Georgia Tech Math 2551

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1 Module 1: Three Dimensional Space, Vectors, Lines, Planes

1.1 WEEK 1: Geometry of space and vectors (Readings 12.1-12.4)

The three dimensional coordinate system: $\mathbb{R}^3$, a coordinate system with x, y, and z axes where coordinates are represented by an ordered tuple of three numbers. We use a right-hand system such that z is vertical.

Octants: the three-dimensional analogue of quadrants

First octant: the octant where all three coordinates are positive

The three planes:

- xy-plane: $z = 0$ ("the floor")
- xz-plane: $y = 0$ ("a wall")
- yz-plane: $x = 0$

Distance in three dimensions:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example: "Find the distance from the point (2, -3, 1) to the plane $y = 2$."

Solution: The plane $y = 2$ spans all values of x and z so the only coordinate changing is y itself. $d = 2 - (-3) = 5\blacksquare$

Sphere: a three-dimensional object where every point on its surface is equidistant from its centre. The centre has coordinates x_0, y_0, z_0 and the distance from the centre to each point on the surface is the radius.

$$x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

Example: "Identify the centre C and radius a for the sphere given by $2x^2 + 2y^2 + 2x^2 - 8x + 12y - 20z = 22$

Solution:

$$\begin{aligned}
2x^2 + 2y^2 + 2z^2 - 8x + 12y - 20z &= 22 \\
x^2 + y^2 + z^2 - 4x + 6y - 10z &= 11 \\
x^2 - 4x + y^2 + 6y + z^2 - 10z &= 11 \\
(x-2)^2 + (y+3)^2 + (z-5)^2 &= 11 + 4 + 9 + 25 \\
(x-2)^2 + (y+3)^2 + (z-5)^2 &= 49
\end{aligned}$$

$$C = (2, -3, 5) \quad a = 7$$

Example: "Give a geometric description of the sets defined by $y^2 + z^2 = 4, x = 2$"

Solution: A circle of radius 2 on the plane $x = 2$

Vector: an object with a direction and a length, represented by a directed line segment

If point P has coordinates (x_1, y_1, z_1) and point Q has coordinates (x_2, y_2, z_2), then

$$vecPQ = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Magnitude of a vector: the length or norm of a vector, found using the distance formula. Written $||\vec{v}||$

Vectors can be represented by arrow notation ($\vec{PQ}$), component notation ($\langle x, y, z \rangle$), or bold typeface notation ($\mathbf{v}$)

Example: "Find the component form and length of the vector whose intial point is $R(-1, 3, 0)$ and terminal point is $S(-3, -2, 4)$"

Solution:

$$\begin{aligned}
\vec{RS} &= \langle -2, -5, 4 \rangle \\
||\vec{RS}|| &= \sqrt{45} = 3\sqrt{5}
\end{aligned}$$

Vector Addition: Given $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$,

$$vecu + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

Vector multiplication by a scalar: Given $\vec{u} = \langle u_1, u_2, u_3 \rangle$,

$$k\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$$

All the normal properties (commutivity, associativity, additive identity, additive inverse, multiplicative identity, zero multiplication, and distributivity) hold for vectors.

Geometric representation of vector addition: For a vector $\vec{u}$,

- $-\vec{u}$ is a vector of the same magnitude in the opposite direction
- $k\vec{u}$ is a vector in the same direction but k times the length

Addition of vectors $\vec{v}$ and $\vec{w}$ corresponds to the diagonal of their composite parallelogram starting at the common point. For subtraction, the diagonal begins at the tip of the vector being subtracted.

Unit vector: a vector whose length is 1. For a non-zero vector, $\hat{v} = \frac{\vec{v}}{||v||}$

The standard unit vectors:

- $\hat{i} = \langle 1, 0, 0 \rangle$
- $\hat{j} = \langle 0, 1, 0 \rangle$
- $\hat{k} = \langle 0, 0, 1 \rangle$

Any vector can be written as a linear combination of the standard unit vectors

Example: "Find a unit vector i the xy=plane that makes an angle $\theta = -\frac{\pi}{3}$ with the positive x-axis"

Solution: From the unit circle

$$\hat{r} = \frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}$$

The Dot Product: a scalar value

$$\vec{u} \cdot \vec{v} = u_1 v_1 + ... + u_n v_n = ||\vec{u}|| \; ||\vec{v}|| \cos\theta$$

This can give us the angle between two vectors

$$\theta = \cos^{-1}(\frac{\vec{u} \cdot \vec{c}}{||\vec{u}||\ ||\vec{v}||})$$

Vectors are orthogonal if $\vec{v} \cdot \vec{u} = 0$

Vector projection:

$$\text{proj}_b \vec{a} = (\frac{\vec{a} \cdot \vec{b}}{||\vec{b}||})\frac{\vec{b}}{||\vec{b}||}$$

Scalar component of a in the direction of b: The value

$$\text{comp}_b \vec{a} = \frac{\vec{a} \cdot \vec{b}}{||\vec{b}||} = ||\vec{a}|| \cos\theta$$

Work: $W = F \cdot D$ for a force F acting through a distance D

Cross product: the vector

$$\vec{a} \times \vec{b} = ||\vec{a}||\ ||\vec{b}|| \sin\theta \hat{n}$$

Where θ is the angle between the vectors and $\hat{n}$ is the unit vector perpendicular to the plane containing vectors $\vec{a}$ and $\vec{b}$

The product $||\vec{a}||\ ||\vec{b}|| \sin\theta$ also corresponds to the area of the parallelogram formed by the vectors

Parallel vectors: two vectors are parallel iff $\vec{a} \times \vec{b} = \vec{0}$

Properties of the cross product:

- $(r\vec{u}) \times (r\vec{v}) = (rs)(\vec{u} \times \vec{v})$
- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{0} \times \vec{u} = \vec{0}$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$
- $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

Calculating the Cross Product as a determinant: For $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ and $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$, then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example: "Find the area of the parallelogram with vertices A(2, 1, 4), B(1, 4, 3), C(1, 0, 2), D(2, -3, 3)"

Solution:

$$\begin{aligned} \vec{AB} &= \langle -1, 3, -1 \rangle \\ \vec{BC} &= \langle 0, -4, -1 \rangle \\ \vec{CD} &= \langle 1, -3, 1 \\ \vec{AD} &= \langle 0, -4, -1 \end{aligned}$$

Because $\vec{AB}$ and $\vec{CD}$ are parallel but in opposite directions, we can know to calculate the cross of any two adjacent sides to find the area.

$$A = ||\vec{AB} \times \vec{AD}||$$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & -1 \\ 0 & -4 & -1 \end{vmatrix} = -7\hat{i} + \hat{j} + 4\hat{k}$$

$$A = || -7\hat{i} - \hat{j} + 4\hat{k}|| = \sqrt{66}\blacksquare$$

The absolute value of the triple scalar product $((\vec{u} \times \vec{v}) \cdot \vec{w})$ is the volume of the parallelepiped determined by the three vectors.

Helpfully,

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

1.2 WEEK 2: Curves, Tangents, Motion (Readings 12.5-13.2)

A vector equation for the line L through the point $P_0(x_0, y_0, z_0)$ parallel to the vector $\vec{v}$ is given by

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \quad -\infty < t < \infty$$

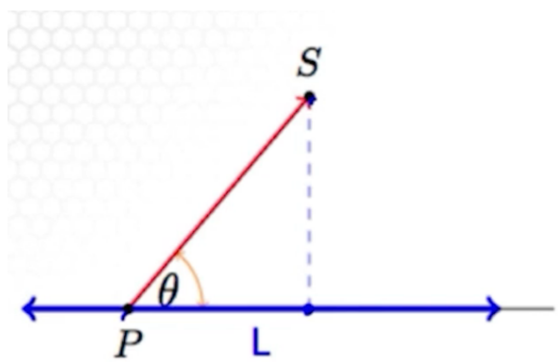
where $\vec{r}$ is the position vector of a point $P(x, y, z)$ on L and $\vec{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

The *standard parameterisation* of the line L through the point $P_0(x_0, y_0, z_0)$ parallel to $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is given by

$$\begin{aligned} x(t) &= x_0 + tv_1 \\ y(t) &= y_0 + tv_2 \\ z(t) &= z_0 + tv_3 \end{aligned}$$

for $-\infty < t < \infty$.

Distance from a point to a line in space:



$$d = \frac{||\vec{PS} \times \vec{v}||}{||\vec{v}||}$$

Example: "Find the distance from S(2,0,2) to the line through P(3,-1,1) parallel to the vector $\vec{v} = \hat{i} - 2\hat{j} - 2\hat{k}$"

Solution:

$$\begin{aligned} d &= \frac{||\vec{PS} \times \vec{v}||}{||\vec{v}||} \\ \vec{PS} &= \langle -1, 1, 1 \rangle \\ \vec{PS} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -2 & -2 \end{vmatrix} = -\hat{j} + \hat{k} \\ d &= \frac{\sqrt{1+1}}{\sqrt{1+4+4}} = \frac{\sqrt{2}}{3} \quad \blacksquare \end{aligned}$$

Equations for a plane: The plane through the point $P_0(x_0, y_0, z_0)$ normal to $\vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$ is given by the *vector equation*

$$\vec{n} \cdot \vec{P_0P} = 0$$

and the *component equation*

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

In other words, the direction of the plane is defined by the vector normal to it.

Example: "Find an equation for the plane which passes through P(1,3,4) and contains the line $l : x(t) = 3t,\ y(t) = 4t,\ z(t) = 2 + 2t$

Solution: If the plane contains a point and a line distinct from each other, then the cross product of the vector for that point and the equation for the line will be normal to both of them and thus the plane.

$$\ell \to \ell(0, 0, 2)$$

$$\vec{QP} = \langle 1, 3, 2\rangle$$
$$\vec{d} = \langle 3, 4, 2\rangle$$

$$\vec{n} = \vec{QP} \times \vec{d} = -2\hat{i} + 4\hat{j} - 5\hat{k}$$
$$-2(x - 1) + 4(y - 3) - 5(z - 4) = 0$$

$$2x - 4y - 5z = 10$$

Angle between two planes: the acute angle between their normal vectors

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{||\vec{n}_1||\ ||\vec{n}_2||}$$

Distance from a point to a plane:

$$d = \left|\vec{PS} \cdot \tfrac{\vec{n}}{||\vec{n}||}\right|$$

Example: "Determine whether the lines l_1 and l_2 are parallel, coincident, skew, or intersecting."

$$l_1 : x_1(t) = 1 + t, \quad y_1(t) = -1 - t, \quad z_1(t) = -4 + 2t$$
$$l_2 : x_2(s) = 1 - s, \quad y_2(s) = 1 + 3s, \quad z_2(s) = 2s$$

Solution:

$$\vec{v_1} = \langle 1, -1, 2 \rangle$$
$$\vec{v_2} = \langle -1, 3, 2 \rangle$$

Therefore the lines are not parallel or coincident.

$$\begin{cases} 1 + t = 1 - s \\ -1 - t = 1 + 3s \\ -4 + 2t = 2s \end{cases}$$

Because we have only two variables, only the first equations are needed to solve. If when checking, however, the third is true, the lines are intersecting. Else, they are skew. We add the first two equations:

$$0 = 2 + 2s \implies s = -1$$

Plugging in s:

$$1 + t = 1 - (-1) \implies t = 1$$

We check with the third equation:

$$4 + 2(1) = 2(-1)$$

which is true so the lines are intersecting. To find the point of intersection, we plug either variable in to the original lines:

$$l_1(1) = (2, -2, -2) \quad \blacksquare$$

Intersecting planes: Two planes are parallel if their normal vectors are parallel. Otherwise, they intersect at a line. The direction vector for that line of intersection is the cross product of the normal vectors from the two planes.

