Georgia Tech Math 2551

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1 Module 1: Three Dimensional Space, Vectors, Lines, Planes

1.1 WEEK 1: Geometry of space and vectors (Readings 12.1-12.4)

The three dimensional coordinate system: $\mathbb{R}^3$, a coordinate system with x, y, and z axes where coordinates are represented by an ordered tuple of three numbers. We use a right-hand system such that z is vertical.

Octants: the three-dimensional analogue of quadrants

First octant: the octant where all three coordinates are positive

The three planes:

- xy-plane: $z = 0$ ("the floor")
- xz-plane: $y = 0$ ("a wall")
- yz-plane: $x = 0$

Distance in three dimensions:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example: "Find the distance from the point (2, -3, 1) to the plane $y = 2$."

Solution: The plane $y = 2$ spans all values of x and z so the only coordinate changing is y itself. $d = 2 - (-3) = 5$■

Sphere: a three-dimensional object where every point on its surface is equidistant from its centre. The centre has coordinates x_0, y_0, z_0 and the distance from the centre to each point on the surface is the radius.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

Example: "Identify the centre C and radius a for the sphere given by $2x^2 + 2y^2 + 2x^2 - 8x + 12y - 20z = 22$

Solution:

$$\begin{aligned}
2x^2 + 2y^2 + 2z^2 - 8x + 12y - 20z &= 22 \\
x^2 + y^2 + z^2 - 4x + 6y - 10z &= 11 \\
x^2 - 4x + y^2 + 6y + z^2 - 10z &= 11 \\
(x-2)^2 + (y+3)^2 + (z-5)^2 &= 11 + 4 + 9 + 25 \\
(x-2)^2 + (y+3)^2 + (z-5)^2 &= 49
\end{aligned}$$

$$C = (2, -3, 5) \quad a = 7$$

Example: "Give a geometric description of the sets defined by $y^2 + z^2 = 4, x = 2$"

Solution: A circle of radius 2 on the plane $x = 2$

Vector: an object with a direction and a length, represented by a directed line segment

If point P has coordinates (x_1, y_1, z_1) and point Q has coordinates (x_2, y_2, z_2), then

$$\vec{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Magnitude of a vector: the length or norm of a vector, found using the distance formula. Written $||\vec{v}||$

Vectors can be represented by arrow notation ($\vec{PQ}$), component notation ($\langle x, y, z \rangle$), or bold typeface notation ($\mathbf{v}$)

Example: "Find the component form and length of the vector whose intial point is $R(-1, 3, 0)$ and terminal point is $S(-3, -2, 4)$"

Solution:

$$\begin{aligned}
\vec{RS} &= \langle -2, -5, 4 \rangle \\
||\vec{RS}|| &= \sqrt{45} = 3\sqrt{5}
\end{aligned}$$

Vector Addition: Given $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$,

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

Vector multiplication by a scalar: Given $\vec{u} = \langle u_1, u_2, u_3 \rangle$,

$$k\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$$

All the normal properties (commutivity, associativity, additive identity, additive inverse, multiplicative identity, zero multiplication, and distributivity) hold for vectors.

Geometric representation of vector addition: For a vector $\vec{u}$,

- $-\vec{u}$ is a vector of the same magnitude in the opposite direction
- $k\vec{u}$ is a vector in the same direction but k times the length

Addition of vectors $\vec{v}$ and $\vec{w}$ corresponds to the diagonal of their composite parallelogram starting at the common point. For subtraction, the diagonal begins at the tip of the vector being subtracted.

Unit vector: a vector whose length is 1. For a non-zero vector, $\hat{v} = \frac{\vec{v}}{||v||}$

The standard unit vectors:

- $\hat{i} = \langle 1, 0, 0 \rangle$
- $\hat{j} = \langle 0, 1, 0 \rangle$
- $\hat{k} = \langle 0, 0, 1 \rangle$

Any vector can be written as a linear combination of the standard unit vectors

Example: "Find a unit vector i the xy=plane that makes an angle $\theta = -\frac{\pi}{3}$ with the positive x-axis"

Solution: From the unit circle

$$\hat{r} = \frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}$$

The Dot Product: a scalar value

$$\vec{u} \cdot \vec{v} = u_1 v_1 + ... + u_n v_n = ||\vec{u}|| \, ||\vec{v}|| \cos\theta$$

This can give us the angle between two vectors

$$\theta = \cos^{-1}(\frac{\vec{u} \cdot \vec{c}}{||\vec{u}|| \, ||\vec{v}||})$$

Vectors are orthogonal if $\vec{v} \cdot \vec{u} = 0$

Vector projection:

$$\text{proj}_b\vec{a} = (\frac{\vec{a} \cdot \vec{b}}{||\vec{b}||})\frac{\vec{b}}{||\vec{b}||}$$

