Black-Litterman model for continuous distributions

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Abstract

The Black-Litterman methodology of portfolio optimization, developed at the turn of the 1990s, combines statistical information on asset returns with investor's views within the Markowitz mean-variance framework. The main assumption underlying the Black-Litterman model is that asset returns and investor's views are multivariate normally distributed. However, empirical research demonstrates that the distribution of asset returns has fat tails and is asymmetric, which contradicts normality. Recent advances in risk measurement advocate replacing the variance by risk measures that take account of tail behavior of the portfolio return distribution. This paper extends the Black-Litterman model into general continuous distributions and deviation measures of risk. Using ideas from the Black-Litterman methodology, we design numerical methods (with variance reduction techniques) for the inverse portfolio optimization that extracts statistical information from historical data in a stable way. We introduce a quantitative model for stating investor's views and blending them consistently with the market information. The theory is complemented by efficient numerical methods with the implementation distributed in the form of publicly available R packages. We conduct practical tests, which demonstrate significant impact of the choice of distributions on optimal portfolio weights to the extent that the classical Black-Litterman procedure cannot be viewed as an adequate approximation. (*JEL*: C52, C61, C63, G11)

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1. Introduction

Black & Litterman (1992) show how to combine statistical information on asset returns with investor's views (private information/beliefs) in the framework of Markowitz portfolio optimization. They contribute to the asset management literature in two distinct ways. Firstly, they postulate that there exists an equilibrium portfolio with which one can associate an *equilibrium distribution* of asset returns (referred to as a *prior distribution* in the present paper). This equilibrium assumption, which follows from the Capital Asset Pricing Model, is used to replace the most unstable parameter of returns, the mean vector, with a vector reverse-engineered from the market portfolio. The prior distribution summarizes neutral information and is significantly less sensitive to estimation errors than estimates purely based on time-series analysis since it utilizes a directly observable quantity – the market portfolio. The second contribution of Black & Litterman (1992) is the process that twists the prior distribution according to *investor's views/opinions* (private information). Views are represented as uncertain predictions about returns of combinations of assets. An application of a Bayesian argument gives a returns distribution (*posterior distribution*) that is subsequently used in a standard Markowitz optimization procedure (all details of the model and its implementation can be found in Litterman et al. (2004) and are also shortly discussed in Section 2). The Black-Litterman model

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considerably improves statistical properties of portfolio recommendations and allows for intuitive incorporation of private information. However, its main assumption is that asset returns and investor's views are multivariate normally distributed, which contradicts observations, e.g., the asymmetry and fat tails of empirical returns (Mandelbrot, 1963; Fama, 1965; Ane & Geman, 2000). Last decade has seen a multitude of extensions and adaptations of the model which go beyond this normality assumption.

The first attempt to extend the Black-Litterman procedure beyond normal markets can be attributed to Meucci (2006), who blends non-normal prior distributions and views using copula approach. Giacometti et al. (2007) extend the computation of prior distributions (in fact the mean of prior distribution) to Student's t and stable distributions and alternative risk measures (VaR and CVaR) using results of Stoyanov et al. (2006). Xiao & Valdez (2015) provide a partial adaptation of Black-Litterman's Bayesian step to elliptical distributions. By assuming that the prior distribution and the distribution of investor's views are jointly elliptical[1] they derive the posterior distribution which is also elliptical. However, the special choice of distributions leads to solutions of optimal portfolio problem identical (modulo a multiplicative adjustment to CVaR) to those obtained in the original Black-Litterman model, c.f. Xiao & Valdez (2015, Proposition 3.1) and Palczewski & Palczewski (2017b). Martellini & Ziemann (2007) consider an extension of the original Black-Litterman methodology to higher moments of the returns distribution (up to the fourth). This extension is distribution independent and is based on the Cornish-Fisher approximation of VaR using higher moments. Gaussian factor models, with investor's predictions affecting factors as well as returns, are incorporated into the Black-Litterman framework by Giacometti & Mignacca (2010); Cheung (2013); Kolm & Ritter (2017).

A series of papers by Meucci (see Meucci (2008b, 2009) and references therein) advocates an ad-hoc method to blend investor's opinions about the market without restrictions on distributions. He uses a technique called distribution pooling to build coordinate-wise mixtures of distributions and applies copulas to enforce dependence between coordinates. This approach, however, lacks any statistical model-based explanation and does not recover the original Black-Litterman's results when distributions are normal. A different approach presented in Cheung (2009) extends the construction of posterior distribution of the Black-Litterman model into a generalized factor view blending framework with non-linear views and general distributions. The paper offers a numerical algorithm to compute a discrete distribution approximating the posterior distribution, similar to the method employed in this paper.

The construction of the prior distribution is equally important for practical applications and has not received much attention except the original paper Black & Litterman (1992). In particular, Meucci (2008b) and Cheung (2009) assume that the prior distribution is known and do not address its calculation in their papers. In the original Black-Litterman model the prior distribution is obtained by an inverse optimization motivated by CAPM. Giacometti et al. (2007) reconstruct prior distribution for Conditional Value-at-Risk (CVaR) and stable distributions for which they use a special property of stable distributions that allows them to simplify the problem to that of the original Black-Litterman procedure but with a modified risk aversion coefficient.

In this work we employ deviation measures due to Rockafellar et al. (2006a,b) and build on a generalization of CAPM provided by Rockafellar et al. (2007). Without any restrictions on the choice of distributions, we characterize the prior distribution as a solution of an inverse optimization problem for which we provide an explicit formula in terms of risk identifiers for the market portfolio. We design efficient numerical methods employing importance sampling for CVaR deviation measure. Those importance sampling results are of their own interest: we derive explicitly the optimal (minimizing the variance of the estimator) importance sampling distribution for CVaR of a Delta approximation to portfolio with normally distributed risk factors.

In the Bayesian step we use a market-based Black-Litterman framework (Meucci, 2005) in which investor's views are expressed for future asset returns. This is in contrast to the original framework and its extension by Kolm & Ritter (2017), where investor's views concern parameters of the future returns distribution. Our choice of the market-based model is motivated by its mathematical tractability and transparency beyond normal distributions. A general framework of Kolm & Ritter (2017) is very elegant for

[1]This implies that the investor's views follows a generalized elliptical distribution, and its covariance matrix depends on a realization of the prior returns.

Gaussian distributions but its extension to non-normal distributions cannot be achieved by a minor modification of the original arguments. Indeed, they assume that asset returns are normally distributed with the mean being a normally distributed random variable leading to a normal unconditional distribution of asset returns. Beyond normal distributions, the unconditional distribution in such hierarchical models does not usually belong to any known family of distributions, hence its estimation is difficult and financial interpretation hard.[2]

Theoretical advances in the paper are complemented by extensive empirical tests in which we explore how the choice of distribution classes for prior distribution and investor's views affect the posterior distribution, optimal portfolio weights and realized portfolio returns. We conclude that different distributions fitted to the same data lead to significantly different optimal portfolios. In particular, portfolios obtained in non-Gaussian frameworks cannot be computed in the Gaussian Black-Litterman model with a tweak of parameters. This emphasizes the importance of choosing a correct distributions for modeling of market risk factors and investor's views. To aid further research in this area, we have made implementations of our numerical algorithms available in the CRAN repository for statistical programming language R (Palczewski & Palczewski, 2017a; Palczewski, 2017).

The paper is organized as follows. Section 2 introduces the market-based Black-Litterman approach and our extension to general distributions and deviation measures. In Section 3 we present numerical methods for computation of the prior and posterior distributions. Empirical examples and the performance of our numerical methods are discussed in Section 4. Section 5 concludes. Electronic Appendix contains additional material, in particular, extensive numerical tests for 5 and 12 Industry Portfolios from Kenneth French's library.

2. The Black-Litterman methodology

In this section we extend a market-based version of the Black-Litterman model due to Meucci (2005) (see also Meucci (2008a) for a discussion) to general continuous distributions and deviation measures of risk. We begin with a short presentation of this approach for normal distributions to pave the ground for our contribution. In the Gaussian market-based Black-Litterman model, the prior distribution of future returns follows $R \sim \mathcal{N}(\mu_{eq}, S)$ with S usually coinciding with the covariance matrix Σ estimated from market data and μ_{eq} derived from CAPM equilibrium theory by the inverse optimization: $\mu_{eq} = \gamma_M \Sigma x_M$, where x_M denotes the market portfolio (tangency portfolio of the mutual fund theorem) and γ_M is the known market risk aversion. The set of investor's views is represented by a 'pick matrix' P collecting in rows combinations of assets, the vector v of forecasted excess returns for those combinations and a covariance matrix Q expressing the uncertainty (the lack of confidence) in the forecasts. The resulting Bayesian model is

$$\begin{aligned} &\text{prior:} \quad R \sim \mathcal{N}(\mu_{eq}, S), \\ &\text{observation:} \quad V|[R = r] \sim \mathcal{N}(Pr, Q). \end{aligned}$$

The posterior distribution of future returns R given $V = v$ is given by the Bayes formula $f_{R|[V=v]} \propto f_{V|[R=r]} f_R$ and is normal with mean $\mu_{BL} = \mu_{eq} + SP^T(Q + PSP^T)^{-1}(v - P\mu_{eq})$ and covariance matrix $\Sigma_{BL} = S - SP^T(Q + PSP^T)^{-1}PS$. In the market-based Black-Litterman approach, these values are directly applied in portfolio optimization in the mean-variance setting.

Our extension of the Black-Litterman model to non-Gaussian distributions is based on the framework introduced by Rockafellar et al. (2006a,b). We assume that the market consists of a riskless asset with a constant return R_0 and n risky assets with random returns $R = (R_1, \ldots, R_n)$. Investor's portfolio is represented by an n-dimensional vector $x = (x_1, \ldots, x_n)$ with coordinates representing fractions of wealth invested in risky assets. The fraction of wealth invested in the riskless asset is $1 - x^T e$, where $e = (1, \ldots, 1)$. Denoting by $L^2(\Omega)$ the space of square integrable random variables, let $\mathcal{D} : L^2(\Omega) \to [0, \infty)$ be a deviation

[2] In a simple case of Student's t distributed returns with the location parameter being itself Student's t distributed with the same degrees of freedom, the unconditional distribution of returns is Student's t if and only if both variables are *jointly* Student's t distributed. In particular, they are not independent, which contradicts the main assumption of classical hierarchical models.

measure which fulfills the conditions of Definition 1 in Rockafellar et al. (2006a); see Section A in Electronic Appendix for a precise definition and properties. We are interested in a portfolio which solves the optimization problem

$$\begin{aligned} &\text{minimize } \mathcal{D}(x^T R^e), \\ &\text{subject to: } x^T \mu^e \geq \bar{r}, \end{aligned} \tag{1}$$

where $R^e = R - R_0 e$ is the vector of excess returns of risky assets and $\mu^e = \mathbb{E}[R^e]$. An important particular solution for this optimization problem is a *master fund* (of positive type): a portfolio $\hat{x}$ such that $\hat{x}^T e = 1$. The master fund plays for optimization problem (1) a similar role as the tangency portfolio in the mean-variance framework. Theorem 2.1 in Rockafellar et al. (2007) guarantees that for problem (1) with $\bar{r} > 0$ there exists the master fund $\hat{x}$ such that $\hat{x} \geq 0$.

Rockafellar et al. (2007) develop an analogue of CAPM which we sketch here. There are I investors who have preference functions of the form

$$U^i(Y^i) = \mathbb{E}[Y^i] - \gamma_i(\mathcal{D}(Y^i))^{q_i}, \tag{2}$$

where Y^i is investor's i wealth at the end of the investment period, γ_i is investor's i risk aversion with respect to deviation measure $\mathcal{D}$ and $q_i > 1$ is a constant. The same deviation measure is shared by all investors. Each of them invests in a portfolio x^i maximizing their preference function U^i subject to portfolio weights being non-negative. Theorems 3.1 and 3.2 in Rockafellar et al. (2007) assert the existence of an equilibrium and the form of optimal portfolios: investor's i portfolio x^i is a positive multiplicity of the master fund corresponding to the deviation measure $\mathcal{D}$.[3] These results are analogous to CAPM and the mutual fund theorem for mean-variance portfolio optimization and, when choosing standard deviation as $\mathcal{D}$, one recovers those well-known results. Following this lead we will construct a prior distribution in the Black-Litterman model.

2.1. *Prior distribution*

Acting in the spirit of the original Black-Litterman approach we assume that the prior distribution is known up to a deterministic shift, i.e., we know its centered version (with the mean zero). The only parameter of the prior distribution which is unknown is its location. To recover the latter, following the market equilibrium rationale presented in the previous paragraph, we interpret the market portfolio as the master fund and calculate the location parameter via an inverse optimization problem: knowing the solution x_M to problem (1) we find the mean excess return vector μ_{eq} for a given expected market return $\bar{r} = r_M$. We defer the discussion on how to obtain r_M to Section 4.

To solve the inverse optimization problem stated above for general deviation measures we use the concept of *risk envelope*. By Rockafellar et al. (2006a, Theorem 1), every lower semicontinuous deviation measure $\mathcal{D}$ can be represented in the form

$$\mathcal{D}(X) = \mathbb{E}[X] + \sup_{Q \in \mathcal{Q}} E[-XQ], \tag{3}$$

where $\mathcal{Q} \subset L^2(\Omega)$ is called risk envelope and can be recovered from $\mathcal{D}$ by

$$\mathcal{Q} = \{\, Q \in L^2(\Omega) \,\big|\, \mathbb{E}[X(1-Q)] \leq \mathcal{D}(X) \ \ \forall X \in L^2(\Omega)\}. \tag{4}$$

Moreover, the set $\mathcal{Q}$ is closed and convex in $L^2(\Omega)$. Elements $Q \in \mathcal{Q}$ for which supremum in (3) is attained are called *risk identifiers* of X and will be denoted by $\mathcal{Q}(X)$. The theorem below is a convenient reformulation of Theorem 4 in Rockafellar et al. (2006c).

[3]The equivalence between maximization of the preference function (2) and the optimization problem (1) follows from Krokhmal et al. (2002, Theorem 3).

Theorem 2.1. *For a continuous deviation measure $\mathcal{D}$ a portfolio x_M is a nonnegative solution to* (1) *with $\bar{r} > 0$ if and only if $x_M^T R^e$ is a random variable (is nonconstant) and there is a risk identifier Q^* for $x_M^T R^e$ such that*

$$\mu_{eq} = \frac{r_M}{\mathcal{D}(x_M^T R^e)} \mathbb{E}[-\hat{R}Q^*] = \frac{r_M}{x_M^T \mathbb{E}[-\hat{R}Q^*]} \mathbb{E}[-\hat{R}Q^*], \tag{5}$$

where $\hat{R} = R^e - \mathbb{E}[R^e]$ is the centered prior distribution. For the solution x_M the constraints in (1) *are binding, i.e. they are all equalities.*

Remark 2.2. Note that the solution in Theorem 2.1 to the inverse portfolio optimization problem can only be interpreted in view of the market equilibrium theory presented at the beginning of the present section: if appropriate assumptions stated in Section 2 of Rockafellar et al. (2007) are satisfied, which in terms of excess returns reads $\mu_{eq} > 0$. Fulfillment of this inequality depends on the distribution of excess returns and the form of market portfolio. Hence, only if the outcome of the inverse optimization $\mu_{eq} > 0$, there is a theoretical backing for the market portfolio x_M being a master fund which is shared by all investors.

To make the results of Theorem 2.1 more transparent to the reader we present explicit computations of μ_{eq} for three deviation measures: the mean absolute deviation, CVaR deviation, and standard deviation. Their risk envelopes are taken from Rockafellar et al. (2006c).

