Numerical Functional Analysis and Optimization, 36:748–776, 2015

ISSN: 0163-0563 print/1532-2467 online
DOI: 10.1080/01630563.2015.1031379

A GLOBALLY CONVERGENT ALGORITHM FOR A PDE-CONSTRAINED OPTIMIZATION PROBLEM ARISING IN ELECTRICAL IMPEDANCE TOMOGRAPHY

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□ *We study the convergence properties of an algorithm for the inverse problem of electrical impedance tomography, which can be reduced to a partial differential equation (PDE) constrained optimization problem. The direct problem consists of the potential equation* $\mathrm{div}(\varepsilon\nabla u) = 0$ *in a circle, with Neumann condition describing the behavior of the electrostatic potential in a medium with conductivity given by the function* $\varepsilon(x, y)$. *We suppose that at each time a current* ψ_i *is applied to the boundary of the circle (Neumann's data), and that it is possible to measure the corresponding potential* φ_i *(Dirichlet data). The inverse problem is to find* $\varepsilon(x, y)$ *given a finite number of Cauchy pairs measurements* (φ_i, ψ_i), $i = 1, \ldots, N$. *The problem is formulated as a least squares problem, and the developed algorithm solves the continuous problem using descent iterations in its corresponding finite element approximations. Wolfe's conditions are used to ensure the global convergence of the optimization algorithm for the continuous problem. Although exact data are assumed, measurement errors in data and regularization methods shall be considered in a future work.*

Keywords Elliptic equations; Finite element discretization; Global convergence algorithm; Inverse problem; Least squares method; Wolfe's conditions.

2010 Mathematics Subject Classification 65N21; 90C53; 65N30.

1. INTRODUCTION

Electrical impedance tomography (EIT) is the recovery of the conductivity at the interior of a conducting body from a knowledge of currents and voltages applied to its surface. This problem is a well known

Received 17 January 2014; Revised 15 March 2015; Accepted 15 March 2015.
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inverse problem, to which an increasing amount of literature about its theoretical and computational aspects has been developed (see [3], [22]).

Knowledge of the ε conductivity allows pictures of the inside of the body under study, and leads to many applications in various fields. For example, in geophysics the method is used in mineral prospecting and in archaeology for locating buried archaeological remains. In industrial processing, tomography is used in nondestructive testing ([8], [27], [10], [31]). In medical imaging, it is still a experimental technique rather than a common clinical practice [30].

Recently, another method has been developed known as "Current Density Impedance Imaging" (CDII) which, unlike the EIT, reconstructs the conductivity ε using knowledge of the magnitude of the current density field $a = |\varepsilon \nabla u_\varepsilon|$ inside the object. This allows higher resolution images to be obtained away from the boundary $\partial\Omega$ ([22], [23]). In our work, we follow the classical approach, where data are taken from boundary measures and no information exists about inside the body.

The mathematical formulation of the inverse problem is the following:

Given $\Omega_0 = B(0, R)$, the ball centered at the origin and with radius $R > 0$, N pairs of functions (φ_i, ψ_i), representing applied currents ψ_i and the corresponding measured potentials φ_i on the boundary $\Gamma_0 = \partial\Omega_0$, with $\varphi_i \in H^{\frac{1}{2}}(\Gamma_0)$, $\psi_i \in H^{-\frac{1}{2}}(\Gamma_0)$, find the function $\varepsilon(x, y)$ such that the potential functions $u_i(x, y)$ satisfy the following Neumann elliptic problems:

$$\begin{cases} \operatorname{div}(\varepsilon \nabla u_i) = 0, & \text{on } \Omega_0 \\ \varepsilon \dfrac{\partial u_i}{\partial n}_{|\Gamma_0} = \psi_i \end{cases} \tag{1}$$

where $i = 1, \ldots, N$.

We assume that $\varepsilon \in L^\infty(\Omega_0)$ satisfies:

$$\varepsilon(x, y) \geq K > 0, \quad a.e \text{ on } \Omega_0, \tag{2}$$

and $\psi_i \in H^{-\frac{1}{2}}(\Gamma_0)$ satisfies the compatibility condition:

$$\int_{\Gamma_0} \psi_i \, dS = 0. \tag{3}$$

It is known that the direct Neumann problem (1)–(2)–(3) has a unique solution except for an additive constant, which we fix requiring that:

$$\int_{\Gamma_0} u_i dS = \int_{\Gamma_0} \varphi_i dS. \tag{4}$$

The variational formulation of the direct problem is the following:

$$\begin{aligned}&\text{Find } u_i \in H^1(\Omega) \text{ satisfying (4), such that:}\\ &a(u_i, v) = F_i(v), \quad \forall v \in H^1(\Omega),\end{aligned} \tag{5}$$

where

$$\begin{aligned} a(u,v) &= \int_{\Omega_0} \varepsilon \nabla u \nabla v d\overline{x},\\ F_i(v) &= \int_{\Gamma_0} \psi_i v dS,\end{aligned} \tag{6}$$

and $\psi_i \in H^{-\frac{1}{2}}(\Gamma_0)$ satisfies (3). This problem has a unique solution $u_i \in H^1(\Omega)$, $1 \leq i \leq N$.

Remark 1. Indeed, the problem is formulated in the quotient space $V = H^1(\Omega)/\mathbb{R}$, defined by the equivalence classes with respect to the relation:

$$u \simeq v \Longleftrightarrow u - v = \text{constant}, \quad \forall u, v \in H^1(\Omega).$$

V is a Hilbert space with norm:

$$\|\overline{u}\|_V = \|\nabla u\|_{L_2(\Omega)}, \quad \forall u \in \overline{u},\ \overline{u} \in V,$$

where $\overline{u}$ is the equivalence class represented by u.

The bilinear form a is defined in $V \times V$ as coercive, and the Lax-Milgram lemma ensures that the unique solution $u_i \in H^1(\Omega)$ of (5)–(6) satisfies the inequality:

$$\|\nabla u_i\|_{L^2(\Omega)} \leq \frac{C_\gamma(\Omega)}{\alpha} \|\psi_i\|_{H^{-\frac{1}{2}}(\Gamma_0)},$$

where $\alpha > 0$ is the elipticity constant and $C_\gamma(\Omega) > 0$ is the constant associated to the trace operator γ.

Remark 2. To fix the additive constant, equality (4) can be replaced by

$$u_i(\bar{x}) = \varphi_i(\bar{x}), \quad \text{for some } \bar{x} \in \Gamma_0.$$

With respect to the numerical approximation of the inverse problem, the most efficient methods proved to be the iterative methods. They are classified into variational and output least squares methods [3]. Both methods reduce the problem to a quadratic minimization problem (PDE constrained problem). The first one uses variational principles, and the minimization is implicitly performed in $H^{\frac{1}{2}}(\partial\Omega)$-norm, where $\partial\Omega$ is the boundary of the domain Ω. In case of output least squares, the $H^{\frac{1}{2}}(\partial\Omega)$ norm is changed by the $L^2(\partial\Omega)$ norm and the minimization is done over the discretization problem by Newton-type optimization algorithms ([16], [18], [32]). The discretization can be done by finite differences or finite elements. Despite their differences, both methods approximate the solution of the discretized problem ([4], [5], [15], [16], [18], [30]).

Here we propose a different approach. We approximate the *continuous solution* with solutions of approximated problems using the finite element method, but ensuring the fulfillment of continuous Wolfe's global convergent conditions at each iteration of the discrete problems. In Section 2 we formulate the least squares problem and its finite element discretization. In Section 3 the general global convergence results for multi-directional algorithms, given in [10], are recalled and it is shown its dependence of gradient convergence. In Section 4, we calculate the continuous and discrete gradient, and in Section 5 we prove the convergence property between gradients. In Section 6, the relations between Wolfe's conditions of both continuous and discrete problems are examined. The algorithm is described in Section 7 and in Section 8 some numerical results for simplified problems are given as illustration. In Section 9, some conclusions and future work are discussed.

Additional Hypothesis

As usual in finite element discretization, we assume that Ω_0 is replaced by a near enough convex polygonal domain Ω, which shall be considered fixed in the rest of the article. It is known [26] that if the boundary of Ω_0 is smooth enough, then the solution of the variational problem on $\Omega \subset \Omega_0$ is a good approximation of the solution on Ω_0. In our case, we have $\Gamma_0 \in C^\infty$.

We denote $\Gamma := \partial\Omega$ the boundary of Ω, and we assume φ_i, ψ_i to be continuous and piecewise linear functions on Γ.

2. LEAST SQUARES FORMULATION AND DISCRETIZATION

The least squares formulation of the inverse problem is:
Given Cauchy pairs (φ_i, ψ_i), $i = 1, \ldots, N$, consider the problem:

$$\min_{\varepsilon \in L^\infty(\Omega)} J(\varepsilon) = \frac{1}{2} \sum_{i=1}^{N} \int_\Gamma |u_i - \varphi_i|^2 \, dS, \tag{7}$$

$$s.t. \text{ for } i = 1, \ldots, N$$

$$a(\varepsilon, u_i) = \int_\Omega \varepsilon \nabla u_i \nabla v d\overline{x} = F_i(v) = \int_\Gamma \psi_i v dS, \quad \forall v \in H^1(\Omega), \tag{8}$$

$$\int_\Gamma u_i dS = \int_\Gamma \varphi_i dS, \tag{9}$$

with $u_i \in H^1(\Omega)$ and φ_i satisfying (3).

We want to find $\varepsilon \in L^\infty(\Omega)$, minimizing the residual between the potential data φ_i and the potential u_i on Γ obtained by the model. In fact, the minimization should be done over the closed convex set $M \cap Q$, where:

$$M := \{\varepsilon \in L^\infty(\Omega) : \varepsilon \geq K > 0,\ a.e\},$$

and Q is some class of uniqueness of the inverse problem, and it is assumed to be closed and convex with nonempty interior in $L^\infty(\Omega)$. Our aim is to use first order optimality conditions, and although in a first approach we consider the problem (7)–(8)–(9) without constraints for ε, we later shall return to this restriction.