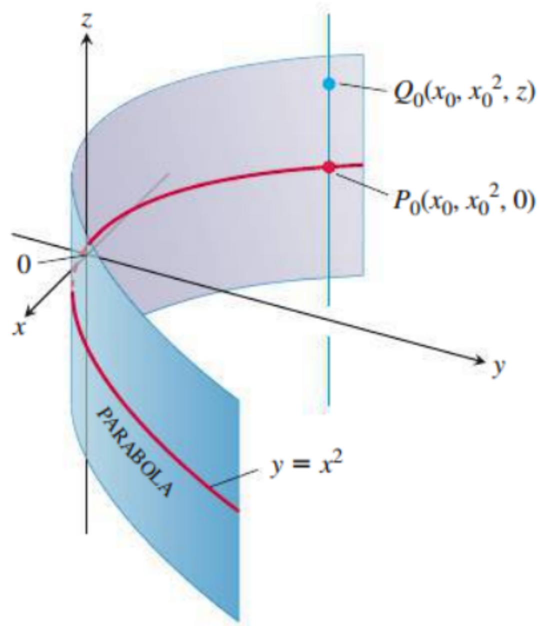
1.2.1 Cylinders and Quadric Surfaces

Cylinder: a surface that is generated by moving a straight line along a given planar curve (*the generating curve*) while holding the line parallel to a given fixed line.

Unlike in solid geometry, the generating curves are not limited to circles.

Example: "Find an equation for the cylinder made by the lines parallel to the z-axis that pass through the parabola $y = x^2,\ z = 0$"

Solution: $P_0(x_0, x_0^2, 0)$ lies on the parabola $y = x^2$ in the xy-plane. Then, $\forall z, Q(x_0, x_0^2, z)$ is on the cylinder because it will lie on the line $y = x^2$ through P_0 parallel to the z-axis.



Quadric surfaces: the graph in space of a second-degree equation in x, y, z. The most simple form is

$$Ax^2 + By^2 + Cz^2 + Dz = E$$

for A, B, C, D, and E are constants.

The basic quadric surfaces are ellipsoids, paraboloids, elliptical cones, and hyperboloids.

1.2.2 The Ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

cuts the coordinate axes at $(\pm a, 0, 0)$, $(0, \pm b, 0)$, $(0, 0, \pm c)$ and lies within the rectangular box defined by $|x| \leq a$, $|y| \leq b$, $|z| \leq c$ and the surface is symmetric.

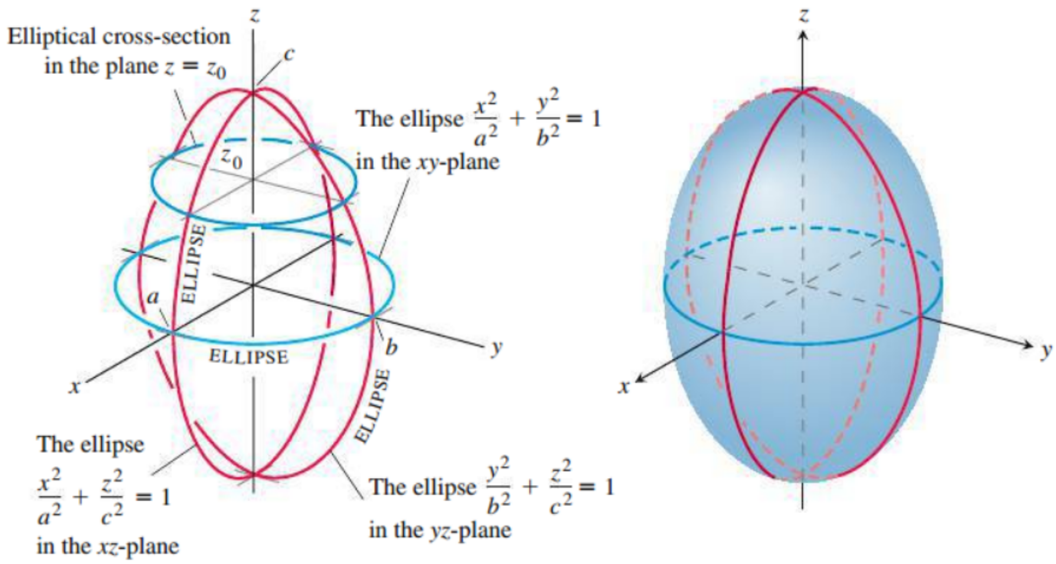


FIGURE 12.46 The ellipsoid

The cross sections for each of the three coordinate planes are ellipses.

If any two of the semiaxes a, b, c are equal, the surface is an ellipsoid of revolution. If all three are equal, the surface is a sphere.

1.2.3 The Hyperbolic Paraboloid:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0$$

which has symmetry with respect to $x = 0$ and $y = 0$. The cross sections in these planes are

$$\begin{aligned} x = 0: &\quad \text{the parabola } z = \tfrac{c}{b^2}y^2 \\ y = 0: &\quad \text{the parabola } z = -\tfrac{c}{a^2}x^2 \end{aligned}$$

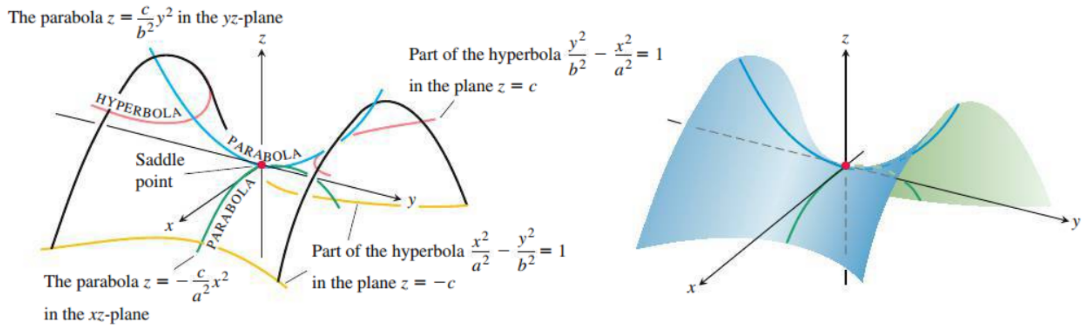


FIGURE 12.47 The hyperbolic paraboloid $(y^2/b^2) - (x^2/a^2) = z/c, c > 0$. The cross-sections in planes perpendicular to the z-axis above and below the xy-plane are hyperbolas. The cross-sections in planes perpendicular to the other axes are parabolas.

If we cut the surface by a plane $z = z_0 > 0$, the cross section is a hyperbola with its focal axis parallel to the y-axis and its vertices on the parabola for $x = 0$ above. If z_0 is negative, the focal axis is parallel to the x-axis and the vertices lie on the parabola for $y = 0$ above.

Near the origin, the surface is shaped like a saddle or mountain pass. Travelling along the surface in the yz-plane, the origin looks like a minimum. Travelling along the xz-plane, the origin looks like a maximum. This is a *saddle point*

1.2.4 General Quadric Surfaces

The general equation for a quadric surface in three variables is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gz + Hy + Iz + J = 0$$

Terms of the type Gx, Hy, or Iz lead to translations.

Example 4: "Identify the surface given by the equation $x^2+y^2+4z^2-2x+4y+1 = 0$"

Solution: Complete the square

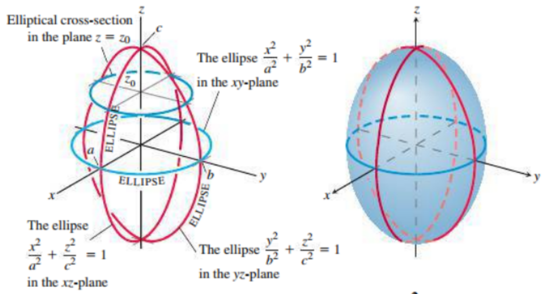
$$x^2 + y^2 + 4z^2 - 2x + 4y + 1 = (x-1)^2 + (y+2)^2 + 4z^2 - 4$$

We can rewrite the original equation as

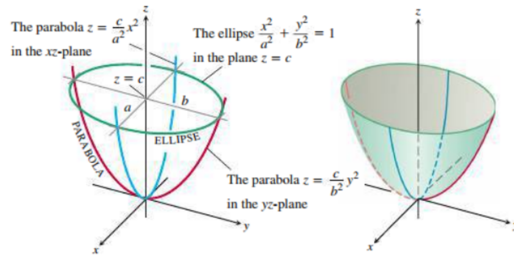
$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{4} + \frac{z^2}{1} = 1$$

This is the equation of an ellipsoid whose three semiaxes have lengths 2, 2, and 1 which is centered at the point (1, -2, 0).

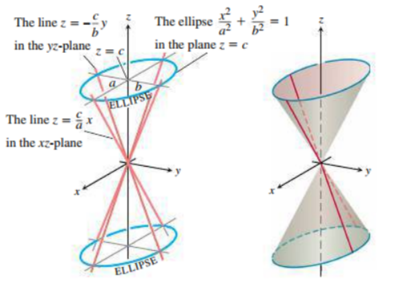
1.2.5 Graphs of Quadric Surfaces:



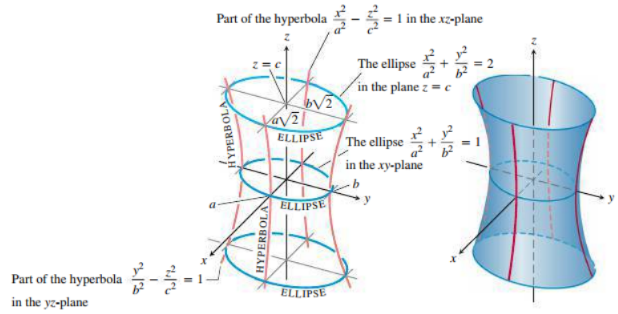
ELLIPSOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



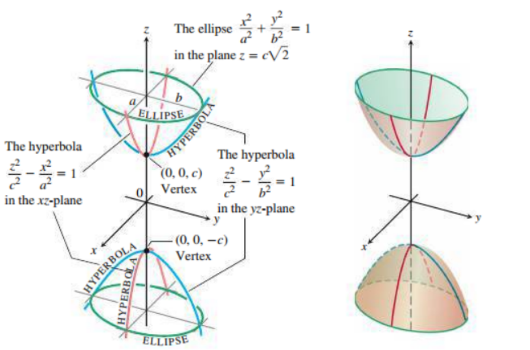
ELLIPTICAL PARABOLOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



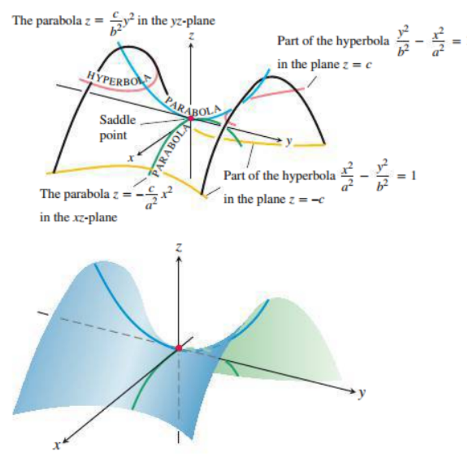
ELLIPTICAL CONE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

1.3 Curves in space

We can describe the path of a particle's motion through space by defining its coordinates as functions on I:

$$x = f(t),\ y = g(t),\ z = h(t)$$

In vector form:

$$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

In general, f, g, and h are *component functions* of the position vector.