Example 2.3 (Mean absolute deviation)**.** Let $X \in L^2(\Omega)$ be a random variable. apThe Mean absolute deviation of X is defined as

$$MAD(X) = \mathbb{E}[|X - \mathbb{E}[X]|].$$

Mean absolute deviation is a finite, continuous deviation measure. The set of risk identifiers is given by

$$\mathcal{Q}(X) = \{Q = 1 + \mathbb{E}[Z] - Z \colon Z \in \operatorname{sign}(X - \mathbb{E}[X])\},$$

where

$$\operatorname{sign}(Y) = \begin{cases} +1, \text{for } Y(\omega) > 0, \\ -1, \text{for } Y(\omega) < 0, \\ \xi \in [-1, 1], \text{for } Y(\omega) = 0. \end{cases}$$

For $X = x_M^T \hat{R}$, using $\mathbb{E}[\hat{R}] = 0$, we obtain

$$\mathbb{E}[-\hat{R}Q^*] = \mathbb{E}[\hat{R}Z] \quad \text{for } Z \in \operatorname{sign}(x_M^T \hat{R}).$$

When $\mathbb{P}[x_M^T \hat{R} = 0] = 0$, there is a unique Z and

$$\mathbb{E}[\hat{R}Z] = \mathbb{E}[\hat{R}\mathbb{1}_{\{x_M^T \hat{R} > 0\}}] - \mathbb{E}[\hat{R}\mathbb{1}_{\{x_M^T \hat{R} < 0\}}] = 2\mathbb{E}[\hat{R}\mathbb{1}_{\{x_M^T \hat{R} > 0\}}],$$

where $\mathbb{1}_A$ denotes the indicator function of an event A, i.e. $\mathbb{1}_A = 1$ if A holds and $\mathbb{1}_A = 0$ otherwise. Hence

$$\mu_{eq} = \frac{r_M}{\mathbb{E}[x_M^T \hat{R}\mathbb{1}_{\{x_M^T \hat{R} > 0\}}]} \mathbb{E}[\hat{R}\mathbb{1}_{\{x_M^T \hat{R} > 0\}}].$$

Example 2.4 (Deviation CVaR)**.** Deviation CVaR on confidence level α for a random variable X is defined as

$$\mathrm{CVaR}_\alpha^D(X) = \mathbb{E}[X] - \frac{1}{\alpha} \int_0^\alpha q_X(s) ds,$$

where $q_X(\beta)$ is a β-quantile of X. The set of risk identifiers $\mathcal{Q}(X)$ comprises random variables Q with values in $[0, 1]$, $\mathbb{E}[Q] = 1$, and

$$Q = \begin{cases} 0 \ \text{ if } \ X > -VaR_\alpha(X), \\ 1/\alpha \ \text{ if } \ X < -VaR_\alpha(X). \end{cases}$$

For $X = x_M^T \hat{R}$, if $\mathbb{P}(x_M^T \hat{R} = VaR_\alpha(x_M^T \hat{R})) = 0$ (this holds, for example, when the distribution of $\hat{R}$ is continuous) then there is a unique risk identifier Q^* and

$$\mu_{eq} = \frac{1}{\alpha} \frac{r_M}{\mathrm{CVaR}_\alpha^D(x_M^T \hat{R})} \mathbb{E}\left[-\hat{R}\mathbb{1}_{\{x_M^T \hat{R} \le -VaR_\alpha(x_M^T \hat{R})\}}\right] = \frac{r_M}{\mathbb{E}\left[-x_M^T \hat{R}\mathbb{1}_{\{x_M^T \hat{R} \le -VaR_\alpha(x_M^T \hat{R})\}}\right]} \mathbb{E}\left[-\hat{R}\mathbb{1}_{\{x_M^T \hat{R} \le -VaR_\alpha(x_M^T \hat{R})\}}\right]. \tag{6}$$

To relate this framework to a classical measure of risk, we provide the following example.

Example 2.5 (Standard deviation)**.** For a random variable X, the set of risk identifiers for standard deviation $\sigma(X)$ is a singleton

$$\mathcal{Q}(X) = \left\{1 - \frac{1}{\sigma(X)}(X - \mathbb{E}[X])\right\}.$$

For $X = x_M^T \hat{R}$, using $\mathbb{E}[\hat{R}] = 0$, we obtain

$$\mathbb{E}[-\hat{R}Q^*] = \mathbb{E}[\hat{R}Z] \quad \text{for } Z = \frac{1}{\sigma(x_M^T \hat{R})} x_M^T \hat{R}.$$

Further,

$$\mathbb{E}[\hat{R}Z] = \frac{1}{\sigma(x_M^T \hat{R})}\mathbb{E}[\hat{R}x_M^T\hat{R}] = \frac{1}{\sigma(x_M^T \hat{R})}\mathbb{E}[\hat{R}\hat{R}^T x_M] = \frac{1}{\sigma(x_M^T \hat{R})}\Sigma x_M,$$

where $\Sigma = Cov(\hat{R})$, which recovers the classical result

$$\mu_{eq} = \frac{r_M}{x_M^T \Sigma x_M}\Sigma x_M.$$

2.2. Blending of views for continuous distributions

The market-based Black-Litterman methodology reduces to Bayesian inference in the model of the following type:

$$\begin{aligned} \text{prior } Y &\sim f_Y(\cdot), \\ \text{observation } V &\sim f_V(\cdot|Y), \end{aligned} \tag{7}$$

where f_Y is a density function of the prior distribution (the prior knowledge derived from market equilibrium) and $f_V(\cdot|Y)$ is a family of density functions parametrized with Y (corresponding to investor's opinions). As explained above, we postulate $f_Y(\cdot) \sim \mu_{eq} + \hat{R}$. We will usually assume the distribution of observations f_V to be normal (or elliptical) with mean PY and variance Q proportional to $P^T \Sigma P$ (or the diagonal of thereof if investor's opinions are thought to be burdened with uncorrelated errors) with $\Sigma = Cov(R)$.

The posterior distribution of Y given $V = v$ forms a basis for portfolio optimization. Bayes theorem implies that this posterior distribution is continuous with the density

$$f(y|V = v) \propto f_V(v, y) f_Y(y), \tag{8}$$

where the sign $\propto$ means 'proportional to'. The following section is devoted to the development of efficient numerical procedures for representing the posterior distribution in a form suitable for portfolio optimization.

3. Numerical algorithms for continuous distributions

Inverse optimization as well as computation of the posterior distribution require numerical methods. In a particular case of elliptically distributed asset returns R^e, the inverse optimization can be performed analytically, but even then the posterior distribution is not elliptical and one needs to resort to numerical computations (Palczewski & Palczewski, 2017b). Therefore, this section presents methods which can be used for each step of the generalized Black-Litterman procedure. We make a standing assumption that prior distributions and distributions of observations are continuous with strictly positive density. We will also restrict exposition to CVaR deviation measure although some of the presented algorithms (e.g., Algorithm 3.1) can be adapted to other deviation measures of risk. Implementations for deviation CVaR, MAD and LSAD (Lower Semi Absolute Deviation) are collected in the R package BLModel (Palczewski & Palczewski, 2017a).

3.1. Plain Monte Carlo calculation of z

```
Input: N (Monte Carlo iterations), f_R̂ (density of R̂)

For i = 1,2,...,N do
   Draw r_i ~ f_R̂
   Set y_i = x_M^T r_i
EndFor

Sort (y_i)_{i=1,2,...,N} in ascending order

Set i_VaR = ⌊(1 − α)N⌋
Set z = −(1/i_VaR) Σ_{i=1}^{i_VaR} r_i

Output: z
```

3.1. Prior distribution

By (6), the prior mean return μ_{eq} is given by $\frac{r_M}{x_M^T z} z$ with

$$z = \mathbb{E}\left[-\hat{R}\mathbf{1}_{\{x_M^T \hat{R} \le -VaR_\alpha(x_M^T \hat{R})\}}\right],$$

where x_M is the market portfolio and $\hat{R} = R^e - \mathbb{E}[R^e]$ denotes the centered prior returns. Algorithm 3.1 shows plain Monte Carlo estimate of z. One generates a large number of returns $r_1, \ldots, r_N$ from the distribution $\hat{R}$ and then averages those for which the portfolio return is below $-\mathrm{VaR}_\alpha(x_M^T \hat{R})$ which itself is approximated by a $(1-\alpha)$-quantile of the empirical distribution $\{x_M^T r_i,\ i = 1, \ldots, N\}$. Standard arguments show that as N increases, the estimates converge to the true value of z.

Algorithm 3.1 can be significantly improved by exploiting the link between z and $\mathrm{CVaR}_\alpha^D(x_M^T \hat{R}) = x_M^T z$ adapting techniques used in Monte Carlo estimates of CVaR. The estimate of z depends largely on those returns r_i for which the portfolio return falls below $-\mathrm{VaR}(x_M^T \hat{R})$. With α large, this is just a small fraction of generated returns. This fraction can be increased by employing *importance sampling*. Algorithm 3.2 provides details of implementation. There are a few features of the algorithm which are crucial for its practical performance. Firstly, the importance sampling weights w_i are not normalized to 1 contrary to what is often done in similar cases. In practice, the sum of weights may significantly differ from 1 and only approach it for a very large number of samples. This is brought by a small number of disproportionately large weights that arise when a point r_i drawn from the tail of the importance sampling distribution f_{IS} falls into the centre of the original distribution $f_{\hat{R}}$. This small probability event is particularly affected by numerical errors in calculation of densities and in the generation of samples from f_{IS} so it may happen more often or result in higher weights in practical calculations than in theory. Normalization of weights would allow those outliers to affect all other weights and we have seen its significant negative effect in numerical experiments: the empirical variance of the CVaR estimator was over 6 orders of magnitude larger than the theoretical value. Another feature of Algorithm 3.2 is that the value-at-risk is computed as $(1-\alpha)$-quantile of the distribution of portfolio return against an α quantile of the distribution of losses. This again minimizes the effect of the outliers mentioned above which generally fall below VaR_α.

As hinted above, there is a strong link between the vector z and $\mathrm{CVaR}_\alpha(x_M^T \hat{R})$. This motivates designing the importance sampling distribution which minimizes the variance of the estimator of $\mathrm{CVaR}_\alpha(x_M^T \hat{R})$. In our portfolio problem the dependence on risk factors is linear making the task similar to Delta based importance sampling used for VaR and CVaR (see Glasserman et al. (2002) for the multivariate elliptical case). The aforementioned paper and others that study non-linear portfolios (see Glasserman (2003, Chapter 9) for classical results) do not minimize the variance of VaR and CVaR but rather design an importance sampling distribution which positions the expectation of the portfolio return at the best guess of $-\mathrm{VaR}_\alpha(x_M^T \hat{R})$.

3.2. Importance Sampling Monte Carlo calculation of z

```
Input: N (Monte Carlo iterations), f_R̂ (density of R̂), f_IS (importance sampling
density)

For i = 1,2,...,N do
    Draw r_i ~ f_IS
    Set y_i = x_M^T r_i
    Set w_i = (1/N) f_R̂(r_i)/f_IS(r_i)
EndFor

Sort (y_i, w_i)_{i=1,2,...,N} in ascending order of the first coordinate

Set i_VaR = max{i : sum_{k=1}^i w_k <= 1 - alpha}
Set z = -( sum_{i=1}^{i_VaR} w_i )^{-1} sum_{i=1}^{i_VaR} w_i r_i

Output: z
```

Input: N (Monte Carlo iterations), $f_{\hat{R}}$ (density of $\hat{R}$), f_{IS} (importance sampling density)

For $i = 1, 2, \dots, N$ do
  Draw $r_i \sim f_{IS}$
  Set $y_i = x_M^T r_i$
  Set $w_i = \frac{1}{N} f_{\hat{R}}(r_i)/f_{IS}(r_i)$
EndFor

Sort $(y_i, w_i)_{i=1,2,\dots,N}$ in ascending order of the first coordinate

Set $i_{\mathrm{VaR}} = \max\{i : \sum_{k=1}^{i} w_k \leq 1 - \alpha\}$
Set $z = -\left(\sum_{i=1}^{i_{\mathrm{VaR}}} w_i \right)^{-1} \sum_{i=1}^{i_{\mathrm{VaR}}} w_i r_i$

Output: z

3.3. Importance Sampling algorithm when f_{IS} is obtained by shifting $f_{\hat{R}}$ by m

Input: $f_{\hat{R}}$ (density of $\hat{R}$), $r_1, \dots, r_N$ (sample from $f_{\hat{R}}$), m

For $i = 1, 2, \dots, N$ do
  Set $r_i = r_i + m$
  Set $y_i = x_M^T r_i$
  Set $w_i = \frac{1}{N} \frac{f_{\hat{R}}(r_i)}{f_{\hat{R}}(r_i - m)}$
EndFor

The remaining steps are identical as in Algorithm 3.2

Recently, Sun & Hong (2010) derived an exact formula for the variance of CVaR. Since CVaR can be numerically approximated in a number of ways, we present a short summary of necessary notations and definitions.

Let $(r_1, \dots, r_N)$ be a sample from f_{IS}. Denote an empirical importance sampling cumulative distribution function

$$F_N(z) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{x_M^T r_i \leq z\}} l(r_i),$$

where $l(r) = f_{\hat{R}}(r)/f_{IS}(r)$. An importance sampling estimator of VaR is given by

$$\hat{v}_\alpha^N = -F_N^{-1}(1 - \alpha)$$

and an importance sampling estimator of CVaR is

$$\hat{c}_\alpha^N = \hat{v}_\alpha^N + \frac{1}{n(1 - \alpha)} \sum_{i=1}^{N} (-x_M^T r_i - \hat{v}_\alpha^N)^+ l(r_i).$$

Extending results of Sun & Hong (2010) to a multivariate setting, we obtain the following formula for asymptotic variance of $\hat{c}_\alpha^N$.

Theorem 3.1. *Assume that l is bounded on every set of the form $\{r \in \mathbb{R}^n : |x^T r - a| < \varepsilon\}$ for any $a \in \mathbb{R}$ and a sufficiently small $\varepsilon > 0$ depending on a. If $\mathbb{E}_{f_{IS}}[l^p(\hat{R})] < \infty$ for some $p > 2$, then*

$$\sqrt{N}\Big(\hat{c}_\alpha^N - CVaR_\alpha(x^T\hat{R})\Big) \implies \mathcal{N}\left(0, \frac{Var_{f_{IS}}\big[(-x^T\hat{R} - VaR_\alpha(x^T\hat{R}))^+ l(\hat{R})\big]}{(1-\alpha)^2}\right),$$

where $\implies$ denotes convergence in distribution.

