Math 1820A Spring 2024 - Homework 7

Instructions: This assignment is worth twenty points. Please complete the following problems assigned below. Submissions with insufficient explanation may lose points due to a lack of reasoning or clarity. If you are handwriting your work, please ensure it is readable and well-formatted for the grader.

Be sure when uploading your work to **assign problems to pages**. Problems with pages not assigned to them **may not be graded**.

Textbook Problems:

Additional Problems: For these problems let $\mathbb{H}$ denote the algebra of Hamiltonians and $\mathbb{H}^\times$ denote all the non-zero elements as a *group* under multiplication. For the context of this homework, assume a 'rotation' is orientation preserving, and a 'reflection' is preserves a codimension 1 subspace.

Problem 1

Construct a faithful representation of $\mathbb{H}^\times$ into $\mathrm{GL}(4,\mathbb{R})$. (Hint: The set $\beta = \{1, i, j, k\} \subset \mathbb{H}$ forms a real-basis for $\mathbb{H}$ as a vector-space. Find a matrix representation for L_q where L_q denotes left-multiplication by the quaternion $q \in \mathbb{H}^\times$).

Let $q = a + bi + cj + dk$ and $p = x + yi + zj + wk$ be elements in $\mathbb{H}^\times$. We want a representation of q as a matrix L_q acting on p such that

$$L_q \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = pq = \begin{pmatrix} ax - by - cz - dw \\ ay + bx + cw - dz \\ az - bw + cx + dy \\ aw + bz - cy + dx \end{pmatrix}$$

(note that pq here means the quaternion product of p and q represented as a vector in $\mathbb{R}^4$).

Clearly, then,

$$L_q = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$$

To show that this is, in fact, a faithful representation, it suffices to show that the map

$$\phi : \begin{aligned} \mathbb{H}^\times &\to \mathrm{GL}(4,\mathbb{R}) \\ q &\mapsto L_q \end{aligned}$$

is an injective homomorphism.

Consider $q = a_1 + b_1 i + c_1 j + d_1 k$ and $p = a_2 + b_2 i + c_2 j + d_2 k$. Then

$$
\begin{aligned}
\phi(qp) &= \phi((a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)j + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2) \\
&= \begin{pmatrix} a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 & -a_1b_2 - b_1a_2 - c_1d_2 + d_1c_2 & -a_1c_2 + b_1d_2 - c_1a_2 - d_1b_2 & -a_1d_2 - b_1c_2 + c_1b_2 - d_1a_2 \\ a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2 & a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 & -a_1d_2 - b_1c_2 + c_1b_2 - d_1a_2 & a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2 \\ a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2 & a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2 & a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 & -a_1b_2 - b_1a_2 - c_1d_2 + d_1c_2 \\ a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2 & -a_1c_2 + b_1d_2 - c_1a_2 - d_1b_2 & a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2 & a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \end{pmatrix} \\
\phi(q)\phi(p) &= \begin{pmatrix} a_1 & -b_1 & -c_1 & -d_1 \\ b_1 & a_1 & -d_1 & c_1 \\ c_1 & d_1 & a_1 & -b_1 \\ d_1 & -c_1 & b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & -b_2 & -c_2 & -d_2 \\ b_2 & a_2 & -d_2 & c_2 \\ c_2 & d_2 & a_2 & -b_2 \\ d_2 & -c_2 & b_2 & a_2 \end{pmatrix} \\
&= \begin{pmatrix} a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 & -a_1b_2 - b_1a_2 - c_1d_2 + d_1c_2 & -a_1c_2 + b_1d_2 - c_1a_2 - d_1b_2 & -a_1d_2 - b_1c_2 + c_1b_2 - d_1a_2 \\ a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2 & a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 & -a_1d_2 - b_1c_2 + c_1b_2 - d_1a_2 & a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2 \\ a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2 & a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2 & a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 & -a_1b_2 - b_1a_2 - c_1d_2 + d_1c_2 \\ a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2 & -a_1c_2 + b_1d_2 - c_1a_2 - d_1b_2 & a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2 & a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \end{pmatrix}
\end{aligned}
$$