Vector function: a function on a domain set D that assigns a vector in space to each element in D.

When the domain is an interval of real numbers, the graph represents a curve in space. When domains are regions in the plane, the graph will be a surface in space.

Scalar functions: real valued functions such as the components of a vector function

The domain of a vector-valued function is the common domain of its components.

1.3.1 Limits of vector functions

Let $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$ be a vector function with domain D, and let $\vec{L}$ be a vector. We say that $\vec{r}$ has limit $\vec{L}$ as $t \to t_0$:

$$\lim_{t\to t_0} \vec{r}(t) = \vec{L}$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$|\vec{r}(t) - \vec{L}| < \varepsilon \quad \text{when} \quad 0 < |t - t_0| < \delta$$

Further, if $\vec{L} = L_1\hat{\imath} + L_2\hat{\jmath} + L_3\hat{k}$, then $\lim_{t\to t_0} \vec{r}(t) = \vec{L}$ when

$$\lim_{t\to t_0} f(t) = L_1,\ \lim_{t\to t_0} g(t) = L_2,\ \lim_{t\to t_0} h(t) = L_3$$

Definition: A vector function $\vec{r}(t)$ is continuous at a point $t = t_0$ in its domain if

$$\lim_{t\to t_0} \vec{r}(t_0)$$

The function is *continuous* if it is continuous at every point in its domain.

1.3.2 Derivatives and motion:

The vector function $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ is differentiable at t if f, g, and h have derivatives at t.

$$\vec{r}'(t) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{df}{dt}\hat{i} + \frac{dg}{dt}\hat{j} + \frac{dh}{dt}\hat{k}$$

Differentiable: a property of a function if it has a derivative at every point of its domain

Smooth: a property of the curve traced by $\vec{r}$ if $\frac{d\vec{r}}{dt}$ is continuous and never $\vec{0}$ On a smooth curve, there are no sharp corners or cusps.

Tangent line: the line through a point $(f(t_0), g(t_0), h(t_0))$ parallel to $\vec{r}'(t)$.

Piecewise smooth: a curve that is made up of a finite number of smooth curves pieced together in a continuous fashion

If $\vec{r}$ is the position vector of a particle moving along a smooth curve,

1. Velocity is the derivative of position: $\vec{v} = \frac{d\vec{r}}{dt}$
2. Speed is the magnitude of velocity: Speed $= |\vec{v}|$
3. Acceleration is the derivative of velocity: $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$
4. The unit vector $\vec{v}/|\vec{v}|$ is the direction of motion at time t

1.3.3 Differentiation rules:

- $\frac{d}{dt}\vec{C} = \vec{0}$
- $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$
- $\frac{d}{dt}[f(t)\vec{u(t)}] = f'(t)\vec{u(t)} + f(t)\vec{u}'(t)$
- $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$
- $\frac{d}{dt}[\vec{u}(t) - \vec{v}(t)] = \vec{u}'(t) - \vec{v}'(t)$
- $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
- $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$
- $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\vec{u}'(f(t))$

If $\vec{r}$ is a differentiable vector function of t and the length of $\vec{r}(t)$ is constant, then

$$\vec{r} \cdot \frac{d\vec{r}}{dt} = 0$$

1.3.4 Integrals of vector functions:

Indefinite integral: the set of all anti-derivatives of $\vec{r}$, denoted by $\int \vec{r}(t)dt$. If $\vec{R}$ is any antiderivative of $\vec{r}$, then

$$\int \vec{r}(t)dt = \vec{R}(t) + \vec{C}$$

If the components of $\vec{t} = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ are integrable over [a, b] then so is $\vec{r}$, forming the definite integral of $\vec{r}$ from a to b:

$$\int_a^b \vec{r}(t)dt = \left(\int_a^b f(t)dt\right)\hat{i} + \left(\int_a^b g(t)dt\right)\hat{j} + \left(\int_a^b h(t)dt\right)\hat{k}$$

The Fundamental Theorem of Calculus:

$$\int_a^b \vec{r}(t)dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

An antiderivative of a vector function is also a vector function but a definite integral of a vector function is a single constant vector.

Ideal projectile motion equation:

$$\vec{r} = (v_0 \cos\alpha)t\hat{i} + \left((v_0 \sin\alpha)t - \frac{1}{2}gt^2\right)\hat{j}$$

where α is the launch angle and $v_0 = |\vec{v}|$, the initial speed.

Formulas:

- $y_{\max} = \frac{(v_0 \sin\alpha)^2}{2g}$
- $t = \frac{2v_0 \sin\alpha}{g}$
- $R = \frac{v_0^2}{g} \sin 2\alpha$

1.4 WEEK 3: Arclength, curvature, acceleration (Readings 13.3-13.6)

The lengths of a smooth curve $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k},\ a \le t \le b$ that is traced exactly once as t increases from $t = a$ to $t = b$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt$$

If we choose a base point $P(t_0)$ on a smooth curve C parameterised by t, each value of t determines a point $P(t) = (x(t), y(t), z(t))$ on C and a *directed distance*

$$s(t) = \int_{t_0}^{t} |\vec{v}(\tau)| d\tau$$

and

$$\frac{ds}{dt} = |\vec{v}(t)|$$

Each value of s determines a point on C, parameterising C with respect to s, making s an *arc length parameter* for the curve. The parameter's value increases in the direction of increasing t.

Arc length parameter with Base Point $P(t_0)$:

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)^2] + [y'(\tau)^2] + [z'(\tau)^2]} d\tau = \int_{t_0}^{t} |\vec{v}(\tau)| d\tau$$

If a given curve $\vec{r}(t)$ is already given in terms of some parameter and the arclength is $s(t)$ then we can solve for t as a function of $s : t = t(s)$. Then the curve can be parameterised in terms of s by substituting for $t : \vec{r} = \vec{r}(t(s))$. This identifies a point on the curve with its directed distance along the curve from the base point.

The speed with which a particle moves along its path is the magnitude of $\vec{v}$:

$$\frac{ds}{dt} = |\vec{v}(t)|$$

Because the velocity vector $\vec{v} = \frac{d\vec{r}}{dt}$ is tangent to $\vec{r}(t)$ so the vector

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

is a unit vector tangent to the curve. $\vec{T}$ is a differentiable function of t whenever $\vec{v}$ is a differentiable function of t.

1.4.1 Curvature of a curve

Curvature:

$$k = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

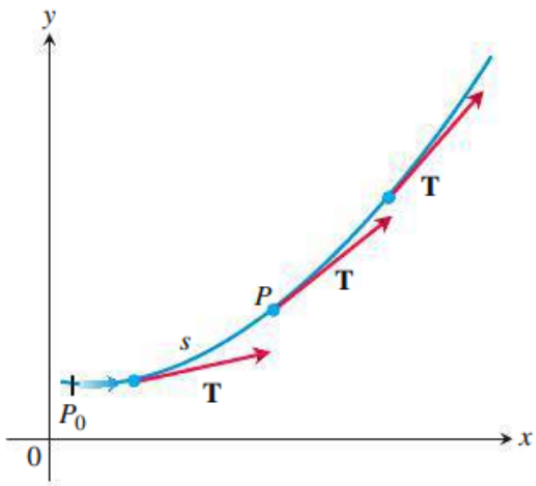
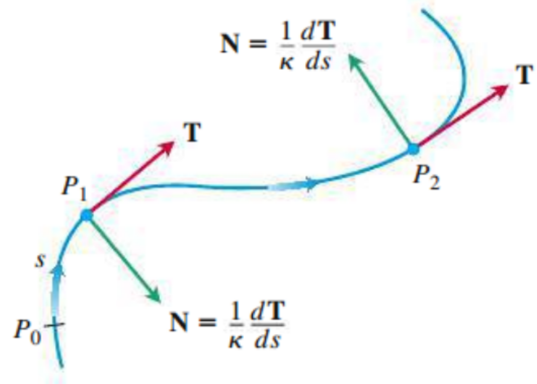


FIGURE 13.17 As P moves along the curve in the direction of increasing arc length, the unit tangent vector turns. The value of $|d\mathbf{T}/ds|$ at P is called the *curvature* of the curve at P.

If $|d\vec{T}/ds|$ is large, $\vec{T}$ turns sharply as the particle passes through P and the curvature is large. If the length is small, $\vec{T}$ turns more slowly

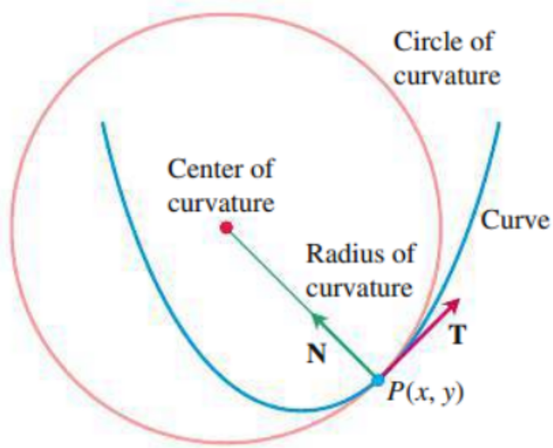
Principal unit normal vector: a unit vector orthogonal to $\vec{T}$ which points in the direction the curve is turning (towards the concave side of the curve)

$$\vec{N} = \frac{1}{k}\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$



Osculating circle: the circle of curvature at a point P on a plane curve that

1. is tangent to the curve at P (has the same tangent line as the curve)
2. has the same curvature at P
3. has center that lies toward the concave (inner) side of the curve



Radius of curvature: the radius of the circle of curvature $\rho = \frac{1}{k}$

Center of curvature: the center of the circle of curvature

1.4.2 Acceleration in space

Travelling along a curve, the **IJK** reference frame is generally less importnat than the vector of forward motion ($\vec{T}$), the direction the path is turning ($\vec{N}$), and the tendency of the to "twist" out of the plane of those vectors (*the unit binormal vector* $\vec{B} = \vec{T} \times \vec{N}$)

These vectors together form a moving right-handed vector frame called the *Frenet Frame* or the TNB frame.

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \vec{T}\frac{ds}{dt}$$
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2s}{dt^2}\vec{T} + k\left(\frac{ds}{dt}\right)^2\vec{N}$$

Also,

$$|v \times a| = k\left|\frac{ds}{dt}\right|^3 |\vec{B}| = k|\vec{v}|^3$$

so

$$k = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

If the acceleration vector is written as $\vec{a} = a_T\vec{T} + a_N\vec{N}$, then the tangential and normal scalar components of acceleration are

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\vec{v}|$$
$$a_N = k\left(\frac{ds}{dt}\right)^2 = k|\vec{v}|^2 = \sqrt{|\vec{a}|^2 - a_T^2}$$

Further, acceleration will always line in the plane of $\vec{T}$ and $\vec{N}$. The tangential component a_T measures the rate of change of the length of $\vec{v}$ (the change in the speed). The normal component a_N measures the rate of change of the direction of $\vec{v}$

1.4.3 Torsion

Torsion: the rate at which the osculating plane turns about $\vec{T}$ as P moves along the curve. In other words, torsion measures how the curve twists for $\vec{B} = \vec{T} \times \vec{N}$, torsion is

$$\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}$$

A space curve is a helix if and only if it has constant nonzero curvature and constant nonzero torsion.