Scalar component of a in the direction of b: The value

$$\text{comp}_b\vec{a} = \frac{\vec{a} \cdot \vec{b}}{||\vec{b}||} = ||\vec{a}|| \cos\theta$$

Work: $W = F \cdot D$ for a force F acting through a distance D

Cross product: the vector

$$\vec{a} \times \vec{b} = ||\vec{a}|| \, ||\vec{b}|| \sin\theta\hat{n}$$

Where θ is the angle between the vectors and $\hat{n}$ is the unit vector perpendicular to the plane containing vectors $\vec{a}$ and $\vec{b}$

The product $||\vec{a}|| \, ||\vec{b}|| \sin\theta$ also corresponds to the area of the parallelogram formed by the vectors

Parallel vectors: two vectors are parallel iff $\vec{a} \times \vec{b} = \vec{0}$

Properties of the cross product:

- $(r\vec{u}) \times (r\vec{v}) = (rs)(\vec{u} \times \vec{v})$
- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{0} \times \vec{u} = \vec{0}$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$
- $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

Calculating the Cross Product as a determinant: For $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ and $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$, then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example: "Find the area of the parallelogram with vertices A(2, 1, 4), B(1, 4, 3), C(1, 0, 2), D(2, -3, 3)"

Solution:

$$\begin{aligned} \vec{AB} &= \langle -1, 3, -1 \rangle \\ \vec{BC} &= \langle 0, -4, -1 \rangle \\ \vec{CD} &= \langle 1, -3, 1 \\ \vec{AD} &= \langle 0, -4, -1 \end{aligned}$$

Because $\vec{AB}$ and $\vec{CD}$ are parallel but in opposite directions, we can know to calculate the cross of any two adjacent sides to find the area.

$$A = ||\vec{AB} \times \vec{AD}||$$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & -1 \\ 0 & -4 & -1 \end{vmatrix} = -7\hat{i} + \hat{j} + 4\hat{k}$$

$$A = || -7\hat{i} - \hat{j} + 4\hat{k}|| = \sqrt{66}\blacksquare$$

The absolute value of the triple scalar product $((\vec{u} \times \vec{v}) \cdot \vec{w})$ is the volume of the parallelepiped determined by the three vectors.

Helpfully,

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

1.2 WEEK 2: Curves, Tangents, Motion (Readings 12.5-13.2)

A vector equation for the line L through the point $P_0(x_0, y_0, z_0)$ parallel to the vector $\vec{v}$ is given by

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \quad -\infty < t < \infty$$

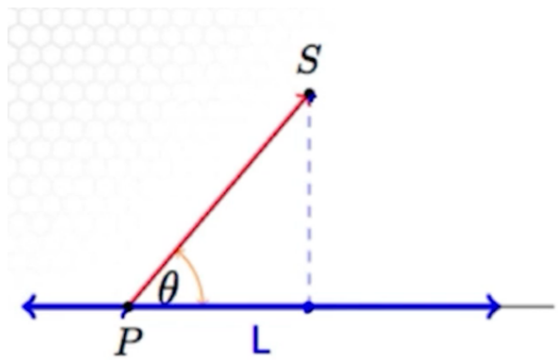
where $\vec{r}$ is the position vector of a point $P(x, y, z)$ on L and $\vec{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

The *standard parameterisation* of the line L through the point $P_0(x_0, y_0, z_0)$ parallel to $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is given by

$$\begin{aligned} x(t) &= x_0 + tv_1 \\ y(t) &= y_0 + tv_2 \\ z(t) &= z_0 + tv_3 \end{aligned}$$

for $-\infty < t < \infty$.

Distance from a point to a line in space:



$$d = \frac{||\vec{PS} \times \vec{v}||}{||\vec{v}||}$$

Example: "Find the distance from S(2,0,2) to the line through P(3,-1,1) parallel to the vector $\vec{v} = \hat{i} - 2\hat{j} - 2\hat{k}$"

Solution:

$$\begin{aligned} d &= \frac{||\vec{PS} \times \vec{v}||}{||\vec{v}||} \\ \vec{PS} &= \langle -1, 1, 1 \rangle \\ \vec{PS} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -2 & -2 \end{vmatrix} = -\hat{j} + \hat{k} \\ d &= \frac{\sqrt{1+1}}{\sqrt{1+4+4}} = \frac{\sqrt{2}}{3} \quad \blacksquare \end{aligned}$$

Equations for a plane: The plane through the point $P_0(x_0, y_0, z_0)$ normal to $\vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$ is given by the *vector equation*

$$\vec{n} \cdot \vec{P_0P} = 0$$

and the *component equation*

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

In other words, the direction of the plane is defined by the vector normal to it.