Discretization of problem (7)–(8)–(9) is done by the finite element method [28]. We denote by τ_h the family of regular triangulations of domain Ω, where $h > 0$ is the diameter of τ_h [29]. We have:

$$\overline{\Omega} = \bigcup_{k=1}^{N_{T_h}} T_k,$$

for any triangulation τ_h, where N_{T_h} is the number of elements and T_k is the $k-$th triangle of the mesh.

We define the subspace of $H^1(\Omega)$:

$$V_h = \left\{v_h \in C(\overline{\Omega}) \cap H^1(\Omega) : v_{h|T} \in P_1(T), \forall T \in \tau_h\right\} \tag{10}$$

as the space of solutions of the discretized problem, where $P_1(T)$ is the set of polynomials of degree less or equal to one defined on T. We also denote by:

$$N_h : \text{number of } \tau_h - \text{nodes},$$
$$N_{\Gamma_h} : \text{number of } \tau_h - \text{nodes in } \Gamma,$$
$$N_{\Omega_h} : \text{number of } \tau_h - \text{nodes in } \Omega,$$

therefore: $N_h = N_{\Gamma_h} + N_{\Omega_h}$. We will assume a node numbering of the form:

$$\left\{1, 2, \ldots N_{\Omega_h}, N_{\Omega_h} + 1, \ldots N_h\right\},$$

that is, we first consider the inner nodes.

We also give a basis $B_h = \{\phi_1, \ldots, \phi_{N_h}\}$ of V_h (see [19]) satisfying:

$$\phi_i(\overline{x}_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j, \end{cases}$$

where $\overline{x}_j$ is the $j-th$ node of $\overline{\Omega}$, $1 \leq j \leq N_h$. Then, the discrete solution $u_{i_h} \in V_h$ can be written as follows:

$$u_{i_h}(\overline{x}) = \sum_{j=1}^{N_h} \alpha_j \phi_j(\overline{x}), \quad \overline{x} \in \Omega \tag{11}$$

with $\alpha_j = u_{i_h}(\overline{x}_j)$.

$\mathbb{R}^{N_{T_h}}$ denotes the space of discretized coefficients, i.e., $\varepsilon \in L^\infty(\Omega)$ shall be discretized by a $N_{T_h}-$vector $\varepsilon_h = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{N_{T_h}})$, and ε is considered constant $= \varepsilon_j^h$ at each triangle T_j of τ_h.

We also consider vector ε_h as a function of $L^\infty(\Omega)$ by its canonical extension:

$$\varepsilon_h(x, y) = \varepsilon_j^h, \ \text{ if } (x, y) \in \overset{\circ}{T}_j, \quad j = 1, \ldots, N_{T_h},$$

defined almost everywhere (a.e) on $\overline{\Omega}$. We denote $\varepsilon_h(.)$ the canonical extension of ε_h to $L^\infty(\Omega)$. Moreover, we denote $\mathbb{R}^{N_{T_h}}(.)$ the set of canonical extensions to $L^\infty(\Omega)$ of all possible $N_{T_h}-$vectors ε_h defined on a triangulation τ_h. Hence, $\mathbb{R}^{N_{T_h}}(.)$ is a subspace of $L^\infty(\Omega)$ and its elements $\varepsilon_h(.) \in \mathbb{R}^{N_{T_h}}(.)$ shall be called τ_h-piecewise functions.

The double discretization of the variational problem (7)–(8)–(9) in both, equation and parameter, generates the discretized problem:

Find $u_{i_h} \in V_h$, such that: (12)

$$a(u_{i_h}, v_h) = \int_\Omega \varepsilon_h \nabla u_{i_h} \nabla v_h d\overline{x} = F_i(v_h) = \int_\Gamma \psi_i v_h dS, \quad \forall v \in V_h, \tag{13}$$

$$\int_\Gamma u_{i_h} dS = \int_\Gamma \varphi_i dS, \tag{14}$$

for $1 \leq i \leq N$, with V_h defined in (10).

(13) gives a $N_h \times N_h-$system of linear equations:

$$K^{(h)} \alpha^{(i)} = b^{(i)} \tag{15}$$

for each $i = 1, 2, \ldots, N$, where:

$$\alpha_k^{(i)} = u_{i_h}(\overline{x}_k), \quad K_{jk}^{(h)} = \int_\Omega \varepsilon_h \nabla\phi_j \nabla\phi_k d\overline{x}, \quad b_j^{(i)} = \int_\Gamma \psi_i \phi_j dS, \tag{16}$$

and $1 \leq j, k \leq N_h$. This is a consistent and undetermined system. On the other hand, condition (14) generates the additional equation:

$$\sum_{j=N_{\Omega_h}+1}^{N_h} K_j^{(h)} \alpha_j^{(i)} = b^{(i)}, \tag{17}$$

where

$$K_j^{(h)} = \int_\Gamma \phi_j dS \quad \text{and} \quad b^{(i)} = \int_\Gamma \varphi_i dS.$$

Finally, we obtain a consistent and unique solvable system of equations which can be solved by usual methods.

Remark 3. The last equation is a consequence of replacing:

$$u_{i_h}(\overline{x}) = \sum_{j=N_{\Omega_h}+1}^{N_h} u_{i_h}(\overline{x}_j)\phi_j(\overline{x}),$$

for u_{i_h} restricted to the boundary Γ for all nodes $j = N_{\Omega_h} + 1, \ldots, N_h$ into (14).

The discretized version of (7)–(8)–(9) in functional form, is given by:

$$\min_{\varepsilon_h \in \mathbb{R}^{N_{T_h}}} J_h(\varepsilon_h) = \frac{1}{2}\sum_{i=1}^{N} \int_\Gamma \left|u_{i_h} - \varphi_i\right|^2 dS, \tag{18}$$

$$a(u_{i_h}, v_h) = \int_\Omega \varepsilon_h \nabla u_{i_h} \nabla v_h d\overline{x} = F_i(v_h) = \int_\Gamma \psi_i v_h dS, \quad \forall v_h \in V_h. \tag{19}$$

$$\int_\Gamma u_{i_h} dS = \int_\Gamma \varphi_i dS. \tag{20}$$

From finite element theory, it is well known (see [26]) that we have convergence conditions, i.e., if $\varepsilon \in L^\infty(\Omega)$, $\varepsilon \geq K$, and if $u_{i_h}^\varepsilon$ is the solution of (13)–(14) for fixed $\varepsilon_h = \varepsilon$, $\forall h > 0$, then:

$$\lim_{h \longrightarrow 0} \left\| u_i - u_{i_h}^\varepsilon \right\|_V = 0, \quad \text{i.e.} \lim_{h \longrightarrow 0} \left\| \nabla u_i - \nabla u_{i_h}^\varepsilon \right\|_{L^2(\Omega)} = 0 \tag{21}$$

for all $1 \leq i \leq N$, where $u_i \in V$ is the solution of (8)–(9) corresponding to ε and $V = H^1(\Omega)/\mathbb{R}$. Moreover, in Section 5 we shall prove that:

$$\lim_{h \to 0} \|\varepsilon_h - \varepsilon\|_{L_\infty(\Omega)} = 0 \Longrightarrow \lim_{h \longrightarrow 0} \left\| u_i - u_{i_h} \right\|_V = 0,$$

where u_{i_h} is the solution of (13)–(14) corresponding to ε_h.

In addition, if $u_i \in H^2(\Omega)$, then:

$$\left\| u_i - u^{\varepsilon}_{i_h} \right\|_V \leq Ch \left\| u_i \right\|_{H^2(\Omega)}. \tag{22}$$

3. DESCENT ALGORITHM AND GLOBAL CONVERGENCE

Let be X a normed space and $J : X \longrightarrow \mathbb{R}$. Denote X' the topological dual space of X. We recall that a general descent algorithm searching local minima of the functional $J \in \mathscr{C}^1$, is given by the following steps [25]:

Descent Algorithm:

$A1)$ Choose $\varepsilon^0 \in X$ and set $k = 0$.
$A2)$ Stopping test: If $\left\| \nabla J(\varepsilon^k) \right\| = 0$, take ε^k as a solution.
$A3)$ Descent direction search: Find $d_k \in X$, such that $\left\langle \nabla J(\varepsilon^k), d_k \right\rangle_{X',X} < 0$.
$A4)$ Step length search: Find $\lambda_k > 0$ such that $J\left(\varepsilon^k + \lambda_k d_k\right) < J(\varepsilon^k)$.
$A5)$ Iteration: Take $\varepsilon^{k+1} = \varepsilon^k + \lambda_k d_k$, $k = k + 1$ and go to step $A2)$.

For each ε^0 this algorithm generates a finite or infinite sequence $\{\varepsilon^k\}$, and we say it is globally convergent if for all $\varepsilon^0 \in X$, the sequence is finite and stops in $A2)$ or it is infinite and any accumulation point $\bar{\varepsilon}$ of $\{\varepsilon^k\}$ satisfies $\nabla J(\bar{\varepsilon}) = \theta$.

When $\dim X = n < \infty$, step $A3)$ is implemented looking for a direction within the non-orthogonal directions set:

$$G_\rho(\varepsilon^k) = \left\{ d : \left\langle \nabla J(\varepsilon^k), d \right\rangle \leq -\rho \left\| \nabla J(\varepsilon^k) \right\| \|d\| \right\}, \quad \text{where } \rho \in (0, 1). \tag{23}$$

To this set belong all the descent directions used in common optimization algorithms, for example, in Quasi-Newton's versions, where a symmetric positive definite matrix H_k is used to approximate the Hessian $H_k \approx \nabla^2 J(\varepsilon^k)$, obtaining a descent direction $d_k = -H_k^{-1} \nabla J(\varepsilon^k)$ of Newton's type. Several successful iterative formulae have been developed for H_k (see for example [20], [24]).