Torsion can also be calculated directly in terms of the parameters

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} \quad (\text{if } \vec{v} \times \vec{a} \neq \vec{0})$$

1.4.4 Motion in polar coordinates

When a particle moves along a curve in the polar plane, we express its position, velocity, and acceleration in terms of the moving unit vectors

$$\vec{u}_r = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$$
$$\vec{u}_\theta = -(\sin\theta)\hat{i} + (\cos\theta)\hat{j}$$

Then, Position:

$$\vec{r} = r\vec{u}_r + z\hat{k}$$
$$\vec{v} = \dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta + \dot{z}\hat{k}$$
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{u}_\theta + \ddot{z}\hat{k}$$

This forms a right handed frame where

$$\vec{u}_r \times \vec{u}_\theta = \hat{k}$$
$$\vec{u}_\theta \times \hat{k} = \vec{u}_r$$
$$\hat{k} \times \vec{u}_r = \vec{u}_\theta$$

Newton's law of universal gravitation:

$$\vec{F} = -\frac{GmM}{|\vec{r}|^2}\frac{\vec{r}}{|\vec{r}|}$$

where $\vec{r}$ is the radius vector from the centre of a sun of mass M to the centre of a planet of mass m.

Kepler's Laws:

1. A planet's path is an ellipse with the sun at one focus. The eccentricity is

$$e = \frac{r_0 v_0^2}{GM} - 1$$

and the polar equation is

$$r = \frac{(1+e)r_0}{1 + e\cos\theta}$$

2. The radius vector from the sun to a planet sweeps out equal areas in equal times. If the initial line is $\theta = 0$, the direction $\vec{r}$ when $|\vec{r}| = r$ is a minimum value. Then,

$$\dot{r}\Big|_{t=0} = \frac{dr}{dt}\Big|_{t=0} = 0, \quad v_0 = |\vec{v}|_{t=0} = [r\dot{\theta}]_{t=0}$$

3. The orbital period and the orbit's semimajor axis a are related by

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$$

2 Module 2: Partial Derivatives and Applications

2.1 WEEK 4: Calculus of multivariable functions (Readings 14.1-14.4)

Real-valued function: a rule that assigns a single, unique real number $w = f(x_1, x_2, ..., x_n)$ to each element in a domain D (a set of n-tuples of real numbers). The set of w-values is the range.

In choosing the domain, we largely exclude inputs which lead to complex numbers or division by zero. The domain is assumed to be the largest set for which the defining rule generates real numbers, unless the domain is otherwise specified explicitly.

Interior point: a point (x_0, y_0) in a region R if it is the center of a disk of positive radius that lies entirely in R

Boundary point: a point of a region R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R

A region is *open* if it consists is entirely of interior points. The region is *closed* if it contains all its boundary points A region is *bounded* if it lies inside a disk of finite radius. It is *unbounded* otherwise

Level curve: the set of points where a function $f(x, y)$ has a constant value. For three variables, the analog set of points for which $f(x, y, z) = c$ is the *level surface*

Graph: the set of all points $(x, y, f(x, y))$ in space for (x, y) in the domain of f; also called the *surface* $z = f(x, y)$

2.1.1 Limits

A function $f(x, y)$ approaches the limit L as (x, y) approaches (x_0, y_0)

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if $\forall \varepsilon > 0, \forall (x, y) \in D : \quad \exists \delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{when} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \varepsilon$$

THEOREM 1—Properties of Limits of Functions of Two Variables
The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$,
   n a positive integer, and if n is even, we assume that $L > 0$.

2.1.2 Continuity

A function f(x, y) is continuous at (x_0, y_0) if

1. f is defined at (x_0, y_0)
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

The function is continuous if it is continuous at every point of its domain

Algebraic combinations of continuous function are continuous at every point at which all the functions involved are defined

Two-Path test for Nonexistence of a limit: If a function has different limits along two different paths in the domain of f as (x, y) approaches $(x_0, y_0$, then the limit does not exist

Note: having the same limit along all straight lines approaching (x_0, y_0) does not imply that a limit exists at (x_0, y_0)

Continuity of compositions: If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$ then the composition $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0)

Extreme Value Theorem of Continuous functions on closed, bounded sets: a function that is continuous on a closed, bounded set R in the plane takes an absolute maximum and an absolute minimum at some points in R.

2.1.3 Partial derivatives

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = \frac{d}{dx} f(x_0, y_0)\big|_{x=x_0}$$

When we do not specify a point at which to evaluate the partial, the partial derivative becomes a function whose domain is the set of points where the partial derivative exists.

$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ is the slope of the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ which is the tangent line to that point.

We can say that $\frac{\partial f}{\partial x}$ is the "rate of change of f with respect to x when y is held fixed. This idea can be generalised to all dimensions.

For differentiable functions, the plane defined by $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}$ defines a tangent plane to the surface at a particular point

Note: a function can have partial derivatives with respect to both x and y at a point without the function being continuous there. The function is continuous iff the partials exist and are continuous throughout a disk centered at (x_0, y_0)

2.1.4 Second partials

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Mixed Derivative Theorem (Clairaut's Theorem): If f(x, y) and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing P(a,b) and are all continuous at P(a, b), then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

2.1.5 Differentiability

A function $z = f(x,y)$ is differentiable at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta_y \to 0$

A function is differentiable if it is differentiable at every point in its domain and the graph of a differentiable equation is a smooth surface.

If the partials f_x and f_y are continuous throughout an open region R, then f is differentiable at every point of R. If a function is differentiable at a point, it is continuous at that point.

2.1.6 The Chain Rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}\bigg|_{x=x_0,\ t=t_0}$$

Implicit differentiation: supposing F(x, y) is differentiable and that $F(x,y) = 0$ defines y as a differentiable function of x, then $\forall F_y \neq 0$:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

2.2 WEEK 5: Gradients, tangent planes, and extreme values (Readings 14.5-14.7)

Suppose $f(x,y)$ is defined on R in the xy-plane, that $P_0(x_0, y_0)$ is a point in R, and that $\vec{u} = u_1\hat{i} + u_2\hat{j}$. Then:

$$x = x_0 + su_1, \quad y = y_0 + su_2$$

parameterise the line through P_0 parallel to $\vec{u}$. If the parameter measures arc length from P_0 in the direction of $\vec{u}$, df/ds at P_0gives the rate of change of f at a point in a direction

2.2.1 The Directional Derivative

Directional Derivative: the derivative at a point in the direction of a unit vector

$$D_{\hat{u}}f(P_0) = D_{\hat{u}}f|_{P_0} = \left(\frac{df}{ds}\right)_{\hat{u},P_0} = \lim_{s\to 0}\frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s}$$

If $z = f(x,y)$ is a surface, the vertical plane that passes through a point $P(x_0, y_0, z_0)$ and $_0(x_0, y_0)$ parallel to $\hat{u}$ intersects S in a curve C, then the directional derivative towards $\hat{u}$ is the slope of the tangent to C at P in the right handed $\hat{u}\hat{k}$-system

Gradient vector:

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

Then:

$$\left(\frac{df}{ds}\right)_{\hat{u},P_0} = \nabla f|_{P_0} \cdot \hat{u} \ = |\nabla f| \cos\theta$$

Properties of the directional derivative:

1. f increaes most rapidly in the direction of the gradient vector ($D_{\hat{u}}f = |\nabla f|$)
2. f decreases most rapidly in the direction of $-\nabla f$ (where $D_{\hat{u}} = -|\nabla f|$)
3. Any direction $\hat{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of no change in f

At every point (x_0, y_0) in the domain of a differentiable function f(x, y), the gradient of f is normal to the level curve through (x_0, y_0). Thus, the tangent lines to level curves are the lines normal to the gradients.

Tangent line to a level curve:

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0$$

Algebra Rules for Gradients:

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(f-g) = \nabla f - \nabla g$
3. $\nabla(kf) = k\nabla f$

4. $\nabla(fg) = f\nabla g + g\nabla f$
5. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Derivative along a path: For $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$$

Tangent plane: the plane through a point P_0 normal to $\nabla f|_{P_0}$ for a differentiable level surgace

Normal line: the line through P_0 parallel to $\nabla f|_{P_0}$ For a point $P_0(x_0, y_0, z_0)$:

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$$

Plane tangent to a surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$:

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Estimating the change in f in a direction $\hat{u}$:

$$df = (\nabla f|_{P_0} \cdot \hat{u})ds$$

2.2.2 Linearisation

Linearisation of $f(x, y)$:

$$f(x, y) \approx L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Error in the standard linear approximation: If f has continuous first and second partial derivatives throughout an open set contianing a rectangle R centered at (x_0, y_0) and if M is any upper bound for $|f_x x|, |f_y y|, |f_x y|$ on R, then

$$|E(x, y)| \leq \frac{1}{2}M\left(|x - x_0| + |y - y_0|\right)^2$$

The total differential of f: The change in the linearisation of f moving from (x_0, y_0) to $(x_0 + dx, y_0 + dy)$:

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

2.2.3 Extreme Values

Extreme values on a closed interval can happen only at the boundary points or at interior domain points where both first partials are zero or at least one partial does not exist.

Local maximum: a value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centred at (a, b)

Local minimum: a value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centred at (a, b)

Critical point: an interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where at least one of f_x and f_y does not exist

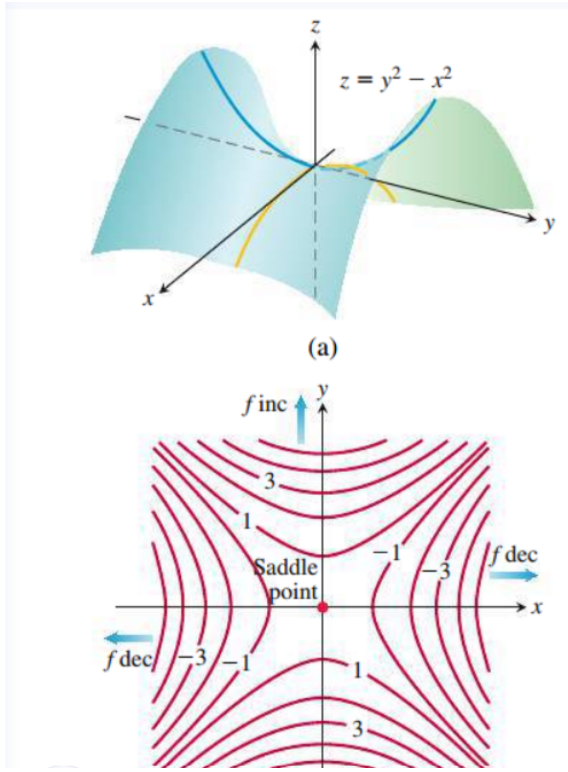
Saddle point: a critical point where in every open disk centred at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and where $f(x, y) < f(a, b)$

Second Derivative test: For f(x, y) with continuous second partial derivatives with $f_x(a, b), f_y(a, b) = 0$. Then:

1. f has a local max if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$
2. f has a local min if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$
3. f has a saddle point if $f_{xx}f_{yy} - f_{xy}^2 < 0$
4. the test is inconclusive if $f_{xx}f_{yy} - f_{xy}^2 = 0$

Hessian (discriminant): the expression

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$



Finding absolute extrema on closed regions:

1. List and evaluate the critical points
2. List and evaluate the boundary points
3. Identify the extrema from those list

2.3 WEEK 6: Lagrange multipliers and Taylor's formula (Readings 14.8-14.10)

Method of Lagrange Multipliers: says that local extrema of a function $f(x, y, z)$ whose variables are subject to a constraint $g(x, y, z) = 0$ are to be found on the surface $g = 0$ among the points where

$$\nabla f = \lambda \nabla g$$

for some scalar λ (the *lagrange multiplier*)

The Orthogonal Gradient Theorem: if $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

$$C: \ \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

if P_0 is a point on C where f has a local max or min relative to its values on C, then ∇f is orthogonal to C at P_0

Thus, to find the local extrema of f subject to some constraint: find x, y, z, and λ that simultaneously satisfy

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g(x, y, z) &= 0 \end{aligned}$$

If there are two constraints $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$, the simultaneous solutions are

$$\begin{aligned} \nabla f &= \lambda \nabla g_1 + \mu \nabla g_2 \\ g_1(x, y, z) &= 0 \\ g_2(x, y, z) &= 0 \end{aligned}$$

for some new Lagrange multiplier μ

2.3.1 Taylor's Formula

Suppose $f(x, y)$ and its partial derivatives through order $n+1$ are continuous throughout an open rectangular region R centred at a point (a, b). Then, throughout R,

$$\begin{aligned} f(a+h, b+k) &= f(a,b) + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(a+ch, b+ck)} + \sum_{i=1}^{n} \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n} f\Big|_{(a,b)} \\ &= f(a,b) + (hf_x + kf_y)\big|_{(a,b)} + \frac{1}{2}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\big|_{(a,b)} \\ &+ \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})\big|_{(a,b)} + \ldots \end{aligned}$$

The first n derivative terms are evaluated at (a, b). The last term is evaluated at some point $(a + ch, b + ck)$ on the line segment joining (a, b) and $(a + h, b + k)$

Taylor's formula provides polynomial approximations for two variable function. The first n derivative terms give the polynomial; the last term gives the approximation error. THe first three terms of Taylor's formula give the linearisation.

