Math 1820A Spring 2024 - Homework 1

Instructions: This assignment is worth twenty points. Please complete the following problems assigned below. Submissions with insufficient explanation may lose points due to a lack of reasoning or clarity. If you are handwriting your work, please ensure it is readable and well-formatted for the grader.

Be sure when uploading your work to **assign problems to pages**. Problems with pages not assigned to them **may not be graded**.

Textbook Problems:

Additional Problems: For these problems, let e_i denote the i-th coordinate vector in $\mathbb{R}^n$.

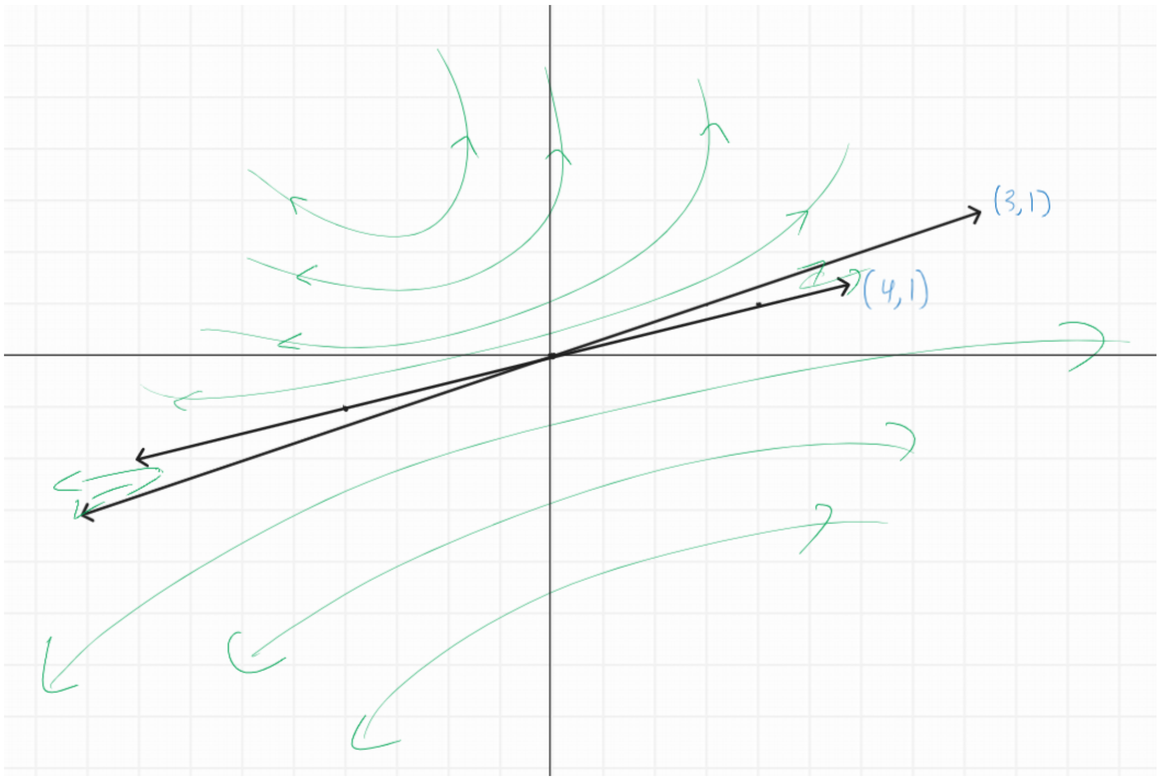
1. Solve the following linear system of differential equations.

$$\frac{dx_1}{dt} = -2x_1 + 12x_2$$
$$\frac{dx_2}{dt} = -x_1 + 5x_2$$

where $x_1(0) = x_2(0) = 1$.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 12 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\implies (-2-\lambda)(5-\lambda) + 12 = 0$$
$$\implies \lambda^2 - 3\lambda + 2 = 0$$
$$\implies \lambda_1 = 1,\ \lambda_2 = 2$$
$$\implies \begin{pmatrix} -2-\lambda_1 & 12 \\ -1 & 5-\lambda_1 \end{pmatrix} = \begin{pmatrix} -3 & 12 \\ -1 & 4 \end{pmatrix} \implies \begin{cases} -3v_1 + 12v_2 = 0 \\ -v_1 + 4v_2 = 0 \end{cases} \implies v_{\lambda_1} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$
$$\implies \begin{pmatrix} -2-\lambda_2 & 12 \\ -1 & 5-\lambda_2 \end{pmatrix} = \begin{pmatrix} -4 & 12 \\ -1 & 3 \end{pmatrix} \implies \begin{cases} -4v_1 + 12v_2 = 0 \\ -v_1 + 3v_2 = 0 \end{cases} \implies v_{\lambda_2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
$$\implies x(t) = C_1 e^t \begin{pmatrix} 4 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
$$\implies x(0) = C_1\begin{pmatrix} 4 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies \implies \begin{cases} 4C_1 + 3C_2 = 1 \\ C_1 + C_2 = 1 \end{cases} \implies C_1 = -2,\ C_2 = 3$$
$$\implies \boxed{x(t) = -2e^t\begin{pmatrix} 4 \\ 1 \end{pmatrix} + 3e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}}$$