So the mapping is a genuine representation. To show that it is injective, it suffices to show that the only quaternion which maps to the identity matrix is the identity quaternion. Clearly,

$$\phi(1 + 0i + 0j + 0k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and by definition of L_q, $1 \in \mathbb{H}^\times$ is the only quaternion which maps to the identity matrix. Therefore, the representation is faithful. $\blacksquare$

Problem 2

Using your result from Problem 1, let $\mathfrak{h}$ denote the Lie-algebra of $\mathbb{H}^\times$ inside $\mathrm{GL}(4,\mathbb{R})$. Write out a basis for the Lie-algebra and the corresponding brackets.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$B = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$
$$D = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Provides a natural basis. Using Matlab, we can calculate the bracket $[X,Y] = XY - YX$:

$$[A,B] = 0$$
$$[A,C] = 0$$
$$[A,D] = 0$$
$$[B,C] = 2D$$
$$[B,D] = -2C$$
$$[C,D] = 2B$$

Problem 3

Define $\mathfrak{q} = \mathfrak{h}/\mathfrak{z}(\mathfrak{h})$ where $\mathfrak{z}(\mathfrak{h})$ denotes the center of $\mathfrak{h}$. Determine if $\mathfrak{q}$ is solvable. Is it simple?
We can create the short exact sequence

$$\mathfrak{z}(\mathfrak{h}) \hookrightarrow \mathfrak{h} \twoheadrightarrow \mathfrak{q}$$

$\mathfrak{q}$ is thus solvable iff $\mathfrak{h}$ is solvable.
From Problem 2, we have

$$\mathfrak{h}^1 = [\mathfrak{h}, \mathfrak{h}] = \langle 2D, -2C, 2B \rangle$$

so

$$\mathfrak{h}^2 = [\mathfrak{h}^1, \mathfrak{h}^1] = [\langle 2D, -2C, 2B \rangle, \langle 2D, -2C, 2B \rangle] = 4 \langle B, C, D \rangle$$

Since this is just a scalar multiple of $\mathfrak{h}^1$, the derived series will not terminate. Therefore, $\mathfrak{h}$ is not solvable so $\mathfrak{q}$ is not solvable.
Further, $\mathfrak{q}$ is not simple since $\langle B, C, D \rangle \subseteq \mathfrak{q}$ and

$$\begin{aligned} [\langle B, C, D \rangle, \langle B, C, D \rangle] &= \langle B, C, D \rangle \\ [\langle B, C, D \rangle, A] &= 0 \subseteq \langle B, C, D \rangle \end{aligned}$$

so $\mathfrak{q}$ has a non-trivial ideal. $\blacksquare$

Problem 4

Consider the natural group action of $\mathrm{SO}(3,\mathbb{R})$ on $\mathbb{R}^3$, namely you pair a matrix $A \in \mathrm{SO}(3,\mathbb{R})$ with a vector $v \in \mathbb{R}^3$ and get $Av \in \mathbb{R}^3$. Prove that for any $A \in \mathrm{SO}(3,\mathbb{R})$, there exists a line $L_A \subset \mathbb{R}^3$ through the origin, such that A fixes all points along L_A. Is the same result true for $\mathrm{SO}(4,\mathbb{R})$?

It suffices to show that for all $A \in \mathrm{SO}(3,\mathbb{R})$, there exists a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

i.e. there exists an eigenvector with eigenvalue 1 for all $A \in \mathrm{SO}(3,\mathbb{R})$.
Since the characteristic polynomial of A will be cubic, it will have at least one real root so there exists at least one real eigenvalue λ of A.
Therefore, $Av = \lambda v$ for some vector v. Without loss of generality, we can assume that v is a unit vector, i.e. $||v|| = v^T v = 1$. Further, since $A \in \mathrm{SO}(3,\mathbb{R})$, we have that $A^T A = I$. Thus

$$1 = v^T v = v^T(A^T A)v = (Av)^T(Av) = (\lambda v)^T(\lambda v) = \lambda^2 v^T v = \lambda^2 \implies \lambda = \pm 1$$