Calculation of the asymptotic variance in the above theorem is impossible for general distributions. We propose therefore a procedure which involves approximation of the distribution of centred returns $\hat{R}$ with a centred elliptical distribution R^a (e.g., normal or Student's t-distribution). We optimize the calculation of $\mathrm{CVaR}_\alpha(x_M^T R^a)$ by minimizing the asymptotic variance over a family of importance sampling distributions which are affine transformations $r \mapsto m + Ar$ of R^a with A being an invertible $n \times n$ matrix and $m \in \mathbb{R}^n$.[4] We then apply this affine transformation to the original density of $\hat{R}$, i.e., postulate to sample from the distribution of $m^* + A^*\hat{R}$, where m^*, A^* are the minimizers of the asymptotic variance for R^a. This procedure can be applied to any distribution for which the density $f_{\hat{R}}$ is known and a sampling scheme is available. The importance sampling density can be expressed in terms of $f_{\hat{R}}$ by using the change of variables formula

$$f_{IS}(r) = f_{\hat{R}}\Big((A^*)^{-1}(r - m^*)\Big)\,\Big|\det A^*\Big|^{-1}.$$

We have developed and empirically tested this importance sampling procedure, but the results were not sufficiently improved in comparison to those obtained by only shifting the distribution by m (i.e., taking $A = \mathrm{Id}$) to merit inclusion of substantially more complicated calculations in this paper. This was particularly true when an elliptical distribution was used to approximate a non-elliptical one, for example, a mixture of Gaussian distributions. Moreover, the linear transformation A which may benefit the computation of CVaR by scaling and rotating the coordinates in response to portfolio composition does not necessarily improve the calculation of z which averages the multivariate returns for which the portfolio return falls below $-$VaR. In conclusion, we restrict our attention to importance sampling distributions arising by shifting $\hat{R}$ by a vector m. Algorithm 3.3 provides the details of implementation. The calculation of optimal m is developed in the following paragraphs.

Recall that the density of R^a has the form $y \mapsto g(y^T D^{-1} y)$ for a dispersion matrix D, and a density generator $g : [0,\infty) \to (0,\infty)$. Section 4.2 in Stoyanov et al. (2006) gives the formula for the value-at-risk of $x_M^T R^a$ (since R^a is centered $\mu = \mathbb{E}(R^a) = 0$): $\mathrm{VaR}(x_M^T R^a) = \beta := \mathcal{B}_\alpha \sqrt{x_M^T \Sigma^a x_M}$, where Σ^a is the covariance matrix of R^a. Hence, $\mathrm{CVaR}_\alpha(x_M^T R^a) = \mathbb{E}\left[x_M^T R^a \mathbb{1}_{\{x_M^T R^a \le -\beta\}}\right]$. Minimizing the variance of this estimator over shifts $m \in \mathbb{R}^n$ is equivalent to minimizing the expression

$$\int_{\mathbb{R}^n} (x_M^T y + \beta)^2 \mathbb{1}_{\{x_M^T y \le -\beta\}} \frac{g^2(y^T D^{-1} y)}{g((y-m)^T D^{-1}(y-m))} dy. \tag{9}$$

The above problem has to be solved numerically and requires numerical evaluation of the integral. Interested reader is referred to Glasserman et al. (2002) for discussion of more direct albeit approximate formulas for transformations reducing the variance of VaR (but not to an optimal level), and, as the authors argue, also improving estimation of CVaR. Here, we pursue exact results for the normal distribution, i.e., $g(z) = e^{-z/2}$, D is the covariance matrix of R^a and (9) simplifies to

$$\int_{\mathbb{R}^n} (x_M^T y + \beta)^2 \mathbb{1}_{\{x_M^T y \le -\beta\}} \frac{1}{(2\pi)^{n/2}\sqrt{\det D}} e^{-\frac{1}{2}\left(y^T D^{-1} y - m^T D^{-1} m + 2 y^T D^{-1} m\right)} dy. \tag{10}$$

Under the assumption of normality of R^a we obtain the following theorem.

[4] This family includes exponential twisting widely applied in literature in the framework of normal and, more general, elliptical distributions, see e.g.Glasserman et al. (2002), Glasserman (2003, Chapter 9), Sun & Hong (2010).

Theorem 3.2. *The asymptotic second moment* (10) *of* $\hat{c}_\alpha^N$ *with the importance sampling distribution obtained by shifting* $\hat{R}$ *by* $m \in \mathbb{R}^n$ *is given by*

$$\frac{1}{(1-\alpha)^2} e^{m^T D^{-1} m} \left(\Phi\left(-\frac{\mu}{\sigma}\right)(\mu^2 + \sigma^2) - \frac{\mu\sigma}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}} \right), \tag{11}$$

where $\sigma^2 = x_M^T D x_M$ *and* $\mu = \beta - x_M^T m$.

Proof. Notice that

$$y^T D^{-1} y - m^T D^{-1} m + 2y^T D^{-1} m = (y+m)^T D^{-1}(y+m) - 2m^T D^{-1} m.$$

Hence

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left((x_M^T y + \beta)^+\right)^2 \mathbb{1}_{\{x_M^T y \le -\beta\}} \frac{1}{(2\pi)^{n/2}\sqrt{\det D}} e^{-\frac{1}{2}\left(y^T D^{-1} y - m^T D^{-1} m + 2y^T D^{-1} m\right)} dy \\
&= e^{m^T D^{-1} m} \int_{\mathbb{R}^n} \left((x_M^T y + \beta)^+\right)^2 \mathbb{1}_{\{x_M^T y \le -\beta\}} \frac{1}{(2\pi)^{n/2}\sqrt{\det D}} e^{-\frac{1}{2}\left((y+m)^T D^{-1}(y+m)\right)} dy \\
&= e^{m^T D^{-1} m} \mathbb{E}\left[(x_M^T Y + \beta)^2 \mathbb{1}_{\{x_M^T Y + \beta \le 0\}}\right],
\end{aligned}$$

where $Y \sim N(-m, D)$. Denoting $Z = x_M^T Y + \beta$ we get $Z \sim N(\beta - x_M^T m, x_M^T D x_M)$. We conclude the proof by observing that for $X \sim N(\mu, \sigma^2)$ the following equalities hold

$$\mathbb{E}\left[X^2 \mathbb{1}_{\{X \le 0\}}\right] = \Phi\left(-\frac{\mu}{\sigma}\right)(\mu^2 + \sigma^2) - \frac{\mu\sigma}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}}, \tag{12}$$

$$\mathbb{E}\left[X \mathbb{1}_{\{X \le 0\}}\right] = \mu\Phi\left(-\frac{\mu}{\sigma}\right) - \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}}. \tag{13}$$

For the reader's convenience the above elementary results are derived in Section B in Electronic Appendix. □

Although numerical minimization of the non-linear function (11) seems complicated, one should notice that the function is smooth in m and fast optimization methods (e.g., Newton-Rawson) can be applied. Moreover, it is worth employing importance sampling algorithm if the choice of m reduces the variance of the estimator, i.e., the value of the function (11), so great benefits can be reaped if one finds m which is not necessarily optimal but in a reasonable vicinity of optimality.

3.2. Posterior distribution

In the majority of cases, the posterior distribution of Y given $V = v$ in the Bayesian inference problem (7) cannot be written in a closed form. Formula (8) defines it up to a multiplicative normalizing constant. Finding this constant by numerical integration suffers from the course of dimensionality. Furthermore, the market-based Black-Litterman approach requires optimization with respect to the full posterior distribution. Given non-parametric form of this distribution, we represent it by a finite but arbitrarily large sample.

It is appealing to use importance sampling idea to generate samples from the posterior distribution. Taking as a sampling density f_Y, the ratio of the posterior density to the sampling density equals f_V up to an unknown normalizing constant, see Algorithm 3.4.[5] The output is a discrete distribution $(y_i, p_i)_{i=1,2,\ldots,N}$ approximating the posterior distribution. Probabilities p_i vary with i but this is easily accommodated in portfolio optimization with CVaR, c.f. Rockafellar & Uryasev (2002) and our implementation Palczewski (2017). This approach performs well when the posterior density is not 'too far' from the prior f_Y, which we expect to be true in most practical applications. If the posterior puts most of its weight where the prior density is small, there are large variations in the magnitude of weights w_i. This results in the approximating distribution (y_i, p_i) having many probabilities p_i close to 0 meaning that very large values of N are required

[5] Algorithm 3.4 is also employed by Cheung (2009) in his extension of the Black-Litterman procedure to factor models.

3.4. Plain Monte Carlo sampling from $f(y|V = v)$

```
Input: N (Monte Carlo iterations)

For i = 1,2,...,N do
   Draw y_i ~ f_Y(y)
   Set w_i = f_V(v, y_i)
EndFor

Set (p_1,...,p_N) = (w_1 / Σ_{i=1}^N w_i, ..., w_N / Σ_{i=1}^N w_i)

Output: (y_i, p_i)_{i=1,2,...,N}
```

to obtain a reasonable closeness to the posterior distribution $f(y|V = v)$.[6] Another feature of Algorithm 3.4 is that estimators of functionals of posterior distribution (such as the expectation used in the classical Black-Litterman procedure) are consistent but exhibit bias. This is due to the normalization of weights by their sum instead of using the exact normalizer $\left(\int f_V(v, y) f_Y(y) dy\right)^{-1}$. However, we found Algorithm 3.4 adequate in the numerical experiments in Section 4: the bias was negligible and we did not experience the degeneracy of weights (results not reproduced in the paper).

4. Illustration

This section puts our theory to a test. In particular, we show that

(A) the choice of prior and views distributions has a profound impact on optimal portfolio weights and posterior distributions;

(B) Gaussian distributions (i.e., the original Black-Litterman approach) do not provide a sufficiently close approximation for practical portfolio optimisation tasks;

(C) the numerical methods presented in previous sections perform well in practice.

We will prove the above points on an example of real portfolio problems for 5 US Industry Portfolios (corresponding to industry sectors ”Consumer”, ”Manufacturing”, ”HiTech”, ”Healthcare” and ”Other”) and 12 US Industry Portfolios (”Consumer NonDurables”, ”Consumer Durables”, ”Manufacturing”, ”Energy”, ”Chemicals”, ”Business Equipment”, ”Phone and TV”, ”Utilities”, ”Shops”, ”Healthcare”, ”Finance” and ”Other”) from Kenneth French’s library. As mentioned in the introduction, the choice of a relatively small number of assets has two reasons. Black-Litterman approach is widely used for making strategic investment decisions about splitting funds between small number of classes of assets with views produced by analytical teams, c.f. Litterman et al. (2004); Morgan Stanley Investment Management (2015); Nystedt et al. (2016). The relatively small number of assets in 5 US Industry Portfolios aids clarity in comparison of posterior distributions and optimal portfolios arising under different distributional assumptions. In computations we measure risk using dispersion CVaR but qualitatively similar results are obtained for mean absolute deviation (MAD) and lower semi absolute deviation (LSAD).

We perform two studies. In the first one, we test our extension of the Black-Litterman model at a single date. We compute optimal portfolios for four different scenarios and compare portfolio weights for that

[6] The degeneracy of weights of Algorithm 3.4 may be alleviated by applying Monte Carlo Markov Chain (MCMC) methods. The idea behind MCMC is to construct a Markov chain with a stationary distribution equal to the distribution in question. However, our experience shows that only recent advances, such as MALA, Metropolis-adjusted Langevin algorithm (Roberts & Rosenthal, 1998), may provide benefits in terms of running time and accuracy in practical applications. Their presentation and analysis is beyond the scope of this paper.

scenario	prior distribution	views distribution
BL	normal	normal
SS	Student's t	Student's t
TSK	skew Student's t	Student's t
BM	Bernoulli mixture	normal

Table 1: Four scenarios analyzed in Section 4.

scenarios. We test also accuracy of our computational algorithms. In the second study we compare performance of the Black-Litterman approach with different classes of distributions in a multi-period investment using variety of views.

Scenarios and calibration We consider 4 scenarios for the choice of the prior distribution and the distribution of investor's views, see Table 1. Scenario BL follows the original Black-Litterman paper: prior distribution and distribution of views are normal which leads to a normal posterior with parameters given by analytical formulas. In scenario SS, prior distribution is Student's t (as suggested in many empirical papers). For investor views we take Student's t-distribution with the same number of degrees of freedom as the prior distribution. The reader might notice the similarity between scenario SS and the setting of Xiao & Valdez (2015), where the authors provide analytical formulas for parameters of the posterior distribution. We recall two reasons why we do not use their results referring the reader to Palczewski & Palczewski (2017b) for a detailed discussion. Firstly, Xiao & Valdez (2015) require that the prior distribution and distribution of views are *jointly* Student's t, which implies that the variance of views depends on the realization of prior returns unlike in the original Black-Litterman framework (see Section 2) therefore losing the interpretation of confidence in investor's forecasts. Secondly, optimal portfolios for their posterior distribution lie on the efficient frontier obtained in the original Black-Litterman model for normal distributions.

Adding skewness to Student's t multivariate distribution is a natural extension beyond elliptical distributions. There are many skew extensions of Student's t-distribution used for financial data (cf. Adcock (2010), Zhu & Galbraith (2010) and Adcock et al. (2015) and references therein). In this paper in scenario TSK we use the skew t-distribution of Azzalini & Capitanio (2003) for the prior and Student's t-distribution for investor's views as in scenario SS.

Behr & Pötter (2009) suggest that a Bernoulli mixture of normal distributions is a preferred model for stock returns over medium time horizons. This distribution is multi-modal unlike distributions in the other scenarios. Scenario BM comprises Bernoulli mixture prior distribution with normal distribution of investor's views as in scenario BL.

For all scenarios, prior distributions are calibrated to the daily data from from Kenneth French's library. For scenario BL the covariance matrix Σ of the prior distribution is computed as a sample covariance. For scenarios SS, TSK and BM, we fitted the parameters of corresponding distributions to market data using Expectation Maximization algorithm (Lee & McLachlan, 2013; Wang et al., 2017). We calculated the market expected return r_M assuming that the market Sharpe ratio SR is 0.5 (c.f. He & Litterman (2002)): $r_M = 0.5\sqrt{x_M^T \Sigma x_M}$, where x_M is the market portfolio. In the first two scenarios, Theorem 2.1 provides an analytical expression for the mean of the prior distribution (see Theorem 4.4 in Palczewski & Palczewski (2017b) for an explicit formula), while scenarios TSK and BM require numerical calculations.

Computation of posterior distribution. In our scenarios, the distribution of views requires specification of the covariance matrix Q, c.f. Section 2. The literature offers a number of approaches for constructing Q, but this is not of prime importance in this section. We, therefore, follow the spirit of the original Black-Litterman approach, i.e., we assume that Q is a diagonal matrix with the diagonal elements proportional to the corresponding assets variances. In scenarios BL and BM the distribution of investor's views is normal with the covariance matrix $Q = diag\left(\frac{1}{\tau}P\Sigma P^T\right)$, where Σ is the covariance matrix of the prior distribution and P is the pick matrix. The Student's t-distribution of views in scenarios SS and TSK employs diagonal dispersion matrix D_Q

$$D_Q = diag\left(\frac{1}{\tau}PDP^T\right),$$

assets	excess return
Consumer	-1.78
Manufacturing	10.74
HiTech	-46.25
Healthcare	26.49
Other	8.63

Table 2: Realized excess returns over the period from Jan-01-2000 to Dec-31-2000 (%).

absolute views	assets are predicted to have the excess return as in Table 2
relative views	the relative performance of assets is predicted as follows: return of asset 2 – return of asset 1 = 12.52% return of asset 4 – return of asset 3 = 72.74% return of asset 5 – return of asset 3 = 54.88%

Table 3: Two types of views considered in Section 4.1.

where D is the dispersion matrix of the prior Student's t-distribution of scenario SS. The constant τ represents confidence in the views and is decided by an investor.

As we remarked earlier scenario BL follows the original Black-Litterman approach which leads to a normal posterior with parameters given by analytical formulas. In scenario SS parameters of the prior distribution are given by analytical formulas (c.f. Palczewski & Palczewski (2017b)), while the posterior distribution, which is not elliptical, requires numerical methods developed in Section 3.2. Prior distributions for scenarios TSK and BM are computed using the results of Theorem 2.1 and the Monte Carlo procedure of Section 3.1, and posterior distributions are obtained by numerical methods from Section 3.2.