Partial derivatives with multiple constraints:

1. Decide which variables are dependent and independent (in practice, based on the physical or theoretical context)
2. Eliminate the other dependent variables to solve for a single dependent. If this is not possible, differentiate as they are then try to solve for $\partial w/\partial x$ after
3. Differentiate as usual.

Notation:

- $\left(\frac{\partial w}{\partial x}\right)_y$ Derivative with x and y independent
- $\left(\frac{\partial w}{\partial y}\right)_{x,t}$ Derivative with x, y, and t independent

3 Module 3: Integrals

3.1 WEEK 7: Double integrals and area (Readings 15.1-15.3)

Double integral: the unique number I that is between the lower and upper Riemann sums for all partitions in a region

- Lower sum: $L_f(\mathcal{P}) = \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta x_i \Delta y_i m_{ij}$, for $\mathcal{P}$ a partition of $\mathcal{R}$ and m_{ij} the minimum of f on the i, j sub-rectangle of $\mathcal{R}_{ij}$
- Lower sum: $L_f(\mathcal{P}) = \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta x_i \Delta y_i m_{ij}$, where M_{ij} is the max of f on the sub-rectangle $\mathcal{R}_{ij}$

Double integral notation:

$$I = \int\int_{\mathcal{R}} f(x,y)dxdy = \int\int_{\mathcal{R}} f(x,y)dA$$

Fubini's Theorem (First Form): If $f(x, y)$ is continuous throughout the rectangular region $R : a \leq x \leq b, c \leq y \leq d$, then

$$\int\int_{R} f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

Example: Solve $\int_0^3 \int_{-2}^0 (x^2 y - 2xy)dydx$

Solution:

$$\begin{aligned}
\int_0^3 \int_{-2}^0 (x^2 - y - 2xy)dydx &= \int_0^3 \left[\frac{1}{2}x^2y^2 - xy^2\right]_{-2}^0 dx \\
&= \int_0^3 [0 - 2x^2 + 4x]dx \\
&= \left[-\frac{2}{3}x^3 + 2x^2\right]_0^3 = -18 + 18 = 0
\end{aligned}$$

Volume: if $f(x, y)$ is a positive function over a rectangular region R in the xy-plane, the volume of the solid region over the xy-plane bounded below by R and above by

$f(x, y)$ is given by:

$$V = \int\int_R f(x, y)dA$$

Fubini's theorem (Stronger form:) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$ with g_1 and g_2 continuous on $[a, b]$, then
$$\int\int_R f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$$

2. If If R is defined by $c \leq y \leq c, h_1(y) \leq x \leq h_2(y)$ with h_1 and h_2 continuous on $[c, d]$, then
$$\int\int_R f(x, y)dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x, y)dxdy$$

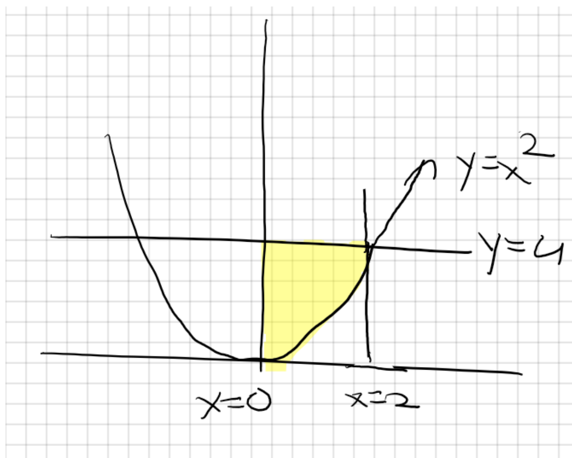
Limits of integration for general regions:

1. Using vertical cross sections:
$$\int\int_R f(x, y)dA = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} f(x, y)dydx$$

2. Using horizontal cross sections:
$$\int\int_R f(x, y)dA = \int_{y=c}^{y=d} \int_{x=g(y)}^{x=h(y)} f(x, y)dxdy$$

Example: Evaluate $\int_0^2 \int_{x^2}^4 2x\cos(y^2)dydx$ by changing the order of integration

Solution:



$$\int_0^2 \int_{x^2}^4 2x\cos(y^2)dydx = \int_0^4 \int_0^{\sqrt{y}} 2x\cos(y^2)dxdy$$

$$= \int_0^4 \left[x^2\cos(y^2)\right]_0^{\sqrt{y}} = \int_0^4 y\cos(y^2)dy$$

$$= \int_0^4 \left[x^2\cos(y^2)\right]_0^{\sqrt{y}} = \int_0^4 y\cos(y^2)dy$$

$$= \int_0^4 \left[x^2\cos(y^2)\right]_0^{\sqrt{y}} = \int_0^4 y\cos(y^2)dy$$

$$= \frac{1}{2}\sin(y^2)\Big|_0^4 = \frac{1}{2}\sin 16$$

Properties of Double Integrals:

1. Constant Multiple:

$$\int\int_R cf(x,y)dA = c\int\int_R f(x,y)dA$$

2. Sum and difference:

$$\int\int_R (f(x,y) \pm g(x,y))dA = \int\int_R f(x,y)dA \pm \int\int_R g(x,y)dA$$

3. Domination:

   (a) $\int\int_R f(x,y)dA \geq 0 \quad \text{if } f(x,y) \geq 0 \in R$

   (b) $\int\int_R f(x,y)dA \geq \int\int_R g(x,y)dA \quad \text{if } f(x,y) \geq g(x,y) \in R$

4. Additivity: If R is the union of two nonoverlapping regions R_1 and R_2,

5.
$$\int\int_R f(x,y)dA = \int\int_{R_1} f(x,y)dA + \int\int_{R_2} f(x,y)dA$$

Area: the area of a closed, bounded region R is

$$A = \int\int_R dA$$

Average value:

$$\frac{1}{\text{area of R}}\int\int_R f(x,y)dA$$

3.2 WEEK 8: Double and tripe integrals (Readings 15.4-15.6)

Polar coordinates: an alternative to rectangular coordinate system. Uses two parameters θ, an angle, and r, a radius. Measurements begin from the polar axis to draw the angle in the counterclockwise direction if $\theta < 0$ (clockwise otherwise,), then go a distance r along the ray *theta* (or along the ray $\theta + \pi$ if $r < 0$).

Polar formulas:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2 + y^2 &= r^2 \\ \theta &= \arctan(\frac{y}{x}) \end{aligned}$$

Area of a sector of a curve: $\frac{\theta}{2\pi r} \cdot \pi r^2 = \frac{1}{2}r^2\theta$

Area under a curve in polar coordinates comes from summing infinite sectors rather than rectangles.

Area of polar regions:

$$A = \int\int_R r dr d\theta$$

Example: Find a formula for the area of a circle with radius a. Solution:

$$0 \leq r \leq a,\ 0 \leq \theta \leq 2\pi$$

$$
\begin{aligned}
A &= \int_0^{2\pi}\int_0^a r dr d\theta = \int_0^{2\pi} \frac{1}{2}a^2 d\theta \\
&= \frac{1}{2}a^2\theta\Big|_0^{2\pi} = \pi a^2
\end{aligned}
$$

3.2.1 Double integration with Polar coordinates:

Given $F = F(r,\theta)$ is continuous on $\Gamma : a \le r \le b, \alpha \le \theta \le \beta$ we have

$$\int\int_\Gamma F(r,\theta) r dr d\theta = \int_\alpha^\beta \int_a^b F(r,\theta) r dr d\theta$$

And the double integral of $F(r,\theta)$ over the polar region $\Omega : \alpha \le \theta \le \beta, \rho_1(\theta) \le r \le \rho_2(\theta)$ is

$$\int\int_\Omega F(r,\theta) r dr d\theta = \int_\alpha^\beta \int_{\rho_1(\theta)}^{\rho_2(\theta)} F(r,\theta) r dr d\theta$$

Volume: if $F(r,\theta) \ge 0$ over a region R, the volume of the solid with R as base, bounded above by F is

$$V = \int\int_R F(r,\theta) r dr d\theta$$

3.2.2 Triple integrals

Integration over a "Box": Given $f = f(x,y,z)$ is continuous on the rectangular box B, where $B : a_1 \le x \le b_1, a_2 \le y \le b_2, a_3 \le z \le b_3$

$$\int\int\int_B f(x,y,z) dV = \int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y,z) dx dy dz$$

Volume: the volume of a closed, bounded region D in space is

$$V = \int\int\int_D dV$$

Average Value of F over a region D:

$$\frac{1}{\text{volume of D}}\int\int\int_D F dV$$

3.2.3 Applications of Iterated Integrals

For three-dimensional solids

- Mass:

$$M = \int\int\int_D \delta dV$$

where $\delta = \delta(x, y, z)$, the density function and D is the solid

- First moments about the coordinate planes:

$$M_{yz} = \int\int\int_D x\delta vD$$
$$M_{xz} = \int\int\int_D y\delta dV$$
$$M_{xy} = \int\int\int_D z\delta dV$$

- Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}$$
$$\bar{y} = \frac{M_{xz}}{M}$$
$$\bar{z} = \frac{M_{xy}}{M}$$

- Moments of Inertia:

$$I_x = \int\int\int (y^2 + z^2)\delta dV$$
$$I_y = \int\int\int (x^2 + z^2)\delta dV$$
$$I_z = \int\int\int (x^2 + y^2)\delta dV$$
$$I_L = \int\int\int r^2(x, y, z)\delta dV$$

For two-dimensional plates:

- Mass:

$$M = \int\int_R \delta dA$$

where $\delta = \delta(x, y, z)$, the density function and D is the solid

- First moments about the coordinate planes:

$$M_y = \int\int_R x\delta dA$$
$$M_x = \int\int_R y\delta dA$$

- Center of mass:

$$\bar{x} = \frac{M_y}{M}$$
$$\bar{y} = \frac{M_x}{M}$$

- Moments of Inertia:

$$I_x = \int\int y^2\delta dA$$
$$I_y = \int\int x^2\delta dA$$
$$I_L = \int\int r^2(x, y)\delta dA$$
$$I_O = \int\int (x^2 + y^2)\delta \; dA = I_x + I_y$$

