Math 1820A Spring 2024 - Homework 2

Instructions: This assignment is worth twenty points. Please complete the following problems assigned below. Submissions with insufficient explanation may lose points due to a lack of reasoning or clarity. If you are handwriting your work, please ensure it is readable and well-formatted for the grader.

Be sure when uploading your work to **assign problems to pages**. Problems with pages not assigned to them **may not be graded**.

Here's a helpful trick if you haven't seen it before. If you have a continuous function $F : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, and a sequence of points $x_n \longrightarrow x$, then $f(x_n) \longrightarrow f(x)$. This sort of thing is helpful for proving identities, e.g. Problem 4 and 6.

1. Let $\alpha : \mathbb{R} \longrightarrow \mathrm{GL}(n, \mathbb{R})$ be a one parameter subgroup. Prove that $\alpha'(t) = \alpha(t)\alpha'(0)$. Use this to show that if α is a one-parameter subgroup such that $\alpha(0) = 1$ and $\alpha'(0) = A$, then $\alpha(t)$ must commute with A for all t. (Do not assume that every one parameter subgroup is of the form a matrix exponential)

1. As α is a 1-parameter subgroup, it satisfies $\alpha(t+s) = \alpha(t)\alpha(s)$. Then,

$$\frac{d}{ds}\bigg|_{s=0}(\alpha(t+s)) = \frac{d}{ds}\bigg|_{s=0}\alpha(t)\alpha(s) = \alpha(t)\alpha'(0)$$

2. Since α is a homomorphism and addition is commutative in $\mathbb{R}$, we have that

$$\alpha(s+t) = \alpha(s)\alpha(t) = \alpha(t+s) = \alpha(t)\alpha(s)$$

Deriving,

$$\alpha'(t) = \alpha'(0)\alpha(t) = \alpha(t)\alpha'(0) = A\alpha(t) = \alpha(t)A$$

So for all t, $\alpha(t)$ commutes with A $\blacksquare$

2. If A is any matrix in $M_n(\mathbb{C})$, prove that there exists a sequence of diagonalizable matrices A_n such that A_n converges to A.

If A is diagonalizable, we are done as there exists a diagonalizable matrix A_n such that $|A_n - A| = 0 < \varepsilon$

If A is not diagonalizable, then we know that its eigenvalues are not distinct. By Hall Theorem B.7, every matrix is similar to an upper triangular matrix, i.e. $A = PTP^{-1}$ for some upper triangular matrix T. As T is upper triangular, its eigenvalues are on the diagonal (call them $\lambda_1, \ldots, \lambda_n$).

We may then construct a sequence of diagonal matrices D_n with eigenvalues $\lambda_i + \varepsilon_i$ with $(\varepsilon_i)_n \to 0$ chosen such that $\lambda_i + \varepsilon_i \neq \lambda_j + \varepsilon_j$ for $i \neq j$. Then, since diagonal matrices are dense in $M_n(\mathbb{C})$, $D_n \to T$ and $P^{-1}D_nP \to A$ $\blacksquare$

3. Consider the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Show that there is no sequence of real diagonalizable matrices converging to A. (Hint: Look at the discriminant of the characteristic polynomial)

The characteristic polynomial of A is $\lambda^2 + 1$, so the discriminant of the characteristic polynomial is -4. As the discriminant is negative, A has no real eigenvalues. Thus, A is not diagonalizable over $\mathbb{R}$. Further, as diagonal matrices are not dense in $M_n(\mathbb{R})$, there is no sequence of real diagonalizable matrices converging to A $\blacksquare$

4. For any $A \in M_n(\mathbb{C})$ satisfying $||A - I_n|| < 1$, define the *matrix logarithm* to be

$$\log(A) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{1}{m} (A - I_n)^m$$

Prove that $e^{\log A} = A$ and if in addition $||A|| < \log 2$ then $\log e^A = A$.

If A is diagonalizable, then $A = PDP^{-1}$ and $A - I = P(D - I)P^{-1}$. Then,

$$(A - I)^m = P \begin{pmatrix} (\lambda_1 - 1)^m & & \\ & \ddots & \\ & & (\lambda_n - 1)^m \end{pmatrix} P^{-1}$$

Then since $||A - I_n|| < 1$, $|\lambda_i - 1| < 1$ (Hall Th 2.7) so we may apply the usual analytic log function,

$$\log A = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (A - I_n)^m = P \begin{pmatrix} \log \lambda_1 & & \\ & \ddots & \\ & & \log \lambda_n \end{pmatrix} P^{-1}$$

So by Hall Lemma 2.5,

$$e^{\log A} = P \begin{pmatrix} e^{\log \lambda_1} & & \\ & \ddots & \\ & & e^{\log \lambda_n} \end{pmatrix} P^{-1} = P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^{-1} = A$$

In the case where A is not diagonalizable, we simply need to invoke Problem 2 above and construct a sequence of diagonalizable matrices A_n converging to A. When $||A_n - I|| < 1$, we have that $e^{\log A_n} = A_n$ and as $A_n \to A$, $e^{\log A_n} \to e^{\log A}$. Thus, $e^{\log A} = A$.