4.1. Single date analysis

In this subsection we analyze optimal portfolios obtained for Jan-01-2000 for 5 US Industry portfolios. We calibrate the distributions for the four scenarios using daily data from Jan-01-1990 to Dec-31-1999 (rescaled to a yearly basis). The choice of this data period for testing the model accuracy and performance of numerical algorithms is motivated by the fact that it was relatively quiet period on the market without big crashes or enormous growth periods. The market portfolio x_M is based on the market value of the 5 US Industry Portfolios on Jan-01-2000 (which can be obtained from data provided by Kenneth French).

Views. A key ingredient in the Black-Litterman methodology is investor's views – subjective forecasts of future returns of assets. For illustrative purposes, we replace subjective judgments with those based on realized excess returns[7] over the period of one year starting on Jan-01-2000, see Table 2. Extreme returns, in particular for HiTech, are caused by the burst of the dotcom bubble and highlight differences in accommodation of extreme views for various combinations of the prior distribution and the distribution of investor's views. We will explore two types of views: absolute views which prescribe future returns for each asset independently, and relative views which provide forecasts of relative behaviour of asset returns, see Table 3. In all computations in this subsection, we choose the confidence parameter of views $\tau = 0.02$.

Tables 4-5 collect expected returns for the prior and posterior distributions. They do not provide a full description of posterior distributions because the dispersion of returns around the mean is different for the prior and posterior distributions.

[7]The usage of realized returns do not impose any bias in discussion as we do not test financial performance of optimal portfolios. This choice of investor's views provides a feasible example on which we can demonstrate main points of this section.

		abs. views		rel. views	
assets	μ^{eq}	μ^{BL}	μ^{SS}	μ^{BL}	μ^{SS}
Consumer	6.58	6.14	5.86	6.5	6.45
Manufacturing	5.21	4.88	4.68	5.25	5.26
HiTech	8.44	7.6	7.01	7.05	6.42
Healthcare	7.17	6.84	6.66	7.76	8.01
Other	7.94	7.41	7.09	8.18	8.28

Table 4: Expected excess return vectors for 2 scenarios with elliptical priors: absolute and relative views compared with the prior mean return vector (annualized (%)). For non-elliptical posterior SS computations were performed for $\tau = 0.02$ and with the sample of size 1 000 000.

	skew Student's t			Bernoulli mix		
assets	μ^{eq}	μ^{TSK}_{abs}	μ^{TSK}_{rel}	μ^{eq}	μ^{BM}_{abs}	μ^{BM}_{rel}
Consumer	5.38	4.81	5.15	7.19	6.85	7.36
Manufacturing	4.5	4.02	4.44	5.45	5.14	5.57
HiTech	7.06	5.79	5.07	7.29	5.99	4.69
Healthcare	5.98	5.58	6.61	9.12	9.42	11.26
Other	6.35	5.7	6.61	8.8	8.51	9.52

Table 5: Expected excess return vectors for scenarios with non-elliptical priors TSK and BM: absolute and relative views compared with the prior mean return vector (annualized (%)). Computations were performed for dispersion CVaR with $\alpha = 0.95$, $\tau = 0.02$ and sample of size 1 000 000.

Posterior distribution of excess returns. Explaining differences between optimal portfolios for different scenarios is a difficult task as these portfolios result from a nonlinear interplay between distributions of single assets and their dependencies. In order to shed more light on the effect of the choice of prior distribution and distribution of views, we analyze mean returns of posterior distribution for all scenarios. Although this does not provide a complete representation of the effects of distributions, it enables a simple comparison with the original Black-Litterman approach. Table 4 contains posterior mean excess returns for scenarios BL and SS for absolute and relative views. As prior distributions are elliptical in these cases they share the same mean (as follows from Theorem 2.1) displayed in the second column. Scenarios TSK and BM with their different prior mean returns are shown in Table 5. What beams from these data is a clear impact of forecasts and variation in how forecasts are accommodated in each scenario. For absolute views, when the forecast is for a low/negative return of an asset we observe a substantial decrease of the posterior mean compared to the prior mean. This is particularly visible for the sector HiTech where the absolute view predicts a high negative return. A similar pattern is shared under the relative views, however, relative forecasts do not pull the market down as a whole but rather change the relative performance between assets. It is interesting to notice that the posterior means corresponding to absolute views in Table 4 are all smaller than the prior mean return, while the bi-modality of Bernoulli mixture model allows a positive forecast for Healthcare to counter this trend, see Table 5.

In Table 4, all posterior expected excess returns for absolute views are lower than the prior mean returns. The drop is the most significant for scenario SS, followed by BL. When the absolute views are Student's t-distributed, fatter tails mean higher probability of even more extreme forecasts driving the posterior distribution to lower values. For scenario TSK the prior mean returns of all assets are smaller than for elliptical priors. Comparing the posterior returns we observe a similar pattern as for scenarios BL and SS, i.e., the posterior returns for both absolute and relative views are smaller than the prior mean returns.

assets	X^{eq}	absolute views X^{BL}	X^{SS}	X^{TSK}	X^{BM}	relative views X^{BL}	X^{SS}	X^{TSK}	X^{BM}
Consumer	14.6	25.1	23.3	15.7	27.2	25.8	25.1	19.9	27.5
			(0.43)	(0.48)	(0.42)		(0.54)	(0.48)	(0.38)
Manufacturing	18.3	81.6	73	33.1	78.3	93.6	97.8	76.8	87.1
			(0.26)	(0.37)	(0.32)		(0.35)	(0.46)	(0.32)
HiTech	38.8	5.2	5.5	15.6	16.6	-2.3	-8.1	-22.4	9.7
			(0.27)	(0.36)	(0.21)		(0.39)	(0.40)	(0.23)
Healthcare	8.9	6.7	9.4	17.7	-0.7	6.9	8.8	20.8	-1
			(0.24)	(0.24)	(0.20)		(0.28)	(0.29)	(0.20)
Other	19.4	-18.6	-11.3	17.9	-21.3	-23.9	-23.5	4.8	-23.3
			(0.30)	(0.34)	(0.29)		(0.30)	(0.44)	(0.29)

Table 6: Optimal portfolios for annualized target portfolio excess return of 5% and dispersion CVaR with $\alpha = 0.95$, $\tau = 0.02$ and constraints $e^T x = 1$ for four considered scenarios with absolute and relative views compared with the market portfolio. Standard deviation of portfolio weights computed using Monte Carlo approximation given in parentheses (all values are in percentage points). The sample size in Monte Carlo approximations is 1 000 000.

The assumption of Bernoulli mixture priors changes not only posterior expected excess returns but also the mean of prior distribution. Comparing prior mean returns of Tables 4 and 5 reveals that for all assets apart from HiTech the prior mean return in scenario BM is higher than in the other three. The effect of relative views is similar and pushes up expected returns of all assets but HiTech. However, as remarked earlier, the absolute views rise the expected posterior return of Healthcare contrary to other scenarios.

Computations of optimal portfolios. We solve the following portfolio optimisation problem:

$$\begin{aligned} &\text{minimize } \mathrm{CVaR}^D_\alpha(x^T R_p), \\ &\text{subject to: } x^T \mathbb{E}[R_p] \geq \bar{r}, \quad e^T x = 1, \end{aligned} \tag{14}$$

where CVaR^D_α denotes the deviation CVaR, R_p is the posterior distribution of excess returns, x is a vector of portfolio weights in risky assets, and $\bar{r} = 5\%$ is the target expected excess return. Excluding Table 7 which explores dependence on α, in all remaining computations we set the quantile α in CVaR to be 0.95. In scenario BL, the optimisation problem (14) simplifies to a quadratic program because deviation CVaR can be written as a linear function of the standard deviation of $x^T R_p$. In other scenarios, the posterior distribution of excess returns R_p is not in a parametric form and is represented as a weighted sample of the size 1 000 000. We reformulate problem (14) as a linear programme (Rockafellar & Uryasev, 2000) and compute optimal portfolio weights using Benders decomposition-based approach (see Palczewski (2018) for the details of the algorithm and Palczewski (2017) for implementation).

Optimal portfolios. Table 6 displays optimal portfolios for each scenario and two groups of investor's views. The second column presents the market portfolio to provide a reference point. Then four columns give asset weights for the absolute views followed by the relative views, cf. Table 3. Results for scenario BL are exact while optimization for the remaining scenarios involves Monte Carlo approximations of posterior distributions. Values in brackets are the standard deviations of portfolio weights estimators obtained as follows. We generated 100 independent samples of the size 1, 000, 000 approximating the posterior distribution as discussed earlier in this section. For each sample, we found an optimal portfolio. A square root of the sample variance of each portfolio weight approximates its standard deviation. If the difference between portfolio weights is larger than two standard deviations we will conclude that the difference is statistically significant.[8]

The portfolios corresponding to scenarios BL and SS are the most alike, although there are still significant differences between them. The portfolios corresponding to scenario TSK, which combines skew

[8]This corresponds to 95% two-sided Student's t test whose critical value for the sample size of 100 is 1.984. The 99% critical value is 2.626.

assets	SS			TSK			BM		
α values	0.7	0.8	0.95	0.7	0.8	0.95	0.7	0.8	0.95
Consumer	23.7	23.0	23.6	15.5	15.9	15.7	33.9	32.3	26.9
Manufacturing	72.6	73.0	72.6	32.7	32.5	32.5	84.4	82.6	78.4
HiTech	6.3	6.2	5.7	16.6	16.4	15.6	4.7	7.9	16.5
Healthcare	8.9	9.2	9.4	18.0	18.0	17.9	1.0	0.4	-0.6
Other	-11.5	-11.3	-11.4	17.3	17.3	18.3	-23.9	-23.2	-21.3

Table 7: Comparison of optimal portfolios with absolute views for three scenarios with non-elliptical posterior distribution for different confidence values α in the risk measure CVaR. Computations are performed with the Monte Carlo sample size 1 000 000, $\tau = 0.02$ and annualized target expected excess return 5%.

Student's t-distribution as a prior with symmetric Student's t views, stand out the most. For absolute views this portfolio is the most balanced but for relative views the picture is less clear. The difference between this scenario and the other three can be attributed to asymmetric structure of the prior distribution (and as a consequence asymmetric posterior distribution). As it is widely claimed that market returns are not symmetric, our results for scenario TSK indicate possible lines of divergence from symmetric Gaussian Black-Litterman framework when accounting for this feature.

Dependence of portfolios on α. For elliptical distributions neither the prior distribution nor the optimal portfolio depend on the choice of the confidence level α (Palczewski & Palczewski, 2017b, Theorem 3.6), as is the case for scenario BL. We expect that the further distributions are from the ellipticity the bigger the difference between optimal portfolios for different values of α. Table 7 collects optimal portfolios with absolute views for scenarios SS, TSK and BM. Let us remark that the portfolios for $\alpha = 0.95$ differ slightly from those presented in Table 6 due to Monte Carlo errors but lie confidently within 95% confidence intervals.

For each scenario, optimal portfolios for three different values of α are calculated from the same sample of returns to reduce the variance of differences between columns. We have not estimated variances of these differences due to computational burden and because they would not benefit the discussion. Notice that the differences for scenarios SS and TSK are rather small suggesting that posterior distributions are not far from ellipticity. Results for the last scenario with the Bernoulli mixture of normals as the prior distribution shows a markedly different pattern. Portfolio weights of all assets apart from "Healthcare" display significant variations. Investments in "Consumer", "Manufacturing" and "HiTech" decrease when α increases while "Other" shows an opposite trend. This behaviour cannot then be attributed to a change in the risk appetite due to changing value of α but follows from a different risk assessment implied by the choice of a quantile.

Performance of Monte Carlo estimators of prior mean returns. Calibration of the market equilibrium for general prior distributions requires numerical methods. We proposed two algorithms in Section 3.1 and here we evaluate their performance. Algorithm 3.1 employs a plain Monte Carlo principle without variance reduction techniques. Importance sampling based on optimal choice of the mean of the sampling distribution (with the covariance matrix unchanged) is presented in Algorithm 3.3. Recall that we calculated the optimum under the assumption that the prior distribution is normal and with the objective of minimising the variance of market portfolio's CVaR which is closely related to an estimator of the mean of the prior distribution.

We perform analysis for four prior distributions calibrated in this section: (a) normal distribution, (b) Student's t-distribution, (c) skew Student's t-distribution and (d) Bernoulli mixture of normal distributions. Although there are analytical formulas for the first two (see Palczewski & Palczewski (2017b)), we include them in this numerical analysis due to insights that they provide for our numerical methods. For each distribution fitted to the data (see the discussion of calibration earlier in this section), we generate samples of returns (r_i) of size 500, 5 000 and 50 000 and centre them by deducting the sample mean. For each sample, we apply Algorithm 3.1 directly. To find an importance sampling distribution, we compute a

	normal distr.		Student's t		skew Student's t		Bernoulli mix	
sample size	plain MC	IS MC	plain MC	IS MC	plain MC	IS MC	plain MC	IS MC
500	7.80E-05	1.18E-05	1.32E-04	5.17E-05	8.29E-05	2.43E-05	1.71E-04	4.66E-05
5000	7.25E-06	1.20E-06	1.46E-05	4.73E-06	7.53E-06	2.28E-06	1.62E-05	5.08E-06
50000	8.24E-07	1.19E-07	1.35E-06	4.82E-07	7.74E-07	2.29E-07	1.75E-06	4.86E-07

Table 8: Comparison of sums of variances of 5 coordinates of prior mean returns for plain Monte Carlo algorithm and importance sampling Monte Carlo algorithm.

sample covariance matrix D from the generated sample and calculate numerically the shift m that minimizes expression (11). This shift is then used in Algorithm 3.3. For each sample size we repeat the above 500 times in order to compute the empirical variance of Monte Carlo simulations. Table 8 compares sums of variances of 5 coordinates of the prior mean return μ_{eq} for the plain Monte Carlo algorithm ("plain MC") and Monte Carlo implementing importance sampling improvement ("IS MC") for the 3 sample sizes and the 4 prior distributions. The importance sampling algorithm improves the variance of estimator of prior mean returns (i.e., reduces the computation time) about 7-8 times for normal distribution and 3-4 times for Student's t, skew Student's t and Bernoulli mixture distributions. The algorithm is optimized for normal distribution so a good performance there should be expected. However, our importance sampling idea provided substantial improvements for other three distributions reducing the time needed to obtain accurate estimates of prior mean returns μ_{eq}. Interestingly, improvements in the variance of CVaR of market portfolio computations is about 40−45 times for the normal distribution, 8 times for the Student's t and skew Student's t-distributions and 12 times for the Bernoulli mixture distribution. Under weak investor's views, this improvement will precipitate into a higher accuracy of estimates of CVaR for posterior distributions.

4.2. Scenarios performance in multiperiod analysis

In this section we compare the performance of the four distributional scenarios described in Table 1. We use 12 US Industry Portfolios data for the period from Jan-01-1970 to Dec-31-2015; results for 5 US Industry Portfolios are reported in Electronic Appendix. For every month between January 1975 and December 2015 we calibrate the four scenarios to market data of previous 5 years and extract market portfolios x_M. In this rolling-window procedure we obtain for each scenario a sets of 492 distributions. From every distribution we simulate a sample of the size 100 000 and compute a corresponding prior distribution by performing the inverse optimization. We further compute the posterior distribution using one of the following 3 types of views:

1. *accurate views,* which are absolute views equal to the realized returns in the next month; to mitigate the effect of enormous jumps in monthly returns (which do occur at the time of large clashes on stock exchange) we scaled the monthly returns to limit the maximum return to ±50% on a yearly basis.
2. *momentum views,* which are relative views based on momentum strategy as described by Fabozzi et al. (2006); we used MOM2-12 momentum which is considered to be a good prediction of future asset behavior (see Asness et al. (2013) and references herein).
3. *past views,* which are absolute views equal to the realized returns in the previous month. Due to the observation that short past returns (up to one month) are inversely related to future average returns (Grinblatt & Moskowitz, 2004) these views can be considered as giving inaccurate predictions particularly while trend changes. We rescale the returns as in *accurate* views to constrain the maximum return to ±50%.

