_{k+1} \subseteq Z(G/G_{k+1})$$

(this is a much stronger statement than that quotients are abelian!)

Further, up to diffeomorphism, each of the quotients is of the form $\mathbb{R}^k$

Lemmas:

- Nilpotent implies solvable
- Each $G_n \triangleleft G$ and $G_{i+1} \triangleleft G_i$
- G nilpotent $\implies$ the Killing form of G is identically zero

Commutator: $[x, y] = xyx^{-1}y^{-1} \in G$ is the *commutator* of x, y.

The *commutator subgroup* $[G, G]$ is the subgroup generated by all commutators

$$[G, G] = \{[x, y] : x, y \in G\}$$

Abelianization $G^{\text{ab}} = G/[G, G]$, the *abelianization of* G is the largest abelian quotient of G.

Jordan-Holder Theorem: For finite groups, a composition series with "nice" quotients always exists

- *Ascending sequence:* $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = G$
- *Descending sequence:* $1 \triangleleft \cdots \triangleleft G_2 \triangleleft G_1 \triangleleft G$

Notice, we can write the series of extensions of a nilpotent group as either an ascending or descending series of normal subgroups.

The lower central series gives the descending sequence

$$1 = G_n \triangleleft G_{n-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$$

with central quotients.

However, we can also write G as a series of central *extensions* using the upper central series

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = G$$

such that as the generalized centers converge to G, the quotient subgroups converge to 0

Intuitively, the lower central series yields the largest central quotient while the upper central series yields the largest central *subgroup*. (Of course, is simply $G_1 = Z(G)$)

We say a nilpotent group is *k-steps* if we need k extensions of G for the series to terminate.

Theorem: Let G be a connected lie group. Then G is a nilpotent group iff $\mathfrak{g}$ is a nilpotent lie algebra. Further, $\mathfrak{g}$ is solvable iff G is solvable.

Recall that if $\mathfrak{n} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{h}$, $\mathfrak{g}$ is solvable iff $\mathfrak{n}$ and $\mathfrak{h}$ are solvable.

However, this is *not* true for nilpotency: consider $\mathbb{R} \hookrightarrow \mathfrak{aff}(1, \mathbb{R}) \twoheadrightarrow \mathbb{R}$

Claim: The upper triangular matrices U_n are solvable

Representations

Representation: a *representation* of a group into a vector space V is a homomorphism

$$\Phi : \begin{array}{l} G \to \mathrm{Aut}(V) \\ g \mapsto \{\Phi(g) : V \hookrightarrow\!\!\!\!\twoheadrightarrow V\} \end{array}$$

where $\Phi(g)(v)$ is another vector in V.

Differentiating, we have

$$\phi : \begin{array}{l} \mathfrak{g} \to \mathfrak{gl}(V) \\ X \mapsto \frac{d}{dt}\Big|_{t=0} \Phi(e^{tX}) \end{array}$$

which is a lie-algebra homomorphism ($\phi([X, Y]) = [\phi(X), \phi(Y)]$)

Ado's Theorem: Let $\mathfrak{g}$ be a lie algebra. Then there exists a sufficiently large vector space V such that $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$

Fact: If G is connected, $\exp(\mathfrak{g})$ generates G as a group, i.e. $\forall g \in G$,

$$g = \exp(x_1)\exp(x_2)\cdots\exp(x_n)$$

for some $x_i \in \mathfrak{g}$

Lie's Theorem: Let $\mathfrak{g}$ be a solvable lie algebra and let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ with real eigenvalues for all $\pi(X)$, $X \in \mathfrak{g}$. Then there exists a $v \neq 0 \in V$ such that v is an eigenvector of $\pi(X)$ for all $X \in \mathfrak{g}$

Corollary: In the same context as the theorem, there exists a sequence of subspaces

$$0 = V \subset V_m \subset V_{m-1} \subset \cdots \subset V_0 = V$$

such that V_i is stable under $\pi(X) \in \mathfrak{gl}(V)$ and each V_i is codimension 1 in the next.