Because $A \in \mathrm{SO}(3,\mathbb{R})$, $\det A = \lambda_1\lambda_2\lambda_3 = 1$. If all the eigenvalues are real, there must be 0 negative eigenvalues or two eigenvalues – therefore one eigenvalue must be 1. Similarly, if the eigenvalues are complex, then they must be conjugate pairs with positive product, so the final eigenvalue must be 1.
Therefore, $\exists v \in \mathbb{R}^3$ which is an eigenvector with eigenvalue 1 for any $A \in \mathrm{SO}(3,\mathbb{R})$. Take $L_A = \mathrm{Span}_{\mathbb{R}}\{v\}$ and we have a line through the origin which is invariant under A. ■

Elements of $\mathrm{SO}(4,\mathbb{R})$ may not have a real eigenvalue so the result does not hold.

Problem 5

Prove that every rotation in $\mathbb{R}^2$ can be expressed as the composition of two reflections in $\mathbb{R}^2$. Prove that every rotation in $\mathbb{R}^3$ can be expressed as the composition of two reflections in $\mathbb{R}^3$.
In $\mathbb{R}^2$, rotations can be represented by matrices in $\mathrm{SO}(2,\mathbb{R})$ of the form

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Similarly, reflections can be written

$$F_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

Consider the composition of two reflections F_ϕ, F_ψ:

$$\begin{aligned} F_\phi F_\psi &= \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \begin{pmatrix} \cos 2\psi & \sin 2\psi \\ \sin 2\psi & -\cos 2\psi \end{pmatrix} \\ &= \begin{pmatrix} \cos 2\phi \cos 2\psi + \sin 2\phi \sin 2\psi & \cos 2\phi \sin 2\psi - \sin 2\phi \cos 2\psi \\ \sin 2\phi \cos 2\psi - \cos 2\phi \sin 2\psi & \sin 2\phi \sin 2\psi + \cos 2\phi \cos 2\psi \end{pmatrix} \\ &= \begin{pmatrix} \cos(2(\phi-\psi)) & -\sin(2(\phi-\psi)) \\ \sin(2(\phi-\psi)) & \cos(2(\phi-\psi)) \end{pmatrix} \\ &= R_{2(\phi-\psi)} \end{aligned}$$

Therefore, for any rotation R_θ, we can write it as the composition of two reflections $F_{\frac{\theta}{4}}, F_{-\frac{\theta}{4}}$.

Let A and be B be lines in $\mathbb{R}^3$ with L_A and L_B their respective line symmetries.
In the trivial case $A = B$, we have $L_B \circ L_A = L_A^2 = 1$.
Now suppose $A \neq B$. Clearly, there is a non-zero angle between the two.
We can consider a rotated reference frame such that L_A and L_B both lie within the xz-plane. Since these are elements in $\mathrm{SO}(3,\mathbb{R})$, we claim that a vector parallel to $A \times B$ is fixed by $L_B \circ L_A$:

$$(L_B \circ L_A)(A \times B) = L_B(L_A(A \times B)) = L_B(-A \times B) = A \times B$$

(since $A \times B$ is perpendicular to both A and B)
Therefore, it suffices to analyze the action of $L_B \circ L_A$ on the orthogonal complement of $A \times B$. From the 2-d case, we know that two reflection in this plane will generate a rotation – and indeed, we know that this relation is precisely twice the angle from A to B. Therefore, given any rotation in $\mathrm{SO}(3,\mathbb{R})$ (say specified by its fixed axis and its angle of rotation θ), we may represent it as the composition of two line symmetries $\theta/2$ degrees apart which are orthogonal to the fixed axis. ■

Problem 6

Prove that $\exp : \mathbf{so}(3, \mathbb{R}) \longrightarrow SO(3, \mathbb{R})$ is surjective. (Hint: You can do a tedious calculation, but there's a slicker way using Problem 4)

Since every rotation in $\mathrm{SO}(3, \mathbb{R})$ fixes one axis, we may represent any element of $\mathrm{SO}(3, \mathbb{R})$ by

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

relative to some basis.

