Problem 1: Temperature of a satellite shadowed by the Earth

A black satellite is orbiting at an altitude H=2000km above the ground.

1) Compute the solid angle subtended by the Earth.

2) If the Earth irradiates as a black body at T=255 K, compute the Earth radiation flux impinging onto the satellite (assumed as a sphere of radius R_{sat}=1m).

3) Compute the radiative equilibrium temperature of the satellite when it is in the shadow of the Earth.

4) How does your answer change when the satellite is also illuminated by the Sun?

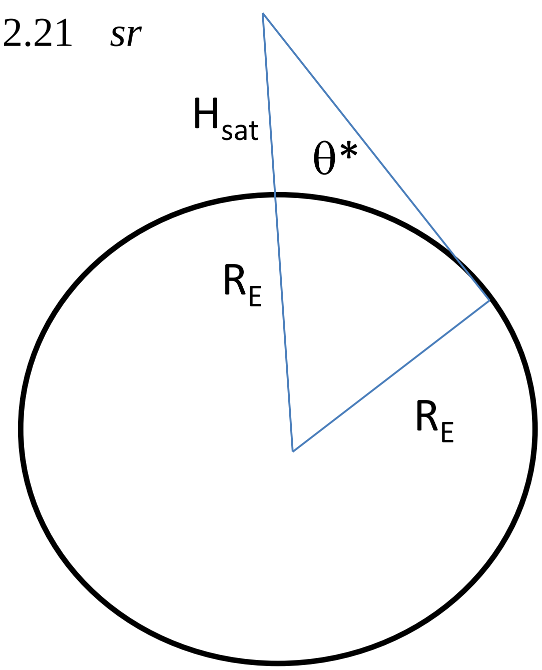
Computation of the solid angle: [2p]

Solid angle subtended by a spherical cap

$$d\Omega = d\phi \sin\theta \, d\theta$$

1st method

$$\Omega_{spherical \quad cap} = \int d\Omega = \iint_{\varphi,\theta} d\phi \, \sin(\theta) \, d\theta = 2\pi \left[1 - \cos(\theta^*)\right] = 2.21 \quad sr$$



$$\sin(\theta^*) = \frac{R_E}{H_{sat} + R_E} = \frac{6400}{8400} = 0.762$$

$$\cos(\theta^*) = \sqrt{1 - \sin^2(\theta^*)} = 0.648$$

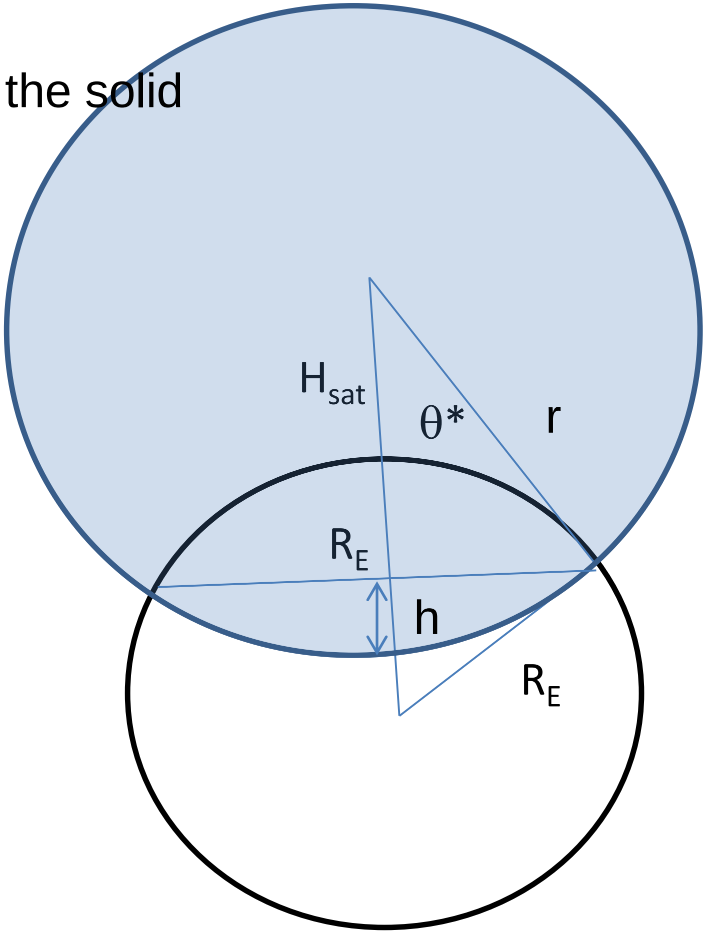
$$\theta^* = \arcsin(0.762) = 0.8662 = 49.6^o$$