For the second part, we have that $||A|| < \log 2$ so by the sub-multiplicativity of the operator norm,

$$\begin{aligned} \left|\left|e^A - I\right|\right| &\leq \left|\left|e^A\right|\right| - ||I|| \\ &= \left|\left|e^A\right|\right| - 1 \\ &= \left|\left|I + A + \frac{1}{2}A^2 + \dots\right|\right| - 1 \\ &\leq 1 + ||A|| + \frac{1}{2}\left|\left|A^2\right|\right| + \dots - 1 \\ &\leq \log 2 + \frac{1}{2}(\log 2)^2 + \dots \\ &= e^{\log 2} - 1 = 2 - 1 = 1 \end{aligned}$$

so $\left|\left|e^A - I\right|\right| < 1$.

Now we may apply exactly the same argument as above. First suppose e^A is diagonalizable, then

$$e^A = Pe^D P^{-1}$$

so (now since $\left|\left|e^A - I_n\right|\right| < 1$),

$$\log e^A = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (e^A - I_n)^m = P \begin{pmatrix} \log(e^{\lambda_1}) & & \\ & \ddots & \\ & & \log(e^{\lambda_n}) \end{pmatrix} P^{-1}$$

and since $\left|\left|e^{\lambda_i} - 1\right|\right| < 1$,

$$e^{\log A} = P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^{-1} = A \quad \blacksquare$$

5. Hall 2.6.5: Calculate e^A where $A \in M_n(\mathbb{R})$ is defined below. (Hint: be careful when calculating!)

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

Notice first that $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ but

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \neq \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

so we *cannot* say that $e^A = e^{S+N} = e^S e^N$.

Instead, we will proceed by explicit terms:

$$\begin{aligned}
A &= \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \\
A^2 &= \begin{pmatrix} a^2 & b(a+d) \\ 0 & d^2 \end{pmatrix} \\
A^3 &= \begin{pmatrix} a^3 & b(a^2+ad+d^2) \\ 0 & d^3 \end{pmatrix} \\
A^4 &= \begin{pmatrix} a^4 & b(a^3+a^2d+ad^2+d^3) \\ 0 & d^4 \end{pmatrix}
\end{aligned}$$

We notice that the diagonals are simply a^n and d^n, while the upper right entry is the telescoping sum $b\left(\frac{a^{n-1}-b^{n-1}}{a-b}\right)$.

So,

$$\begin{aligned}
e^A &= I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} + \frac{1}{2}\begin{pmatrix} a^2 & b(a+d) \\ 0 & d^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} a^3 & b(a^2+ad+d^2) \\ 0 & d^3 \end{pmatrix} + \dots \\
&= \begin{pmatrix} 1 + a + \frac{a^2}{2} + \frac{a^3}{3!} + \dots & b + \frac{ab}{2} + \frac{a^2b}{3!} + \dots \\ 0 & 1 + d + \frac{d^2}{2} + \frac{d^3}{3!} + \dots \end{pmatrix} \\
&= \begin{pmatrix} \sum_{n=1}^{\infty} \frac{a^n}{n!} & b\left(\frac{a^n-d^n}{a-d}\right) \\ 0 & \sum_{n=1}^{\infty} \frac{d^n}{n!} \end{pmatrix} \\
&= \boxed{\begin{pmatrix} e^a & b\left(\frac{e^a-e^d}{a-d}\right) \\ 0 & e^d \end{pmatrix}}
\end{aligned}$$

6. For any matrix $A \in M_n(\mathbb{C})$ define $\sin(A)$ and $\cos(A)$ as you would guess.

$$\sin(A) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}A^{2n+1} \text{ and } \cos(A) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}A^{2n}$$

Show that $\sin^2(A) + \cos^2(A) = I_n$ (Hint: Show it works for diagonal matrices first).