In step $A4)$, an inexact line search is done, i.e., for fixed $\alpha \in (0, 1)$ and $\beta \in (\alpha, 1)$, a real $\lambda_k > 0$ is found such that:

$$\begin{aligned} &w1) J(\varepsilon^k + \lambda_k d_k) \leq J(\varepsilon^k) + \alpha \lambda_k \langle \nabla J(\varepsilon^k), d_k \rangle \\ &w2) \langle \nabla J(\varepsilon^k + \lambda_k d_k), d_k \rangle \geq \beta \langle \nabla J(\varepsilon^k), d_k \rangle. \end{aligned} \tag{24}$$

$w_1)$ and $w_2)$ are called the α and β Wolfe's conditions. According to classical results (see [20]), the resulting algorithm using (23) and (24) is globally convergent to the set:

$$\Gamma = \{ x \in X : \nabla J(x) = \theta \}.$$

The search criteria (23) and (24) can be extended to a finite number of directions. Given $\rho, \alpha, \beta \in (0,1)$, $\beta > \alpha$, $p \geq 1$, in steps $A3$) and $A4$), we look for p directions $D_k = (d_1^k, \ldots, d_p^k) \in X^p$ and p step lengths $\lambda_k = (\lambda_{k1}, \ldots, \lambda_{kp}) \in \mathbb{R}_+^p$, such that:

$$\left\langle \nabla J(\varepsilon^k), \lambda_k D_k \right\rangle \leq -\rho \left\| \nabla J(\varepsilon^k) \right\|_{X'} \left\| \lambda_k D_k \right\|_X, \tag{25}$$

and

$$\begin{aligned} &w1)\ J(\varepsilon^{k+1}) \leq J(\varepsilon^k) + \alpha \left\langle \nabla J(\varepsilon^k), \lambda_k D_k \right\rangle_{X',X} \\ &w2)\ \left\langle \nabla J(\varepsilon^{k+1}), \lambda_k D_k \right\rangle \geq \beta \left\langle \nabla J(\varepsilon^k), \lambda_k D_k \right\rangle_{X',X} \end{aligned} \tag{26}$$

where:

$$\begin{aligned} \varepsilon^{k+1} &= \varepsilon^k + \lambda_k D_k, \\ \lambda_k D_k &= \sum_{j=1}^{p} \lambda_{kj} d_j^k. \end{aligned}$$

For $p > 1$ a multi-directional and multi-step search algorithm is obtained. Note that this approach allows the case when one or several directions in D_k do not belongs to $G_\rho(\varepsilon^k)$ and/or the intermediate points

$$\varepsilon_r^{k+1} = \varepsilon^k + \sum_{j=1}^{r} \lambda_{kj} d_j^k, \quad r < p,$$

do not satisfy $w1$) and $w2$); but the overall step $\lambda_k D_k$ does. Furthermore, the number p of finite directions can be different at each iteration k, and the algorithm becomes a method with variable multi-directional searches.

Theorem 4. *If $J \in \mathscr{C}^1$ and is bounded below, then the variable multi-directional and multi-step search algorithm is globally convergent to the set*

$$\Gamma = \{x \in X : \nabla J(x) = \theta\}.$$

Proof. (See [10]). □

The algorithm can be applied to the continuous problem (7)–(8)–(9) through its discrete approximations (18)–(19)–(20) as follows:

At the beginning of $k+1-$step, a current point $\varepsilon_h^k \in \mathbb{R}^{N_{T_h}}$ is given at the discrete problem, and its corresponding canonical extension $\varepsilon_h^k(.) \in L^\infty(\Omega)$ is implicitly given at the continuous problem. Working in the discrete problem, a $\rho-$descent direction $\delta\varepsilon_h^k \in \mathbb{R}^{N_{T_h}}$ is sought and an inexact line search is performed to find a step length λ_k and a new point $\varepsilon_h^{k+1} = \varepsilon_h^k + \lambda_k \delta\varepsilon_h^k \in \mathbb{R}^{N_{T_h}}$ satisfying the $\alpha, \beta-$Wolfe conditions $w1$), $w2$) for J_h. At the same time, we verify if the corresponding canonical extensions $\delta\varepsilon_h^k(.)$, $\varepsilon_h^{k+1}(.) \in L^\infty(\Omega)$ satisfy Wolfe's conditions for J. If not, new iterations are done in the discrete problem (say p_k iterations) until $\nabla J_h = \theta$ or Wolfe's conditions are fulfilled in the continuous one, i.e., $\lambda_k D_k(.)$ and $\varepsilon_h^{k+1}(.)$ satisfy (25) and (26). Afterward, the step h is decreased (for example, to $h' = h/2$) and a new discrete problem is defined with a given initial solution $\varepsilon_{h'}^{k+1}$.

Actually, this process is a variable multi-directional and multi-step algorithm for problem (7)–(8). The number p_k of iterations performed in the $k-th$ discrete problem corresponds to the number of directions taken in D_k for a single iteration of the multi-directional algorithm in the continuous problem. The number p_k can be different at each iteration, but remains bounded. This approach was proposed by Gómez [10] in the context of nonlinear optimal control algorithms.

In Section 6, we shall see when the fulfilling of non-orthogonal and Wolfe's conditions in the discrete problem imply the same properties in the continuous problem. It depends on the relations between ρ, α and β conditions in both problems and, therefore, depends on $J(\varepsilon), J_h(\varepsilon_h)$ and on the continuos $\nabla J(\varepsilon)$ and discrete $\nabla J_h(\varepsilon_h)$ gradients of the objective functions.

Obviously, we cannot expect both are satisfied for the same parameters ρ, α and β, but some criteria for its appropriate selection shall also be given in Section 6. These are similar results to those given in [10], and the only difference shall be the specific form adopted for the gradients at each respective problem, and its convergence property when $h \to 0$.

4. COMPUTATION OF GRADIENTS

Functions J in (7) and J_h in (18) implicitly depend on $\varepsilon, \varepsilon_h$, then we shall use the Lagrangean method [11] in order to calculate the respective gradients $\nabla J(\varepsilon), \nabla J_h(\varepsilon_h)$.

4.1. Continuous Gradient

Define $J_i(\varepsilon) = \frac{1}{2} \| u_i - \varphi_i \|^2_{L^2(\Gamma)}$, then $\nabla J(\varepsilon) = \sum_{i=1}^N \nabla J_i(\varepsilon)$. We will work in the quotient space $V = H^1(\Omega)/\mathbb{R}$ to avoid fixing the constant, and we will skip the coset notation.

We define the Lagrangean $\mathscr{L}_i : L^\infty(\Omega) \times V \times V \longrightarrow \mathbb{R}$ as follows:

$$\mathscr{L}_i(\varepsilon, u_i, p_i) = \frac{1}{2} \| u_i - \varphi_i \|^2_{L^2(\Gamma)} + \langle A_i(\varepsilon, u_i), p_i \rangle_{V',V}, \tag{27}$$

where $A_i(\varepsilon, u_i) \in V'$ is the linear operator defined by:

$$\langle A_i(\varepsilon, u_i), v\rangle_{V',V} = \int_\Omega \varepsilon \nabla u_i \nabla v d\overline{x} - \int_\Gamma \psi_i v dS, \quad v \in V.$$

If $\bar{u}_i = u_i(\varepsilon) \in V$ is the solution of (8) corresponding to ε, then $A_i(\varepsilon, \bar{u}_i) = \theta$, and:

$$\mathscr{L}_i(\varepsilon, \bar{u}_i, p_i) = J_i(\varepsilon).$$

As a consequence:

$$\nabla J_i(\varepsilon)\delta\varepsilon = \partial_\varepsilon \mathscr{L}_i(\varepsilon, \bar{u}_i, p_i)\delta\varepsilon + \partial_{u_i}\mathscr{L}_i(\varepsilon, \bar{u}_i, p_i)\partial_\varepsilon \bar{u}_i(\varepsilon)\delta\varepsilon. \tag{28}$$

Since $\delta u_i = \partial_\varepsilon \bar{u}_i(\varepsilon)\delta\varepsilon \in V$ is arbitrary, we choose $\bar{p}_i \in V$ such that:

$$\partial_{u_i}\mathscr{L}_i(\varepsilon, \bar{u}_i, \bar{p}_i)\delta u_i = 0, \quad \forall \delta u_i \in V.$$

Differentiation in (27) with respect to u_i gives:

$$\partial_{u_i}\mathscr{L}_i(\varepsilon, \bar{u}_i, \bar{p}_i)\delta u_i = \langle \bar{u}_i - \varphi_i, \delta u_i\rangle_{L^2(\Gamma)} + \left\langle \partial_{u_i} A(\varepsilon, \bar{u}_i)\delta u_i, \bar{p}_i\right\rangle = 0, \forall \delta u_i \in V,$$

which implies

$$\int_\Gamma (\bar{u}_i - \varphi_i)\delta u_i dS + \int_\Omega \varepsilon \nabla \delta u_i \nabla \bar{p}_i d\overline{x} = 0, \quad \forall \delta u_i \in V.$$

Then, the $i-th$ adjoint equation for $\bar{p}_i \in V$ is:

$$\int_\Omega \varepsilon \nabla p_i \nabla \delta u_i d\overline{x} = \int_\Gamma (\varphi_i - \bar{u}_i)\delta u_i dS, \quad \forall \delta u_i \in V, \tag{29}$$

which corresponds to a variational formulation of the elliptic problem:

$$\begin{cases} \operatorname{div}(\varepsilon \nabla p_i) = 0 & \text{in } \Omega, \\ \varepsilon \dfrac{\partial p_i}{\partial n}_{|\Gamma} = (\varphi_i - \bar{u}_i) & \text{on } \Gamma, \end{cases}$$

where $p_i \in V,\ i = 1, 2, \ldots, N_{T_h}$, and compatibility condition:

$$\int_\Gamma (\varphi_i - \bar{u}_i) dS = 0 \Longleftrightarrow \int_\Gamma \varphi_i dS = 0. \tag{30}$$

Problem (29)–(30) has a unique solution in V by the same reasons given above for the problem (5). From (27) and (28) we have:

$$\begin{aligned}\nabla J_i(\varepsilon)\delta\varepsilon &= \partial_\varepsilon \mathscr{L}_i(\varepsilon, \bar{u}_i, \bar{p}_i)\delta\varepsilon \\ &= \left\langle \partial_\varepsilon A(\varepsilon, \bar{u}_i)\delta\varepsilon, \bar{p}_i \right\rangle \\ &= \int_\Omega \delta\varepsilon \nabla \bar{u}_i \nabla \bar{p}_i \; d\overline{x}, \quad i = 1, 2, \ldots, N.\end{aligned}$$

Hence,

$$\nabla J(\varepsilon)\delta\varepsilon = \sum_{i=1}^{N} \int_\Omega \delta\varepsilon \nabla \bar{u}_i \nabla \bar{p}_i d\overline{x}, \quad \forall \delta\varepsilon \in L^\infty(\Omega). \tag{31}$$

We denote by:

$$\nabla J(\varepsilon) = \sum_{i=1}^{N} \nabla \bar{u}_i \nabla \bar{p}_i, \tag{32}$$

the function of $L^1(\Omega)$, representing the integral linear functional (31).