Example: "Find an equation for the plane which passes through P(1,3,4) and contains the line $l : x(t) = 3t,\ y(t) = 4t,\ z(t) = 2 + 2t$

Solution: If the plane contains a point and a line distinct from each other, then the cross product of the vector for that point and the equation for the line will be normal to both of them and thus the plane.

$$\ell \to \ell(0, 0, 2)$$

$$\vec{QP} = \langle 1, 3, 2 \rangle$$
$$\vec{d} = \langle 3, 4, 2 \rangle$$

$$\vec{n} = \vec{QP} \times \vec{d} = -2\hat{i} + 4\hat{j} - 5\hat{k}$$
$$-2(x - 1) + 4(y - 3) - 5(z - 4) = 0$$

$$2x - 4y - 5z = 10$$

Angle between two planes: the acute angle between their normal vectors

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{||\vec{n}_1||\ ||\vec{n}_2||}$$

Distance from a point to a plane:

$$d = \left| \vec{PS} \cdot \tfrac{\vec{n}}{||\vec{n}||} \right|$$

Example: "Determine whether the lines l_1 and l_2 are parallel, coincident, skew, or intersecting."

$$l_1 : x_1(t) = 1 + t,\ y_1(t) = -1 - t,\ z_1(t) = -4 + 2t$$

$$l_2 : x_2(s) = 1 - s,\ y_2(s) = 1 + 3s,\ z_2(s) = 2s$$

Solution:

$$\vec{v_1} = \langle 1, -1, 2 \rangle$$
$$\vec{v_2} = \langle -1, 3, 2 \rangle$$

Therefore the lines are not parallel or coincident.

$$\begin{cases} 1 + t = 1 - s \\ -1 - t = 1 + 3s \\ -4 + 2t = 2s \end{cases}$$

Because we have only two variables, only the first equations are needed to solve. If when checking, however, the third is true, the lines are intersecting. Else, they are skew. We add the first two equations:

$$0 = 2 + 2s \implies s = -1$$

Plugging in s:

$$1 + t = 1 - (-1) \implies t = 1$$

We check with the third equation:

$$-4 + 2(1) = 2(-1)$$

which is true so the lines are intersecting. To find the point of intersection, we plug either variable in to the original lines:

$$l_1(1) = (2, -2, -2) \quad \blacksquare$$

Intersecting planes: Two planes are parallel if their normal vectors are parallel. Otherwise, they intersect at a line. The direction vector for that line of intersection is the cross product of the normal vectors from the two planes.

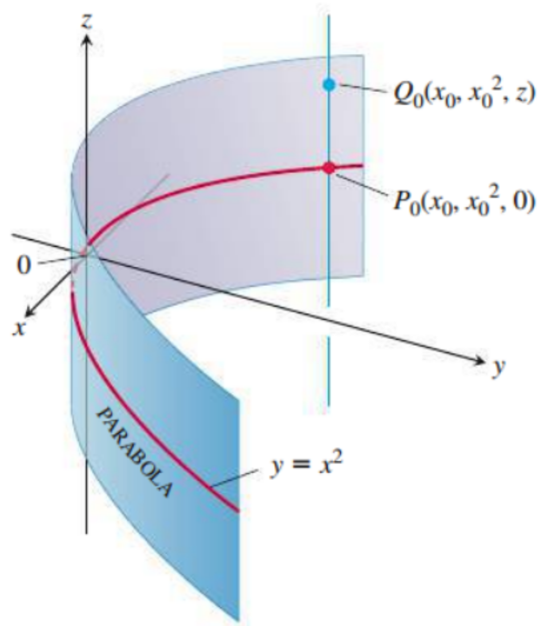
1.2.1 Cylinders and Quadric Surfaces

Cylinder: a surface that is generated by moving a straight line along a given planar curve (*the generating curve*) while holding the line parallel to a given fixed line.

Unlike in solid geometry, the generating curves are not limited to circles.

Example: "Find an equation for the cylinder made by the lines parallel to the z-axis that pass through the parabola $y = x^2,\ z = 0$"

Solution: $P_0(x_0, x_0^2, 0)$ lies on the parabola $y = x^2$ in the xy-plane. Then, $\forall z, Q(x_0, x_0^2, z)$ is on the cylinder because it will lie on the line $y = x^2$ through P_0 parallel to the z-axis.



Quadric surfaces: the graph in space of a second-degree equation in x, y, z. The most simple form is

$$Ax^2 + By^2 + Cz^2 + Dz = E$$

for A, B, C, D, and E are constants.

The basic quadric surfaces are ellipsoids, paraboloids, elliptical cones, and hyperboloids.

1.2.2 The Ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

cuts the coordinate axes at $(\pm a, 0, 0),\ (0, \pm b, 0),\ (0, 0, \pm c)$ and lies within the rectangular box defined by $|x| \leq a,\ |y| \leq b,\ |z| \leq c$ and the surface is symmetric.