$$\begin{aligned}
\sin^2(A) &= \left(\sum_{n=0}^{n}(-1)^n \frac{1}{(2n+1)!}A^{2n+1}\right)^2 \\
&= \left(A - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 - \dots\right)\left(A - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 - \dots\right) \\
&= A^2 + (-\frac{1}{3!}A^4 - \frac{1}{3!}A^4) + (\frac{1}{3!\,3!}A^6 + \frac{1}{5!}A^6 + \frac{1}{5!}A^6) + (-\frac{1}{3!\,5!}A^8 - \frac{1}{3!\,5!}A^8) + \dots \\
&= A^2 - \frac{2}{3!}A^4 + \frac{1}{3!\,3!}A^6 + \frac{2}{5!}A^6 + \dots \\
&= A^2 - \frac{1}{3}A^4 + \frac{2}{45}A^6 + \dots \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,2^{2n-1}}{(2n)!}A^{2n} \\
\cos^2(A) &= \left(\sum_{n=0}^{\infty} \frac{1}{(2n)!}A^{2n}\right)^2 \\
&= \left(I - \frac{1}{2}A^2 + \frac{1}{4!}A^4 - \frac{1}{6!}A^6 + \dots\right)\left(I - \frac{1}{2}A^2 + \frac{1}{4!}A^4 - \frac{1}{6!}A^6 + \dots\right) \\
&= I + (-\frac{1}{2}A^2 - \frac{1}{2}A^2) + (\frac{1}{2!\,2!}A^4 + \frac{1}{4!}A^4 + \frac{1}{4!}A^4) + (-\frac{1}{6!}A^6 - \frac{1}{6!}A^6 - \frac{1}{2!\,4!}A^6 + \frac{1}{2!\,4!}) + \dots \\
&= I - A^2 + \frac{3}{4!}A^4 - \frac{2}{6!}A^6 - \frac{2}{2!\,4!}A^6 + \dots \\
&= I - A^2 + \frac{1}{3}A^4 - \frac{2}{45}A^6 + \dots \\
&= I + \sum_{n=1}^{\infty} \frac{(-1)^n\,2^{2n-1}}{(2n)!}A^{2n} \\
\sin^2(A) + \cos^2(A) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,2^{2n-1}}{(2n)!}A^{2n} + I + \sum_{n=1}^{\infty} \frac{(-1)^n\,2^{2n-1}}{(2n)!}A^{2n} \\
&= I - \sum_{n=1}^{\infty} \frac{(-1)^n\,2^{2n-1}}{(2n)!}A^{2n} + \sum_{n=1}^{\infty} \frac{(-1)^n\,2^{2n-1}}{(2n)!}A^{2n} \\
&= I \quad \blacksquare
\end{aligned}$$

7. Hall 2.6.11: Show that for all $A \in M_n(\mathbb{C})$, $\lim_{m\to\infty}\left(I_n + \frac{A}{m}\right)^m = e^A$. (Hint: Use matrix logarithms.)

Suppose A is diagonalizable ($A = PDP^{-1}$). Then

$$\lim_{m\to\infty}(I_n + \frac{A}{m})^m = \lim_{m\to\infty}(I_n + \frac{PDP^{-1}}{m})^m = \lim_{m\to\infty} P(\frac{D}{m} + I_n)^m P^{-1}$$

so the entries of $(\frac{D}{m} + I_n)^m$ look like

$$(1 + \frac{\lambda_i}{m})^m$$

where λ_i are the eigenvalues of A, i.e.

$$\lim_{m\to\infty}(I_n + \frac{A}{m})^m = P\begin{pmatrix} \lim_{m\to\infty}(1+\frac{\lambda_1}{m})^m & & \\ & \ddots & \\ & & \lim_{m\to\infty}(1+\frac{\lambda_n}{m})^m \end{pmatrix} P^{-1}$$

But $\lim_{m\to\infty}(1+\frac{\lambda_n}{m})^m$ is the definition of e^{λ_n}, so

$$\lim_{m\to\infty}(I_n + \frac{A}{m})^m = P\begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix} P^{-1} = e^A$$

In the case where A is not diagonalizable, we just need to invoke Problem 2 above and construct a sequence of diagonalizable matrices A_n converging to A. So $e^{\log A} = A$ $\blacksquare$

8. Show that there is no matrix $A \in M_2(\mathbb{R})$ for which $e^A = B$ where

$$B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

(Note this shows the matrix exponential is not surjective in the real case.)

We seek a matrix B for which $\log B = A$. Suppose that $\exists\, A \in M_2(\mathbb{R})$ s.t. $e^A = B$.

First notice that the eigenvalues of B are ± 1 so B is diagonalizable. Then $B = PDP^{-1} \implies e^A = PDP^{-1}$. So $A = \log B = P \log DP^{-1}$. However,

$$\log D = \log \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \log(1) & 0 \\ 0 & \log(-1) \end{pmatrix}$$

But $\log -1$ is not defined in $\mathbb{R}$, so there is no matrix A such that $e^A = B$ $\blacksquare$