Equivalently, there exists a basis for V for which $(\pi(X))_\beta$ is upper triangular for all $X \in \mathfrak{g}$.

Conclusion: If we have a solvable lie algebra, then it is embeddable in the upper triangular lie-algebras (with sufficiently many eigenvalues). *The upper triangular lie algebras are the only ones.*

Proposition: Let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a nilpotent lie algebra $\mathfrak{g}$. Then each operator $\pi(X)$ with $X \in \mathfrak{g}$ is nilpotent.

Engel's Theorem: Let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation such that each $\pi(X)$ is nilpotent (vanishes under powers). Then $\pi(\mathfrak{g})$ is a nilpotent lie-algebra (lower central series converges).

The Quaternions

The Unit Quaternions:

$$Q_8 = \{1, i, j, k \mid i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j\}$$

is a group of order 8 with $Z(Q_8) = \{1, -1\}$

The General Quaternion Group (Hamiltonians):

$$\mathbb{H} = \{a + b\hat{\imath} + c\hat{\jmath} + d\hat{k} \mid a, b, c, d \in \mathbb{R}\}$$

As a vector space, $\mathbb{H} \simeq \mathbb{R}^4$ with basis $\{1, \hat{\imath}, \hat{\jmath}, \hat{k}\}$. As a group, $\mathbb{H}$ is non-abelian with multiplication rule

$$\begin{aligned} &(a+b\hat{\imath} + c\hat{\jmath} + d\hat{k})(w + x\hat{\imath} + y\hat{\jmath} + z\hat{k}) \\ &\quad = aw + ax\hat{\imath} + ay\hat{\jmath} + az\hat{k} + bw\hat{\imath} + bx\hat{\imath}^2 + by\hat{\imath}\hat{\jmath} + bz\hat{\imath}\hat{k} + cw\hat{\jmath} + cx\hat{\jmath}\hat{\imath} + cy\hat{\jmath}^2 + cz\hat{\jmath}\hat{k} + dw\hat{k} + dx\hat{k}\hat{\imath} + dy\hat{k}\hat{\jmath} + dz\hat{k}^2 \\ &\quad = (aw - bx - cy - dz) + (ax + bw + cz - dy)\hat{\imath} + (ay - bz + cw + dx)\hat{\jmath} + (az + by - cx + dw)\hat{k} \end{aligned}$$

Given $q = a + b\hat{\imath} + c\hat{\jmath} + d\hat{k}$, we define

- $\mathrm{Scal}(q) = a$ is the *scalar part* of q
- $\mathrm{Vec}(q) = b\hat{\imath} + c\hat{\jmath} + d\hat{k}$ is the *vector part* of q
- $\overline{q} = a - b\hat{\imath} - c\hat{\jmath} - d\hat{k} = \mathrm{Scal}(q) - \mathrm{Vec}(q)$ is the *conjugate* of q
- $||q|| = \sqrt{a^2 + b^2 + c^2 + d^2}$ is the *norm* of q
- $p^{-1} = \frac{\overline{p}}{||p||^2}$ is the *inverse* of p

We say q is a *unit quaternion* if $||q|| = 1$ and *pure* if $\mathrm{Scal}(q) = 0$.

Properties:

- For two pure quaternions,
$$pq = -p \cdot q + p \times q$$
- Similarly, for $X = x + \vec{p}$ and $Y = y + \vec{q}$,
$$XY = (x + \vec{p})(y + \vec{q}) = xy + x\vec{q} + y\vec{p} + \vec{p}\,\vec{q} = (xy - \vec{p} \cdot \vec{q}) + (x\vec{q} + y\vec{p} + \vec{p} \times \vec{q})$$
- $p\overline{p} = ||p||^2 = p \cdot p$
- For u pure and unital, $u^4 = 1$
- $\overline{pq} = \overline{q}\,\overline{p}$

Note that $||\cdot|| : \mathbb{H}^\times \to \mathbb{R}^+$ given by $p \mapsto ||p||$ is multiplicative:

$$||pq|| = (pq)(\overline{pq}) = pq\,\overline{q}\,\overline{p} = p\,||q||^2\,\overline{p} = ||p||^2\,||q||^2$$

We can also look at its kernel:

$$S^3 = \{p \in \mathbb{H}^\times \mid ||p|| = 1\} = \{w + x\hat{\imath} + y\hat{\jmath} + z\hat{k} \mid \sqrt{w^2 + x^2 + y^2 + z^2} = 1\}$$

Group Actions

There exists a group action of S^3 on $\mathbb{R}^3$.