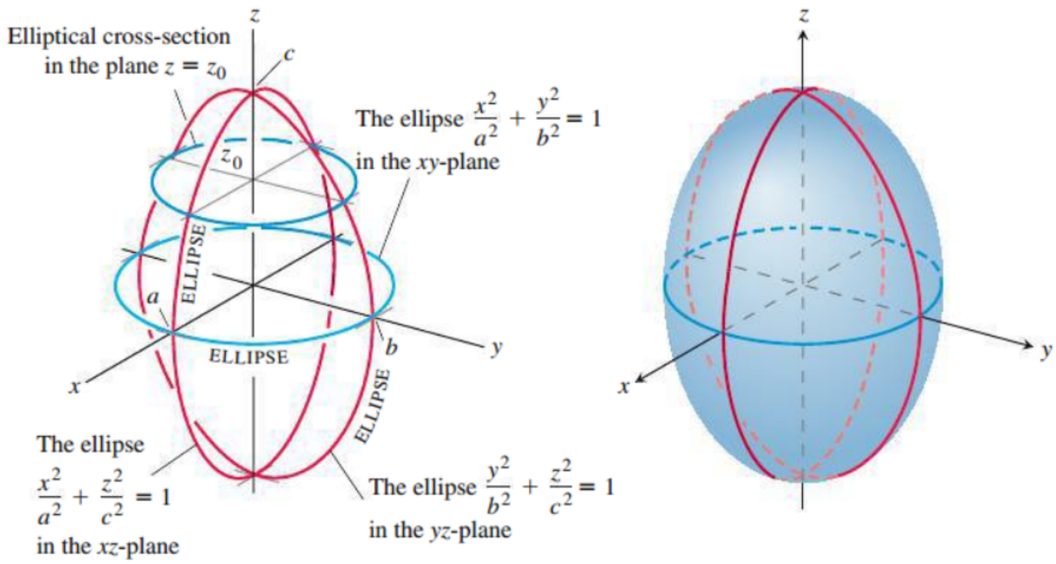


FIGURE 12.46 The ellipsoid

The cross sections for each of the three coordinate planes are ellipses.

If any two of the semiaxes a, b, c are equal, the surface is an ellipsoid of revolution. If all three are equal, the surface is a sphere.

1.2.3 The Hyperbolic Paraboloid:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0$$

which has symmetry with respect to $x = 0$ and $y = 0$. The cross sections in these planes are

$$x = 0: \quad \text{the parabola } z = \tfrac{c}{b^2}y^2$$
$$y = 0: \quad \text{the parabola } z = -\tfrac{c}{a^2}x^2$$

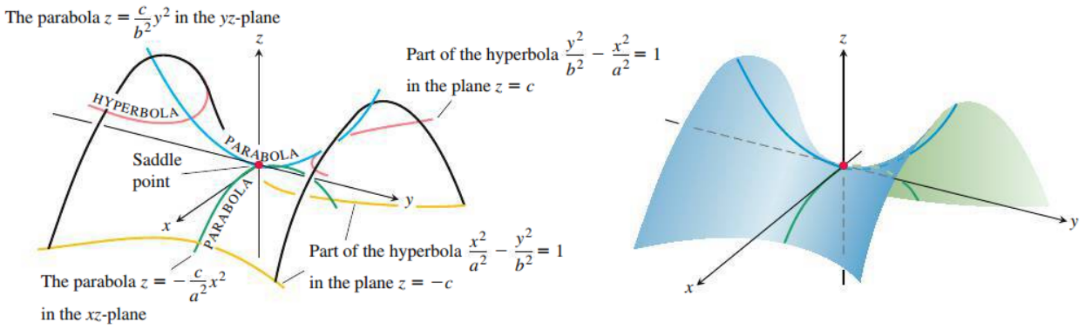


FIGURE 12.47 The hyperbolic paraboloid $(y^2/b^2) - (x^2/a^2) = z/c, c > 0$. The cross-sections in planes perpendicular to the z-axis above and below the xy-plane are hyperbolas. The cross-sections in planes perpendicular to the other axes are parabolas.

If we cut the surface by a plane $z = z_0 > 0$, the cross section is a hyperbola with its focal axis parallel to the y-axis and its vertices on the parabola for $x = 0$ above. If z_0 is negative, the focal axis is parallel to the x-axis and the vertices lie on the parabola for $y = 0$ above.

Near the origin, the surface is shaped like a saddle or mountain pass. Travelling along the surface in the yz-plane, the origin looks like a minimum. Travelling along the xz-plane, the origin looks like a maximum. This is a *saddle point*

1.2.4 General Quadric Surfaces

The general equation for a quadric surface in three variables is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gz + Hy + Iz + J = 0$$

Terms of the type Gx, Hy, or Iz lead to translations.