Note that $\quad \Omega_{spherical \quad cap} \to 2\pi \quad for \quad H_{sat} \to 0$

Computation of the solid angle from area of spherical cap

$$\Omega = \frac{A_{\perp}}{r^2}$$

The solid angle subtended by the Earth is the same as the solid angle subtended by the
Yellow spherical cap



Computation of the solid angle from area of spherical cap

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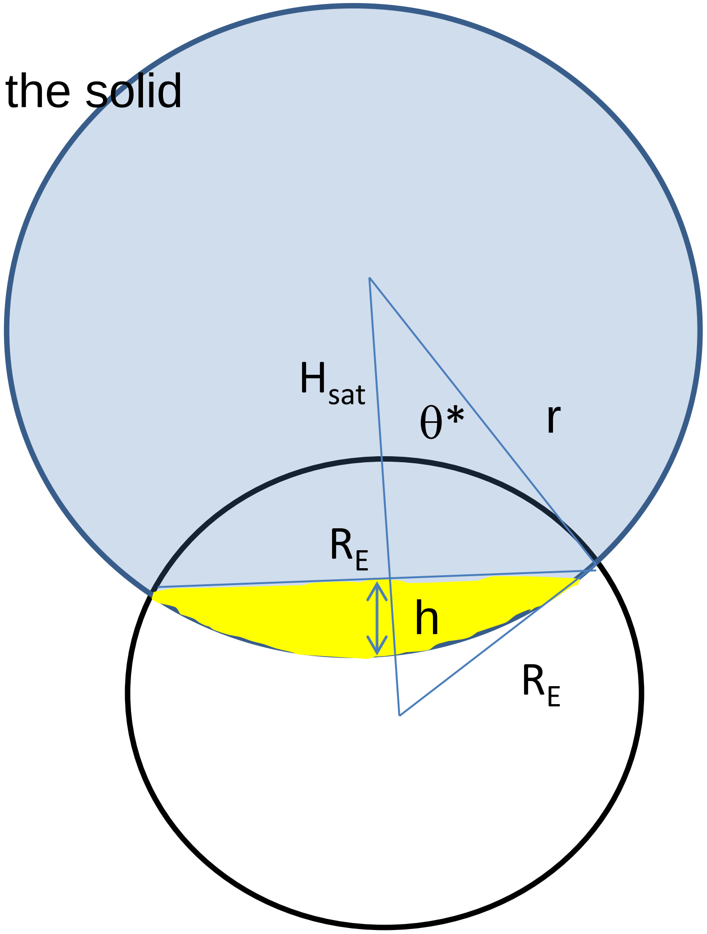
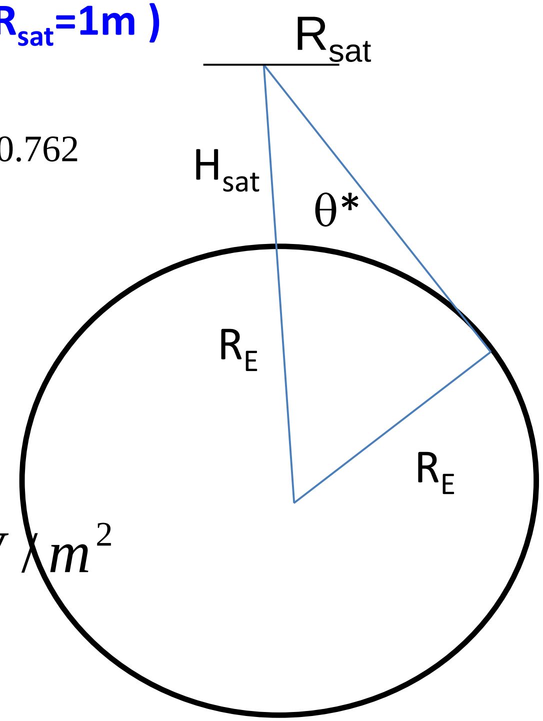
$$A_\perp = 2\pi r h = 2\pi r r \left[1 - \cos(\theta^*)\right] = 2\pi r^2 \left[1 - \cos(\theta^*)\right]$$