Since

$$\exp \begin{pmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbf{so}(3, \mathbb{R})$$

every element of $\mathrm{SO}(3, \mathbb{R})$ is in the image of exp. Therefore, exp is surjective. $\blacksquare$

Problem 7

Let $X \in \mathbf{so}(3, \mathbb{R})$ such that $||X|| = \sqrt{2}$ where $||\cdot||$ denotes the sum of squares norm. Let $t \in \mathbb{R}$ and prove that

$$\exp(tX) = I_3 + \sin(t)X + (1 - \cos(t))\, X^2$$

Since $X \in \mathbf{so}(3, \mathbb{R})$, we can write

$$X = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

Notice that for all $X \in \mathbf{so}(3, \mathbb{R})$ we have

$$\left(\frac{X}{\sqrt{a^2+b^2+c^2}}\right)^3 = -\frac{X}{||X||}, \quad \left(\frac{X}{||X||}\right)^4 = -\left(\frac{X}{||X||}\right)^2, \quad \left(\frac{X}{||X||}\right)^5 = \frac{X}{||X||}, \quad \dots$$

Define

$$\begin{aligned} ||X|| &= \sqrt{a^2+b^2+c^2+(-a)^2+(-b)^2+(-c)^2} \\ &= \sqrt{2a^2+2b^2+2c^2} = \sqrt{2} \\ &\implies 2a^2+2b^2+2c^2 = 2 \\ &\implies a^2+b^2+c^2 = 1 \\ &\implies \sqrt{a^2+b^2+c^2} = 1 \end{aligned}$$

so we can say

$$\begin{aligned} \exp(tX) &= \exp(t\frac{X}{\sqrt{a^2+b^2+c^2}}) \\ &= I_3 + t\frac{X}{\sqrt{a^2+b^2+c^2}} + \frac{t^2}{2}\left(\frac{X}{\sqrt{a^2+b^2+c^2}}\right)^2 + \frac{t^3}{3!}\left(\frac{X}{\sqrt{a^2+b^2+c^2}}\right)^3 + \dots \\ &= I_3 + t\frac{X}{\sqrt{a^2+b^2+c^2}} + \frac{t^2}{2}\left(\frac{X}{\sqrt{a^2+b^2+c^2}}\right)^2 - \frac{t^3}{3!}\left(\frac{X}{\sqrt{a^2+b^2+c^2}}\right) - \frac{t^4}{4!}\left(\frac{X}{\sqrt{a^2+b^2+c^2}}\right)^2 + \dots \\ &= I_3 + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots\right)\frac{X}{\sqrt{a^2+b^2+c^2}} + \left(\frac{t^2}{2} - \frac{t^4}{4!} + \frac{t^6}{6!} + \dots\right)\frac{X^2}{(\sqrt{a^2+b^2+c^2})^2} \\ &= I_3 + \sin(t)\frac{X}{\sqrt{a^2+b^2+c^2}} + (1-\cos(t))\,\frac{X^2}{a^2+b^2+c^2} \\ &= I_3 + \sin(t)X + (1-\cos(t))X^2 \quad \blacksquare \end{aligned}$$

Problem 8

Let R_A denote the rotation about the line generated by the vector $(1,1,1) \in \mathbb{R}^3$ by an angle of $\pi/2$ according to the right-hand rule. Let R_B denote the rotation about the line generated by $(1,0,-1) \in \mathbb{R}^3$ by an angle of $\pi/3$. Determine the rotation axis and angle of $R_A \circ R_B$.

In general, we may represent a rotation about a line v by an angle θ as a quaternion of the form

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\frac{v}{||v||}$$

In our case,

$$R_A \sim q_A = \cos\frac{\pi}{4} + \sin\frac{\pi}{4}(\frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k) = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{6}i + \frac{\sqrt{6}}{6}j + \frac{\sqrt{6}}{6}k$$

and

$$R_B \sim q_B = \cos\frac{\pi}{6} + \sin\frac{\pi}{6}(\frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}}k) = \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{4}i - \frac{\sqrt{2}}{4}k$$