4.2. Discrete Gradient

As in the previous case, we will work on $\widetilde{V_h} = V_h/\mathbb{R}$, where V_h is given in (10), and we will simplify notation writing $\widetilde{V_h}$ as V_h and $\bar{v}_h$ as v_h. Define $J_{i_h}(\varepsilon_h) : \mathbb{R}^{N_{T_h}} \longrightarrow \mathbb{R}$ such that:

$$J_{i_h}(\varepsilon_h) = \frac{1}{2} \left\| u_{i_h} - \varphi_i \right\|^2_{L^2(\Gamma)},$$

where:

$$\begin{aligned}&\left\langle A_i(\varepsilon_h, u_{i_h}), v_h \right\rangle_{V_h', V_h} = \int_\Omega \varepsilon_h(.)\nabla u_{i_h} \nabla v_h \; d\overline{x} - \int_\Gamma \psi_i v_h dS = 0, \\ &\forall v_h \in V_h, \; 1 \le i \le N,\end{aligned} \tag{33}$$

then:

$$\nabla J_h = \sum_{i=1}^{N} \nabla J_{i_h}.$$

Taking into account that $\varepsilon_h = (\varepsilon_k)_{1\leq k\leq N_{T_h}} \in \mathbb{R}^{N_{T_h}}$, integration in (33) over each mesh triangle gives:

$$\begin{aligned}\langle A_i(\varepsilon_h, u_{i_h}), v_h\rangle_{V_h',V_h} &= \sum_{k=1}^{N_{T_h}} \varepsilon_k \int_{T_k} \nabla u_{i_h} \nabla v_h \, d\overline{x} - \int_\Gamma \psi_i v_h \, dS \\ &= \sum_{k=1}^{N_{T_h}} \varepsilon_k (\nabla u_{i_h}, \nabla v_h)_{|T_k} - \int_\Gamma \psi_i v_h \, dS, \end{aligned} \tag{34}$$

where in the last equality we use that the gradients are constant over each triangle, since functions u_{i_h} and v_h are supposed to be piecewise linear on τ_h.

We define the Lagrangean $\mathscr{L} : \mathbb{R}^{N_{T_h}} \times V_h \times V_h$ by:

$$\mathscr{L}(\varepsilon_h, u_{i_h}, p_{i_h}) = \frac{1}{2} \left\| u_{i_h} - \varphi_i \right\|^2_{L^2(\Gamma)} + \langle A_i(\varepsilon_h, u_{i_h}), p_{i_h}\rangle_{V_h',V_h}. \tag{35}$$

If $\bar{u}_{i_h} = u_{i_h}(\varepsilon_h) \in V_h$ is the unique solution of the equation (33) corresponding to ε_h, then:

$$\mathscr{L}(\varepsilon_h, \bar{u}_{i_h}, p_{i_h}) = J_{i_h}(\varepsilon_h), \quad \forall p_{i_h} \in V_h. \tag{36}$$

Differentiation with respect to ε_h, in (36) gives:

$$\begin{aligned}&\nabla J_{i_h}(\varepsilon_h)\delta\varepsilon_h = \partial_{\varepsilon_h}\mathscr{L}(\varepsilon_h, \bar{u}_{i_h}, p_{i_h})\delta\varepsilon_h + \partial_{u_{i_h}}\mathscr{L}(\varepsilon_h, \bar{u}_{i_h}, p_{i_h})\partial_\varepsilon \bar{u}_{i_h}\delta\varepsilon_h, \\ &\forall \delta\varepsilon_h \in \mathbb{R}^{N_{T_h}}.\end{aligned} \tag{37}$$

Define $\delta u_{i_h} = \partial_\varepsilon \bar{u}_{i_h}\delta\varepsilon_h \in V_h$, then the operator $\partial_{u_{i_h}}\mathscr{L}(\varepsilon_h, \bar{u}_{i_h}, p_{i_h}) \in V_h'$ and we choose $\bar{p}_{i_h} \in V_h$ such that:

$$\partial_{u_{i_h}}\mathscr{L}(\varepsilon_h, \bar{u}_{i_h}, \bar{p}_{i_h})\delta u_{i_h} = 0, \quad \forall \delta u_{i_h} \in V_h. \tag{38}$$

Computing differential $\partial_{u_{i_h}}$ in (35), we obtain:

$$\partial_{u_{i_h}}\mathscr{L}(\varepsilon_h, \bar{u}_{i_h}, \bar{p}_{i_h})\delta u_{i_h} = \langle \bar{u}_{i_h} - \varphi_i, \delta u_{i_h}\rangle_{L^2(\Gamma)} + \left\langle \partial_{u_{i_h}} A_i(\varepsilon_h, \bar{u}_{i_h})\delta u_{i_h}, \bar{p}_{i_h}\right\rangle_{V_h',V_h} = 0,$$

or equivalently:

$$\int_\Gamma (\bar{u}_{i_h} - \varphi_i)\delta u_{i_h} \, dS + \int_\Omega \varepsilon_h(.)\nabla\delta u_{i_h}\nabla\bar{p}_{i_h} \, d\overline{x} = 0,$$

Then, $\bar{p}_{i_h}$ should satisfy the equation:

$$\int_{\Omega} \varepsilon_h(.)\nabla p_{i_h} \nabla \delta u_{i_h} \, d\overline{x} = \int_{\Gamma} (\varphi_i - \bar{u}_{i_h}) \delta u_{i_h} \, dS, \quad \forall \delta u_{i_h} \in V_h \tag{39}$$

for $1 \leq i \leq N$ and $p_{i_h} \in V_h$. This is the adjoint equation for p_{i_h}, which gives the $N_h \times N_h$ linear system:

$$K^{(h)} \beta^{(i)} = c^{(i)},$$

for each $i = 1, 2, \ldots, N$, where:

$$\begin{aligned} \beta_k^{(i)} &= p_{i_h}(\overline{x}_k) \\ K_{jk}^{(h)} &= \int_{\Omega} \varepsilon_h \nabla \phi_j \nabla \phi_k \, d\overline{x} \\ c_j^{(i)} &= \int_{\Gamma} (\varphi_i - \bar{u}_{i_h}) \phi_j \, dS \end{aligned}$$

and $1 \leq j, k \leq N_h$.

From (37), (38) and (34) we have:

$$\begin{aligned} \nabla J_{i_h}(\varepsilon_h)\delta\varepsilon_h &= \partial \varepsilon_h \mathscr{L}(\varepsilon_h, \bar{u}_{i_h}, \bar{p}_{i_h})\delta\varepsilon_h \\ &= \sum_{k=1}^{N_{T_h}} \delta\varepsilon_k \, |T_k| \, (\nabla \bar{u}_{i_h}, \nabla \bar{p}_{i_h})_{|T_k} \end{aligned} \tag{40}$$

for all $\delta\varepsilon_h = (\delta\varepsilon_k)_{1 \leq k \leq N_{T_h}} \in \mathbb{R}^{N_{T_h}}$ and $1 \leq i \leq N$.

Finally,

$$\begin{aligned} \nabla J_h(\varepsilon_h)\delta\varepsilon_h &= \sum_{i=1}^{N} \sum_{k=1}^{N_{T_h}} \delta\varepsilon_k \, |T_k| \left(\nabla \bar{u}_{i_h}, \nabla \bar{p}_{i_h}\right)_{|T_k} \\ &= \sum_{k=1}^{N_{T_h}} \delta\varepsilon_k \sum_{i=1}^{N} |T_k| \left(\nabla \bar{u}_{i_h}, \nabla \bar{p}_{i_h}\right)_{|T_k} . \end{aligned} \tag{41}$$

Therefore, the partial derivatives of J_h are given by:

$$\frac{\partial J_h(\varepsilon_h)}{\partial \varepsilon_k} = \sum_{i=1}^{N} |T_k| \left(\nabla \bar{u}_{i_h}, \nabla \bar{p}_{i_h}\right)_{|T_k} \tag{42}$$

for $k = 1, 2 \ldots, N_{T_h}$. We use the same notation $\nabla J_h(\varepsilon_h)(.) \in \mathbb{R}^{N_{T_h}}(.)$ for the canonical extension of the gradient vector $\nabla J_h(\varepsilon_h) = \left(\frac{\partial J_h}{\partial \varepsilon_k}\right)_{1 \le k \le N_{T_h}} \in \mathbb{R}^{N_{T_h}}$ to $L^\infty(\Omega)$, which is equal to the constant (42) at each open triangle $\mathring{T}_k$ of τ_h.

Note that (41) can be written as:

$$\nabla J_h(\varepsilon_h)\delta\varepsilon_h = \left\langle \nabla J_h(\varepsilon_h)(.), \delta\varepsilon_h(.) \right\rangle_{V',V} = \sum_{i=1}^{N} \int_\Omega \delta\varepsilon_h(.) \nabla \bar{u}_{i_h} \nabla \bar{p}_{i_h} \, d\overline{x} \tag{43}$$

for all $\delta\varepsilon_h \in \mathbb{R}^{N_{T_h}}$. Therefore, we can consider the gradient $\nabla J_h(\varepsilon_h)$ as an integral linear functional defined on the subspace $\mathbb{R}^{N_{T_h}}(.) \subset L^\infty(\Omega)$, which can be extended to $L^\infty(\Omega)$ with the same integral formula:

$$\nabla J_h(\varepsilon_h)\,[\delta\varepsilon] = \sum_{i=1}^{N} \int_\Omega \delta\varepsilon \; \nabla \bar{u}_{i_h} \nabla \bar{p}_{i_h} \, d\overline{x}, \quad \forall \delta\varepsilon \in L^\infty(\Omega). \tag{44}$$

We denote by:

$$\nabla J_h(\varepsilon_h)[.] = \sum_{i=1}^{N} \nabla \bar{u}_{i_h} \nabla \bar{p}_{i_h} \in L^1(\Omega) \tag{45}$$

the function representing this integral linear functional over $L^\infty(\Omega)$.