For each class of views we compute posterior distributions corresponding to scenarios BL, SS, TSK and BM and 3 different views confidence levels $\tau = 0.01, 0.02, 0.04$. For accurate views and $\tau = 0.04$ we observe very high realized returns, even in periods of large negative stock returns. This indicates that $\tau = 0.04$ corresponds to overconfidence of forecasts. On the other hand, we observe that the returns

views	τ	return (μ) BL	SS	TSK	BM	SR BL	SS	TSK	BM	$\mu/\text{CVaR}^D_{95\%}$ BL	SS	TSK	BM
accurate	0.04	1.75	2.81	2.08	0.99	1.19	3.26	1.64	0.67	0.112	0.484	0.170	0.066
	0.02	0.87	2.06	1.86	0.83	0.49	1.50	1.25	0.52	0.050	0.149	0.134	0.053
	0.01	0.78	0.90	1.33	0.73	0.44	0.50	0.76	0.44	0.047	0.051	0.083	0.047
momentum	0.04	0.63	0.75	0.81	0.76	0.37	0.44	0.42	0.45	0.045	0.054	0.050	0.052
	0.02	0.59	0.63	0.84	0.74	0.33	0.35	0.43	0.44	0.040	0.043	0.053	0.051
	0.01	0.53	0.54	0.69	0.68	0.30	0.31	0.36	0.41	0.036	0.037	0.042	0.047
past	0.04	0.60	0.50	0.68	0.74	0.37	0.42	0.49	0.51	0.041	0.049	0.055	0.057
	0.02	0.73	0.65	0.83	0.70	0.39	0.43	0.54	0.44	0.044	0.048	0.064	0.049
	0.01	0.75	0.66	0.86	0.60	0.42	0.36	0.50	0.36	0.048	0.040	0.058	0.040

Table 9: Performance of different considered scenarios for annualized target excess return 10%. Table presents average monthly realized excess returns (in percentage points), Sharpe ratios and reward to downside risk measure (ratio of average realized excess return and empirical deviation $\text{CVaR}^D_{95\%}$) for 3 different types of views and given views confidence values τ.

of optimal portfolios for $\tau = 0.01$ follow the returns of the market portfolio which indicates a very low confidence in views.

For each posterior distribution we solve the optimization problem

$$\min_x \text{CVaR}^D_{95\%}(x^T R_p) \qquad \text{subject to } \; x^T \mathbb{E}[R_p] \geq \bar{r},\; x^T e = 1,\; x \geq 0,$$

where $\bar{r} = 10\%$. For an optimal portfolio x_i^* obtained in month i we compute its realized excess return RR_i in the forthcoming month. Table 9 presents the averages of RR_i over the whole data set (492 months) for each type of views, each scenario and the three values of τ. We also report Sharpe ratios computed as follows:

$$SR = \frac{\text{mean of realized excess returns}}{\text{std. deviation of realized excess returns}}$$

and reward-to-downside risk metric $\mu/\text{CVaR}^D_{95\%}$ (the ratio of the mean of realized excess returns and empirical deviation CVaR). The latter attribute is argued to be better suited than Sharpe ratio to express the trade-off between risk and return for asymmetric distributions (Harris et al., 2017). Detailed results (including additional data for target annualized excess return $\bar{r} = 5\%$, and cases with/without short sales constraints) are presented in Section C in Electronic Appendix. We report the mean, standard deviation, skewness, kurtosis and downside risk measured by $\text{CVaR}_{95\%}$ computed from the empirical distribution of the realized portfolio excess returns RR_i. For the evaluation of performance of portfolios we show the Sharpe ratio and the reward to downside risk ratio. We measure portfolio stability over time with the average turnover per month (we use the turnover definition provided in DeMiguel et al. (2009)). We report the average Herfindahl-Hirschman index (sum of squares of portfolio weights) to assess the diversification, where smaller values of the index correspond to more diversified portfolios. Section D in Electronic Appendix contains analogous results for 5 US Industry portfolios.

The majority of excess returns for accurate views reported in Table 9 are higher than the target excess return per month $10\%/12 \approx 0.84\%$. This indicates that in some months, the portfolio chosen is the minimum-risk portfolio and the expected return constraint is not active. This is much more prevalent for the target excess return of 5%, see Section D Electronic Appendix. Notice also that during market downturns the expected return constraint is hard to satisfy, hence it takes the leading role in determining an optimal portfolio whose risk is therefore high. At those times the realized portfolio returns are often negative and may affect significantly the average portfolio return. One could then relax the portfolio problem allowing for disinvestment in risky assets and for adjustment to the target excess return, but this is beyond the scope of this paper as our aim here is to show that distributional assumptions have significant consequences for optimal portfolios in the Black-Litterman framework.

The average realized excess returns are increasing with τ in the case of accurate and momentum views confirming the intuition that increasing trust in correct views improves portfolio performance. This trend is even more evident in Sharpe ratios and reward to downside risk metric. Given the interpretation of past views provided at the beginning of this section, it is not surprising that we observe generally decreased performance when τ increases. Effects observed for momentum views are much smaller and differences between distributional scenarios less distinct. We conjecture that the mixing effect of a single momentum view that combines all assets in the portfolio lowers possible asymmetry of the prior distribution and increases dependence between coordinates alleviating the effect of fat tails of distributions. This also lowers the turnover, see Table C.4.

For all views, the fat-tailed distributions SS and TSK outperform the Gaussian setting BL both in terms of average return as well as in terms of Sharpe ratio and reward to downside risk metric. The latter particularly highlights better accounting for tail behavior of asset returns and shows more distinctly outperformance of the aforementioned fat-tailed distributions. Furthermore, the performance of scenario BL falls behind benchmarks (market portfolio and equally-weighted portfolio, see Table C.1 in Electronic Appendix) for all but strongest accurate views providing further support for using fat-tailed distributions in Black-Litterman analysis.

Scenario BM gives poor portfolios in the case of accurate views with $\tau = 0.4$. We conjecture that this can be related to estimation errors which were linked to the fact that for the analyzed data the observed mix of the two normal distributions (in Bernoulli mixture) was highly unbalanced with the majority of weight put on one of them. Application of strong views amplified the estimation errors and resulted in portfolio performance at par with the less informative views.

Scenarios TSK and BM excel in regimes with less informative views: momentum and past views. We would attribute it to their ability to express asymmetry of portfolio returns and therefore exploit better the historical information in the portfolio optimization step.

As reported in Section C in Electronic Appendix the portfolios without short-sale constraint are significantly more extreme which is shown by the turnover and the Herfindahl-Hirschman index. This amplifies the importance of accuracy of tail behaviour but also of estimation errors. The fat-tailed scenarios show superior performance with scenario SS usually better as symmetric distributions have more stable estimation procedures. Dropping the target annualized excess return from 10% to 5% does not change the above drawn picture. We would only like to reiterate that the excess return constraint of 5% is often not binding at optimum resulting in many minimum risk portfolios.

4.3. Computation time

Our algorithms were implemented in the statistical language R and computations were performed on a mid-range laptop with 8 GB RAM. Our discussion here is based on running times for the empirical example of Subsection 4.2 and reported times are averages per one portfolio optimization round which includes (a) fitting of a distribution, (b) generation of 100 000 samples and computation of the prior distribution from inverse optimization, (c) application of views and (d) the final computation of portfolio weights. Our experience showed that steps (a)-(c) are relatively quick compared to (d). To evaluate the effect of the number of assets on the running time, Table 10 reports results for 5 US Industry Portfolios analyzed in Subsection 4.1 and 12 US Industry Portfolios of Subsection 4.2. Different running times of the optimization step with short sale constraints and without them is related to the speed of convergence of the Bender's decomposition-based algorithm (Palczewski, 2018); in the latter case, the feasible set is unbounded so finding an optimum is harder.

Steps (b)-(c) are easily parallelisable with the running time inversely proportional to the number of parallel instances. Parallelization can also benefit the optimization step (d) through parallel linear solvers such as IBM's CPLEX.

5. Conclusions

In this paper we extend the complete methodology of Black and Litterman (Black & Litterman, 1990) beyond normal distributions. We design an inverse optimization procedure in the framework of generalized CAPM for general deviation measures. This inverse optimization procedure together with an appropriate

	no short sales		no constraints	
	5 assets	12 assets	5 assets	12 assets
step (a)	0.03s	0.12s	0.03s	0.12s
steps (b)-(d)	0.52s	1.04s	0.83s	3.9s

Table 10: Average running time for the following steps of portfolio optimization procedure (a) fitting of a distribution from 5 years of daily data, (b) generation of 100 000 samples and computation of the prior distribution from inverse optimization, (c) application of views and (d) the final portfolio optimization.

Bayesian argument allows for arbitrary continuous distributions of returns and investor's views. In this extension we preserve many of the desired properties of the original formulation. The views are described through a deterministic forecast and a probability distribution representing its uncertainty. Optimal portfolio weights are well diversified and do not exhibit extreme values which plagues the Markowitz mean-variance approach. To allow for arbitrary distributions of views, we ground our approach in numerical computations. We design and implement[9] algorithms for inverse inference of prior distribution and calculation of posterior distribution for a number of deviation measures. Extensive numerical tests show that our procedures based on Monte Carlo ideas are stable and calculated portfolios have small variance of weights.

Our methods are not directly applicable to portfolios with 100s of assets for two reasons. Firstly, the estimation of model parameters from market data is burdened with unacceptably large errors. Extant literature on portfolio optimization employs factor models for large portfolios to overcome this limitation. Secondly, our Monte Carlo algorithms are not able to provide sufficiently good coverage of the asset return space to enable risk estimates of sufficiently high accuracy within acceptable running time. An extension of our modeling and algorithmic framework to factor models, which lower the effective dimensionality of the problem (Cheung, 2013), is left to future research.

The most significant conclusion of our numerical tests is that the choice of the prior distribution and the distribution of investor's views has a profound influence on the resulting optimal portfolios. Rolling-window simulations prove that a correct choice of the prior distribution and the distribution of views can substantially improve the realized optimal portfolio performance. Our results also clearly indicate that the classical Black-Litterman formulas do not offer an acceptable approximation for non-normal market models. It should be remarked that our empirical section explores only a small selection of possible ways of applying our theory to asset management beyond normal markets. In practice, asset managers will equip the model with carefully chosen prior distributions and tailored views.

Our main findings can be summarized in the following points:

1. The Black-Litterman model can be extended to arbitrary prior (equilibrium) distributions and distributions of investor's views.
2. The choice of distributions has a profound impact on posterior distribution, optimal portfolio weights and their performance.
3. Prior mean returns can be robustly computed for a number of deviation risk measures. In the paper we have presented numerical results for deviation CVaR but our approach works for a larger class of deviation measures as indicated in Section 4.
4. Our importance sampling procedure substantially improves computations of portfolio CVaR and the estimation of prior mean returns.

Further research will explore two directions. Firstly, how to extend the present framework to factor models (c.f. Cheung (2013); Kolm & Ritter (2017))? Secondly, how to select the prior distribution, the distribution of uncertainty of views as well as the views themselves to benefit from the flexibility of our model when applied semi-autonomously for regular rebalancing of portfolios over long time horizons?

[9]Palczewski & Palczewski (2017a); Palczewski (2017)

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Electronic Appendix

"Black-Litterman model for continuous distributions" by A. Palczewski and J. Palczewski

Appendix A Deviation measures of Rockafellar et al. (2006a)

Rockafellar et al. (2006a) advertise the replacement of variance in portfolio optimization problems by deviation measures. Denote by $L^2(\Omega)$ the Hilbert space of square integrable random variables.

Definition A.1. A functional $D : L^2(\Omega) \to [0, \infty)$ is a deviation measure if it satisfies the following axioms:

(D1) $\mathcal{D}(X + C) = \mathcal{D}(X)$ for all X and constants C.
(D2) $\mathcal{D}(0) = 0$ and $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all X and all $\lambda > 0$.
(D3) $\mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ for all X and Y.
(D4) $\mathcal{D}(X) \geq 0$ for all X with $\mathcal{D}(X) > 0$ for non constant X.

A deviation measure is lower semicontinuous if

(D5) the set $\{X : \mathcal{D}(X) \leq C\}$ is closed in $L^2(\Omega)$ for every constant C.

Thanks to the above properties, the objective function in (1) simplifies to

$$\mathcal{D}(x^T R^e) = \mathcal{D}(x^T \hat{R}),$$

where $\hat{R} = R^e - \mathbb{E}R^e$ is the centered (de-meaned) version of R^e.

Appendix B Derivation of formulae (12)-(13)

For $Y \sim N(0, 1)$, X has the same distribution as $\mu + \sigma Y$. Hence

$$\begin{aligned}\mathbb{E}\left[X^2 \mathbb{1}_{\{X\leq 0\}}\right] &= \mathbb{E}\left[(\mu + \sigma Y)^2 \mathbb{1}_{\{\mu+\sigma Y\leq 0\}}\right] \\ &= \sigma^2 \mathbb{E}\left[Y^2 \mathbb{1}_{\{Y\leq -\frac{\mu}{\sigma}\}}\right] + 2\mu\sigma \mathbb{E}\left[Y \mathbb{1}_{\{Y\leq -\frac{\mu}{\sigma}\}}\right] + \mu^2 \mathbb{P}\left[Y \leq -\frac{\mu}{\sigma}\right].\end{aligned}$$

Integrating by parts we obtain

$$\begin{aligned}\mathbb{E}\left[Y^2 \mathbb{1}_{\{Y\leq -\frac{\mu}{\sigma}\}}\right] &= \int_{-\infty}^{-\mu/\sigma} y^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = \frac{\mu}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}} + \int_{-\infty}^{-\mu/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\ &= \frac{\mu}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}} + \Phi\left(-\frac{\mu}{\sigma}\right).\end{aligned}$$

Easily,

$$\mathbb{E}\left[Y \mathbb{1}_{\{Y\leq -\frac{\mu}{\sigma}\}}\right] = \int_{-\infty}^{-\mu/\sigma} y \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = -\frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}},$$

and $\mathbb{P}\left[Y \leq -\frac{\mu}{\sigma}\right] = \Phi\left(-\frac{\mu}{\sigma}\right)$. Combining the above calculations proves (12). For (13), it is sufficient to notice

$$\mathbb{E}\left[X \mathbb{1}_{\{X\leq 0\}}\right] = \mathbb{E}\left[(\mu + \sigma Y) \mathbb{1}_{\{Y\leq -\frac{\mu}{\sigma}\}}\right] = \mu \mathbb{P}\left[Y \leq -\frac{\mu}{\sigma}\right] + \sigma \mathbb{E}\left[Y \mathbb{1}_{\{Y\leq -\frac{\mu}{\sigma}\}}\right]$$

and use the above calculations again.