$$\Omega = \frac{2\pi r^2 \left[1 - \cos(\theta^*)\right]}{r^2} = 2\pi \left[1 - \cos(\theta^*)\right]$$



If the Earth irradiates as a black body at T=255 K, compute the Earth radiation flux impinging onto the satellite (assumed as a sphere of radius R_{sat}=1m)



$$F_\lambda = \int_{4\pi} d\Omega \cos\theta \, I_\lambda \quad \left[W\, m^{-2}\, m^{-1}\right]$$

$$\sin(\theta^*) = 0.762$$

$$I_\lambda = B_\lambda(T_{Earth})$$

$$F_\lambda = \int_0^{2\pi} d\phi \int_0^{\theta^*} I_\lambda \cos\theta \sin\theta d\theta = 2\pi\, I_\lambda \sin^2\theta^*/2 = \pi\, I_\lambda \sin^2\theta^*$$

I expect to get less than that!!

$$F = \int F_\lambda \, d\lambda = \sigma \underbrace{T^4_{Earth}}_{255^4} \sin^2(\theta^*) = 240 \times 0.58 = 139\, W/m^2$$

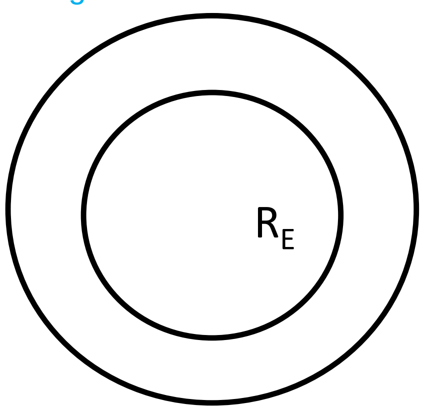
$$\sigma = 5.67 \text{ X } 10^{-8} \; W\, m^{-2} \; K^{-4}$$

Note that for a collimated source, looking straight into it (sun): F=IΔΩ

$$F_{coll\ source} = I\,\Delta\Omega = \frac{240}{\pi} \times 2.21 = 168.8\, W/m^2$$

which is of course larger than F

2^{nd} method



Just imposing conservation of energy:

$$P_\lambda\left[R_E\right] = P_\lambda\left[H_{sat} + R_E\right]$$

$$F_\lambda\left[R_E\right] 4\pi R_E^2 = \pi I_\lambda \; 4\pi \; R_E^2 = F_\lambda\left[H_{sat} + R_E\right] 4\pi \left(H_{sat} + R_E\right)^2$$

Radiative equilibrium [3p]

$$F_\lambda = \pi I_\lambda \sin^2\left(\theta^*\right)$$

$$P_{abs}^{Earth} = \pi R_{sat}^2 \int F_\lambda d\lambda = \pi^2 R_{sat}^2 \underbrace{\int I_\lambda d\lambda}_{\frac{\sigma T_E^4}{\pi}} \sin^2\left(\theta^*\right) = \pi R_{sat}^2\ \sigma T_E^4 \sin^2\left(\theta^*\right) = 437.2\, W$$