In what follows, we shall use the notations $\nabla J_h(\varepsilon_h), \nabla J_h(\varepsilon_h)(.)$ or $\nabla J_h(\varepsilon_h)[.]$ according to our needs. Resume:

- $\nabla J_h(\varepsilon_h)$ is a vector with $N_{T_h}-$components,
- $\nabla J_h(\varepsilon_h)(.)$ is a τ_h-piecewise function on Ω, which is constant $= \frac{\partial J_h(\varepsilon_h)}{\partial \varepsilon_k}$ over each open triangle $\mathring{T}_k$ of τ_h,
- $\nabla J_h(\varepsilon_h)[.]$ is this same function as $\nabla J_h(\varepsilon_h)(.)$, but now considered as the representation in $L^1(\Omega)$ of the integral linear functional (44).

5. CONVERGENCE OF GRADIENTS

Lemma 1. *Let be* $\{\varepsilon_h\} \subset \mathbb{R}^{N_{T_h}}$ *such that* $\{\varepsilon_h(.)\} \subset \mathbb{R}^{N_{T_h}}(.)$ *satisfy:*

$$\varepsilon_h(x, y) \ge K > 0, \quad a.e. \ on \ \Omega,$$

and suppose:

$$\varepsilon_h(.) \longrightarrow \varepsilon \ in \ L^\infty(\Omega), \quad when \ h \to 0.$$

Then:

$$\nabla J_h(\varepsilon_h)\,[.] \longrightarrow \nabla J(\varepsilon) \ in \ L^1(\Omega), \quad when \ h \to 0.$$

Remark 5. From Lemma 1, we conclude that if $\{\varepsilon_{h_k}\}$, $k = 1, 2, \ldots$, is a sequence of stationary solutions of the discrete problems (18)–(19)–(20), corresponding to a triangulation τ_{h_k}, with $h_k \to 0$, i.e.:

$$\nabla J_{h_k}(\varepsilon_{h_k})[.] = \theta, \quad k = 1, 2, \ldots,$$

and if $\bar{\varepsilon}$ is an accumulation point of $\{\varepsilon_{h_k}\}$ then:

$$\nabla J(\bar{\varepsilon}) = \theta,$$

and $\bar{\varepsilon}$ is an stationary solution of the continuous problem.

Moreover, if we now add the convex constraint:

$$\varepsilon \in M = \{\varepsilon \in L^{\infty}(\Omega) : \varepsilon \geq K > 0, \ a.e.\} \cap Q \tag{46}$$

for both, the continuous (7)–(8)–(9) and discrete (18)–(19)–(20) problems, by Lemma 1 and convexity, we deduce from the necessary optimality conditions of discrete problems:

$$\nabla J_{h_k}(\varepsilon_{h_k})[\xi - \varepsilon_{h_k}] \geq 0, \quad \forall \xi \in M \cap Q,$$

(taking $k \to +\infty$) the fulfillment of the necessary optimality condition in $\bar{\varepsilon}$ for the continuous problem:

$$\nabla J(\bar{\varepsilon})(\xi - \bar{\varepsilon}) \geq 0, \quad \forall \xi \in M \cap Q.$$

A more delicate situation appears if we consider a discretization M_h of the constraint set M in (18)–(19)–(20). For example:

$$\varepsilon_h(.) \in M_h = \left\{\varepsilon(.) \in \mathbb{R}^{N_{T_h}}(.) : \varepsilon|_{T_k} = \varepsilon_k^n \geq K, \ k = 1, \ldots, N_{T_h}\right\} \cap Q. \tag{47}$$

Evidently, $M_h \subset M$ and since M and Q are supposed to be closed, any accumulation point of the sequence $\varepsilon_{h_k}(.) \in M_{h_k} \cap Q$, belongs to $M \cap Q$. But the existence of optimal solution for the continuous problem with constraint (46) does not imply the existence of optimal solution for the discrete problem with constraint (47). Moreover, the continuous optimal solution should not necessarily be limit to piecewise constant functions $\varepsilon_h(.)$. This situation would be avoided with a smart selection of the set Q, ensuring existence and uniqueness of the solution for the continuous inverse problems, but set Q should be composed of piecewise constant functions in $L_\infty(\Omega)$ plus their limits in L_∞-norm.

Proof. We use the $L^1(\Omega)$ representations (32), (45), and the notations introduced in Section 4 about the quotient spaces.

$$\nabla J(\varepsilon) = \sum_{i=1}^{N} \nabla \bar{u}_i \nabla \bar{p}_i$$

$$\nabla J_h(\varepsilon_h)[.] = \sum_{i=1}^{N} \nabla \bar{u}_{i_h} \nabla \bar{p}_{i_h}$$

where:

1. $\bar{u}_i \in V$ is the solution of:

$$\int_\Omega \varepsilon \nabla u_i \nabla v d\overline{x} = \int_\Gamma \psi_i v dS, \quad \forall v \in V,$$

2. $\bar{p}_i \in V$ is the solution of:

$$\int_\Omega \varepsilon \nabla p_i \nabla v d\overline{x} = \int_\Gamma (\varphi_i - \bar{u}_i) v dS, \quad \forall v \in V,$$

3. $u_{i_h} \in V_h$ is the solution of:

$$\int_\Omega \varepsilon_h(.) \nabla u_{i_h} \nabla v_h d\overline{x} = \int_\Gamma \psi_i v_h dS, \quad \forall v_h \in V_h,$$

4. $\bar{p}_{i_h} \in V_h$ is the solution of:

$$\int_\Omega \varepsilon_h(.) \nabla p_{i_h} \nabla v_h d\overline{x} = \int_\Gamma (\varphi_i - \bar{u}_{i_h}) v_h dS, \quad \forall v_h \in V_h.$$

We also define:

5. $\bar{u}^\varepsilon_{i_h} \in V_h$ as the solution of:

$$\int_\Omega \varepsilon \nabla u^\varepsilon_{i_h} \nabla v_h d\overline{x} = \int_\Gamma \psi_i v_h dS, \quad \forall v_h \in V_h.$$

From (21), we know that $\bar{u}^\varepsilon_{i_h} \longrightarrow \bar{u}_i$ in V when $h \longrightarrow 0$.
Define $\bar{w}_{i_h} := \bar{u}_{i_h} - \bar{u}^\varepsilon_{i_h} \in V_h$. From above definitions **3.** and **5.** we have:

$$\int_\Omega \varepsilon \nabla \bar{u}^\varepsilon_{i_h} \nabla v_h d\overline{x} = \int_\Omega \varepsilon_h \nabla \bar{u}_{i_h} \nabla v_{i_h} d\overline{x}, \quad \forall v_h \in V_h,$$

and replacing $\bar{u}_{i_h} = \bar{w}_{i_h} + \bar{u}^{\varepsilon}_{i_h}$, we obtain that $\bar{w}_{i_h} \in V_h$ is the solution of:

$$\int_\Omega \varepsilon_h \nabla w_{i_h} \nabla v_h d\overline{x} = \int_\Omega (\varepsilon - \varepsilon_h) \nabla \bar{u}^{\varepsilon}_{i_h} \nabla v_h d\overline{x}, \quad \forall v_h \in V_h,$$

with the continuous and coercive bilinear form:

$$a(w_{i_h}, v_h) = \int_\Omega \varepsilon_h \nabla w_{i_h} \nabla v_h d\overline{x}.$$

Furthermore, $F_h : V_h \longrightarrow \mathbb{R}$ defined by:

$$F_h(v_h) = \int_\Omega (\varepsilon - \varepsilon_h) \nabla \bar{u}^{\varepsilon}_{i_h} \nabla v_h d\overline{x},$$

is linear and satisfies:

$$|F_h(v_h)| \leq \|\varepsilon - \varepsilon_h\|_{L^\infty(\Omega)} \left\| \bar{u}^{\varepsilon}_{i_h} \right\|_V \|v_h\|_V, \quad \forall v_h \in V_h,$$

hence F_h is continuous and:

$$\|F_h\|_{V'} \leq \|\varepsilon - \varepsilon_h\|_{L^\infty(\Omega)} \left\| \bar{u}^{\varepsilon}_{i_h} \right\|_V, \quad \forall h > 0,$$

where $\left\| \bar{u}^{\varepsilon}_{i_h} \right\|_{H^1(\Omega)}$ is uniformly bounded because $\bar{u}^{\varepsilon}_{i_h}$ converges to $\bar{u}_i$ in V.

Then:

$$\|F_h\|_{V'} \longrightarrow 0, \quad \text{if } h \longrightarrow 0.$$

From Lax-Milgram lemma (see [26]):

$$\left\| \bar{w}_{i_h} \right\|_V \leq \frac{1}{\lambda} \|F_h\|_{V'} \longrightarrow 0, \quad \text{if } h \longrightarrow 0,$$

where $\lambda > 0$ is the coercivity constant of a, which is independent of h.