Appendix C Scenarios performance in multiperiod analysis for 12 US industry portfolios

In this section we present performance of multiperiod empirical analysis of Subsection 4.2 for different target excess returns (5% and 10%), and with or without short sale constraints. Table C.1 shows performance of benchmark portfolios. The market portfolio is a value weighted portfolio obtained from data provided by Kenneth French. The equally weighted portfolio 1/N is a common benchmark in portfolio optimization papers.

portfolio	μ	SD	SKEW	KURT	$\mathrm{CVaR}_{95\%}$	SR	$\mu/\mathrm{CVaR}^D_{95\%}$	TURN	HI
market portfolio	.0069	.0464	-0.76	5.17	0.111	0.52	0.059	0.01	0.11
portfolio 1/N	.0074	.0444	-0.74	5.48	0.105	0.58	0.066	0.03	0.08

Table C.1: Performance of benchmarks for 12 Industry Portfolios. Market portfolio is a value weighted portfolio. 1/N portfolio stands for an equally weighted portfolio. For further explanations of columns see the caption to Table C.4.

C.1 With short sales constraint

		return (μ)				SR				$\mu/\mathrm{CVaR}^D_{95\%}$			
views	τ	BL	SS	TSK	BM	BL	SS	TSK	BM	BL	SS	TSK	BM
accurate	0.04	1.75	2.81	2.08	0.99	1.19	3.26	1.64	0.67	0.112	0.484	0.170	0.066
	0.02	0.87	2.06	1.86	0.83	0.49	1.50	1.25	0.52	0.050	0.149	0.134	0.053
	0.01	0.78	0.90	1.33	0.73	0.44	0.50	0.76	0.44	0.047	0.051	0.083	0.047
momentum	0.04	0.63	0.75	0.81	0.76	0.37	0.44	0.42	0.45	0.045	0.054	0.050	0.052
	0.02	0.59	0.63	0.84	0.74	0.33	0.35	0.43	0.44	0.040	0.043	0.053	0.051
	0.01	0.53	0.54	0.69	0.68	0.30	0.31	0.36	0.41	0.036	0.037	0.042	0.047
past	0.04	0.60	0.50	0.68	0.74	0.37	0.42	0.49	0.51	0.041	0.049	0.055	0.057
	0.02	0.73	0.65	0.83	0.70	0.39	0.43	0.54	0.44	0.044	0.048	0.064	0.049
	0.01	0.75	0.66	0.86	0.60	0.42	0.36	0.50	0.36	0.048	0.040	0.058	0.040

Table C.2: Performance of different considered scenarios for annualized target excess return 10% and short sale constraint. Table presents average monthly realized excess returns (in percentage points), Sharpe ratios and reward to downside risk measure (the ratio of average realized excess return and empirical deviation $\mathrm{CVaR}^D_{95\%}$) for 3 different types of views and given views confidence values τ. Table repeated from the article to make the appendix self-contained.

views	τ	return (μ) BL	SS	TSK	BM	SR BL	SS	TSK	BM	$\mu/\mathrm{CVaR}^{D}_{95\%}$ BL	SS	TSK	BM
accurate	0.04	1.32	2.21	1.63	0.78	1.18	2.98	1.70	0.66	0.116	0.491	0.187	0.067
	0.02	0.56	1.43	1.20	0.60	0.44	1.21	1.04	0.50	0.042	0.116	0.104	0.052
	0.01	0.51	0.55	0.77	0.58	0.42	0.41	0.63	0.51	0.043	0.040	0.065	0.055
momentum	0.04	0.79	0.80	0.84	0.79	0.75	0.76	0.75	0.75	0.089	0.090	0.085	0.089
	0.02	0.78	0.78	0.82	0.78	0.74	0.74	0.72	0.74	0.088	0.088	0.082	0.088
	0.01	0.77	0.78	0.79	0.77	0.74	0.74	0.69	0.73	0.088	0.088	0.078	0.087
past	0.04	0.77	0.51	0.64	0.63	0.57	0.48	0.55	0.52	0.064	0.057	0.065	0.060
	0.02	0.65	0.70	0.68	0.69	0.45	0.51	0.55	0.55	0.049	0.057	0.065	0.065
	0.01	0.66	0.63	0.70	0.69	0.53	0.43	0.51	0.61	0.059	0.046	0.056	0.071

Table C.3: Performance of different considered scenarios for target excess return 5% and short sale constraint. For further explanations see the caption to Table C.2.

views	τ	scenario	μ	SD	SKEW	KURT	$CVaR_{95\%}$	SR	$\mu/CVaR^D_{95\%}$	TURN	HI
accurate	0.04	BL	.0174	.0507	-2.25	14.18	0.138	1.19	0.112	1.30	0.53
		SS	.0281	.0298	-0.25	9.58	0.030	3.26	0.484	1.15	0.40
		TSK	.0208	.0440	-1.61	16.79	0.102	1.64	0.170	1.12	0.44
		BM	.0099	.0512	-1.54	8.07	0.139	0.67	0.066	1.17	0.53
	0.02	BL	.0087	.0612	-1.36	6.43	0.165	0.49	0.050	1.13	0.57
		SS	.0205	.0474	-2.18	14.89	0.118	1.50	0.149	1.31	0.52
		TSK	.0186	.0516	-0.74	7.57	0.120	1.25	0.134	1.18	0.53
		BM	.0083	.0559	-1.46	8.32	0.148	0.52	0.053	1.10	0.57
	0.01	BL	.0077	.0615	-1.18	6.90	0.157	0.44	0.047	0.82	0.57
		SS	.0090	.0621	-1.54	7.79	0.168	0.50	0.051	1.19	0.60
		TSK	.0132	.0604	-0.80	5.87	0.146	0.76	0.083	1.00	0.66
		BM	.0072	.0576	-1.31	7.75	0.149	0.44	0.047	0.96	0.60
momentum	0.04	BL	.0063	.0590	-0.35	5.07	0.134	0.37	0.045	0.25	0.61
		SS	.0074	.0586	-0.34	5.12	0.131	0.44	0.054	0.29	0.62
		TSK	.0080	.0660	-0.49	4.78	0.152	0.42	0.050	0.28	0.82
		BM	.0076	.0582	-0.99	7.77	0.139	0.45	0.052	0.44	0.62
	0.02	BL	.0058	.0612	-0.65	7.02	0.140	0.33	0.040	0.19	0.61
		SS	.0062	.0613	-0.66	7.01	0.140	0.35	0.043	0.22	0.62
		TSK	.0084	.0676	-0.45	4.66	0.152	0.43	0.053	0.22	0.85
		BM	.0074	.0586	-0.90	7.57	0.137	0.44	0.051	0.41	0.60
	0.01	BL	.0053	.0605	-0.80	6.86	0.142	0.30	0.036	0.19	0.59
		SS	.0054	.0604	-0.78	6.86	0.140	0.31	0.037	0.18	0.61
		TSK	.0069	.0670	-0.49	4.84	0.157	0.36	0.042	0.20	0.85
		BM	.0068	.0583	-0.92	7.75	0.138	0.41	0.047	0.40	0.57
past	0.04	BL	.0059	.0555	-0.57	6.78	0.140	0.37	0.041	1.31	0.53
		SS	.0049	.0412	-0.03	7.17	0.097	0.42	0.049	1.16	0.40
		TSK	.0068	.0479	-0.23	7.56	0.117	0.49	0.055	1.13	0.44
		BM	.0074	.0504	-0.55	5.84	0.122	0.51	0.057	1.13	0.53
	0.02	BL	.0072	.0640	-0.72	7.30	0.157	0.39	0.044	1.15	0.58
		SS	.0064	.0528	-0.53	6.73	0.129	0.43	0.048	1.32	0.52
		TSK	.0082	.0529	-0.12	5.99	0.121	0.54	0.064	1.17	0.52
		BM	.0069	.0542	-1.17	9.02	0.135	0.44	0.049	1.06	0.57
	0.01	BL	.0074	.0619	-0.78	7.07	0.149	0.42	0.048	0.82	0.57
		SS	.0065	.0638	-0.77	7.30	0.159	0.36	0.040	1.19	0.61
		TSK	.0085	.0600	-0.29	4.78	0.139	0.50	0.058	0.99	0.66
		BM	.0059	.0569	-1.05	7.71	0.142	0.36	0.040	0.94	0.60

Table C.4: Performance of different considered scenarios for target excess return 10% and short sale constraint. Table presents different performance measures for 3 different types of views and given views confidence values τ. We denote: μ – mean of realized excess returns, SD – standard deviation of realized excess returns, SKEW – skewness of realized excess returns, KURT – kurtosis of realized excess returns, $CVaR_{95\%}$ – Conditional Value at Risk of realized excess returns, SR – Sharpe ratio, $\mu/CVaR^D_{95\%}$ – reward to downside risk ratio, TURN – portfolio turnover per month, HI – Herfindahl-Hirschman index.

views	τ	scenario	μ	SD	SKEW	KURT	$\text{CVaR}_{95\%}$	SR	$\mu/\text{CVaR}^{D}_{95\%}$	TURN	HI
accurate	0.04	BL	.0132	.0388	-1.91	14.48	0.101	1.18	0.116	0.96	0.48
		SS	.0221	.0256	0.50	8.99	0.023	2.98	0.491	0.84	0.36
		TSK	.0163	.0332	-0.95	10.62	0.071	1.70	0.187	0.81	0.35
		BM	.0077	.0409	-0.95	6.31	0.108	0.66	0.067	0.88	0.47
	0.02	BL	.0056	.0449	-1.57	9.21	0.128	0.44	0.042	0.90	0.47
		SS	.0143	.0410	-2.49	17.15	0.109	1.21	0.116	0.97	0.46
		TSK	.0119	.0400	-1.45	10.78	0.104	1.04	0.104	0.88	0.42
		BM	.0060	.0416	-0.98	6.62	0.110	0.50	0.052	0.81	0.47
	0.01	BL	.0051	.0419	-1.41	9.65	0.114	0.42	0.043	0.59	0.39
		SS	.0055	.0465	-1.63	8.87	0.133	0.41	0.040	0.98	0.47
		TSK	.0077	.0426	-1.15	7.17	0.111	0.63	0.065	0.96	0.47
		BM	.0058	.0395	-1.11	9.16	0.101	0.51	0.055	0.59	0.40
momentum	0.04	BL	.0078	.0363	-0.36	5.04	0.081	0.75	0.089	0.09	0.43
		SS	.0079	.0363	-0.37	5.05	0.081	0.76	0.090	0.09	0.43
		TSK	.0084	.0389	-0.63	5.72	0.090	0.75	0.085	0.18	0.33
		BM	.0078	.0362	-0.32	4.96	0.080	0.75	0.089	0.17	0.43
	0.02	BL	.0077	.0363	-0.35	4.96	0.081	0.74	0.088	0.09	0.42
		SS	.0078	.0363	-0.36	4.99	0.081	0.74	0.088	0.09	0.42
		TSK	.0082	.0395	-0.66	5.90	0.093	0.72	0.082	0.16	0.29
		BM	.0077	.0365	-0.34	4.96	0.081	0.74	0.088	0.16	0.42
	0.01	BL	.0077	.0363	-0.34	4.95	0.080	0.74	0.088	0.08	0.41
		SS	.0077	.0363	-0.33	4.97	0.080	0.74	0.088	0.09	0.41
		TSK	.0079	.0399	-0.69	6.01	0.094	0.69	0.078	0.15	0.26
		BM	.0077	.0366	-0.35	4.98	0.081	0.73	0.087	0.16	0.42
past	0.04	BL	.0076	.0467	-0.55	8.20	0.112	0.57	0.064	0.96	0.49
		SS	.0050	.0363	-0.12	7.55	0.084	0.48	0.057	0.84	0.36
		TSK	.0063	.0398	-0.12	10.29	0.091	0.55	0.065	0.82	0.35
		BM	.0062	.0418	-0.52	6.08	0.098	0.52	0.060	0.85	0.48
	0.02	BL	.0064	.0496	-1.42	12.33	0.125	0.45	0.049	0.90	0.47
		SS	.0070	.0477	-0.94	10.07	0.115	0.51	0.057	0.96	0.47
		TSK	.0067	.0426	-0.16	8.84	0.098	0.55	0.065	0.88	0.42
		BM	.0068	.0431	-0.54	7.90	0.099	0.55	0.065	0.80	0.47
	0.01	BL	.0065	.0430	-1.17	9.93	0.105	0.53	0.059	0.58	0.39
		SS	.0063	.0507	-1.37	11.85	0.130	0.43	0.046	0.99	0.47
		TSK	.0070	.0480	-0.52	7.36	0.118	0.51	0.056	0.94	0.47
		BM	.0069	.0391	-0.67	6.43	0.091	0.61	0.071	0.57	0.40

Table C.5: Performance of different considered scenarios for target excess return 5% and short sale constraint. For further explanations see the caption to Table C.4.

C.2 *Without short sale constraint*

views	τ	return (μ)				SR				$\mu/\mathrm{CVaR}^D_{95\%}$			
		BL	SS	TSK	BM	BL	SS	TSK	BM	BL	SS	TSK	BM
accurate	0.04	14.23	13.47	4.19	3.28	1.82	2.82	2.55	1.25	0.541	0.895	0.317	0.169
	0.02	1.25	15.85	4.04	1.57	0.37	2.09	2.02	0.74	0.036	0.778	0.251	0.078
	0.01	0.31	3.32	2.93	0.68	0.13	0.60	1.18	0.33	0.012	0.079	0.129	0.033
momentum	0.04	0.85	0.91	1.07	0.84	0.47	0.51	0.49	0.48	0.054	0.059	0.059	0.054
	0.02	0.76	0.79	0.96	0.78	0.42	0.43	0.41	0.43	0.047	0.049	0.049	0.048
	0.01	0.72	0.76	0.90	0.75	0.39	0.41	0.37	0.41	0.044	0.047	0.044	0.046
past	0.04	2.16	2.01	0.98	0.84	0.44	0.59	0.50	0.38	0.053	0.074	0.059	0.050
	0.02	0.42	2.51	1.04	0.73	0.09	0.48	0.45	0.33	0.010	0.060	0.050	0.042
	0.01	0.45	0.49	0.89	0.53	0.19	0.12	0.36	0.27	0.020	0.012	0.039	0.030

Table C.6: Performance of different considered scenarios for target excess return 10% and without short sale constraint. For further explanations see the caption to Table C.2.

views	τ	return (μ)				SR				$\mu/\mathrm{CVaR}^D_{95\%}$			
		BL	SS	TSK	BM	BL	SS	TSK	BM	BL	SS	TSK	BM
accurate	0.04	7.25	7.23	2.39	1.77	1.77	2.83	2.08	1.22	0.548	0.750	0.247	0.153
	0.02	0.76	7.88	2.09	0.86	0.51	2.12	1.65	0.73	0.051	0.717	0.188	0.077
	0.01	0.40	1.66	1.38	0.56	0.33	0.66	1.02	0.50	0.033	0.089	0.105	0.053
momentum	0.04	0.78	0.80	0.82	0.78	0.75	0.76	0.73	0.74	0.089	0.090	0.083	0.088
	0.02	0.76	0.76	0.79	0.76	0.73	0.72	0.69	0.72	0.087	0.086	0.079	0.085
	0.01	0.74	0.75	0.77	0.73	0.71	0.72	0.67	0.69	0.085	0.085	0.075	0.083
past	0.04	1.31	1.46	0.94	0.84	0.50	0.73	0.69	0.60	0.063	0.091	0.085	0.082
	0.02	0.57	1.69	0.87	0.67	0.27	0.60	0.58	0.51	0.029	0.081	0.069	0.064
	0.01	0.60	0.52	0.79	0.63	0.49	0.23	0.56	0.55	0.055	0.025	0.064	0.064

Table C.7: Performance of different considered scenarios for target excess return 5% and without short sale constraint. For further explanations see the caption to Table C.2.

