# Liquidity Risk Theory and Coherent Measures of Risk

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**Abstract**

We discuss liquidity risk from a pure risk–theoretical point of view in the axiomatic context of Coherent Measures of Risk. We propose a formalism for Liquidity Risk which is compatible with the axioms of coherency. We emphasize the difference between "coherent risk measures" (CRM) $\rho(X)$ defined on portfolio values $X$ as opposed to "coherent portfolio risk measures" (CPRM) $\rho(\mathbf{p})$ defined on the vector space of portfolios $\mathbf{p}$, and we observe that in presence of liquidity risk the value function on the space of portfolios is no more necessarily linear. We propose a new nonlinear "Value" function $V^{\mathcal{L}}(\mathbf{p})$ which depends on a new notion of "liquidity policy" $\mathcal{L}$. The function $V^{\mathcal{L}}(\mathbf{p})$ naturally arises from a general description of the impact that the microstructure of illiquid markets has when marking a portfolio to market. We discuss the consequences of the introduction of the function $V^{\mathcal{L}}(\mathbf{p})$ in the coherency axioms and we study the properties induced on CPRMs. We show in particular that CPRMs are convex, finding a result that was proposed as a new axiom in the literature of so called "convex measures of risk". The framework we propose is not a model but rather a new formalism, in the sense that it is completely free from hypotheses on the dynamics of the market. We provide interpretation and characterization of the formalism as well as some stylized example.

## 1 Introduction

Liquidity risk is the unfailing shadow that precedes any severe market crisis. It is always the ultimate fuse carrying the spark which explodes market and credit risks and it is often the catalyst which transforms isolated loss events into systemic contagious breakdowns. Year 2007 makes no exception, witnessing an unprecedented crisis in the US mortgage market

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and global structured credit market where liquidity risk entered in the mispricing of illiquid instruments in the first place and then in the panicky flight-to-quality which followed the massive repricing.

Yet, in year 2007, not only banks have still no resort to any consensus market practice for liquidity risk management, but the state of the art in this research field is still at the starting line of looking for an appropriate notation for quantifying and definiing liquidity risk itself. This is probably no surprise for market practitioners, who perfectly know that this subject is a multi–faceted one and of a somehow elusive kind. It is not a chance that in the market, people correctly use the same term "liquidity risk" for at least three well distinguished phenomena,

*Facet 1:* the risk that our portfolio may run short euros

*Facet 2:* the risk we run trading in a illiquid market

*Facet 3:* the risk of a drainage of the liquidity circulating in our economy

which are respectively a major concern for a treasurer, a trader and a central bank governor. Liquidity risk is in fact a complex reality which manifests through all of these interdependent facets simultaneously. Most of the literature on the subject splits into branches with distinct notations which are often suitable to take only one of these points of view. It would therefore be a fundamental progress to devise a notation which lends itself to the formalization of questions in any of these aspects into quantitative and well stated problems.

As risk scholars, we have often felt like doing solid geometry in a plane. It is clear to everybody that unless we find a formalism encompassing liquidity risk as well, we are missing a fundamental dimension of risk itself. In this paper we face the task of devising a formalism that allows in the first place to capture on a pure risk-theoretical ground the interplay of facets 1 and 2. The same notation, however, will lend itself to describe the link with facet 3 as soon as no-arbitrage pricing arguments will be faced. This is the subject for future work.

## 2 Coherent Measures of Risk and Liquidity Risk

Since their advent [2], Coherent Measures of Risk were cheered as a milestone advance by the financial community and yet soon criticized as manifestly inappropriate for the description of liquidity risk. They appeared, to the eyes of risk experts, as the appropriate scheme for describing the properties of a good risk measure but just in the asymptotic limit of vanishing liquidity risk, e.g. in the case when portfolios' sizes are negligible with respect to the market depth. The axioms of coherence, written for functions $\rho$ of portfolio random variables $X, Y$ read[1]

| | | | |
|---|---|---|---|
| *(M)* | *monotonicity* | $\rho(X) \leq \rho(Y)$ | $X \geq Y$ |
| *(TC)* | *translational covariance* | $\rho(X + e) = \rho(X) - e$ | $e \in \mathbb{R}$ |
| *(PH)* | *positive homogeneity* | $\rho(\lambda X) = \lambda \rho(X)$ | $\lambda \geq 0$ |
| *(S)* | *subadditivity* | $\rho(X + Y) \leq \rho(X) + \rho(Y)$ | |

The liquidity risk argument against these axioms goes

*If I double an illiquid portfolio, the risk becomes more than double as much !*

and the axioms contradicted by this example should be (PH)[2] and (S)[3]. This line of reasoning led to the definition of a weaker set of "convexity axioms", where (PH) and (S) were replaced by a single weaker requirement

*(C)* *convexity* $\rho(\theta X + (1-\theta)Y) \leq \theta\rho(X) + (1-\theta)\rho(Y) \qquad \theta \in (0, 1)$

These weaker axioms define a larger space of "Convex Measures of Risk" introduced and widely described in [8, 3, 6, 7], that allow (PH) and (S) violations. We tend however to be skeptical on this possible route to describe liquidity risk for a very simple reason. With this approach, (PH) and (S) violations are introduced at the level of the chosen risk measures and therefore they apply to all portfolios irrespectively of their size or content. This in turn has the consequence that in the asymptotic limit of vanishing liquidity risk we will not be able to recover the fully coherent scheme, which we consider the appropriate one in this limit.