Example 4: "Identify the surface given by the equation $x^2+y^2+4z^2-2x+4y+1 = 0$"

Solution: Complete the square

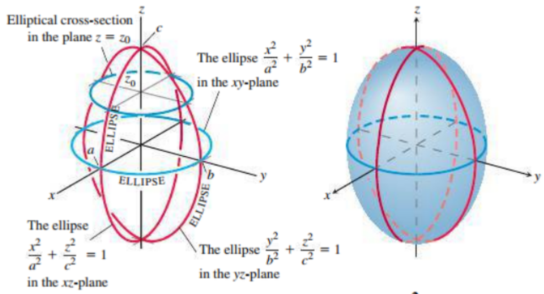
$$x^2 + y^2 + 4z^2 - 2x + 4y + 1 = (x-1)^2 + (y+2)^2 + 4z^2 - 4$$

We can rewrite the original equation as

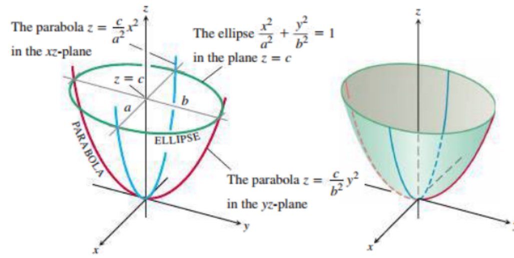
$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{4} + \frac{z^2}{1} = 1$$

This is the equation of an ellipsoid whose three semiaxes have lengths 2, 2, and 1 which is centered at the point (1, -2, 0).

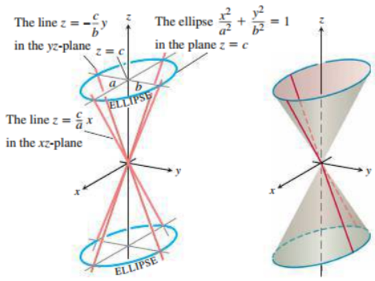
1.2.5 Graphs of Quadric Surfaces:



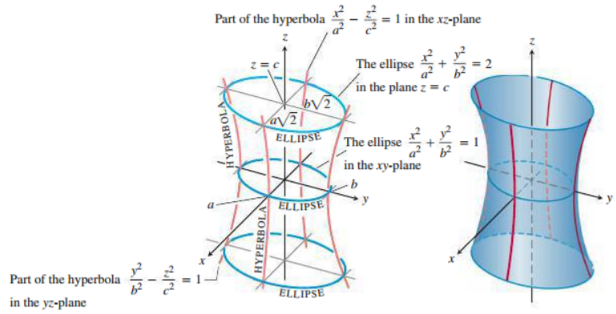
ELLIPSOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



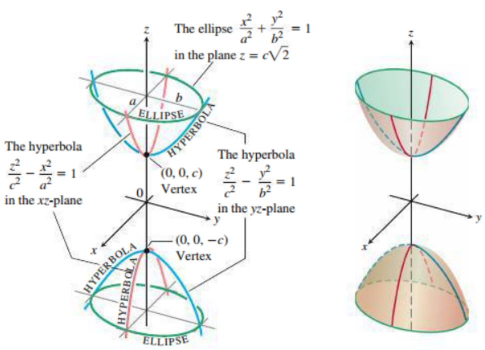
ELLIPTICAL PARABOLOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



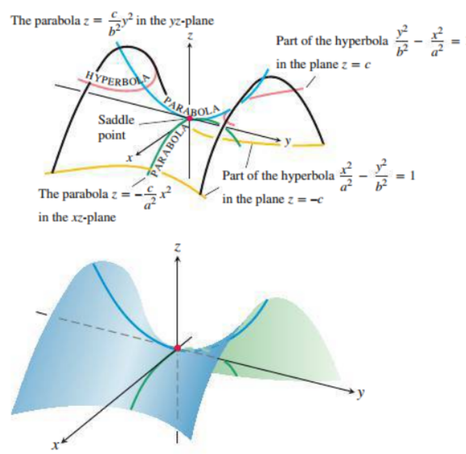
ELLIPTICAL CONE $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\quad \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

1.3 Curves in space

We can describe the path of a particle's motion through space by defining its coordinates as functions on I:

$$x = f(t),\ y = g(t),\ z = h(t)$$

In vector form:

$$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

In general, f, g, and h are *component functions* of the position vector.

Vector function: a function on a domain set D that assigns a vector in space to each element in D.

When the domain is an interval of real numbers, the graph represents a curve in space. When domains are regions in the plane, the graph will be a surface in space.

Scalar functions: real valued functions such as the components of a vector function

The domain of a vector-valued function is the common domain of its components.