views	τ	scenario	μ	SD	SKEW	KURT	$\text{CVaR}_{95\%}$	SR	$\mu/\text{CVaR}^{D}_{95\%}$	TURN	HI
accurate	0.04	BL	.1422	.2715	3.10	14.46	0.120	1.82	0.541	11.88	31.18
		SS	.1347	.1652	1.93	6.81	0.016	2.82	0.895	8.18	12.03
		TSK	.0418	.0568	0.50	9.17	0.090	2.55	0.317	4.55	5.25
		BM	.0327	.0908	2.43	23.61	0.161	1.25	0.169	6.23	8.95
	0.02	BL	.0125	.1169	-3.71	36.27	0.339	0.37	0.036	14.75	16.42
		SS	.1585	.2632	3.06	15.15	0.045	2.09	0.778	11.82	29.43
		TSK	.0403	.0691	0.61	10.52	0.121	2.02	0.251	5.46	7.43
		BM	.0156	.0735	-0.86	22.76	0.186	0.74	0.078	4.92	5.26
	0.01	BL	.0030	.0835	-3.41	24.94	0.254	0.13	0.012	2.08	1.33
		SS	.0332	.1918	2.04	56.39	0.390	0.60	0.079	9.32	17.21
		TSK	.0293	.0857	-0.08	13.12	0.197	1.18	0.129	5.62	8.61
		BM	.0068	.0711	-2.98	21.45	0.199	0.33	0.033	2.73	1.78
momentum	0.04	BL	.0085	.0623	-0.76	6.55	0.148	0.47	0.054	0.48	0.77
		SS	.0091	.0619	-0.72	6.46	0.145	0.51	0.059	0.57	0.91
		TSK	.0107	.0752	-0.59	5.79	0.171	0.49	0.059	1.05	2.37
		BM	.0084	.0613	-0.79	6.43	0.147	0.48	0.054	0.90	0.91
	0.02	BL	.0076	.0633	-0.94	7.34	0.154	0.42	0.047	0.37	0.62
		SS	.0078	.0632	-0.88	7.15	0.152	0.43	0.049	0.43	0.67
		TSK	.0095	.0804	-0.64	6.33	0.186	0.41	0.049	0.83	1.96
		BM	.0077	.0626	-0.96	7.30	0.153	0.43	0.048	0.66	0.69
	0.01	BL	.0072	.0636	-1.02	7.65	0.156	0.39	0.044	0.33	0.57
		SS	.0075	.0635	-0.99	7.49	0.155	0.41	0.047	0.38	0.59
		TSK	.0089	.0831	-0.65	6.69	0.194	0.37	0.044	0.68	1.77
		BM	.0074	.0631	-1.07	7.75	0.156	0.41	0.046	0.56	0.62
past	0.04	BL	.0215	.1716	0.66	11.89	0.382	0.44	0.053	38.58	30.91
		SS	.0201	.1183	0.50	9.59	0.251	0.59	0.074	9.23	11.75
		TSK	.0098	.0684	-0.06	10.54	0.157	0.50	0.059	4.65	5.15
		BM	.0083	.0757	2.13	25.36	0.159	0.38	0.050	6.28	8.23
	0.02	BL	.0041	.1542	-1.40	26.30	0.414	0.09	0.010	9.04	9.68
		SS	.0251	.1804	-0.18	18.54	0.394	0.48	0.060	13.15	26.64
		TSK	.0104	.0809	-0.14	9.65	0.197	0.45	0.050	5.56	7.50
		BM	.0072	.0775	1.13	22.64	0.166	0.33	0.042	4.94	7.20
	0.01	BL	.0045	.0812	-1.71	13.80	0.218	0.19	0.020	1.94	1.30
		SS	.0049	.1431	-1.18	15.94	0.402	0.12	0.012	12.33	16.99
		TSK	.0089	.0855	-0.68	9.15	0.222	0.36	0.039	5.47	8.44
		BM	.0052	.0667	-1.42	12.99	0.168	0.27	0.030	2.49	1.62

Table C.8: Performance of different considered scenarios for target excess return 10% and without short sale constraint. For further explanations see the caption to Table C.4.

views	τ	scenario	μ	SD	SKEW	KURT	$\text{CVaR}_{95\%}$	SR	$\mu/\text{CVaR}^D_{95\%}$	TURN	HI
accurate	0.04	BL	.0724	.1415	3.19	15.31	0.060	1.77	0.548	6.07	9.07
		SS	.0723	.0883	1.90	7.01	0.024	2.83	0.750	4.40	4.41
		TSK	.0238	.0398	0.09	8.42	0.073	2.08	0.247	2.67	2.36
		BM	.0177	.0505	0.75	14.01	0.098	1.22	0.153	3.10	2.73
	0.02	BL	.0075	.0515	-3.19	30.68	0.143	0.51	0.051	2.90	2.91
		SS	.0788	.1289	2.84	12.59	0.031	2.12	0.717	5.89	7.59
		TSK	.0208	.0439	0.00	8.78	0.090	1.65	0.188	3.02	2.68
		BM	.0086	.0407	-1.58	12.70	0.103	0.73	0.077	2.15	1.45
	0.01	BL	.0040	.0430	-1.99	14.82	0.117	0.33	0.033	0.92	0.52
		SS	.0166	.0866	0.24	45.29	0.170	0.66	0.089	3.82	3.94
		TSK	.0137	.0468	-1.11	10.98	0.117	1.02	0.105	2.90	2.22
		BM	.0056	.0392	-1.11	8.86	0.101	0.50	0.053	1.09	0.69
momentum	0.04	BL	.0078	.0362	-0.39	4.96	0.080	0.75	0.089	0.24	0.59
		SS	.0080	.0364	-0.42	4.92	0.081	0.76	0.090	0.26	0.62
		TSK	.0081	.0389	-0.70	5.11	0.090	0.73	0.083	0.33	0.42
		BM	.0078	.0367	-0.35	4.81	0.082	0.74	0.088	0.46	0.67
	0.02	BL	.0076	.0361	-0.37	5.00	0.080	0.73	0.087	0.23	0.56
		SS	.0075	.0362	-0.39	5.01	0.081	0.72	0.086	0.24	0.57
		TSK	.0079	.0396	-0.73	5.44	0.093	0.69	0.079	0.28	0.33
		BM	.0076	.0366	-0.36	4.91	0.082	0.72	0.085	0.42	0.63
	0.01	BL	.0074	.0360	-0.36	5.03	0.080	0.71	0.085	0.22	0.55
		SS	.0074	.0361	-0.39	4.97	0.080	0.72	0.085	0.24	0.55
		TSK	.0077	.0402	-0.77	5.85	0.096	0.67	0.075	0.24	0.29
		BM	.0073	.0366	-0.35	5.05	0.082	0.69	0.083	0.40	0.61
past	0.04	BL	.0130	.0913	1.76	22.67	0.196	0.50	0.063	6.68	8.54
		SS	.0146	.0696	0.47	8.28	0.146	0.73	0.091	4.70	4.26
		TSK	.0094	.0474	0.11	7.54	0.102	0.69	0.085	2.72	2.38
		BM	.0084	.0487	1.54	16.00	0.094	0.60	0.082	3.03	2.59
	0.02	BL	.0056	.0737	-0.85	19.38	0.190	0.27	0.029	2.75	2.10
		SS	.0169	.0970	0.14	19.25	0.191	0.60	0.081	6.55	7.76
		TSK	.0086	.0515	-0.01	8.37	0.117	0.58	0.069	3.03	2.64
		BM	.0067	.0452	0.11	10.07	0.099	0.51	0.064	2.18	1.83
	0.01	BL	.0060	.0424	-1.05	8.44	0.104	0.49	0.055	0.91	0.52
		SS	.0051	.0764	-1.94	21.56	0.205	0.23	0.025	3.99	4.26
		TSK	.0079	.0491	-0.33	6.74	0.117	0.56	0.064	2.82	2.20
		BM	.0063	.0394	-0.61	6.30	0.093	0.55	0.064	1.04	0.66

Table C.9: Performance of different considered scenarios for target excess return 5% and without short sale constraint. For further explanations see the caption to Table C.4.

Appendix D Scenarios performance in multiperiod analysis for 5 US industry portfolios

We repeat here the analysis of the previous section for 5 US industry portfolios, the assets used in Section 4.1. Table D.1 shows performance of benchmark portfolios.

portfolio	μ	SD	SKEW	KURT	$\mathrm{CVaR}_{95\%}$	SR	$\mu/\mathrm{CVaR}^{D}_{95\%}$	TURN	HI
market portfolio	.0070	.0464	-0.76	5.15	0.111	0.52	0.059	0.01	0.24
portfolio 1/N	.0074	.0453	-0.68	4.99	0.105	0.57	0.067	0.02	0.20

Table D.1: Performance of benchmarks for 5 Industry Portfolios. Market portfolio is a value weighted portfolio. 1/N portfolio stands for an equally weighted portfolio. For further explanations of columns see the caption to Table C.4.

D.1 With short sale constraint

		return (μ)				SR				$\mu/\mathrm{CVaR}^{D}_{95\%}$			
views	τ	BL	SS	TSK	BM	BL	SS	TSK	BM	BL	SS	TSK	BM
accurate	0.04	0.89	1.38	1.03	0.72	0.55	1.02	0.72	0.48	0.053	0.104	0.074	0.051
	0.02	0.79	0.89	0.89	0.72	0.48	0.55	0.57	0.46	0.049	0.055	0.060	0.051
	0.01	0.85	0.72	0.96	0.77	0.53	0.43	0.58	0.48	0.059	0.044	0.064	0.054
momentum	0.04	0.85	0.89	0.74	1.00	0.55	0.57	0.43	0.66	0.063	0.065	0.048	0.077
	0.02	0.88	0.84	0.77	0.93	0.57	0.54	0.43	0.61	0.067	0.063	0.049	0.072
	0.01	0.85	0.90	0.59	0.92	0.56	0.59	0.33	0.58	0.067	0.071	0.037	0.068
past	0.04	0.65	0.72	0.61	0.70	0.41	0.49	0.45	0.48	0.044	0.053	0.051	0.055
	0.02	0.70	0.68	0.74	0.71	0.44	0.42	0.50	0.48	0.049	0.044	0.058	0.053
	0.01	0.78	0.69	0.74	0.75	0.51	0.43	0.46	0.49	0.058	0.047	0.052	0.056

Table D.2: Performance of different considered scenarios for target excess return 10% and with short sale constraint. For further explanations see the caption to Table C.2.

		return (μ)				SR				$\mu/\mathrm{CVaR}^{D}_{95\%}$			
views	τ	BL	SS	TSK	BM	BL	SS	TSK	BM	BL	SS	TSK	BM
accurate	0.04	0.84	1.28	1.06	0.74	0.66	1.14	0.92	0.58	0.069	0.121	0.098	0.063
	0.02	0.77	0.80	0.77	0.73	0.61	0.62	0.61	0.57	0.065	0.064	0.066	0.063
	0.01	0.79	0.78	0.72	0.78	0.65	0.62	0.57	0.64	0.073	0.068	0.061	0.073
momentum	0.04	0.85	0.85	0.88	0.82	0.72	0.72	0.73	0.69	0.085	0.085	0.084	0.081
	0.02	0.85	0.85	0.87	0.82	0.72	0.72	0.71	0.69	0.085	0.085	0.082	0.081
	0.01	0.85	0.85	0.84	0.82	0.72	0.72	0.67	0.68	0.085	0.085	0.077	0.081
past	0.04	0.85	0.78	0.72	0.81	0.65	0.61	0.61	0.62	0.074	0.071	0.070	0.072
	0.02	0.82	0.80	0.78	0.83	0.66	0.60	0.62	0.68	0.076	0.068	0.070	0.079
	0.01	0.84	0.82	0.78	0.83	0.70	0.66	0.60	0.69	0.082	0.076	0.068	0.081

Table D.3: Performance of different considered scenarios for target excess return 5% and without short sale constraint. For further explanations see the caption to Table C.2.

views	τ	scenario	μ	SD	SKEW	KURT	$\text{CVaR}_{95\%}$	SR	$\mu/\text{CVaR}^D_{95\%}$	TURN	HI
accurate	0.04	BL	.0088	.0561	-1.49	7.47	0.157	0.55	0.053	1.18	0.73
		SS	.0138	.0468	-1.65	10.80	0.119	1.02	0.104	1.17	0.65
		TSK	.0103	.0494	-1.50	8.68	0.129	0.72	0.074	1.02	0.65
		BM	.0072	.0519	-1.11	5.88	0.134	0.48	0.051	0.96	0.66
	0.02	BL	.0079	.0576	-1.22	6.55	0.154	0.48	0.049	0.89	0.72
		SS	.0089	.0565	-1.38	7.31	0.155	0.55	0.055	1.11	0.75
		TSK	.0089	.0545	-0.94	5.44	0.139	0.57	0.060	0.80	0.73
		BM	.0072	.0541	-0.89	5.47	0.136	0.46	0.051	0.76	0.70
	0.01	BL	.0085	.0554	-0.76	5.17	0.136	0.53	0.059	0.56	0.73
		SS	.0072	.0585	-1.20	6.55	0.156	0.43	0.044	0.80	0.76
		TSK	.0095	.0571	-0.85	5.34	0.141	0.58	0.064	0.50	0.80
		BM	.0076	.0549	-0.78	5.25	0.135	0.48	0.054	0.57	0.72
momentum	0.04	BL	.0084	.0535	-0.52	4.41	0.126	0.55	0.063	0.23	0.76
		SS	.0088	.0538	-0.50	4.58	0.127	0.57	0.065	0.22	0.77
		TSK	.0073	.0598	-0.72	5.30	0.146	0.43	0.048	0.23	0.90
		BM	.0099	.0525	-0.47	4.26	0.119	0.66	0.077	0.20	0.76
	0.02	BL	.0087	.0532	-0.48	4.24	0.122	0.57	0.067	0.18	0.76
		SS	.0084	.0538	-0.51	4.22	0.125	0.54	0.063	0.19	0.76
		TSK	.0076	.0610	-0.70	5.15	0.147	0.43	0.049	0.18	0.93
		BM	.0093	.0532	-0.48	4.16	0.121	0.61	0.072	0.21	0.74
	0.01	BL	.0085	.0528	-0.39	4.23	0.119	0.56	0.067	0.17	0.75
		SS	.0090	.0530	-0.41	4.21	0.119	0.59	0.071	0.17	0.76
		TSK	.0059	.0621	-0.77	5.28	0.153	0.33	0.037	0.19	0.95
		BM	.0091	.0543	-0.61	4.84	0.126	0.58	0.068	0.23	0.74
past	0.04	BL	.0064	.0550	-0.80	5.86	0.143	0.41	0.044	1.15	0.73
		SS	.0071	.0508	-0.81	6.74	0.129	0.49	0.053	1.18	0.65
		TSK	.0061	.0469	-0.55	4.63	0.114	0.45	0.051	1.01	0.65
		BM	.0069	.0503	-0.76	5.26	0.121	0.48	0.055	0.94	0.65
	0.02	BL	.0069	.0550	-0.71	5.63	0.136	0.44	0.049	0.86	0.73
		SS	.0067	.0556	-0.79	5.69	0.145	0.42	0.044	1.11	0.75
		TSK	.0074	.0513	-0.49	4.55	0.121	0.50	0.058	0.84	0.73
		BM	.0071	.0517	-0.90	5.65	0.128	0.48	0.053	0.75	0.70
	0.01	BL	.0078	.0534	-0.61	5.12	0.126	0.51	0.058	0.59	0.73
		SS	.0068	.0555	-0.75	5.64	0.139	0.43	0.047	0.79	0.76
		TSK	.0073	.0555	-0.66	5.01	0.133	0.46	0.052	0.49	0.81
		BM	.0075	.0531	-0.75	5.43	0.127	0.49	0.056	0.56	0.72