Thus the rotation $R_A \circ R_B$ corresponds to quaternion rotation $q_A q_B$ which can be computed:

$$\begin{aligned}
q_A q_B &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{6}i + \frac{\sqrt{6}}{6}j + \frac{\sqrt{6}}{6}k\right)\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{4}i - \frac{\sqrt{2}}{4}k\right) \\
&= \frac{\sqrt{6}}{4} + \frac{2}{8}i - \frac{2}{8}k \\
&\quad + \frac{\sqrt{18}}{12}i + \frac{\sqrt{12}}{24}i^2 - \frac{\sqrt{12}}{24}ik \\
&\quad + \frac{\sqrt{18}}{12}j + \frac{\sqrt{12}}{24}ji - \frac{\sqrt{12}}{24}jk \\
&\quad + \frac{\sqrt{18}}{12}k + \frac{\sqrt{12}}{24}ki - \frac{\sqrt{12}}{24}k^2 \\
&= \frac{\sqrt{6}}{4} + \frac{1}{4}i - \frac{1}{4}k + \frac{\sqrt{2}}{4}i - \frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{12}j + \frac{\sqrt{2}}{4}j - \frac{\sqrt{3}}{12}k - \frac{\sqrt{3}}{12}i + \frac{\sqrt{2}}{4}k + \frac{\sqrt{3}}{12}j + \frac{\sqrt{3}}{12} \\
&= \left(\frac{\sqrt{6}}{4} - \frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{12}\right) + \left(\frac{1}{4} + \frac{\sqrt{2}}{4} - \frac{\sqrt{3}}{12}\right)i + \left(\frac{\sqrt{3}}{12} + \frac{\sqrt{2}}{4} + \frac{\sqrt{3}}{12}\right)j + \left(-\frac{1}{4} - \frac{\sqrt{3}}{12} + \frac{\sqrt{2}}{4}\right)k \\
&= \frac{\sqrt{6}}{4} + \frac{3+3\sqrt{2}-\sqrt{3}}{12}i + \frac{3\sqrt{2}+2\sqrt{3}}{12}j + \frac{-3+3\sqrt{2}-\sqrt{3}}{12}k
\end{aligned}$$

Notice that

$$||q_A q_B|| = \sqrt{\frac{6}{16} + \frac{30+18\sqrt{2}-6\sqrt{3}-6\sqrt{6}}{144} + \frac{30+12\sqrt{6}}{144} + \frac{30-18\sqrt{2}+6\sqrt{3}-6\sqrt{6}}{144}} = \sqrt{\frac{6}{16} + \frac{90}{144}} = 1$$

so

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}u$$

with u pure and unit. If we normalize

$$\frac{3+3\sqrt{2}-\sqrt{3}}{12}i + \frac{3\sqrt{2}+2\sqrt{3}}{12}j + \frac{-3+3\sqrt{2}-\sqrt{3}}{12}k$$

we have

$$\begin{aligned}
u &= \frac{144}{90}\left(\frac{3+3\sqrt{2}-\sqrt{3}}{12}i + \frac{3\sqrt{2}+2\sqrt{3}}{12}j + \frac{-3+3\sqrt{2}-\sqrt{3}}{12}k\right) \\
&= \frac{6+6\sqrt{2}-2\sqrt{3}}{15}i + \frac{6\sqrt{2}+4\sqrt{3}}{15}j + \frac{-6+6\sqrt{2}-2\sqrt{3}}{15}k
\end{aligned}$$

and $\theta = 2\cos^{-1}\left(\frac{\sqrt{6}}{4}\right)$ so the composite rotation axis is

$$u = \begin{pmatrix} \frac{6+6\sqrt{2}-2\sqrt{3}}{15} \\ \frac{6\sqrt{2}+4\sqrt{3}}{15} \\ \frac{-6+6\sqrt{2}-2\sqrt{3}}{15} \end{pmatrix} \approx \begin{pmatrix} 0.735 \\ 1.028 \\ -0.065 \end{pmatrix}$$

through an angle of $\theta = 2\cos^{-1}\left(\frac{\sqrt{6}}{4}\right) \approx 104.45^\circ$

Bonus: [**3 pts**] Prove that every element of $\mathrm{SO}(n,\mathbb{R})$ is the product of at most $(n+2)/2$ reflections.