Centroid: the center of mass for an object with constant density. To find the centroid, use $\delta = 1$

Example: Find the centroid of the region between the x-axis, $y = \sin(x)$ for $0 \leq x \leq \pi$

Solution:

$$
\begin{aligned}
M &= \int_0^\pi \int_0^{\sin x} dydx = \int_0^\pi \sin x \; dx = [-\cos x]_0^\pi = 2 \\
M_x &= \int_0^\pi \int_0^{\sin x} y \; dydx = \int_0^\pi \frac{1}{2} \sin^2(x) dx \\
&= \int_0^\pi \frac{1}{4}(1 - \cos(2x)) \; dx = \frac{1}{4}\left[x - \frac{1}{2}\sin(2x)\right]_0^\pi = \frac{\pi}{4} \\
M_y &= \int_0^\pi \int_0^{\sin x} x dydx = \int_0^\pi x \sin x dx \\
&= [-x\cos x + \sin x] = \pi \bar{x} = \frac{\pi}{2} \bar{y} = \frac{M_x}{M} = \frac{\pi}{8}
\end{aligned}
$$

Joint probability density function: a function that satisfies three conditions

1. $f(x, y) \geq 0$
2. $\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(x, y) \; dxdy = 1$
3. $P((X, Y) \in R) = \int \int_R f(x, y) \; dxdy$

Mean and expected value: when X and Y have joint probability density function f,

$$
\mu_X = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} x f(x, y) \; dxdy
$$

$$
\mu_Y = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} y f(x, y) \; dxdy
$$

Example: verify that f gives a joint probability density function. Then find the expected values μ_X and μ_Y

$$
f(x, y) = \begin{cases} 6x^2 & 0 \leq x \leq 1, \; 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}
$$

Solution:

1. f is positive on $(x, y) \in [0, 1]$ and 0 elsewhere so $f(x, y) \geq 0$

2. We need to check $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x,y)\ dxdy = 1$ but we only need to evaluate it on the positive region:

$$\int_0^1\int_0^1 6x^2y\ dxdy = \int_0^1 \left[2x^3y\right]_0^1 dy = \int_0^1 2y\ dy = y^2\Big|_0^1 = 1$$

So the function is a joint probability function and the expected values are

$$\mu_X = \int_0^1\int_0^1 6x^3y\ dxdy = \int_0^1 \frac{3}{2}y\ dy = \frac{3}{4}y^2\Big|_0^1 = \frac{3}{4}$$
$$\mu_Y = \int_0^1\int_0^1 6x^2y^2\ dxdy = \int_0^1 \left[2x^3y^2\right]_0^1 dy = \int_0^1 2y^2 dy = \left[\frac{2}{3}y^3\right]_0^1 = \frac{2}{3}$$

3.3 WEEK 9: Triple integrals and applications (Readings 15.7-15.8)

Cyclindrical coordinates: represent a point P in space by ordered triples $(r,\theta,z), r \geq 0$

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate

Equations:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ r^2 &= x^2+y^2 \\ \tan\theta &= \frac{y}{x} \end{aligned}$$

In cyclindrical coordinates, $r = a$ describes a cylinder about the z-axis (which is given by $r = 0$). The equation $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis.

Norm: the largest of the values $\Delta r_k, \Delta\theta_k, \Delta z_k$ among all the cylindrical wedges.

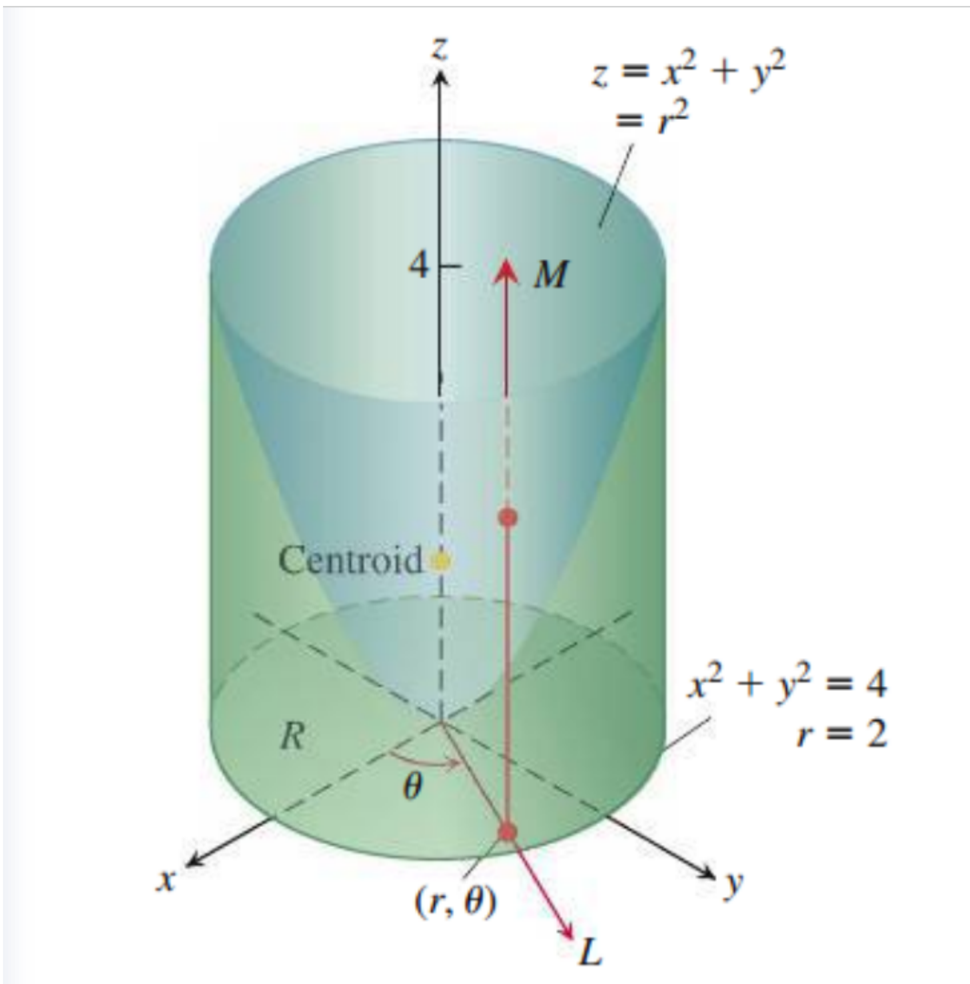
$$\int\int_D\int f\ dV = \int\int_D\int f\ \ dz\, r\, dr\, d\theta$$

To integrate:

1. Sketch the function
2. Find the z limits
3. Find the r limits
4. Find the theta limits
5. Integrate

Example: Find the centroid ($\delta = 1$) of the solid enclosed by the cylinder $x^2 + y^2 = 4$ bounded above by $z = x^2 + y^2$ and below by the xy-plane.

Solution:



Z-limits: $0 \leq z \leq r^2$ R-limits: $0 \leq r \leq 2$ θ-limits: $0 \leq \theta \leq 2\pi$

$$\begin{aligned} M_{xy} &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} z \;\; dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^2 \left[\frac{z^2}{2} \right]_0^{r^2} r \; dr\, d\theta \\ &= \int_0^{2\pi} \int_0^{2\pi} \frac{r^5}{2}\, dr\, d\theta = \int_0^{2\pi} \left[\frac{r^6}{12} \right]_0^2 d\theta = \int_0^{2\pi} \frac{16}{3} d\theta = \frac{32\pi}{3} \end{aligned}$$

$$\begin{aligned} M = \int_0^{2\pi} \int_0^2 \int_0^{r^2} \;\; dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^2 [z]_0^{r^2}\, r\; dr\, d\theta \\ = \int_0^{2\pi} \int_0^{2\pi} r^3\; dr\, d\theta = \int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^2 d\theta = \int_0^{2\pi} 4\; d\theta = \end{aligned}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi/3}{8\pi} = \frac{4}{3}$$

Centroid: $(0, 0, 4/3)$

3.3.1 Spherical Coordinates

Spherical coordinates: represent a point in space by ordered triples (ρ, ϕ, θ) where

1. ρ is the distance from P to the origin
2. ϕ is the angle $\vec{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$
3. θ is the angle from cylindrical coordinates

$\rho = a$ describes a sphere of radius a centred at the origin, $\phi = \phi_0$ describes a single cone with vertex at the origin and axis along the z-axis, $\theta = \theta_0$ describes the half-plane that contains the z-axis and makes an angle θ_0 with the positive x-axis

Equations:

$$r = \rho \sin\phi \tag{1}$$

$$x = r\cos\theta = \rho\sin\phi\cos\theta \tag{2}$$

$$y = r\sin\theta = \rho\sin\phi\sin\theta \tag{3}$$

$$z = \rho\cos\phi \tag{4}$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2} \tag{5}$$

Example: Find a spherical coordinate equation for $x^2 + y^2 + (z-1)^2 = 1$

Solution:

$$\begin{aligned} x^2 + y^2 + (z-1)^2 &= 1 \\ \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2 \sin^2\theta + (\rho\cos\phi - 1)^2 &= 1 \\ \rho^2 \sin^2\phi + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 &= 1 \\ \rho^2 &= 2\rho\cos\phi \\ \rho &= 2\cos\theta \end{aligned}$$

$$\int\int_D\int f(\rho,\phi,\theta)\ dV = \int\int_D\int f(\rho,\phi,\theta)\rho^2\sin\phi\ d\rho\, d\phi\, d\theta$$

To solve, we integrate over solids of revolution about the z-axis (where the θ and ϕ limits are constant)

Example: Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.

Solution:

$$\begin{aligned} 0 \leq \rho &\leq 1 \\ 0 \leq \phi &\leq \frac{\pi}{3} \\ 0 \leq \theta &\leq 2\pi \end{aligned}$$

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^2\sin\phi\ d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{\rho^3}{3}\right]_0^1 \sin\phi\ d\phi d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/3}\frac{1}{3}\sin\phi\ d\phi d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\right.$$

3.3.2 Coordinate Conversion Formulas:

Cylindrical to rectangular:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Spherical to rectangular:

$$\begin{aligned}
x &= \rho \sin\phi \cos\theta \\
y &= \rho \sin\phi \sin\theta \\
z &= \rho \cos\theta
\end{aligned}$$

Spherical to cylindrical:

$$\begin{aligned}
r &= \rho \sin\phi \\
z &= \rho \cos\phi \\
\theta &= \theta
\end{aligned}$$

Corresponding formulas for dV:

$$\begin{aligned}
dV &= dx\, dy\, dz \\
&= dz\, r\, dr\, d\theta \\
&= \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta
\end{aligned}$$

3.3.3 Substitution

Jacobian determinant: function for the coordinate transformation $x = g(u, v), y = h(u, v)$ which measures how much the transformation is expanding or contracting the area around a point (u, v)

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{\partial(x,y)}{\partial(u,v)}$$

Substitution for Double Integrals: For f(x,y) continuous over the region R, let G be the preimage of R under the transformation $x = g(u, v), y = h(u, v)$ (one-to-one on the interior of G). If the functions g and h have continuous first partial derivatives on G, then

$$\int\int_R f(x,y)dxdy = \int\int_G f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

Example: Write the Cartesian integral $\int\int_R f(x,y)dxdy$ as a polar integral.