F=139.2 W/m^2

239.7 W/m^2

Stefan's Constant 5.670 x 10^{-8} W / m^2 / K^4

$$P_{em} = \sigma\, T_{sat}^4\ 4\pi\, R_{sat}^2$$

$$P_{abs}^{Earth} = P_{em} \Rightarrow T_{sat} = T_E \left[\frac{\sin^2\left(\theta^*\right)}{4}\right]^{0.25} = 255\sqrt{\frac{0.762}{2}} = 157.4\ K$$

Contribution from the Sun [3p]

$$P^{Sun}_{abs} = \pi R_{sat}^2 \int F_\lambda \, d\lambda = \pi^2 R_{sat}^2 \underbrace{\int I_\lambda \, d\lambda}_{\frac{\sigma_{SB} T_S^4}{\pi}} \sin^2(\theta_{Sun}) = \pi R_{sat}^2 \underbrace{\sigma_{SB} T_S^4 \left(\frac{R_{Sun}}{d_{Sun}}\right)^2}_{C_0 = 1380\, W/m^2} = 4331\, W$$

<u>10 times bigger than previous!</u>

T_{sun}=5800 K

Black body assumption

Sun Diameter = 1,390,600 km

Sun-Earth distance=1.5 x 10^8 km

$$P^{Earth}_{abs} + P^{Sun}_{ass} = P^{sat}_{em}$$

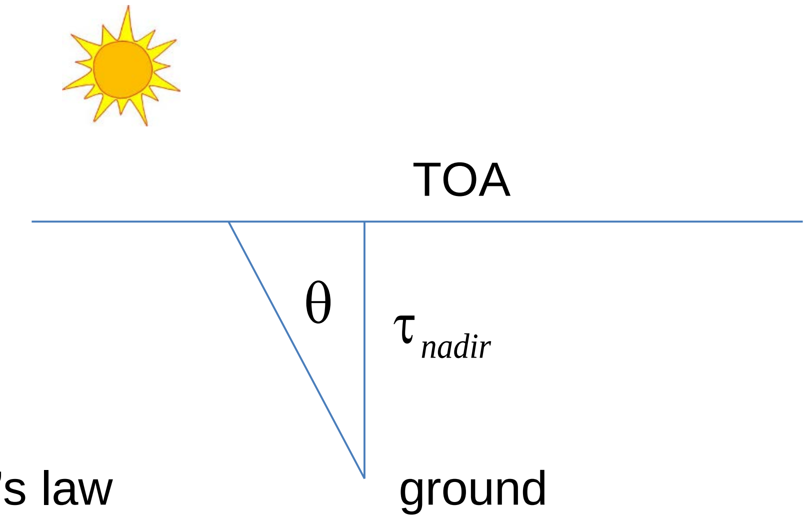
$$\pi R_{sat}^2 \; C_0 + \pi R_{sat}^2 \; \sigma T_E^4 \sin^2(\theta^*) = \sigma \, T_{sat}^4 \; 4\pi \, R_{sat}^2$$

$$T_{sat} = \left[\frac{C_0}{4\sigma} + \frac{T_E^4 \sin^2(\theta^*)}{4} \right]^{1/4} = \left[60.7 \times 10^8 + 6.1 \times 10^8 \right]^{1/4} = 286\, K$$

Strong temperature gradient are therefore expected to be experienced by satellite ➔ proper design needed

Problem 2

A ground-based radiometer operating at $\lambda = 0.45\mu m$ is used to measure the solar intensity at the ground $I_\lambda(0)$. For a solar zenith angle of $\theta = 30°, I_\lambda(0) = 1.74 \times 10^7\ Wm^{-2}\mu m^{-1} sr^{-1}$, and for $\theta = 60°, I_\lambda(0) = 1.14 \times 10^7\ Wm^{-2}\mu m^{-1} sr^{-1}$ is measured. From this information, determine the top-of-the-atmosphere solar intensity S_λ and the atmospheric optical thickness τ_λ.