Table D.4: Performance of different considered scenarios for target excess return 10% and short sale constraint. For further explanations see the caption to Table C.4.

views	τ	scenario	μ	SD	SKEW	KURT	$\text{CVaR}_{95\%}$	SR	$\mu/\text{CVaR}^{D}_{95\%}$	TURN	HI
accurate	0.04	BL	.0084	.0440	-1.14	6.70	0.113	0.66	0.069	0.77	0.66
		SS	.0128	.0389	-1.28	9.34	0.093	1.14	0.121	0.84	0.60
		TSK	.0106	.0398	-0.77	5.61	0.097	0.92	0.098	0.72	0.58
		BM	.0073	.0438	-1.04	6.96	0.110	0.58	0.063	0.57	0.63
	0.02	BL	.0077	.0435	-1.11	7.52	0.111	0.61	0.065	0.44	0.61
		SS	.0079	.0446	-1.30	7.87	0.116	0.62	0.064	0.75	0.63
		TSK	.0076	.0433	-0.90	5.81	0.109	0.61	0.066	0.73	0.63
		BM	.0072	.0439	-1.22	8.49	0.109	0.57	0.063	0.37	0.59
	0.01	BL	.0079	.0422	-0.89	6.69	0.100	0.65	0.073	0.19	0.58
		SS	.0077	.0431	-1.09	7.69	0.106	0.62	0.068	0.37	0.58
		TSK	.0072	.0441	-1.06	7.08	0.112	0.57	0.061	0.64	0.60
		BM	.0077	.0421	-0.84	6.52	0.099	0.64	0.073	0.18	0.59
momentum	0.04	BL	.0084	.0409	-0.62	5.45	0.092	0.72	0.085	0.06	0.59
		SS	.0085	.0410	-0.61	5.41	0.092	0.72	0.085	0.06	0.59
		TSK	.0088	.0420	-0.64	5.40	0.096	0.73	0.084	0.12	0.55
		BM	.0081	.0412	-0.61	5.36	0.092	0.69	0.081	0.09	0.59
	0.02	BL	.0084	.0409	-0.61	5.45	0.092	0.72	0.085	0.06	0.59
		SS	.0084	.0410	-0.62	5.46	0.092	0.72	0.085	0.06	0.59
		TSK	.0087	.0425	-0.61	5.42	0.098	0.71	0.082	0.13	0.54
		BM	.0081	.0412	-0.61	5.34	0.093	0.69	0.081	0.09	0.59
	0.01	BL	.0084	.0409	-0.61	5.45	0.091	0.72	0.085	0.06	0.59
		SS	.0084	.0409	-0.61	5.43	0.092	0.72	0.085	0.06	0.59
		TSK	.0083	.0429	-0.63	5.39	0.100	0.67	0.077	0.12	0.52
		BM	.0081	.0413	-0.61	5.34	0.093	0.68	0.081	0.09	0.59
past	0.04	BL	.0085	.0454	-0.67	5.92	0.106	0.65	0.074	0.77	0.65
		SS	.0078	.0445	-0.55	7.05	0.103	0.61	0.071	0.85	0.60
		TSK	.0072	.0412	-0.55	5.12	0.096	0.61	0.070	0.69	0.58
		BM	.0081	.0452	-0.86	6.53	0.105	0.62	0.072	0.58	0.63
	0.02	BL	.0082	.0434	-0.64	5.95	0.101	0.66	0.076	0.43	0.61
		SS	.0079	.0459	-0.67	5.90	0.110	0.60	0.068	0.73	0.63
		TSK	.0078	.0438	-0.70	5.63	0.104	0.62	0.070	0.71	0.63
		BM	.0083	.0424	-0.75	5.86	0.097	0.68	0.079	0.34	0.59
	0.01	BL	.0084	.0417	-0.70	5.90	0.095	0.70	0.082	0.18	0.58
		SS	.0082	.0429	-0.80	6.23	0.100	0.66	0.076	0.36	0.58
		TSK	.0077	.0449	-0.82	7.07	0.107	0.60	0.068	0.63	0.60
		BM	.0082	.0415	-0.67	5.69	0.094	0.69	0.081	0.18	0.59

Table D.5: Performance of different considered scenarios for target excess return 5% and short sale constraint. For further explanations see the caption to Table C.4.

D.2 *Without short sale constraint*

		return (μ)				SR				$\mu/\text{CVaR}^D_{95\%}$			
views	τ	BL	SS	TSK	BM	BL	SS	TSK	BM	BL	SS	TSK	BM
accurate	0.04	2.89	11.62	2.57	0.04	0.46	1.23	0.63	0.01	0.055	0.208	0.069	0.001
	0.02	-0.22	-0.03	1.19	0.34	-0.07	-0.01	0.34	0.14	-0.007	-0.001	0.032	0.013
	0.01	0.13	-0.16	0.23	0.47	0.06	-0.06	0.06	0.23	0.006	-0.006	0.006	0.023
momentum	0.04	0.92	1.01	1.49	0.97	0.48	0.53	0.49	0.52	0.053	0.059	0.065	0.059
	0.02	0.71	0.78	1.07	0.81	0.37	0.40	0.33	0.43	0.041	0.044	0.042	0.048
	0.01	0.60	0.63	0.84	0.73	0.31	0.33	0.26	0.38	0.035	0.036	0.032	0.043
past	0.04	1.04	2.93	0.72	0.54	0.14	0.36	0.17	0.18	0.016	0.052	0.018	0.019
	0.02	0.10	0.27	0.96	0.68	0.04	0.05	0.26	0.28	0.004	0.006	0.029	0.031
	0.01	0.34	0.18	0.68	0.71	0.15	0.07	0.18	0.34	0.017	0.007	0.021	0.038

Table D.6: Performance of different considered scenarios for target excess return 10% and without short sale constraint. For further explanations see the caption to Table C.2.

		return (μ)				SR				$\mu/\text{CVaR}^D_{95\%}$			
views	τ	BL	SS	TSK	BM	BL	SS	TSK	BM	BL	SS	TSK	BM
accurate	0.04	1.57	5.65	1.63	0.43	0.52	1.39	0.83	0.19	0.064	0.245	0.085	0.023
	0.02	0.52	0.59	0.85	0.69	0.37	0.30	0.52	0.52	0.037	0.028	0.049	0.057
	0.01	0.77	0.62	0.57	0.77	0.63	0.48	0.33	0.65	0.072	0.050	0.034	0.076
momentum	0.04	0.90	0.90	1.03	0.89	0.78	0.78	0.74	0.76	0.094	0.094	0.090	0.094
	0.02	0.89	0.89	0.96	0.88	0.77	0.77	0.68	0.75	0.093	0.093	0.081	0.092
	0.01	0.88	0.88	0.92	0.86	0.76	0.76	0.65	0.74	0.092	0.092	0.077	0.090
past	0.04	0.78	1.81	0.76	0.77	0.27	0.44	0.32	0.45	0.030	0.063	0.034	0.051
	0.02	0.77	0.71	0.94	0.86	0.57	0.34	0.53	0.68	0.066	0.038	0.062	0.081
	0.01	0.86	0.80	0.84	0.84	0.72	0.63	0.54	0.71	0.086	0.074	0.064	0.086

Table D.7: Performance of different considered scenarios for target excess return 5% and without short sale constraint. For further explanations see the caption to Table C.2.

views	τ	scenario	μ	SD	SKEW	KURT	CVaR$_{95\%}$	SR	μ/CVaR$^{D}_{95\%}$	TURN	HI
accurate	0.04	BL	.0289	.2169	-1.31	46.94	0.501	0.46	0.055	15.27	166.51
		SS	.1161	.3280	0.79	30.64	0.442	1.23	0.208	24.52	171.42
		TSK	.0256	.1405	-0.57	26.15	0.345	0.63	0.069	9.14	36.42
		BM	.0003	.1468	-9.89	139.16	0.393	0.01	0.001	7.32	21.21
	0.02	BL	-.0021	.1032	-2.83	15.56	0.332	-0.07	-0.007	5.37	15.96
		SS	-.0003	.1645	-3.41	32.20	0.531	-0.01	-0.001	12.64	63.28
		TSK	.0119	.1200	-1.23	13.90	0.356	0.34	0.032	6.88	27.70
		BM	.0034	.0850	-3.37	24.50	0.252	0.14	0.013	3.35	6.99
	0.01	BL	.0013	.0778	-1.75	8.60	0.225	0.06	0.006	2.30	5.83
		SS	-.0016	.0916	-2.42	12.15	0.286	-0.06	-0.006	4.14	10.81
		TSK	.0023	.1463	-7.33	99.61	0.397	0.06	0.006	4.98	24.09
		BM	.0047	.0725	-2.29	14.66	0.205	0.23	0.023	1.99	4.78
momentum	0.04	BL	.0091	.0667	-0.70	5.14	0.164	0.48	0.053	0.47	3.16
		SS	.0101	.0666	-0.70	5.17	0.163	0.53	0.059	0.55	3.23
		TSK	.0149	.1058	0.35	12.36	0.217	0.49	0.065	1.61	16.94
		BM	.0096	.0647	-0.84	6.39	0.154	0.52	0.059	0.65	2.96
	0.02	BL	.0070	.0671	-0.69	5.26	0.168	0.37	0.041	0.37	3.38
		SS	.0078	.0675	-0.69	5.20	0.168	0.40	0.044	0.42	3.42
		TSK	.0107	.1109	-0.67	13.97	0.244	0.33	0.042	1.74	17.85
		BM	.0080	.0655	-0.94	6.71	0.161	0.43	0.048	0.57	3.19
	0.01	BL	.0059	.0670	-0.70	5.40	0.167	0.31	0.035	0.33	3.44
		SS	.0063	.0673	-0.69	5.37	0.168	0.33	0.036	0.35	3.48
		TSK	.0084	.1121	-0.70	11.94	0.257	0.26	0.032	1.43	18.29
		BM	.0072	.0656	-1.00	7.03	0.163	0.38	0.043	0.52	3.43
past	0.04	BL	.0104	.2525	0.91	21.22	0.626	0.14	0.016	23.90	186.70
		SS	.0292	.2784	3.22	29.41	0.531	0.36	0.052	29.63	165.35
		TSK	.0072	.1458	-0.78	14.93	0.405	0.17	0.018	13.78	37.45
		BM	.0054	.1058	-1.91	23.61	0.275	0.18	0.019	5.40	13.43
	0.02	BL	.0010	.1011	-0.96	10.23	0.288	0.04	0.004	4.89	15.35
		SS	.0026	.1732	0.07	17.33	0.477	0.05	0.006	13.68	67.60
		TSK	.0096	.1257	0.13	11.64	0.316	0.26	0.029	6.90	29.96
		BM	.0067	.0842	-0.77	13.39	0.213	0.28	0.031	3.08	6.31
	0.01	BL	.0033	.0763	-0.78	7.00	0.199	0.15	0.017	2.19	5.78
		SS	.0018	.0904	-0.81	7.81	0.248	0.07	0.007	3.90	10.72
		TSK	.0067	.1297	-2.27	33.91	0.318	0.18	0.021	4.66	22.49
		BM	.0070	.0721	-0.45	11.30	0.177	0.34	0.038	1.87	5.08

Table D.8: Performance of different considered scenarios for target excess return 10% and without short sale constraint. For further explanations see the caption to Table C.4.

views	τ	scenario	μ	SD	SKEW	KURT	$\text{CVaR}_{95\%}$	SR	$\mu/\text{CVaR}^{D}_{95\%}$	TURN	HI
accurate	0.04	BL	.0157	.1040	-1.61	52.19	0.229	0.52	0.064	6.09	40.17
		SS	.0565	.1408	1.13	24.95	0.174	1.39	0.245	15.73	42.01
		TSK	.0162	.0678	-0.79	15.93	0.175	0.83	0.085	3.90	9.34
		BM	.0043	.0766	-9.34	137.67	0.186	0.19	0.023	2.02	3.59
	0.02	BL	.0052	.0491	-2.03	13.95	0.136	0.37	0.037	1.10	1.47
		SS	.0059	.0686	-3.10	26.55	0.202	0.30	0.028	3.44	8.76
		TSK	.0085	.0565	-1.74	12.37	0.167	0.52	0.049	2.78	4.87
		BM	.0068	.0454	-1.65	13.23	0.113	0.52	0.057	0.74	1.45
	0.01	BL	.0076	.0419	-0.82	7.23	0.099	0.63	0.072	0.38	1.25
		SS	.0062	.0451	-1.26	8.72	0.118	0.48	0.050	0.74	1.14
		TSK	.0057	.0609	-5.91	74.05	0.162	0.33	0.034	1.79	2.40
		BM	.0077	.0414	-0.65	7.34	0.094	0.65	0.076	0.41	1.43
momentum	0.04	BL	.0089	.0401	-0.48	5.76	0.087	0.78	0.094	0.14	1.35
		SS	.0090	.0402	-0.49	5.77	0.087	0.78	0.094	0.15	1.38
		TSK	.0102	.0478	0.07	9.44	0.104	0.74	0.090	0.32	1.90
		BM	.0089	.0404	-0.36	6.10	0.086	0.76	0.094	0.23	1.58
	0.02	BL	.0089	.0401	-0.49	5.76	0.087	0.77	0.093	0.14	1.32
		SS	.0089	.0402	-0.48	5.75	0.087	0.77	0.093	0.15	1.34
		TSK	.0096	.0490	-0.80	14.06	0.110	0.68	0.081	0.29	1.77
		BM	.0087	.0404	-0.37	6.14	0.086	0.75	0.092	0.22	1.53
	0.01	BL	.0088	.0401	-0.48	5.75	0.087	0.76	0.092	0.14	1.31
		SS	.0088	.0402	-0.48	5.83	0.087	0.76	0.092	0.15	1.32
		TSK	.0091	.0487	-0.77	12.65	0.110	0.65	0.077	0.26	1.55
		BM	.0086	.0405	-0.35	6.17	0.087	0.74	0.090	0.21	1.50
past	0.04	BL	.0078	.1024	0.18	26.64	0.255	0.27	0.030	5.87	31.08
		SS	.0181	.1440	3.58	31.92	0.272	0.44	0.063	9.17	42.16
		TSK	.0076	.0814	-1.12	15.82	0.218	0.32	0.034	4.12	9.81
		BM	.0076	.0584	-2.40	24.42	0.142	0.45	0.051	1.67	2.63
	0.02	BL	.0076	.0466	-0.94	8.92	0.109	0.57	0.066	1.04	1.39
		SS	.0070	.0709	-0.61	14.82	0.181	0.34	0.038	3.35	8.93
		TSK	.0094	.0617	-0.20	12.06	0.143	0.53	0.062	2.75	4.84
		BM	.0086	.0442	-0.68	7.86	0.098	0.68	0.081	0.69	1.40
	0.01	BL	.0085	.0413	-0.61	6.17	0.091	0.72	0.086	0.36	1.25
		SS	.0079	.0434	-0.79	6.92	0.100	0.63	0.074	0.72	1.14
		TSK	.0083	.0537	-1.66	21.94	0.123	0.54	0.064	1.61	2.08
		BM	.0084	.0412	-0.46	6.36	0.089	0.71	0.086	0.39	1.44

Table D.9: Performance of different considered scenarios for target excess return 5% and without short sale constraint. For further explanations see the caption to Table C.4.