Solution: With $x = r\cos\theta, y = r\sin\theta$, the Jacobian is

$$J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Since $r \geq 0$ in polar coordinates, $|J(r,\theta)| = |r| = r$, so

$$\int\int_R f(x,y)\;dx\,dy = \int\int_G f(r\cos\theta, r\sin\theta)r\;dr\,d\theta$$

Substitutions in triple integrals:

$$\begin{aligned} x &= g(u,v,w) \\ y &= g(u,v,w) \\ z &= g(u,v,w) \end{aligned}$$

Then any function on D in xyz-space is a function

$$F(g(u,v,w), h(u,v,w), k(u,v,w)) = H(u,v,w)$$

on G. If g, h, and k have continuous first partial derivatives, then the integral is

$$\int\int_D\int F(x,y,z)\;dx\,dy\,dz = \int\int_G\int H(u,v,w)|J(u,v,w)|\;du\,dv\,dw$$

4 Module 4: Integrals and Vector Fields

4.1 WEEK 10: Line integrals and vector fields (Readings 16.1-16.3)

Line integral: The curve equivalent of integrating over a straight line distance.

If f is defined on a curve C given parametrically, then

$$\int_C f(x, y, z)\ ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k$$

where s is arc length.

How to evaluate:

1. Find a smooth parametrisation of C

$$\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\ \hat{k}, \quad a \le t \le b$$

2. Evaluate the integral as

$$\int_C f(x, y, z)\ ds = \int_a^b f(g(t), h(t), k(t))\, |\vec{v}(t)|\ dt$$

Example: Evaluate $\int_C (x - y + z + 1)ds$ where C is the straight-line segment given by $z = t, y = 1 - t, z = 1$ from $(0, 1, 1)$ to $(1, 0, 1)$

Solution:

$$0 \le t \le 1$$

$$\begin{aligned} \vec{r}(t) &= t\hat{i}(1-t)\hat{j} + \hat{k} \\ \vec{r}'(t) &= \hat{i} - \hat{j} \\ |\vec{r}'(t)| &= \sqrt{2} \end{aligned}$$

$$\int_0^1 (t - (1-t) + 1 + 1)\sqrt{2}\ dt = \sqrt{2}\int_0^1 (2t+1)\ dt = 2\sqrt{2}$$

4.1.1 Applications of line integrals

Mass:

$$M = \int_C \delta(x, y, z)\ ds$$

for some density δ

First moments:

$$\begin{aligned} M_{yz} &= \int_C x\delta\ ds \\ M_{xz} &= \int_C y\delta\ ds \\ M_{xy} &= \int_C z\delta\ ds \end{aligned}$$

Centroids:

$$\begin{aligned} \bar{x} &= M_{yz}/M \\ \bar{y} &= M_{xz}/M \\ \bar{z} &= M_{xy}/M \end{aligned}$$

Moments of Inertia:

$$\begin{aligned} I_x &= \int_C (y^2 + z^2)\delta\ ds \\ I_y &= \int_C (x^2 + z^2)\delta\ ds \\ I_z &= \int_C (x^2 + y^2)\delta\ ds \\ I_L &= \int_C r^2\delta\ ds \end{aligned}$$

(where $r(x, y, z)$ is the distance from the point (x,y,z) to line L)

Line integrals of vector fields: If $\vec{F}$ is a vector field with continuous components defined along a smooth curve C parametrized by $\vec{r}(t)$, then the line integral of F along C is

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \left(\vec{F} \cdot \frac{d\vec{r}}{ds} \right)\, ds = \int_C \vec{F} \cdot d\vec{r}$$

How to evaluate the line integral of $\vec{F} = M\hat{\imath} + N\hat{\jmath} + P\hat{k}$ along $C:\ \vec{r}(t) = g(t)\hat{\imath} + h(t)\hat{\jmath} + k(t)\hat{k}$

1. Express $\vec{F}$ along the curve as $\vec{F}(\vec{r}(t))$ by substituting the components of $\vec{r}$ into M, N, and P
2. Find $d\vec{r}/dt$
3. Evaluate
$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt$$

Example: Evaluate where $\vec{F}(x,y,z) = xy\hat{\imath} + x^2z\hat{\jmath} + xyz\hat{k}$ along $y = x^2$ from (0,0,0) to (1,1,0) followed by the straight-line segment from (1,1,0) to (1,1,1).

Solution:

$$\begin{aligned} C_1 : \vec{r}_1(t) &= t\hat{\imath} + t^2\hat{\jmath}, \quad 0 \le t \le 1 \\ \vec{r}_1'(t) &= \hat{\imath} + 2t\hat{\jmath} \\ C_2 : \vec{r}_2(t) &= \hat{\imath} + \hat{\jmath} + t\hat{k}, \quad 0 \le t \le 1 \\ \vec{r}_2'(t) &= \hat{k} \end{aligned}$$

$$\begin{aligned} \int_0^1 \langle t^3, 0, 0\rangle \cdot \langle 1, 2t, 0\rangle dt + \int_0^1 \langle 1, t, t\rangle \cdot \langle 0, 0, 1\rangle dt \\ = \int_0^1 t^3\, dt + \int_0^1 t\, dt = \frac{3}{4} \end{aligned}$$

The expression $\int_C M\, dx + N\, dy + P\, dz$ means

$$\int_C M(x,y,z)\, dx + \int_C N(x,y,z)\, dy + \int_C P(x,y,z)\, dz$$

where

$$\int_C M(x,y,z)\,dx = \int_a^b M(g(t),h(t),k(t))\,g'(t)\,dt$$
$$\int_C N(x,y,z)\,dy = \int_a^b N(g(t),h(t),k(t))\,h'(t)\,dt$$
$$\int_C P(x,y,z)\,dz = \int_a^b P(g(t),h(t),k(t))\,k'(t)\,dt$$

Work done by force moving an object: Let C be a smooth curve parameterized by $\vec{r}$ and let $\vec{F}$ be a continuous force field over a region containing C. Then the work from point $A = \vec{r}(a)$ to $B = \vec{r}(t)$ along C is

$$W = \int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}\,dt$$

Flow integral:

$$\text{Flow} = \int_C \vec{F}\cdot\vec{T}\,ds$$

If the curve starts and ends at the same point, the flow is called the circulation around the curve.

Flux: If C is a smooth simple closed curve in the domain of a continuous vector field $\vec{F} = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$ in the plane, and if $\vec{n}$ is the outward-pointing unit normal vector on C, the flux of $\vec{F}$ across C is

$$\text{Flux of F across C} = \int_C \vec{F}\cdot\vec{n}\,ds$$

Flux of $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ can also be calculated

$$C = \oint M\,dy - N\,dx$$

from any smooth parametrization $x = g(t), y = h(t), a \le t \le b$ that traces C counterclockwise exactly once.

Example: find the circulation and flux of the field $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ around and across the closed semicircular path with $\vec{r}_1(t) = 2\cos t\hat{\imath} + 2\sin t\hat{\jmath}, 0 \le t \le \pi$ and $\vec{r}_2(t) =$

$t\hat{i}, -2 \leq t \leq 2$

$$\vec{r}_1'(t) = -2\sin t\hat{i} + 2\cos t\hat{j}$$
$$\vec{r}_2(t) - \hat{i}$$

Flow:

$$\int_0^\pi \langle -2\sin t, 2\cos t\rangle \cdot \langle -2\sin t, 2\cos t\rangle \, dt$$
$$= \int_0^\pi 4\sin^2 t + 4\cos^2 t \, dt = 4\pi$$
$$\int_{-2}^2 \langle 0, 1\rangle \cdot \langle 1, 0\rangle \, dt = \int_{-2}^2 0 \, dt = 0$$

so the total flow is 4π

Flux:

$$\int_0^\pi (-4\sin t\cos t) - (-4\cos t\sin t) \, dt = 0$$
$$\int_{-2}^2 0 - t \, dt = -\frac{t^2}{2}\Big|_{-2}^2 = 0$$

so flux is 0.

Conservative field: a vector field in an open region D in space for which the integral $\int_C \vec{F} \cdot d\vec{r}$ is path independent in D (the line integral along a path C from A to B in D is the same over all paths from A to B)

Potential function: if the vector field $\vec{F} = \nabla f$ for some function f on D

For a vector field whose components are continuous throughout an open connection region in space, $\vec{F}$ **is conservative iff** $\vec{F}$ **is a gradient field** ∇f **for a differentiable function f.**

Component test for conservative fields: $\vec{F} = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Example: Which of the fields are conservative?

- $\vec{F} = y\hat{\imath} + (x+z)\hat{\jmath} - y\hat{k}$
- $\vec{F} = (yz)\hat{\imath} + (xz)\hat{\jmath} + (xy)\hat{k}$
- $\vec{F} = (y\sin z)\hat{\imath} + (x\sin z)\hat{\jmath} + (xy\cos z)\hat{k}$

Solution:

- $\frac{\partial P}{\partial y} = -1, \quad \frac{\partial N}{\partial z} = 1 \therefore$ not conservative
- $\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y} \therefore$ is conservative
- $\frac{\partial P}{\partial y} = x\cos z = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = y\cos z = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \sin z = \frac{\partial M}{\partial y} \therefore$ is conservative

Example: Find a potential function f for $\vec{F} = (y\sin z)\hat{\imath} + (x\sin z)\hat{\jmath} + (xy\cos z)\hat{k}$

$$\frac{\partial f}{\partial x} = y\sin z$$

$$\int y\sin z\, dx = xy\sin z + g(y,z) = f$$

$$\frac{\partial f}{\partial y} = x\sin z + \frac{\partial g}{\partial y} = x\sin z \implies \frac{\partial g}{\partial y} = 0$$
$$\frac{\partial f}{\partial z} = xy\cos z + \frac{\partial g}{\partial z} = xy\cos z \implies \frac{\partial g}{\partial z} = 0$$

Therefore

$$f(x,y,z) = xy\sin z + C$$

Fundamental theorem of line integrals: For a continuous gradient vector $\vec{F} = \nabla f$ on a domain D containing C and a curve C parametrized by $\vec{r}(t)$,

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$$

Loop property of conservative fields: The following statements are equivalent:

1. $\oint_C \vec{F}\cdot d\vec{r} = 0$ around every loop (closed curve C) in D
2. The field $\vec{F}$ is conservative on D

The symbol $\oint$ indicates a closed simple loop (a curve that starts and ends at the same place)

Example: for $\vec{F} = \nabla f$ and $f(x,y,z) = \frac{2x}{y^2+z^2+1}$, finf $\int_C \vec{F} \cdot d\vec{r}$ from (1,2,-1) to (2,3,0).

Solution:

$$\int_C \vec{F} \cdot d\vec{r} = f(2,3,0) - f(1,2,-1) = \frac{4}{10} - \frac{2}{6} = \frac{1}{15}$$

Exact: a differential form $((x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz)$ if, for some scalar function f throughout D,

$$M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

Component Test for exactness: The differential form $M\,dx + N\,dy + P\,dz$ is exact on an open simply connected domain iff

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

which is equivalent to saying $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ is conservative

4.2 WEEK 12: Green's theorem and surfaces (Readings 16.4-16.6)

Spin around an axis Suppose that $\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$ is the velocity field of a fluid and that the first partials of M and N are continuous on R. Let (x,y) be a point in R and A a small rectangle with one corner at (x,y) that lies entirely in R.

Circulation rate: roughly the rate at which a paddle wheel with axis perpendicular to the pkae spins at a point in the flood; how fast the fluid is circulating

The circulation rate of $\vec{F}$ around the boundary of A is the sum of flow rates along the sides in the tangential direction.