1.3.1 Limits of vector functions

Let $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$ be a vector function with domain D, and let $\vec{L}$ be a vector. We say that $\vec{r}$ has limit $\vec{L}$ as $t \to t_0$:

$$\lim_{t\to t_0} \vec{r}(t) = \vec{L}$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$|\vec{r}(t) - \vec{L}| < \varepsilon \quad \text{when} \quad 0 < |t - t_0| < \delta$$

Further, if $\vec{L} = L_1\hat{\imath} + L_2\hat{\jmath} + L_3\hat{k}$, then $\lim_{t\to t_0} \vec{r}(t) = \vec{L}$ when

$$\lim_{t\to t_0} f(t) = L_1,\ \lim_{t\to t_0} g(t) = L_2,\ \lim_{t\to t_0} h(t) = L_3$$

Definition: A vector function $\vec{r}(t)$ is continuous at a point $t = t_0$ in its domain if

$$\lim_{t\to t_0} \vec{r}(t_0)$$

The function is *continuous* if it is continuous at every point in its domain.

1.3.2 Derivatives and motion:

The vector function $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ is differentiable at t if f, g, and h have derivatives at t.

$$\vec{r}'(t) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{df}{dt}\hat{i} + \frac{dg}{dt}\hat{j} + \frac{dh}{dt}\hat{k}$$

Differentiable: a property of a function if it has a derivative at every point of its domain

Smooth: a property of the curve traced by $\vec{r}$ if $\frac{d\vec{r}}{dt}$ is continuous and never $\vec{0}$ On a smooth curve, there are no sharp corners or cusps.

Tangent line: the line through a point $(f(t_0), g(t_0), h(t_0))$ parallel to $\vec{r}'(t)$.

Piecewise smooth: a curve that is made up of a finite number of smooth curves pieced together in a continuous fashion

If $\vec{r}$ is the position vector of a particle moving along a smooth curve,

1. Velocity is the derivative of position: $\vec{v} = \frac{d\vec{r}}{dt}$
2. Speed is the magnitude of velocity: Speed $= |\vec{v}|$
3. Acceleration is the derivative of velocity: $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$
4. The unit vector $\vec{v}/|\vec{v}|$ is the direction of motion at time t

1.3.3 Differentiation rules:

- $\frac{d}{dt}\vec{C} = \vec{0}$
- $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$
- $\frac{d}{dt}[f(t)u\vec{(t)}] = f'(t)u\vec{(t)} + f(t)\vec{u}'(t)$
- $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$
- $\frac{d}{dt}[\vec{u}(t) - \vec{v}(t)] = \vec{u}'(t) - \vec{v}'(t)$
- $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
- $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$
- $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\vec{u}'(f(t))$

If $\vec{r}$ is a differentiable vector function of t and the length of $\vec{r}(t)$ is constant, then

$$\vec{r} \cdot \frac{d\vec{r}}{dt} = 0$$

1.3.4 Integrals of vector functions:

Indefinite integral: the set of all anti-derivatives of $\vec{r}$, denoted by $\int \vec{r}(t)dt$. If $\vec{R}$ is any antiderivative of $\vec{r}$, then

$$\int \vec{r}(t)dt = \vec{R}(t) + \vec{C}$$

If the components of $\vec{t} = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ are integrable over [a, b] then so is $\vec{r}$, forming the definite integral of $\vec{r}$ from a to b:

$$\int_a^b \vec{r}(t)dt = \left(\int_a^b f(t)dt\right)\hat{i} + \left(\int_a^b g(t)dt\right)\hat{j} + \left(\int_a^b h(t)dt\right)\hat{k}$$

The Fundamental Theorem of Calculus:

$$\int_a^b \vec{r}(t)dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

An antiderivative of a vector function is also a vector function but a definite integral of a vector function is a single constant vector.

Ideal projectile motion equation:

$$\vec{r} = (v_0 \cos\alpha)t\hat{i} + \left((v_0 \sin\alpha)t - \frac{1}{2}gt^2\right)\hat{j}$$

where α is the launch angle and $v_0 = |\vec{v}|$, the initial speed.

Formulas:

- $y_{\max} = \frac{(v_0 \sin\alpha)^2}{2g}$
- $t = \frac{2v_0 \sin\alpha}{g}$
- $R = \frac{v_0^2}{g} \sin 2\alpha$

1.4 WEEK 3: Arclength, curvature, acceleration (Readings 13.3-13.6)

The lengths of a smooth curve $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k},\ a \le t \le b$ that is traced exactly once as t increases from $t = a$ to $t = b$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt$$

If we choose a base point $P(t_0)$ on a smooth curve C parameterised by t, each value of t determines a point $P(t) = (x(t), y(t), z(t))$ on C and a *directed distance*

$$s(t) = \int_{t_0}^t |\vec{v}(\tau)| d\tau$$

and

$$\frac{ds}{dt} = |\vec{v}(t)|$$

Each value of s determines a point on C, parameterising C with respect to s, making s an *arc length parameter* for the curve. The parameter's value increases in the direction of increasing t.