We have proved that:

$$\left\| \bar{u}_{i_h} - \bar{u}^{\varepsilon}_{i_h} \right\|_V \longrightarrow 0, \quad \text{if } h \longrightarrow 0,$$

and as a consequence:

$$\left\| \bar{u}_i - \bar{u}_{i_h} \right\|_V \leq \left\| \bar{u}_i - \bar{u}^{\varepsilon}_{i_h} \right\|_V + \left\| \bar{u}_{i_h} - \bar{u}^{\varepsilon}_{i_h} \right\|_V \longrightarrow 0, \quad \text{if } h \longrightarrow 0.$$

Analogously, we define $\bar{p}^{\varepsilon}_{i_h} \in V_h$ satisfying:

$$\int_\Omega \varepsilon \nabla \bar{p}_{i_h} \nabla v_h d\overline{x} = \int_\Gamma (\varphi_i - \bar{u}_i) v_h dS, \quad \forall v_h \in V_h,$$

and $\bar{z}_{i_h} = \bar{p}_{i_h} - \bar{p}^{\varepsilon}_{i_h} \in V_h$ as the solution of:

$$\int_{\Omega} \varepsilon_h \nabla z_{i_h} \nabla v_h d\overline{x} = \int_{\Omega} (\varepsilon - \varepsilon_h) \nabla \bar{p}^{\varepsilon}_{i_h} \nabla v_h, \quad \forall v_h \in V_h.$$

With the same arguments as before, we obtain:

$$\left\| \bar{p}_{i_h} - \bar{p}^{\varepsilon}_{i_h} \right\|_V = \left\| \bar{z}_{i_h} \right\|_V \longrightarrow 0, \quad \text{if } h \longrightarrow 0,$$

then:

$$\left\| \bar{p}_i - \bar{p}_{i_h} \right\|_V \leq \left\| \bar{p}_i - \bar{p}^{\varepsilon}_{i_h} \right\|_V + \left\| \bar{p}_{ih} - \bar{p}^{\varepsilon}_{i_h} \right\|_V \longrightarrow 0, \quad \text{if } h \longrightarrow 0.$$

Therefore, we have:

$$\begin{aligned} \nabla \bar{u}_{i_h} &\longrightarrow \nabla \bar{u}_i \quad \text{in } L^2(\Omega), \\ \nabla \bar{p}_{i_h} &\longrightarrow \nabla \bar{p}_i \quad \text{in } L^2(\Omega), \end{aligned}$$

and, using triangle inequality, we obtain:

$$\begin{aligned} &\left\| \nabla \bar{u}_{i_h} \nabla \bar{p}_{i_h} - \nabla \bar{u}_i \nabla \bar{p}_i \right\|_{L^1(\Omega)} \\ &\quad \leq \left\| \nabla \bar{u}_{i_h} \nabla \bar{p}_{i_h} - \nabla \bar{u}_{i_h} \nabla \bar{p}_i \right\|_{L^1(\Omega)} + \left\| \nabla \bar{u}_{i_h} \nabla \bar{p}_i - \nabla \bar{u}_i \nabla \bar{p}_i \right\|_{L^1(\Omega)} \\ &\quad \leq \left\| \nabla \bar{u}_{i_h} \right\|_{L^2(\Omega)} \left\| \nabla \bar{p}_{i_h} - \nabla \bar{p}_i \right\|_{L^2(\Omega)} + \left\| \nabla \bar{p}_i \right\|_{L^2(\Omega)} \left\| \nabla \bar{u}_{i_h} - \nabla \bar{u}_i \right\|_{L^2(\Omega)} \longrightarrow 0, \\ &\qquad \text{if } h \longrightarrow 0, \end{aligned}$$

and then:

$$\nabla \bar{u}_{i_h} \nabla \bar{p}_{i_h} \longrightarrow \nabla \bar{u}_i \nabla \bar{p}_i \quad \text{in } L^1(\Omega),$$

for all $i = 1, 2, \ldots, N$. Finally:

$$\nabla J_h(\varepsilon_h)[.] = \sum_{i=1}^{N} \nabla \bar{u}_{i_h} \nabla \bar{p}_{i_h} \longrightarrow \nabla J(\varepsilon) = \sum_{i=1}^{N} \nabla \bar{u}_i \nabla \bar{p}_i$$

in $L^1(\Omega)$ when $h \longrightarrow 0$. □

Definition 6. Let τ_{h_1} be a regular triangulation of Ω, with diameter $h_1 > 0$, and $\varepsilon_{h_1}(.) \in \mathbb{R}^{N_{T_{h_1}}}(.)$ be a $\tau_{h_1}-$piecewise function on Ω. We say that $\varepsilon_h(.) \in \mathbb{R}^{N_{T_h}}(.)$ is a *reduction of* $\varepsilon_{h_1}(.)$ *to* τ_h if and only if τ_h is a regular triangulation of Ω, which is finer than τ_{h_1}, with diameter $h > 0$, $h \leq h_1$ and $\varepsilon_h(.)$ is a τ_h-piecewise function on Ω, satisfying:

$$\varepsilon_h(\bar{x}) = \varepsilon_{h_1}(\bar{x}), \quad \forall\ \bar{x} \in \Omega.$$

Lemma 7. *Let* $h_1 > 0$ *and* $\varepsilon_{h_1}(.) \in \mathbb{R}^{N_{T_{h_1}}}(.)$ *be a* τ_{h_1}*−piecewise function defined on* Ω*, and suppose* $\varepsilon_h(.) \in \mathbb{R}_{N_{T_h}}(.)$ *is a reduction of* $\varepsilon_{h_1}(.)$ *to* τ_h*, where* τ_h *is any regular triangulation finer than* τ_{h_1}*. Then:*

$$\left\| \nabla J_h(\varepsilon_h)[.] - \nabla J(\varepsilon_h(.)) \right\|_{L^1(\Omega)} \longrightarrow 0, \quad \text{if } h \to 0.$$

Proof. Clearly, we can write:

$$\varepsilon_h(.) \longrightarrow \varepsilon_{h_1}(.) \quad \text{in } L^\infty(\Omega), \quad \text{if } h \to 0.$$

Then, by Lemma 2:

$$\nabla J_h(\varepsilon_h)[.] \longrightarrow \nabla J(\varepsilon_{h_1}(.)) \quad \text{in } L^1(\Omega), \quad \text{if } h \to 0. \tag{48}$$

In addition:

$$\nabla J(\varepsilon_{h_1}(.)) = \nabla J(\varepsilon_h(.)), \tag{49}$$

for all $h \leq h_1$, since ∇J is continuous and $\varepsilon_h(.)$ and $\varepsilon_{h_1}(.)$ are the same function, piecewise defined in different triangulations. The result follows from (48) and (49). □

6. RELATIONS BETWEEN WOLFE'S CONDITIONS

In what follows, we rewrite in our notations the general results given in [10].

We consider the discrete problem (18)–(19)–(20) defined in the space $\mathbb{R}_\infty^{N_{T_h}}$, i.e., the set $\mathbb{R}^{N_{T_h}}$ provided with the norm:

$$\|v\|_{\mathbb{R}_\infty^{N_{T_h}}} = \max_{1 \leq j \leq N_{T_h}} \left| v_j \right|.$$

It is known that the topological dual of $\mathbb{R}_\infty^{N_{T_h}}$ is $\mathbb{R}_1^{N_{T_h}}$, i.e., $\mathbb{R}^{N_{T_h}}$ provided with the norm:

$$\|v\|_{\mathbb{R}_1^{N_{T_h}}} = \sum_{j=1}^{N_{T_h}} \left| v_j \right|.$$

If $\varepsilon_h(.) \in L^\infty(\Omega)$ is the canonical extension of $\varepsilon_h \in \mathbb{R}_\infty^{N_{T_h}}$, it is easy to see that:

$$\|\varepsilon_h\|_{\mathbb{R}_\infty^{N_{T_h}}} = \|\varepsilon_h(.)\|_{L^\infty(\Omega)}.$$

We use the two representations of the discrete gradient obtained in Section 4:

- the vectorial form:

$$\nabla J_h(\varepsilon_h) = \left\{ \sum_{i=1}^{N_e} |T_k| \, (\nabla u_{i_h}, \nabla p_{i_h})_{|T_k} \right\}_{1\le k\le N_{T_h}} \in \mathbb{R}^{N_{T_h}}, \quad \text{where } \overline{\Omega} = \bigcup_{k=1}^{N_{T_h}} T_k,$$

- and the functional form:

$$\nabla J_h(\varepsilon_h)[.] = \sum_{i=1}^{N_e} \nabla u_{i_h} \nabla p_{i_h} \in L^1(\Omega).$$

Then, we have:

$$\begin{aligned} \left\| \nabla J_h(\varepsilon_h)[.] \right\|_{L^1(\Omega)} &= \int_\Omega \sum_{i=1}^{N} \nabla u_{i_h} \nabla p_{i_h} \, d\overline{x} = \sum_{i=1}^{N} \sum_{k=1}^{N_{T_h}} \int_{T_k} \nabla u_{i_h} \nabla p_{i_h} \, d\overline{x} \\ &= \sum_{k=1}^{N_{T_h}} \sum_{i=1}^{N} |T_k| \, (\nabla u_{i_h}, \nabla p_{i_h})_{|T_k} = \left\| \nabla J_h(\varepsilon_h) \right\|_{\mathbb{R}_1^{N_{T_h}}}, \end{aligned}$$

which implies that:

$$\Longrightarrow \left\| \nabla J_h(\varepsilon_h)[.] \right\|_{L^1(\Omega)} = \left\| \nabla J_h(\varepsilon_h) \right\|_{\mathbb{R}_1^{N_{T_h}}}. \tag{50}$$

In the same way, it can be deduced the equality:

$$\left\langle \nabla J_h(\varepsilon_h)[.], \varepsilon_h(.) \right\rangle_{L^1, L^\infty} = \left\langle \nabla J_h(\varepsilon_h), \varepsilon_h \right\rangle_{\mathbb{R}^{N_{T_h}}}. \tag{51}$$

Definition 8. For the continuous and discrete functional J and J_h introduced in Section 2, we define the following error functions:

$$\begin{aligned} &\Delta J(\varepsilon_1, \varepsilon_2) := J(\varepsilon_1) - J(\varepsilon_2) \\ &\Delta J_h(\varepsilon_{1h}, \varepsilon_{2h}) := J_h(\varepsilon_{1h}) - J_h(\varepsilon_{2h}) \\ &\eta_h = \eta_h(\varepsilon_{1h}, \varepsilon_{2h}) := \Delta J(\varepsilon_{1h}(.), \varepsilon_{2h}(.)) - \Delta J_h(\varepsilon_{1h}, \varepsilon_{2h}) \\ &\omega_h = \omega_h(\varepsilon_h) := \left\| \nabla J(\varepsilon_h(.)) - \nabla J_h(\varepsilon_h)[.] \right\|_{L^1(\Omega)}, \end{aligned} \tag{52}$$

where $\varepsilon_1, \varepsilon_2 \in L^\infty(\Omega)$ and $\varepsilon_{1h}, \varepsilon_{2h} \in \mathbb{R}_\infty^{N_{T_h}}$. Note that $\omega_h \longrightarrow 0$, when $h \longrightarrow 0$ by Lemma 2.