- Top: $\vec{F}(x, y+\Delta y) \cdot (-\hat{i})\Delta x = -M(x, y+\Delta y)\Delta x$
- Bottom: $\vec{F}(x,y) \cdot \hat{i}\Delta x = M(x,y)\Delta x$
- Right: $\vec{F}(x+\Delta x, y) \cdot \hat{j}\Delta y = N(x+\Delta x, y)\Delta y$
- Left: $\vec{F}(x,y) \cdot (-\hat{j})\Delta y = -N(x,y)\Delta x$

We sum opposite pairs to get

- Top and bottom: $\approx -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$
- Right and left: $\approx -\left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$

Thus, *Circulation density of a vector field $\vec{F} = M\hat{i} + N\hat{j}$ at (x,y)*: the k-component of curl (curl $\vec{F}\cdot\hat{k}$)

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Looking down onto the xy-plane, a field spins counterclockwise when the k-component of curl is positive and clockwise when it is negative.

Divergence (flux density):

$$\text{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Green's Theorem (circulation-curl or tangential form): The work around a loop

Let C be a piecewise smooth, simple closed curve enclosing a region R. Let $\vec{F} = M\hat{i} + N\hat{j}$ be a vector field with M and N having continuous first partials on the opne region R. Then the counterclockwise circulation of $\vec{F}$ around C equals the double integral of the k-component of curl over R.

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dx\,dy$$

Green's theorem flux-divergence or Normal form: The outward flux across C equals the double integral of div $\vec{F}$ over the region R

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \int\int_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\,dx\,dy$$

Example: Find the counterclockwise circulation and outward flux of $\vec{F} = \langle x^2+4y, x+y^2\rangle$ over $0 \le x \le 1, 0 \le y \le 1$

Solution: Counterclockwise circulation:

$$\int_0^1\int_0^1 (1-4)\,dx\,dy = -3\int_0^1\int_0^1\,dx\,dy = -3$$

Outward flux:

$$\int_0^1 \int_0^1 (2x+2y)\,dx\,dy = \int_0^1 \left[x^2+2xy\right]_)^1\,dy = \int_0^1 (1+2y)\,dy = \left[y+y^2\right] = 2$$

Green's theorem for Area:

$$A_R = \frac{1}{2}\oint x\,dy - y\,dx$$

Example: Find a formula for the area of an ellipse $\vec{r}(t) = a\cos t\hat{i} + b\sin t\hat{j}, 0 \leq t \leq 2\pi$

Solution:

$$\frac{1}{2}\int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)\,dt = \frac{1}{2}\int_)^{2\pi} ab\,dt = ab\pi$$

4.2.1 Parameterising Surfaces

Some common surfaces:

- Sphere $x^2+y^2+z^2=a^2$

$$\vec{r}(\phi,\theta) = a\sin\phi\cos\theta\hat{i} + a\sin\phi\sin\theta\hat{j} + a\cos\phi\hat{k}, \quad (0 \leq \phi \leq \pi, 0 \leqq \theta \leq 2\pi)$$

- Cylinder: $x^2+y^2=a^2, 0 \leq z \leq b$

$$\vec{r}(\theta,z) = a\cos\theta\hat{i} + a\sin\theta\hat{j} + z\hat{k}, \quad (0 \leq \theta \leq 2\pi, 0 \leq z \leq b)$$

- Cone: $z = \sqrt{x^2+y^2}, 0 \leq z \leq b$

$$\vec{r}(r,\theta) = r\cos\theta\hat{i} + r\sin\theta\hat{j} + r\hat{k}, \quad (0 \leq \theta \leq 2\pi, 0 \leq r \leq b)$$

Smooth surface: a parameterized surface if $\vec{r}_u$ and $\vec{r}_v$ are continuous and $\vec{r}_u \times \vec{r}_v$ is never zero on the interior of the parameter domain

Area of a smooth surface: The area of $\vec{r}(u,v) - f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}, \quad a \leq u \leq b, c \leq v \leq d)$ is

$$A = \int\int |\vec{r}_u \times \vec{r}_v| dA = \int_c^d \int_a^b |\vec{r}_u \times \vec{r}_v| l du\,dv$$

Surface area differential:

$$d\sigma = |\vec{r}_u \times \vec{r}_v| l du\, dv \implies \int\int_S d\sigma$$

Example: area of a surface formed by $y + 2z = 2$ inside $x^2 + y^2 = 9$

$$\vec{r}(u,v) = u\hat{i} + v\hat{j} + (1 - \frac{v}{2})\hat{k}, 0 \leq u^2 + v^2 \leq 9$$

$$\vec{r}_u = \hat{i}, \vec{r}_v = \hat{j} - \frac{1}{2}\hat{k}$$

$$||\vec{r}_u \times \vec{r}_v|| = \sqrt{1/4 + 1} = \sqrt{5}/2$$

$$A = \int\int_R \frac{\sqrt{5}}{2} dA = \frac{\sqrt{5}}{2}(9\pi) =$$

Formula for the surface area of an implicit surface:

$$A = \int\int_R \frac{||\nabla F||}{|\nabla F \cdot \vec{\hat{p}}}$$

where $hatp = \hat{i}, \hat{j},$ or $\hat{k}$ is normal to R and $\nabla F \cdot \hat{p} \neq 0$

Example: Find area of region cut by $x + 2y + 2z = 5$ and the cylinder whose sides are $x = y^2, x = 2 - y^2$

$$\nabla F(x,y,z) = \hat{i} + 2\hat{j} + 2\hat{k}$$
$$||\nabla F|| = 3$$
$$\nabla F \cdot \hat{k} = 2$$

$$\int\int_R \frac{3}{2} dA = \int_{-1}^{1} \int_{y^2}^{2-y^2} \frac{3}{2}\, dx\, dy = 4$$

Formula for the surface area of a graph: For a graph $z = f(x,y)$ over a region R,

$$A = \int\int_R \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$$

4.2.2 Surface integrals

$$d\sigma = |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

Surface integral:

$$\int\int_S G(x,y,z)d\sigma = \lim_{n\to\infty} \sum_{k=1}^{n} G(x_k, y_k, z_k)\Delta\sigma_k$$

Formulas for the surface integral of a scalar function:

1. For a smooth surface S defined $\vec{r}(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}, \quad (u,v) \in R$ and a continuous function $G(x,y,z)$ on S,

$$\int\int_S G(x,y,z) \, d\sigma = \int\int_R G(f(u,v), g(u,v), h(u,v))|\vec{r}_u \times \vec{r}_v| \, du \, dv$$

2. For a surface given implicitly ($F(x,y,z) = c$), where F is continuously differentiable and S lies above its closed and bounded shadow region R,

$$\int\int_S G(x,y,z) \, d\sigma = \int\int_R G(x,y,z)\frac{|\nabla F|}{|\nabla F \cdot \hat{p}|}dA$$

where $\hat{p}$ is a unit vector normal to R

3. For a surface given explicitly ($z = f(x,y)$) where f is continuously differentiable over R,

$$\int\int_S G(x,y,z)d\sigma = \int\int_R G(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy$$

Example: Evaluate $\int\int_S 2xyd\sigma$ over $x + 2y + 2zz = 4$ in the first octant

Solution:

$$f(x,y) = 2 - y - \frac{1}{2}x$$
$$f_x = -1/2$$
$$f_y = -1$$
$$\int\int_S 2xy\sqrt{1/4+1+1} \, dx \, dy = \int_0^2 \int_0^{4-2y} 3xy \, dx \, dy = 8$$

Flux:

$$\int\int_S \vec{F}\cdot\vec{n}\, d\sigma$$

where $\vec{F}$ is a 3d vector field with continuous components over S having a chosen field of normal unit vectors $\vec{n}$ orienting S.

If the surface integral is $g(x,y,z)=c$ then

$$\vec{n}=\pm\frac{\nabla g}{||\nabla g||}$$

If the surface is parametrised $\vec{r}(u,v)$, then the integral is

$$\int\int_S \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\, du\, dv$$

Example: find the flux for $\vec{F}=-x\hat{i}-y\hat{j}+z^2\hat{k}$ outward through the portion of the cone $z=\sqrt{x^2+y^2}, 1\leq z\leq 2$

Solution:

$$\begin{aligned}
\vec{r}(u,v) &= v\cos u\hat{i}+v\sin u\hat{j}+v\hat{k}\\
\vec{r}_u &= -v\sin u\hat{i}+v\cos u\hat{j}\\
\vec{r}_v &= \cos u\hat{i}+\sin u\hat{j}+\hat{k}\\
\vec{r}_u\times\vec{r}_v &= v\cos u\hat{i}+v\sin u\hat{j}-v\hat{k}\\
\int\int_S \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\, du\, dv &= \int\int(-v^2\cos^2 u-v^2\sin^2 u-v^3)\, du\, dv\\
&= \int_0^{2\pi}\int_0^1(-v^2-v^3)\, dv\, du=-\frac{73}{6}\pi
\end{aligned}$$

Mass and moment for thin shells:

- Mass: $M=\int\int_S \delta\, d\sigma$
- First moment about x: $M_{yz}=\int\int_S x\delta\, d\sigma$
- First moment about y: $M_{xz}=\int\int_S y\delta\, d\sigma$
- First moment about z: $M_{xy}=\int\int_S z\delta\, d\sigma$

- Coordinates of center of mass: $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$
- Moment of inertia about x: $\int\int_S (y^2+z^2)\delta \, d\sigma$
- Moment of inertia about y: $\int\int_S (x^2+z^2)\delta \, d\sigma$
- Moment of inertia about z: $\int\int_S (x^2+y^2)\delta \, d\sigma$
- Moment of inertia about a line L at distance r from point (x,y,z): $\int\int_S r^2 \delta \, d\sigma$

4.3 WEEK 13: Stokes' and Divergence theorems (Readings 16.7-16.8)

The del operator:

$$\nabla = \imath \frac{\partial}{\partial x} + \jmath \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

Divergence:

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

Curl:

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

Example: Find the div and curl for $\vec{F} = (x^2 - yx)\imath + ye^x \jmath + (xy+z)\hat{k}$

Solution:

$$\text{div} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(ye^x) + \frac{\partial}{\partial z} = 2x + e^x + 1$$

$$\text{curl } = \nabla \times \vec{F} = \begin{vmatrix} \imath & \jmath & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & ye^x xy + z & \end{vmatrix} = x\imath - 2y\jmath + (ye^x + z)\hat{k}$$

Important Identity:

$$\text{curl grad } \mathbf{f} = \tilde{\mathbf{0}}, \quad \nabla \times \nabla \mathbf{f} = \tilde{\mathbf{0}}$$

Stokes' theorem: Let S be a piecewise smooth oriented surface with a piecewise smooth boundary curve C. If $\vec{F} = M\imath + N\jmath + P\hat{k}$ is a differentiable vector field, then

$$\oint \vec{F} \cdot d\vec{r} = \int\int_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma)$$

Example: calculate the circulation of $\vec{F} = y\imath + xz\jmath + x^2\hat{k}$ where C is the triangle cut by $x + y + z = 1$ in the first octant, counterclockwise when viewed from above

$$
\begin{aligned}
\nabla \times \vec{F} - -x\imath - 2x\jmath + (z-1)\hat{k} \\
\vec{n} = \frac{\imath + \jmath + \hat{k}}{\sqrt{3}} \\
(\nabla \times \vec{F}) \cdot \vec{n} = -\frac{3x}{\sqrt{3}} + \frac{z-1}{\sqrt{3}} \\
d\sigma = \sqrt{(-1)^2_(- 1)^2 + 1} \qquad = \sqrt{3}\, dx\, dy \\
\Longrightarrow \int\int (-3x + z - 1)\, dx\, dy = \int_0^1 \int_0^1 (-4x - y)\, dy\, dx = -\frac{5}{6}
\end{aligned}
$$