Notice that group conjugation ($q \in \mathbb{H}^\times \mapsto \{I_q : H \hookrightarrow\!\!\!\!\rightarrow \mathbb{H}\}$) respects Hamiltonian conjugation:

$$\mathrm{conj}(I_q(p)) = I_q(\mathrm{conj}(p))$$

This means that a quantity expressed in terms of p and $\overline{p}$ will be preserved by I_q.

In particular, for a quaternion p,

$$\mathrm{Scal}(p) = \frac{p + \overline{p}}{2}$$
$$\mathrm{Vec}(p) = \frac{p - \overline{p}}{2} \mathrm{Norm}(p) \qquad = \sqrt{p\overline{p}}$$

so if p is pure, $I_q(p)$ is pure.

If we think of $\mathbb{H} = \mathbb{R} \oplus \mathrm{Span}_{\mathbb{R}}\{\hat{\imath}, \hat{\jmath}, \hat{k}\} = R \oplus P$, where P is the space of pure quaternions, then I_q preserves the 3-dimensional, real subspace P. (In fact, since I_q preserves the norm, it is an isometry of P).

Therefore, we have a map

$$S^3 \to \mathrm{SO}(3, \mathbb{R})$$
$$q \mapsto I_q : P \hookrightarrow\!\!\!\!\rightarrow P$$

What is I_q? By definition $I_q(p) = qp^{-1}q$. Trivially, this is a map in $\mathrm{SO}(3,\mathbb{R})$ and has fixed axis $I_q(q) = qqq^{-1} = q$. Therefore, it suffices to examine the action of I_q on the orthgonal complement C of $\mathbb{R}q$. Take $v \in \mathbb{R}q^\perp$. Using some of the pure quaternion multiplication properties, we find that $I_q(v) = qvq^{-1} = -v$.

Therefore, I_q is a line symmetry with axis of symmetry $\mathbb{R}q$. Relative to $\beta = \{q, v, q \times v\}$,

$$(I_q)_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Cartan-Dieudonné Theorem: Any isometry of $\mathbb{R}^n$ is expressible as at most $(n+1)$-reflections in hyperplanes

Proposition: The composition of two line symmetries $L_B \circ L_A$ in the plane is a a rotation by twice the angle from A to B

Euler-Rodrigues Theorem: Let $q = \cos\frac{\theta}{2} + u \sin\frac{\theta}{2}$ be a unit quaternion in S^3. Then $I_q : P \hookrightarrow\!\!\!\!\rightarrow P$ is a rotation about the axis $u \in P$ by an angle of θ.

Covering Maps

By the Euler-Rodrigues theorem, there exists $\phi : S^3 \twoheadrightarrow \mathrm{SO}(3,\mathbb{R})$ with kernel $\mathbb{Z}_2$ called the *spin double cover of* $SO(3,\mathbb{R})$.

In a sense, $S^3/\mathbb{Z}_2$ is the topology on the space of lines through the origin in $\mathbb{R}^4$:

$$\mathbb{RP}^3 \simeq S^3/\mathbb{Z}_2 \simeq \mathrm{SO}(3,\mathbb{R})$$

SU$(2,\mathbb{C})$ isomorphism:

We have another isomorphism. Consider $\mathbb{H}$ a *right* $\mathbb{C}$-vector space $\mathbb{H}^\mathbb{C}$ with $\mathbb{C}$-basis $\{1, \widehat{\jmath}\}$ so

$$(a + b\widehat{\imath} + c\widehat{\jmath} + d\widehat{k})_\beta = 1 \cdot (a + bi) + \widehat{\jmath}(c - di)$$

and

$$L_q(pc) = q(pc) = (qp)c = L_q(p)c$$

so that $\forall q = z + w\widehat{\jmath} \in \mathbb{H}$, L_q is an honest $\mathbb{C}$-endomorphism of $\mathbb{H}^\mathbb{C}$ so $q \mapsto L_q$ yields

$$\mathbb{H}^\times \hookrightarrow \mathrm{GL}(\mathbb{H}^\mathbb{C}) \simeq \mathrm{GL}(2,\mathbb{C})$$

so relative to $\beta = \{1, \widehat{\jmath}\}$,

$$(L_q)_\beta = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$$

Restricting to S^3 we have $S^3 \hookrightarrow \mathrm{SU}(2,\mathbb{C})$ where

$$\mathrm{SU}(2,\mathbb{C}) = \left\{ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \middle| z\,\overline{z} + w\,\overline{w} = 1 \right\}$$

then $S^3 \hookrightarrow\!\!\!\!\rightarrow \mathrm{SU}(2,\mathbb{C})$ again with $\mathbb{Z}_2$ kernel.

Spin 4 double cover:

Similar to the case of $\mathbb{H}$ as a $\mathbb{C}$-vector space, we can consider $\mathbb{H}$ as an $\mathbb{R}$-vector space which means that we do not have to worry about left/right structure.

With basis $\beta = \{1, \widehat{\imath}, \widehat{\jmath}, \widehat{k}\}$, we have

$$L_q \sim \begin{pmatrix} x & -y & -z & -w \\ y & x & -w & z \\ z & w & x & -y \\ w & -z & y & x \end{pmatrix}, \quad R_q \sim \begin{pmatrix} x & -y & -z & -w \\ y & x & w & -z \\ z & -w & x & y \\ w & z & -y & x \end{pmatrix}$$

Since we have orthogonal columns, if $||q|| = 1$, then $L_q \in \mathrm{SO}(4, \mathbb{R})$ and $R_q \in \mathrm{SO}(4, \mathbb{R})$ so we have a homomorphism

$$\begin{aligned} S^3 \times S^3 &\to \mathrm{SO}(4, \mathbb{R}) \\ (q, p) &\mapsto (L_q \circ R_{p^{-1}})_\beta \end{aligned}$$

Using Euler-Rodrigues, we can show this map is surjective with another $\mathbb{Z}_2$ kernel so

$$S^3 \times S^3 / Z_2 \simeq \mathrm{SO}(4, \mathbb{R})$$

(i.e. $S^3 \times S^3$ is a double cover of $\mathrm{SO}(4, \mathbb{R})$) and

$$S^3 \times S^3 \simeq \mathrm{Spin}(4, \mathbb{R})$$

Fibre Bundles

Fibre Bundle: A *fibre bundle* is a surjective smooth math $p : E \to B$ such that for a small set $U \subseteq B$,

$$p^{-1}(U) = U \times F$$

Fibre: the inverse-image (pullback) of a point under a map. (Heuristically, a parameterization of the total space over a variable base space)

Examples:

- $p(x, y) = x$ is an $\mathbb{R}$-bundle over $\mathbb{R}$ of total space $\mathbb{R}^2$
- A cylinder projected onto its base is an I-bundle over S^1 for interval I
- The Mobius band is a I-bundle over S^1 (but the Mobius band is orientable while the cylinder is not)
- T^2 is a S^1-bundle over S^1
- The Klein bottle is an S^1-bundle over S^1

Principal G-bundle: a fibre bundle E with fibre G such that the fibre is isomorphic to G as a group. Equivalently, a fibre bundle equipped with a right G-action on the total space so the fibres are the orbits of the action.