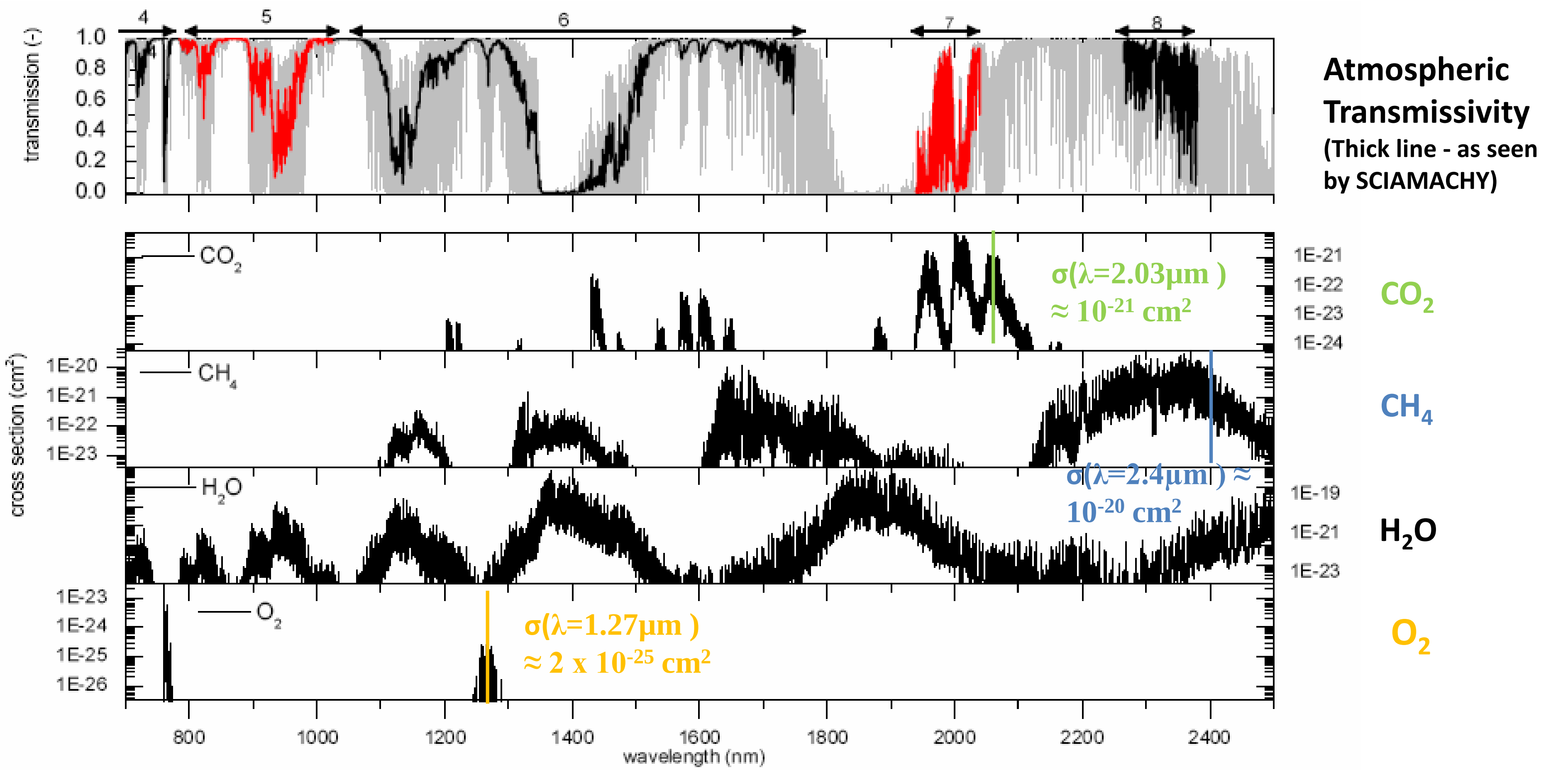
Beer Lambert's law

$$I_{ground}(\theta) = I_{TOA} e^{-\frac{\tau_{nadir}}{\cos(\theta)}} = \begin{cases} I_{TOA} e^{-\frac{2\tau_{nadir}}{\sqrt{3}}} & \theta = 30^o \\ I_{TOA} e^{-2\tau_{nadir}} & \theta = 60^o \end{cases} \Rightarrow \frac{I_{ground}\left(\theta = 30^0\right)}{I_{ground}\left(\theta = 60^0\right)} = e^{\left(2 - \frac{2}{\sqrt{3}}\right)\tau_{nadir}}$$