2. Draw some trajectories of the linear system in Problem 1.



3. Solve the following linear system of differential equations.

$$\frac{dx_1}{dt} = -13x_1 + 17x_2$$
$$\frac{dx_2}{dt} = -10x_1 + 13x_2$$

where $x_1(0) = x_2(0) = 1$.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -13 & 17 \\ -10 & 13 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\implies (-13-\lambda)(13-\lambda) + 170 = 0 \implies \lambda^2 + 1 = 0 \implies \lambda_1 = i,\ \lambda_2 = -i$$

$$\implies \begin{pmatrix} -13-\lambda_1 & 17 \\ -10 & 13-\lambda_1 \end{pmatrix} = \begin{pmatrix} -13-i & 17 \\ -10 & 13-i \end{pmatrix} \implies (-13-i)x_1 + 17x_2 = 0$$

$$\implies v_{\lambda_1} = \begin{pmatrix} 13-i \\ 10 \end{pmatrix}$$

$$\implies x(t) = e^{it}\begin{pmatrix} 13-i \\ 10 \end{pmatrix} = (\cos(t) + i\sin(t))\left(\begin{pmatrix} 13 \\ 10 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)$$

$$x(t) = C_1\begin{pmatrix} 13\cos(t) + \sin(t) \\ 10\cos(t) \end{pmatrix} + C_2\begin{pmatrix} 13\sin(t) - \cos(t) \\ 10\sin(t) \end{pmatrix}$$

$$\implies x(0) = C_1\begin{pmatrix} 13 \\ 10 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies \begin{cases} 13C_1 - C_2 = 1 \\ 10C_1 = 1 \end{cases} \implies C_1 = \frac{1}{10},\ C_2 = \frac{3}{10}$$

$$\implies x(t) = \begin{pmatrix} \frac{13}{10}\cos(t) + \frac{1}{10}\sin(t) + \frac{39}{10}\sin(t) - \frac{3}{10}\cos(t) \\ \cos(t) + 3\sin(t) \end{pmatrix} = \boxed{\begin{pmatrix} \cos(t) + 4\sin(t) \\ \cos(t) + 3\sin(t) \end{pmatrix}}$$

4. Let A be a real 2×2-matrix with non-real complex eigenvalues. Explain why there exists an invertible matrix P such that
$$A = PRP^{-1} \text{ where } R = r\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$
for some $r > 0$ and $\theta \notin 2\pi\mathbb{Z}$ where r and θ are expressible in terms of the eigenvalues of A.

Since A is real, if $\lambda = a + bi$ is an eigenvalue of A, then the conjugate $\lambda = a - bi$ is also an eigenvalue of A. Then since A has two distinct eigenvalues, it is diagonalizable. Further, its eigenvectors are conjugates.

Since we know its eigenvalues are $a \pm bi$, A is similar to $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Then, letting $r = \langle \lambda \rangle = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(\frac{b}{a})$, we have that
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = r\begin{pmatrix} a/r & -b/r \\ b/r & a/r \end{pmatrix} = r\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$
Call this R. Then, $A \sim R$. Equivalently, $A = PRP^{-1}$ for some invertible P.

5. Let A be a real 2×2-matrix with non-real complex eigenvalues where one of them has real part less than 0, i.e. $\lambda = a + ib$, where $a < 0$. Let $x_0 \in \mathbb{R}^2$ and define $\gamma_{x_0}(t)$ be the solution to $x' = Ax$ with $\gamma(0) = x_0$. Calculate $\lim_{t\to\infty} \gamma_{x_0}(t)$. Since A is real and 2×2, we know that it has two (distinct) eigenvalues which are conjugates of each other. We will make the change of representation $\lambda = -a+bi$ with $a > 0$ (so the other eigenvalue is $-a - bi$). Let v_1 be the eigenvector associated with $-a + bi$. Then the solution to the system will be of the form

$$\begin{aligned}
\gamma_{x_0}(t) &= e^{(-a+bi)t} v_1 \\
&= e^{-at} e^{bti} v_1 \\
&= e^{-at}(\cos(bt) + i\sin(bt))v_1 \\
&= e^{-at}\cos(bt)v_1 + ie^{-a}\sin(bt)v_1
\end{aligned}$$

and the general solution will be of the form

$$x(t) = C_1 e^{-at}\cos(bt)v_1 + C_2 e^{-at}\sin(bt)v_1$$

where $\gamma(0) = x_0 = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$.