Examples:

- With $(x, y)t = (x, y + x)$ (an upwards translation), $\mathbb{R}^2$ is an principal $\mathbb{R}$-bundle over I
- With S^1-action $(\theta, \phi)t = (\theta, \phi + t), t \in \mathbb{R}/\mathbb{Z}$, T^2 is a principal S^1-bundle over S^1
- Let $E = \{(p, \{v_1, v_2\}) \mid p \in S^2, v_1, v_2 \in T_pS^2\}$ such that $T_pS^2 = \mathrm{Span}_{\mathbb{R}}\{v_1, v_2\}$ with action $(p, \{v_1, v_2\})t = p$. Then E is a $\mathrm{GL}(2, \mathbb{R})$ bundle over S^2 and our group action is $q^{-1}(N) \simeq \mathrm{GL}(2, \mathbb{R})$ because we just need two linearly independent vectors in $T_NS^2 \simeq \mathbb{R}^2$

The Hopf Fibration: a principal S^1-bundle over S^2 with total space S^3 when S^3 acts on itself by conjugation

Recall that $S^3 \to \mathrm{Aut}(S^2)$ given by $q \mapsto I_q : S^2 \hookrightarrow\!\!\!\!\!\rightarrow S^2$ is a transitive S^3 action on S^2. Consequently, we have $H : S^3 \twoheadrightarrow S^2$ via $q \mapsto I_q(\widehat{k})$ with $\mathrm{Orbit}(\widehat{k}) = S^2$.

By the orbit stabilizer theorem, $S^3/\text{Stab}(\widehat{k}) \simeq \text{Orbit}(\widehat{k}) \simeq S^2$. We know

$$\text{Stab}(\widehat{k}) = \{q \in S^3 \mid I_q(\widehat{k}) = \widehat{k}\} = \{q \in S^3 \mid q\widehat{k}q^{-1} = \widehat{k}\}$$

Certainly we know $q\widehat{k}q^{-1} = \widehat{k}$ for $q \in \text{Span}_{\mathbb{R}}\{1, \widehat{k}\} \simeq \mathbb{C}$. Further, we can show it does not commute with $\text{Span}\{\widehat{\imath}, \widehat{\jmath}\}$ so

$$\text{Stab}(\widehat{k}) = \{\cos\theta + \sin\theta\widehat{k} \mid \theta \in \mathbb{R}\} \simeq S^1$$

is our fibre over $\widehat{k}$.

What about over the rest of S^2?

Notice:

$$H(qp) = qp\widehat{k}p^{-1}q^{-1} = qH(p)q^{-1} = I_q(H(p))$$

so left multiplication is just rotation of $H(p)$ about the vector part of q.

This means that finding the fibre over a point $v \in S^2$ is a matter of finding a single $q \in S^3$ with $H(q) = v$ since $H^{-1}(v) = qS$ where $S = \text{Stab}(\widehat{k})$.

Geometrically, this means we need to find $p \in S^3$ so

$$H(p\widehat{k}) = v \implies I_p(H(\widehat{k})) = v \implies I_p(\widehat{k}) = v$$

and then let $q = p\widehat{k}$ so $H^{-1}(v) = p\widehat{k}S = pS$

To find p, we can use the fact that $p = \cos\phi + u\sin\phi$ where $\phi = \theta/2$ to reduce the problem to finding the fixed axis of rotation which transforms $\widehat{k}$ to v and the angle θ from $\widehat{k}$ to v.

For all $v = (a, b, c) \neq \{\widehat{k}, -\widehat{k}\}$, the obvious choice of axis is

$$u = \frac{v \times \widehat{k}}{\left|\left|v \times \widehat{k}\right|\right|} = \frac{1}{\left|\left|v \times \widehat{k}\right|\right|}\begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{k} \\ a & b & c \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{a^2+b^2}}(b\widehat{\imath} - a\widehat{\jmath})$$

Then

$$v \cdot \widehat{k} = ||v|| \, \left|\left|\widehat{k}\right|\right| \cos\theta \implies c = \cos\theta \implies \cos\phi = \sqrt{\frac{1+c}{2}}$$
$$\sin\phi = \sqrt{\frac{1-c}{2}}$$

so

$$p = \frac{1}{\sqrt{2+2c}}(1 + c - b\widehat{\imath} + a\widehat{\jmath})$$

and the full fibre over S^2 is

$$H^{-1}(v) = \frac{1}{\sqrt{2+2c}}(1 + c - b\widehat{\imath} + a\widehat{\jmath})(\cos\theta + \sin\theta\widehat{k})$$