Theorem 9 ([10]). *Let be* $\varepsilon_{h_1} \in \mathbb{R}_{\infty}^{N_{T_{h_1}}}$ *and* $\rho, \alpha, \beta \in (0,1)$ *with* $\alpha < \beta$. *Choose* $\xi \in (0,1)$ *satisfying:*

$$\xi < \min\left\{\frac{\rho}{2+\rho}, \frac{\rho(1-\beta)}{2+\rho(1-\beta)}, \frac{\rho\alpha}{1+\alpha(2+\rho)}\right\}.$$

Let $h \leq h_1$, be a regular triangulation τ_h and suppose the reduction $\varepsilon_h(.)$ of $\varepsilon_{h_1}(.)$ to τ_h, are such that:

$$\omega_h(\varepsilon_h) = \left\|\nabla J(\varepsilon_h(.)) - \nabla J_h(\varepsilon_h)(.)\right\|_{L^1(\Omega)} \leq \xi \left\|\nabla J(\varepsilon_h(.))\right\|_{L^1(\Omega)}.$$

Let be $\varepsilon_h^* = \varepsilon_h + \delta\varepsilon_h$ with $\delta\varepsilon_h \in \mathbb{R}_{\infty}^{N_{T_h}}$ for which the following inequalities hold:

$$\omega_h(\varepsilon_h^*) = \left\|\nabla J(\varepsilon_h^*(.)) - \nabla J_h(\varepsilon_h^*)(.)\right\|_{L^1(\Omega)} \leq \xi \left\|\nabla J(\varepsilon_h(.))\right\|_{L^1(\Omega)}$$
$$\eta_h(\varepsilon_h^*, \varepsilon_h) = \left|\Delta J(\varepsilon_h^*(.), \varepsilon_h(.)) - \Delta J_h(\varepsilon_h^*, \varepsilon_h)\right| \leq \xi \left\|\nabla J(\varepsilon_h(.))\right\|_{L^1(\Omega)} \|\delta\varepsilon_h(.)\|_{L^\infty(\Omega)}.$$

If vector $\varepsilon_h^* \in \mathbb{R}^{N_{T_h}}$ satisfies $\rho, \alpha, \beta-$conditions for the discrete problem, then there exist numbers $\rho_1, \alpha_1, \beta_1$ with $\alpha_1 < \beta_1$ such that its canonical extension $\varepsilon_h^*(.) \in L^\infty(\Omega)$ satisfies $\rho_1, \alpha_1, \beta_1-$conditions for the continuous problem. Furthermore, $\rho_1, \alpha_1, \beta_1$ can be chosen into the following intervals:

$$\rho_1 \in (\widehat{\rho_1}, \rho - \xi(1+\rho)),$$
$$\beta_1 \in (\beta + \gamma(1+\beta), 1),$$
$$\alpha_1 \in (0, \alpha - \gamma(1+\alpha)),$$

where:

$$\widehat{\rho_1} = \max\left\{\frac{\rho(1+\beta)}{2+\rho(1-\beta)}, \frac{\rho(1+\alpha)}{1+\alpha(2+\rho)}\right\},$$
$$\gamma = \frac{1}{\rho_1}\left(\frac{\rho-\rho_1}{1+\rho}\right).$$

Remark 10. Note that given intervals do not depend on h and only depend on the parameters ρ, α, β.

7. ALGORITHM

1. Choose $h_{inic} > 0$, $\varepsilon_{inic} \in \mathbb{R}^{N_{T_{h_{inic}}}}$.
2. Choose $\rho, \alpha, \beta, \widehat{\sigma}, \widehat{\rho} \in (0,1)$ satisfying:

$$\beta > \alpha, \quad \widehat{\sigma} < \min\left\{\frac{\rho}{2+\rho}, \frac{\rho(1-\beta)}{2+\rho(1-\beta)}, \frac{\rho(1+\alpha)}{1+\alpha(2+\rho)}\right\},$$

$$\widehat{\rho} \geq \max\left\{\frac{\rho(1+\beta)}{2+\rho(1-\beta)}, \frac{\rho(1+\alpha)}{1+\alpha(2+\rho)}\right\}.$$

3. Choose $\sigma_0 \in (0, \widehat{\sigma})$, $\rho_1 \in (\widehat{\rho}, \rho - \sigma_0(1+\rho))$, $\beta_1 \in (\beta + \gamma(1+\beta), 1)$, $\alpha_1 \in (0, \alpha - \gamma(1+\alpha))$ with $\gamma = \frac{1}{\rho_1}\left(\frac{\rho-\rho_1}{1+\rho}\right)$.
4. Set $h_0 = h_{inic}$, $\varepsilon_0(.) = \varepsilon_{h_0}(.) \in L^\infty(\Omega)$, $l = 0$, $k = 0$.
5. If $\left\|\nabla J(\varepsilon_l(.))\right\|_{L^1(\Omega)} = 0$, stop, function $\varepsilon_l(.)$ is a local minima for the continuous problem.
   If $\left\|\nabla J(\varepsilon_l(.))\right\|_{L^1(\Omega)} \neq 0$, go to step 6.
6. Set $k \longrightarrow k+1$, $\sigma_k = \frac{\sigma_{k-1}}{2}$, $h_k = \frac{h_{k-1}}{2}$.
7. Define $\varepsilon_{h_k} = \varepsilon_l$ at the new triangulation τ_{h_k}.
8. Verify if

$$\left\|\nabla J(\varepsilon_{h_k}(.)) - \nabla J_{h_k}(\varepsilon_{h_k})(.)\right\|_{L^1(\Omega)} \leq \sigma_k \left\|\nabla J(\varepsilon_{h_k}(.))\right\|_{L^1(\Omega)}.$$

   If it holds, go to step 9;
   If it does not hold, take $h_k = \frac{h_k}{2}$ and go to step 7.
9. Choose $\delta\varepsilon_{h_k} \in \mathbb{R}^{NT_{h_k}}$ and $\lambda_k > 0$ satisfying ρ, α and $\beta-$conditions for the discrete problem.
10. Define $\varepsilon^*_{h_k} = \varepsilon_{h_k} + \lambda_k \delta\varepsilon_{h_k}$ and verify:

$$\left\|\nabla J(\varepsilon^*_{h_k}(.)) - \nabla J_{h_k}(\varepsilon^*_{h_k})[.]\right\|_{L^1(\Omega)} \leq \sigma_k \left\|\nabla J(\varepsilon_{h_k}(.))\right\|_{L^1(\Omega)}$$
$$\left|\Delta J(\varepsilon^*_{h_k}(.), \varepsilon_{h_k}(.)) - \Delta J_{h_k}(\varepsilon^*_{h_k}, \varepsilon_{h_k})\right| \leq \sigma_k \left\|\nabla J(\varepsilon_{h_k}(.))\right\|_{L^1(\Omega)} \left\|\delta\varepsilon_{h_k}\right\|_{L^\infty(\Omega)}.$$

If inequalities hold, or if $\left\|\nabla J_{h_k}(\varepsilon^*_{h_k})\right\| = 0$, take $\varepsilon_{l+1}(.) = \varepsilon^*_{h_k}(.)$, $l = l+1$, and go to step 5.

If they do not hold and $\left\|\nabla J_{h_k}(\varepsilon^*_{h_k})\right\| \neq 0$, take $\varepsilon_{h_k} = \varepsilon^*_{h_k}$ and go to step 9.

Remark 11. Quasi-Newton methods with inexact line search or trust region routines can be used at step 9.

Remark 12. Estimations for the continuous gradients $\nabla J(\varepsilon_h(.))$, $\nabla J(\varepsilon^*_h(.))$ and for the continuous increment $\Delta J(\varepsilon^*_h(.), \varepsilon_h(.))$ can be obtained using smaller steps in the discrete problems.

Remark 13. The inequalities of step 8 and 10 can be substituted by a direct verification of the Wolfe's conditions for the continuous problem, using estimations of the continuous gradient $\nabla J(\varepsilon_h(.)), \nabla J(\varepsilon^*_h(.))$ and continuous increment $\Delta J(\varepsilon^*_h(.), \varepsilon_h(.))$.

8. NUMERICAL RESULTS

As first numerical experiences for the algorithm, and since we do not have real problems data at hand, we decide to perform a small numerical experiment as follows:

Consider the simplified Neumann $k-$problems:

$$\begin{cases} \operatorname{div}(\varepsilon_k \nabla u) = 0, & \text{on } \Omega_0 \\ \varepsilon_k \dfrac{\partial u}{\partial n}_{|\Gamma_0} = \psi_k, & \end{cases}$$

where Ω_0 denotes the unit ball, Γ_0 its boundary, only one measure ($N = 1$), and functions given by:

$$\text{(P1) } \varepsilon_k(x, y) = xy + k, \quad \text{with } \psi_k(x, y) = 2(xy + k)(1 - 2y^2)$$
$$\text{(P2) } \varepsilon_k(x, y) = kx^2 + 1, \quad \text{with } \psi_k(x, y) = (k + 1 - ky^2)y.$$

First, good approximation $\hat{\varepsilon}_k$ of ε_k is constructed with a fine triangulation $\tau_{h_m} = \bigcup_i T_i$ of Ω_0 with 256 triangles and a discretized function defined by:

$$\hat{\varepsilon}_k(x, y) = \varepsilon_k(V_s), \ \forall (x, y) \in \mathring{T}_i, \text{ where } V_s \text{ is the baricenter of } T_i.$$

Data φ_k on $\Gamma = \partial(\Omega_0 \cap \tau_{h_m})$ is generated by a numerical integration of the Neumann $k-$problem, using $\hat{\varepsilon}_k$ and ψ_k. Then, two least square problems are formulated with the continuous functional:

$$\min_{\varepsilon_k \in L^\infty(\Omega_0)} J(\varepsilon_k) = \frac{1}{2} \int_{\Gamma_0} |u - \varphi_k|^2 \, dS.$$

Each problem is solved in two ways: one way is to use the fine triangulation τ_{h_m} and an optimization method to solve a problem with 256 variables, resulting an estimate ϵ_k of ε_k. The second way is to use our approach, starting with a thicker triangulation τ_{h_0}, which gives a problem with smaller number of variables, and stopping the optimization algorithm when a Wolfe point $\varepsilon_{h_0}(.)$ for the *continuous* problem was identified. Then, increase triangulation τ_{h_1}, $h_1 < h_0$, giving a new discretized (bigger) problem and start the optimization with $\varepsilon^0_{h_1} = \varepsilon_{h_0}$ restricted to τ_{h_1}. Repeat r times until $\tau_{h_r} = \tau_{h_m}$ to obtain an estimate ξ_k of ε_k.