Arc length parameter with Base Point $P(t_0)$:

$$s(t) = \int_{t_0}^t \sqrt{[x'(\tau)^2] + [y'(\tau)^2] + [z'(\tau)^2]} d\tau = \int_{t_0}^t |\vec{v}(\tau)| d\tau$$

If a given curve $\vec{r}(t)$ is already given in terms of some parameter and the arclength is $s(t)$ then we can solve for t as a function of $s : t = t(s)$. Then the curve can be parameterised in terms of s by substituting for $t : \vec{r} = \vec{r}(t(s))$. This identifies a point on the curve with its directed distance along the curve from the base point.

The speed with which a particle moves along its path is the magnitude of $\vec{v}$:

$$\frac{ds}{dt} = |\vec{v}(t)|$$

Because the velocity vector $\vec{v} = \frac{d\vec{r}}{dt}$ is tangent to $\vec{r}(t)$ so the vector

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

is a unit vector tangent to the curve. $\vec{T}$ is a differentiable function of t whenever $\vec{v}$ is a differentiable function of t.

1.4.1 Curvature of a curve

Curvature:

$$k = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

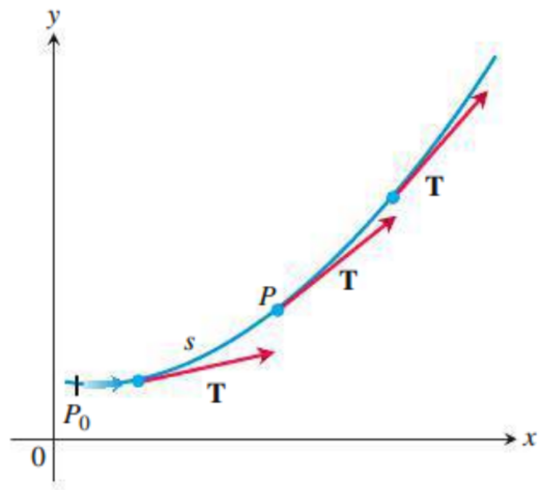
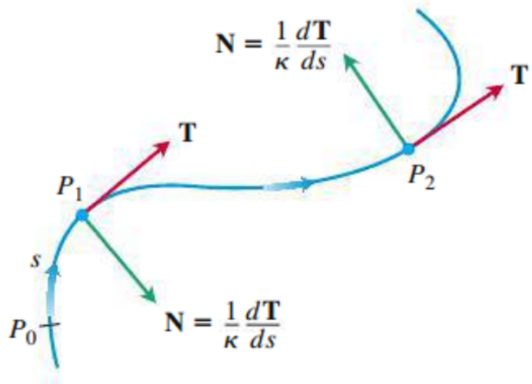


FIGURE 13.17 As *P* moves along the curve in the direction of increasing arc length, the unit tangent vector turns. The value of $|d\mathbf{T}/ds|$ at *P* is called the *curvature* of the curve at *P*.

If $|d\vec{T}/ds|$ is large, $\vec{T}$ turns sharply as the particle passes through P and the curvature is large. If the length is small, $\vec{T}$ turns more slowly

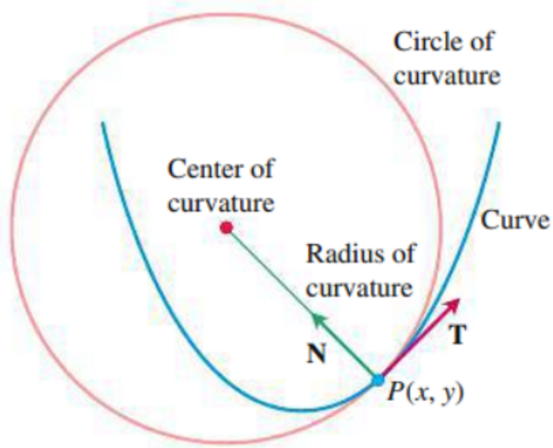
Principal unit normal vector: a unit vector orthogonal to $\vec{T}$ which points in the direction the curve is turning (towards the concave side of the curve)

$$\vec{N} = \frac{1}{k}\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$



Osculating circle: the circle of curvature at a point P on a plane curve that

1. is tangent to the curve at P (has the same tangent line as the curve)
2. has the same curvature at P
3. has center that lies toward the concave (inner) side of the curve



Radius of curvature: the radius of the circle of curvature $\rho = \frac{1}{k}$

Center of curvature: the center of the circle of curvature

1.4.2 Acceleration in space

Travelling along a curve, the **IJK** reference frame is generally less importnat than the vector of forward motion ($\vec{T}$), the direction the path is turning ($\vec{N}$), and the tendency of the to "twist" out of the plane of those vectors (*the unit binormal vector* $\vec{B} = \vec{T} \times \vec{N}$)

These vectors together form a moving right-handed vector frame called the *Frenet Frame* or the TNB frame.