Linked Curves

For two generic points in S^2, their fibres in S^3 will be linked. Explicitly, $p, q \in S^2$ will lead to linked circles $H^{-1}(p), H^{-1}(q)$ via $H^{-1} : S^2 \to S^3$ and stereographic projection $F : S^3/\widehat{k} \hookrightarrow\!\!\!\!\!\rightarrow \mathbb{R}^3$ defined by

$$F(x, y, z, w) = \left(\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w}\right)$$

Example: the fibre at $\widehat{k}$ is a line along the x-axis. The fibre of $-\widehat{k}$ is a circle in the yz-plane. A simple drawing shows that these two circles are linked. But how do we formalize this notion?

Consider a curve $\gamma(t)$ that loops many times around $p \in \mathbb{C}$ with $\gamma(0) = \gamma(1)$. We want to find the **winding number** of oriented wraps around p.

We can consider $\pi_1(s)$, the group of curves at $p \in S$ up to continuous deformation. One way is to imagine picking an orientation on $R^2/0$ and deforming the curve to wrap around a generator n-times where n is the winding number.

We can also think about this geometrically in terms of one-forms. In a sense, we have a function $\mathbb{R}^2/0 \to \mathbb{R}$ which measures the angle relative to 0 at a point p. If we imagine the vector $\vec{OP}$ as the hypotenuse of a triangle with legs x and y, then (locally) the angle is $\theta = \arctan(y/x)$ so (locally)

$$d\theta = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

which means that we can integrate along curves!

$$\gamma \mapsto \int_\gamma d\theta$$

but since $\gamma(0) = \gamma(1)$, we (morally) have

$$\int_\gamma d\theta \approx \theta(\gamma(1)) - \theta(\gamma(0))$$

In fact, there is no honest function $f : \mathbb{R}^2/0 \to \mathbb{R}$ so that $df = \omega$ where $[\omega]$ generates $H^1(\mathbb{R}^2/0)$. If there were, then $\int_\gamma d\theta = 0$ exactly but if we consider the simplest curve $\gamma(t) = (\cos t, \sin t)$, we see

$$\begin{aligned}
\int_\gamma d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \frac{-\sin t}{\sin^2 t + \cos^2 t}\,d(\cos t) + \frac{\cos t}{\sin^2 t + \cos^2 t}\,d(\sin t) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sin t(\sin t) + \cos t(\cos t)\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} 1\,dt = 1
\end{aligned}$$

which would give

$$\omega = \frac{1}{2\pi}\begin{pmatrix} -y & dx \\ x & dy \end{pmatrix}\frac{1}{x^2+y^2}$$

but this is a contradiction.

Our original goal was to understand why linked and unlined curves are fundamentally different in order to understand why $H^{-1}(p), H^{-1}(q)$ are linked.

First reduction: consider $p, q \in S^3$ and their fibres. It suffices to show the case $p = \widehat{k}, q = -\widehat{k}$. Intuitively, the linkage of fibres should be invariant under isotopy (i.e. under deformation, linked curves should remain linked and vice versa). Therefore, if we move p to $\widehat{k}$ and q to $-\widehat{k}$, the fibres should remain linked.

We can imagine two curves a and b, each wrapped around a two parallel poles. Clearly, there is no way to deform a to b. In fact, a, b generate a free group in $\pi_1(\mathbb{R}^3/\text{two parallel lines})$

This problem then resolves to finding a non-trivial relation on the generating circles a, b of two linked curves.

2 Homework Results

For a short exact sequence $\mathfrak{n} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{h}$, $\mathfrak{g}$ is solvable iff $\mathfrak{n}$ and $\mathfrak{h}$ are solvable.