To test the first "direct" way, problems (P1) and (P2) are solved using data φ_k for 9 different values of k, applying an iterative least square optimization method starting from 3 different initial points ε^0_j, and using the triangulation τ_{h_m}. This gives us 54 results (2 problems $\times$ 9 values of $k \times 3$ initial points) for the estimations ϵ^1_k and ϵ^2_k, $k = 1, \ldots, 9$. In several cases, this direct approach was unable to attain convergence (exceeding the maximum number of evaluations/iterations) and ϵ_k^p was taken as the best point found.

In the second way, and for each problem, each k and each initial point ε_j^0, we work successively with meshes of 8, 16, 64 and 256 triangles and take the last solution as estimate ξ_k^1 or ξ_k^2. In almost all cases, this approach shows global convergence to a satisfactory estimation.

In the least square optimization method, lower bounds lb and upper bounds ub for the variables are given, defined as the minimal value and maximal value of $\hat{\varepsilon}_k$, respectively.

We compare 3 important magnitudes, for $k = 1, \ldots, 9$, $p = 1, 2$, $j = 1, 2, 3$:

- The computational runtimes: $T(\xi_{kj}^p)$, $T(\epsilon_{kj}^p)$,
- The errors in $\hat{\varepsilon}_k-$estimation: $\left\| \xi_{kj}^p - \hat{\varepsilon}_k^p \right\|_\infty$, $\left\| \epsilon_{kj}^p - \hat{\varepsilon}_k^p \right\|_\infty$,
- The objective function values: $J(\xi_{kj}^p(.))$, $J(\epsilon_{kj}^p(.))$, where:

$$J(\xi_{kj}^p(.)) \approx J_{h_m}(\xi_{kj}^p) \quad \text{and} \quad J(\epsilon_{kj}^p(.)) \approx J_{h_m}(\epsilon_{kj}^p).$$

Note that index j is associated with the 3 initial points for the optimization method. A total of 108 problems have been solved and, in the following Appendix, comparisons of these magnitudes are shown with 3 graphs of performance profiles.

More than 80% of the results show time savings in computation (see Figure 1), best approximation of $\hat{\varepsilon}_k$ (see Figure 2), and lowest values of the objective function (see Figure 3). We don't claim that this will always be the case. This small experiment is only illustrative, and the results were obtained using the computer codes of the authors themselves. Certainly, more accurate and conclusive results could be obtained with professional codes and with further numerical experiences, especially in practical problems.

9. CONCLUSIONS AND FURTHER STUDY

We have obtained a globally convergent descent algorithm for the infinite dimensional optimization problem (7)–(8)–(9) which interacts step-by-step with approximations of the discrete problems (18)–(19)–(20) using Wolfe's conditions. Its theoretical value supports the idea of using successive triangulation meshes to approximate the continuous problem, obtaining global convergence when Wolfe's conditions are carefully checked.

In our approach, we have assumed no error on data. However, we know that with measurement errors the inverse problem is ill-conditioned, and it requires regularization techniques. In particular, considering Tikhonov's regularization, the problem is defined as:

$$\min_{\varepsilon \in L^{\infty}(\Omega)} J^{\alpha,\delta}(\varepsilon) = \frac{1}{2} \sum_{i=1}^{N} \int_{\Gamma} \left| u_i - \varphi_i^{\delta} \right|^2 dS + \frac{\alpha}{2} \left\| B(\varepsilon - \varepsilon^*) \right\|_{L^2(\Gamma)}^2$$

$$s.t. \ a(\varepsilon, u_i) = \int_{\Omega} \varepsilon \nabla u_i \nabla v d\overline{x} = F_i(v) = \int_{\Gamma} \psi_i v dS, \quad \forall v \in V,$$

where B is a linear bounded operator; for example the identity or the gradient, ε^* is a function representing apriori information on ε, $\alpha > 0$ is the regularization parameter and φ_i^{δ} represent measured data with error such that $\left\| \varphi_i^{\delta} - \varphi_i \right\| \leq \delta$, where φ_i are the exact data. If B is chosen as the identity operator, the discrete gradients with errors assume the form:

$$\nabla J_h(\varepsilon_h) = \sum_{i=1}^{N} \nabla \bar{u}_{i_h} \nabla \bar{p}_{i_h}^{\delta} + \alpha_h(\varepsilon_h - \varepsilon^*)\eta_h,$$

where $\eta_h(x) = \frac{1}{|T_k|}, \mathring{T}_k$, $T_k \in \tau_k$, $k = 1, \ldots, N_{T_h}$, is a normalization function.

An interesting question is if the variable selection of $\alpha = \alpha_h$, regularizing the discrete problem at each triangulation τ_h, can be done in order to regularize the continuous problem in some limit sense. If it is not possible, then try to choose $\alpha_h(\delta)$ as an admissible regularizing strategy for each discrete problem in such a way that $\left\| \nabla J_h(\varepsilon_h^{\alpha_h,\delta}) - \nabla J(\varepsilon) \right\|_{L^1(\Omega)} \longrightarrow 0$, when $h \to 0$, $\delta \to 0$. Here $\left\{ \varepsilon_h^{\alpha_h,\delta} \right\}$ is a set of optimal solutions of the regularized discrete problems which satisfy $\varepsilon_h^{\alpha_h,\delta} \to \varepsilon$ in $L^{\infty}(\Omega)$ when $h, \delta \to 0$.

ACKNOWLEDGEMENTS

The authors would like to thank the Editor-in-Chief and the anonymous referee for their comments and suggestions, which helped to improve the article.

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APPENDIX

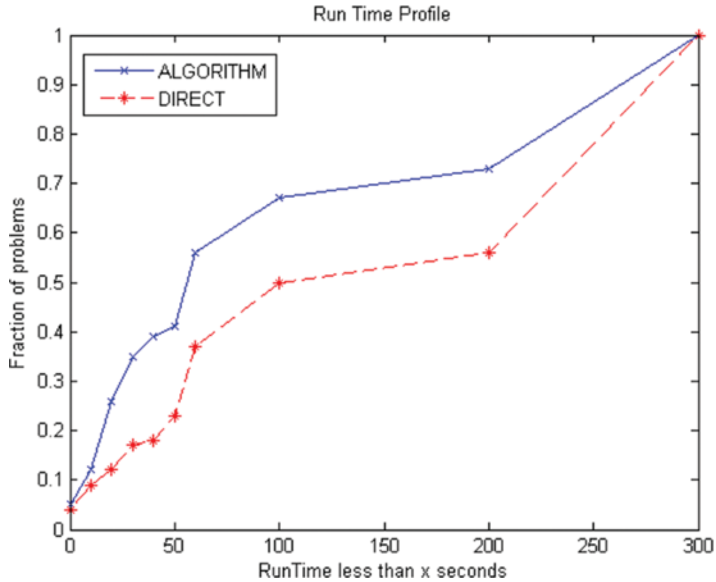

FIGURE 1

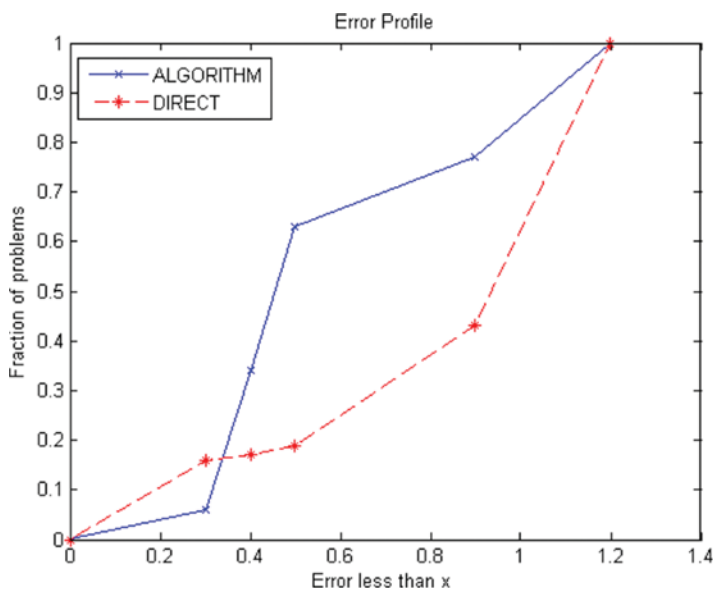

FIGURE 2

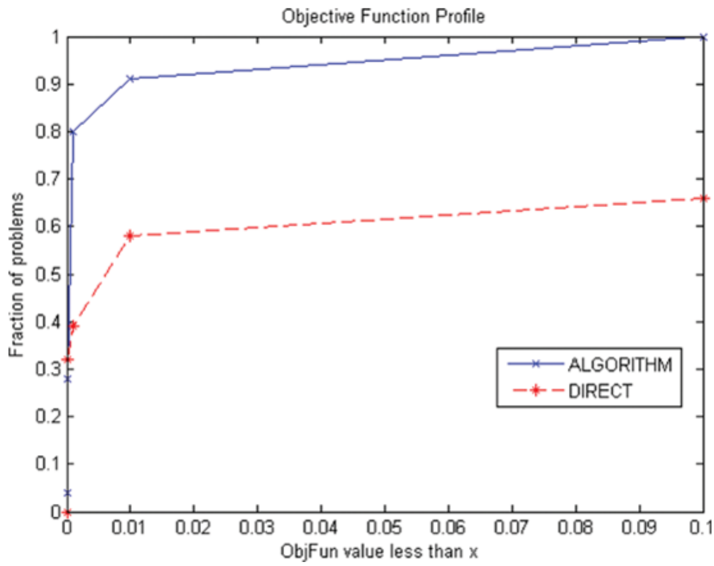
Objective Function Profile
Fraction of problems
ObjFun value less than x
ALGORITHM
DIRECT
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1

FIGURE 3