$$\tau_{nadir} = \ln\left[\frac{I_{ground}\left(\theta = 30^0\right)}{I_{ground}\left(\theta = 60^0\right)}\right] / \left(2 - \frac{2}{\sqrt{3}}\right) = \ln\frac{1.74}{1.14} / 0.85 = 0.5$$

$$I_{TOA} = I_{ground}\left(\theta = 60^o\right) e^{2\tau_{nadir}} = I_{ground}\left(\theta = 60^o\right) e = 3.1 \times 10^7 \; W / (m^2 \; \mu m \; sr)$$

Estimate the transmissivity of well mixed gases CO_2, CH_4 and O_2 in the near-infrared



Estimate the transmissivity of well mixed gases CO_2, CH_4 and O_2 in the near-infrared

Transmissivity $\mathcal{T}$ is given by (ignoring scattering)

$$\mathcal{T}(\lambda) = \exp - \{\sigma(\lambda)\, c\, L\}$$

Or more accurate: $\mathcal{T}(\lambda) = \exp - \int_L \sigma(\lambda, p(L), T(L)) \times c(L)\, dL$

Which can be approximated by : $\mathcal{T}(\lambda) = \exp - \{\sigma(\lambda, \bar{p}, \bar{T}) \times \chi \times VCD_{air}\}$

With vertical column density of air (molecules per area):

VCD = 2.14E25 molecule/cm^2

Well Mixed Gases:

- CO_2: χ = 380 ppm and $\sigma(\lambda = 2.03\ \mu m)$ = 1E-21 cm^2 -> $\mathcal{T}$ = exp{-8.1} = 3E-3
- CH_4: χ = 1800 ppb and $\sigma(\lambda = 2.4\ \mu m)$ = 1E-20 cm^2 -> $\mathcal{T}$ = exp{-0.39} = 0.68
- O_2: χ = 0.2 and $\sigma(\lambda = 1.27\ \mu m)$ = 2E-25 cm^2 -> $\mathcal{T}$ = exp{-0.86} = 0.42

Compare the atmospheric transmissivity for Rayleigh scattering in UV with larger wavelength (600nm, 900 nm)

Rayleigh Cross Section

λ, nm	σ, cm^2
300	6.00 E-26
400	1.90 E-26
600	3.80 E-27
1000	4.90 E-28
10,000	4.85 E-32

Compare the atmospheric transmissivity for Rayleigh scattering in UV with larger wavelength (600nm, 900 nm)

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λ, nm	σ, cm^2
300	6.00 E-26
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1000	4.90 E-28
10,000	4.85 E-32

Transmissivity (ignoring absorption or Mie scattering)

$$\mathcal{T}(\lambda) = I_t(\lambda) / I_0(\lambda)$$

$$= \exp{-\tau} = \exp\{- k_s(\lambda)\, L\}$$

$$= \exp\{- \sigma\, c\, L\} = \exp\{- \sigma\, VCD\}$$

with Vertical Column Density of air (VCD) = 2.14 E25 molec/cm2

Transmissivity: $\mathcal{T}(\lambda\text{=300nm}) = \exp\{-6E\text{-}26 \times 2.14E25\} = \exp\{-1.3\} = 0.27$

$\mathcal{T}(\lambda\text{=600nm}) = \exp\{-0.08\} = 0.92$

$\mathcal{T}(\lambda\text{=1000nm}) = \exp\{-0.01\} = 0.99$

Only 25% of direct light reaches surfaces in UV but 99% at 1000nm.