Because the exponential argument is negative for both terms, we have that

$$\boxed{\lim_{t\to\infty} \gamma_{x_0} = 0}$$

6. Solve the following linear system of differential equations.

$$\frac{dx_1}{dt} = -x_1 + x_2 + x_3$$
$$\frac{dx_2}{dt} = -x_1 + 2x_2$$
$$\frac{dx_3}{dt} = -2x_1 + 5x_2 - x_3$$

where $x_1(0) = x_2(0) = x_3(0) = 1$. (Hint: the characteristic polynomial is $-\lambda^3$).

Let $A = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 2 & 0 \\ -2 & 5 & -1 \end{pmatrix}$. Then, we seek to solve $x' = Ax$ with $x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Immediately, we see that $x(t) = e^{tA} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Since the characteristic polynomial is $-\lambda^3$, we have that $\lambda = 0$ is an eigenvalue of A. But by the Cayley Hamilton Theorem, we also have that $A^3 = 0$.

Thus expanding,

$$e^{tA} = I + tA + \frac{1}{2}t^2A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -t & t & t \\ -t & 2t & 0 \\ -2t & 5t & -t \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2t^2 & 6t^2 & -2t^2 \\ -t^2 & 3t^2 & -t^2 \\ -t^2 & 3t^2 & -t^2 \end{pmatrix} = \begin{pmatrix} 1-t-t^2 & t+3t^2 \\ -t-\frac{1}{2}t^2 & 1+2t+\frac{3}{2}t^2 \\ -2t-\frac{1}{2}t^2 & 5t+\frac{3}{2}t^2 \end{pmatrix}$$

Adding the initial condition,

$$e^{tA}x_0 = \begin{pmatrix} 1-t-t^2 & t+3t^2 & t-t^2 \\ -t-\frac{1}{2}t^2 & 1+2t+\frac{3}{2}t^2 & -\frac{1}{2}t^2 \\ -2t-\frac{1}{2}t^2 & 5t+\frac{3}{2}t^2 & 1-t-\frac{1}{2}t^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\boxed{x(t) = \begin{pmatrix} 1+t^2 \\ 1+t+\frac{1}{2}t^2 \\ 1+2t+\frac{1}{2}t^2 \end{pmatrix}}$$

7. Write $x(t) = M(t)x_0$ where $M(t)$ is some smooth family of matrices in $\mathbb{R}^{n\times n}$ parametrized by the variable t. What differential equation and initial condition must $M(t)$ satisfy in order for $x(t)$ to be a solution to the linear system $x' = Ax$ for each choice of $x(0) = x_0$? (Hint: Just plug it in and take derivatives).

   To be a solution to $x' = Ax$, $x(t)$ must itself satisfy the differential equation. Substituting,

   $$x' = M'(t)x_0 = Ax(t) = AM(t)x_0 \implies M'(t) = AM(t)$$

   But we also have that $x(0) = x_0$ so $x(0) = M(0)x_0 = x_0$ which implies that $M(0) = I$. Thus, $M(t)$ must satisfy:

   $$\boxed{M'(t) = AM(t), \quad M(0) = I}$$

8. Let $s(t)$ be a solution to $x' = Ax$ where A is of the form

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

Let $s_1(t) = e^{t\lambda} e_1$. Let $s_2(t) = e^{t\lambda}(te_1 + e_2)$. Figure out a third solution $s_3(t)$ to $x' = Ax$. (You should think about what this looks like for larger Jordan blocks and how this relates to the Jordan decomposition)

First note that

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We will denote the diagonal matrix D and the nilpotent matrix N. Further note that $ND = DN$ so solving $x' = Ax$ gives

$$x(t) = e^{tA} x_0$$

and

$$e^{tA} = e^{t(D+N)} = e^{tD} e^{tN}$$

As it is diagonal, e^{tD} is easy:

$$e^{tD} = \begin{pmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & 0 \\ 0 & 0 & e^{\lambda t} \end{pmatrix}$$

For e^{tN}, we notice that $N^3 = 0$ so

$$e^{tN} = I + tN + \frac{1}{2} t^2 N^2 = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$e^{tA} = \begin{pmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & 0 \\ 0 & 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = e^{t\lambda} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Now to get the third column, we introduce the initial condition $x_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,

$$\begin{aligned} s_3(t) &= e^{tA} x_0 \\ &= e^{t\lambda} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \boxed{e^{t\lambda}(\frac{t^2}{2} e_1 + te_2 + e_3)} \end{aligned}$$