A **derivation** of a lie algebra $\mathfrak{n}$ is any linear map $D : \mathfrak{n} \to \mathfrak{n}$ such that

$$D[X, Y] = [DX, Y] + [X, DY]$$

for all $X, Y \in \mathfrak{n}$.

Example: $\mathrm{ad}_X(Y) = [X, Y]$ is a derivation of $\mathfrak{g}$ for any $X \in \mathfrak{g}$

A **representation** of a lie algebra $\mathfrak{h}$ into $\mathfrak{n}$ is a Lie-algebra homomorphism $\phi : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{n})$

To show that $\mathfrak{g} \simeq \mathfrak{n} \rtimes_\phi \mathfrak{h}$, it suffices to show there exists a split exact sequence $\mathfrak{n} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{h}$.

If $\mathfrak{n} \subset \mathfrak{g}$ is an ideal and $\mathfrak{h} \subset \mathfrak{g}$ is a sub-algebra for which $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ as vector spaces, then $\mathfrak{g} \simeq \mathfrak{n} \rtimes \mathfrak{h}$

$\mathfrak{g}$ is solvable iff there exists a sequence of subalgebras

$$0 = \mathfrak{h}_n \subset \mathfrak{h}_{n-1} \subset \cdots \subset \mathfrak{h}_1 \subset \mathfrak{h}_0 = \mathfrak{g}$$

such that each $\mathfrak{h}_{i+1}$ is an ideal of $\mathfrak{h}_i$ and $\mathfrak{h}_i / \mathfrak{h}_{i+1}$ is abelian.

The sum of solvable ideas is solvable.

For two ideals $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$,

$$(\mathfrak{a} + \mathfrak{b})/\mathfrak{a} = \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$$

$\exp : \mathfrak{so}(3, \mathbb{R}) \to \mathrm{SO}(3, \mathbb{R})$ is surjective.

Let $X \in \mathfrak{so}(3, \mathbb{R})$ such that $||X|| = \sqrt{2}$. Then

$$\exp(tX) = I_3 + \sin(t)X + (1 - \cos t)X^2$$

Every element of $\mathrm{SO}(n, \mathbb{R})$ is the product of at most $(n + 2)/2$ reflections.

$$\mathfrak{so}(3, \mathbb{R}) \oplus \mathfrak{so}(3, \mathbb{R}) \simeq \mathfrak{so}(4, \mathbb{R})$$

The linking number of two smooth curves $\gamma : I \to \mathbb{R}^3$ with no self-intersections and where $\gamma(0) = \gamma(1)$ is given by

$$L(\gamma_1, \gamma_2) = \frac{1}{4\pi} \oint \oint_{S^1 \times S^1} \frac{1}{|\gamma_1(s) - \gamma_2(t)|^3} \det \begin{pmatrix} \frac{d\gamma_1}{ds}(s) & \frac{d\gamma_2}{dt}(t) & \gamma_1(s) - \gamma_2(t) \end{pmatrix} \, ds \, dt$$

$$S^3/S^1 \simeq S^1 \times \mathbb{R}^2$$

3 Important Formulae

Group action:

1. $\alpha(1, x) = x$
2. $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$

One parameter subgroup: $\alpha : \mathbb{R} \to \mathrm{GL}(n, \mathbb{R})$

1. $\alpha(0) = I$
2. $\alpha(s + t) = \alpha(s)\alpha(t)$
3. $\alpha'(t) = \alpha(t)\alpha'(0)$

Matrix exponential:

$$e^{tA} = I_n + At + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \dots$$

If $AB = BA$,

$$e^A e^B = e^B e^A = e^{A+B}$$

With A diagonalizable,

$$e^{tA} = e^{tPDP^{-1}} = Pe^{tD}P^{-1}$$

For any $A \in M_n(\mathbb{C})$,

$$\det(e^A) = e^{\operatorname{tr} A}$$

For $A(t)$ a smooth family of matrices in $\mathrm{GL}(2, \mathbb{R})$ and $A(0) = I$,

$$\left.\frac{d}{dt}\right|_{t=0} \det A(t) = \operatorname{tr} A'(0)$$

Lie Algebra: A vector space with a pairing $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