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \vec{T}\frac{ds}{dt}$$
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2s}{dt^2}\vec{T} + k\left(\frac{ds}{dt}\right)^2\vec{N}$$

Also,

$$|v \times a| = k\left|\frac{ds}{dt}\right|^3 |\vec{B}| = k|\vec{v}|^3$$

so

$$k = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

If the acceleration vector is written as $\vec{a} = a_T\vec{T} + a_N\vec{N}$, then the tangential and normal scalar components of acceleration are

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\vec{v}|$$
$$a_N = k\left(\frac{ds}{dt}\right)^2 = k|\vec{v}|^2 = \sqrt{|\vec{a}|^2 - a_T^2}$$

Further, acceleration will always line in the plane of $\vec{T}$ and $\vec{N}$. The tangential component a_T measures the rate of change of the length of $\vec{v}$ (the change in the speed). The normal component a_N measures the rate of change of the direction of $\vec{v}$

1.4.3 Torsion

Torsion: the rate at which the osculating plane turns about $\vec{T}$ as P moves along the curve. In other words, torsion measures how the curve twists for $\vec{B} = \vec{T} \times \vec{N}$, torsion is

$$\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}$$

A space curve is a helix if and only if it has constant nonzero curvature and constant nonzero torsion.

Torsion can also be calculated directly in terms of the parameters

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} \quad (\text{if } \vec{v} \times \vec{a} \neq \vec{0})$$

1.4.4 Motion in polar coordinates

When a particle moves along a curve in the polar plane, we express its position, velocity, and acceleration in terms of the moving unit vectors

$$\vec{u}_r = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$$
$$\vec{u}_\theta = -(\sin\theta)\hat{i} + (\cos\theta)\hat{j}$$

Then,

$$\text{Position:} \quad \vec{r} = r\vec{u}_r + z\hat{k}$$
$$\text{Velocity:} \quad \vec{v} = \dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta + \dot{z}\hat{k}$$
$$\text{Acceleration:} \quad \vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{u}_\theta + \ddot{z}\hat{k}$$

This forms a right handed frame where

$$\vec{u}_r \times \vec{u}_\theta = \hat{k}$$
$$\vec{u}_\theta \times \hat{k} = \vec{u}_r$$
$$\hat{k} \times \vec{u}_r = \vec{u}_\theta$$

Newton's law of universal gravitation:

$$\vec{F} = -\frac{GmM}{|\vec{r}|^2}\frac{\vec{r}}{|\vec{r}|}$$

where $\vec{r}$ is the radius vector from the centre of a sun of mass M to the centre of a planet of mass m.

Kepler's Laws:

1. A planet's path is an ellupse with the sun at one focus. The eccentricity is

$$e = \frac{r_0 v_0^2}{GM} - 1$$

and the polar equation is

$$r = \frac{(1+e)r_0}{1 + e\cos\theta}$$

2. The radius vector from the sun to a planet sweeps out equal areas in equal times. If the initial line is $\theta = 0$, the direction $\vec{r}$ when $|\vec{r}| = r$ is a minimum value. Then,

$$\dot{r}\Big|_{t=0} = \frac{dr}{dt}\Big|_{t=0} = 0, \quad v_0 = |\vec{v}|_{t=0} = [r\dot{\theta}]_{t=0}$$

3. The orbital period and the orbit's semimajor axis a are related by

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$$

2 Module 2: Partial Derivatives and Applications

2.1 WEEK 4: Calculus of multivariable functions (Readings 14.1-14.4)

2.2 WEEK 5: Gradients, tangent planes, and extreme values (Readings 14.5-14.7)

2.3 WEEK 6: Lagrange multipliers and Taylor's formula (Readings 14.8-14.10)

3 Module 3: Integrals

3.1 WEEK 7: Double integrals and area (Readings 15.1-15.3)

3.2 WEEK 8: Double and tripe integrals (Readings 15.4-15.6)

3.3 WEEK 9: Triple integrals and applications (Readings 15.7-15.8)

4 Module 4: Integrals and Vector Fields

4.1 WEEK 10: Line integrals and vector fields (Readings 16.1-16.3)

4.2 WEEK 12: Green's theorem and surfaces (Readings 16.4-16.6)

4.3 WEEK 13: Stokes' and Divergence theorems (Readings 16.7-16.8)