1. $[A, B] = -[B, A]$ (skew-symmetry)
2. $[A + cB, D] = [A, D] + c[B, D]$ (bilinearity)
3. $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ (Jacobi identity)

and where

$$[A, B] = \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} e^{tA}e^{sB}e^{-tA} = AB - BA$$

Ideal: $\mathfrak{h} \subseteq \mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$

Short exact sequence:

$$N \overset{i}{\hookrightarrow} G \overset{p}{\twoheadrightarrow} H$$

where $\ker p = \mathrm{im}\ i$

Split exact sequence:

$$\begin{tikzcd} N \arrow[r, hook, "i"] & G \arrow[r, two heads, "p"] & H \arrow[l, bend right, "\sigma"'] \end{tikzcd}$$

with $\ker p = \mathrm{im}\ i$ and $p\sigma = 1_H$.

Equivalently, $G \simeq N \rtimes_\phi H$ where $\phi : H \to \mathrm{Aut}(N)$ by

$$(n, h)(a, b) = (n\phi_h(a), hb)$$

(if $\phi = \mathrm{id}$, then $G \simeq N \times H$)

Lie Algebra Homomorphism:

$$\psi([X, Y]) = [\psi(X), \psi(Y)]$$

Adjoint Representation: $\mathrm{Ad}_g(X) = gXg^{-1}$ for $g \in G$.

$$\begin{tikzcd} G \arrow[r, "Ad"] & \mathrm{Aut}(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{ad}"] & \mathfrak{gl}(\mathfrak{g}) \arrow[u, "\exp"'] \end{tikzcd}$$

adjoint Representation: $\mathrm{ad}_x(Y) = [X, Y]$

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{g(t)} = \mathrm{ad}(g'(0)) = \mathrm{ad}(X)$$

Killing Form: $B(X, Y) = \operatorname{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$

One may assign a symmetric matrix A such that $B(X, Y) = x^T Ay$. The signs of the eigenvalues of A determine the signature of the Killing form.

Polarization identity:

$$B(x + y, x + y) = B(x, x) + 2B(x, y) + B(y, y)$$

If $\mathfrak{g}$ is nilpotent, the Killing form is 0.

A lie algebra is semi-simple iff the Killing form is non-degenerate.

Solvable: $g^0 = g$, $g^{i+1} = [g^i, g^i]$

1. $H \subseteq G \implies H$ solvable if G solvable
2. $\phi : G \twoheadrightarrow H \implies H$ solvable if G solvable
3. $N \hookrightarrow G \twoheadrightarrow H \implies G$ solvable iff N and H are solvable
4. G connected implies $\mathfrak{g}$ solvable if G solvable

Nilpotent: $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i]$

1. G connected implies $\mathfrak{g}$ nilpotent if G nilpotent
2. $\mathfrak{g}$ nilpotent implies $\mathfrak{g}$ solvable

If G is connected, for all $g \in G$,

$$g = \exp(x_1)\exp(x_2)\cdots\exp(x_n)$$

for $x_1, \ldots, x_n \in \mathfrak{g}$

Pure quaternions:

$$pq = -p \cdot q + p \times q$$

If u pure and unital, $u^4 = 1$

Euler-Rodrigues:

$$q = \cos\frac{\theta}{2} + u\sin\frac{\theta}{2}$$

is a rotation about fixed axis u by angle θ

SO$(4, \mathbb{R})$ embedding: Left multiplication action of $q = x + y\widehat{\imath} + z\widehat{\jmath} + w\widehat{k}$ can be written

$$L_q \sim \begin{pmatrix} x & -y & -z & -w \\ y & x & -w & z \\ z & w & x & -y \\ w & -z & y & x \end{pmatrix}$$

Fibre Bundle: $p : E \twoheadrightarrow B$ such that $p^{-1}(U) = U \times F$

Principal G-bundle: a fibre bundle with fibre G such that the fibre is isomorphic to G as a group