We think that the definition of convex measures of risk not only was unsatisfactory for describing liquidity risk, but also not necessary. Coherent measures of risk are in fact fully compatible with liquidity risk. The argument above, although certainly true, does not contradict the coherency axioms. The point is that $X, Y$ are not *"portfolios"* but *"portfolio values"*, and although we have been always used to think that the relation among the two concepts is linear, in the presence of liquidity risk this linearity will be naturally lost as we will see later. The axioms of coherency (TC), (PH) and (S) lend themselves to this double interpretation, which is harmless and even useful in the case of liquid markets, where $X, Y$ can be thought of as portfolios or portfolio values likewise. For instance, reading (TC) on portfolios one can mean

[1](TC) was originally termed *"translational invariance"* which however can be misunderstood as the property $f(x + e) = f(x)$.

[2]Put $\lambda = 2$

[3]Put $X = Y$.

*If we add to a portfolio a (risk–free) cash portfolio of $e$ euros, risk decreases by $e$*

while on portfolio values

*If a portfolio value is augmented by a fixed quantity $e$, risk decreases by $e$*

But axiom (M) can be read only thinking of $X$ and $Y$ as portfolio values, as one can realize noting that $X \leq Y$ is meaningless otherwise. This axiom reminds us what is the only orthodox way of reading those axioms. Hence the argument above doesn't contradict the coherency axioms, because the portfolio–doubling experiment is not represented in any axiom unless we make the additional assumption that the portfolio value $X$ is a linear function or at least a positive homogeneous function defined in the vector space of portfolios.

We will see, looking at the microstructure of illiquid markets, and in particular at the interplay of facets 1 and 2, that an appropriate definition of the *value* function on the space of portfolios $\mathbf{p}$ naturally defines a non–linear function $X = V(\mathbf{p})$ which explains this seeming paradox. We will discover that coherent measures of risk and liquidity risk are perfectly compatible.

# 3 Key observations on liquidity risk

In order to choose a suitable formalism, we enumerate some important facts that we would like to be accounted for by our notation. The first fact has been discussed in the previous section. Here we just give it the status of

**Key Observation 1.** In the presence of liquidity risk, the value of portfolios will no more be necessarily supposed to be a linear map in the vector space of portfolios

The second aspect we want to stress comes from the close interplay between facets 1 and 2 that the reader certainly already knows well. The need for euros for paying out debts may force us to sell off our assets probing the liquidity of the market. Therefore, a sensible mark-to-market policy on the illiquid assets we own, should depend on the amount of liquidity we already have and on the amount of debts we will have to repay. The more the liquidity we have, the less prudent we need to be in our mark-to-market policy, for a fixed level of debts to pay.

But then, it is clear that we should ask ourselves in the first place if the problem of a portfolio liquidity risk is a well–stated one or not. In other words, we see from the above reasoning that in the presence of liquidity risk, when asked *"what is the value (or the risk) of this portfolio ?"*, we should always reply with another question *"what do you need to do with that portfolio"* ?

The same portfolio, say a very illiquid portfolio of structured bonds or of issued mortgages, can bear considerable liquidity risk if the owner may need to liquidate parts of it for future payments, but can be liquidity riskless (by definition we are tempted to say) if the owner has no debts at all and therefore can afford to lock these assets in a drawer and forget about them until their maturity.

Collecting all these facts together, we get to the conclusion that in order to speak of "the value" or "the liquidity risk" of an illiquid portfolio, we need to define first a notion which is the mathematical counterpart of a "liquidity policy".

**Key Observation 2.** In the presence of liquidity risk, we cannot speak of the "value" or of the "liquidity risk" of a portfolio unless we fix a liquidity policy.

We address the task of a correct mathematical definition of a liquidity policy in section 4. The appropriate choice for this definition is one of the most subtle and fundamental steps to take for devising an appropriate notation for liquidity risk.

# 4 The formalism

We don't know who coined the name "mark–to–market" (MtM). But he/she certainly had good intuition about liquidity risk. We believe that very seldom a term has been so appropriate and in our case, inspiring. Marking to market means: *"if you want to assess the value of your portfolio you have to confront it with true prices, which are the only physical reality of the market"*. Just an experimental Galilean prescription, if you prefer. In what follows we will just pedantically translate this common-sense recipe into mathematical terms after having looked into detail at the structure of illiquid markets. We adopt a simplified notation for ease of exposition. A more technical publication will follow [1] where we account for a more rigorous mathematical treatment and give all proofs in full generality.

## 4.1 Assets and Portfolios

The first important thing we want to stress is the conceptual difference between *assets* and *portfolios*. The notation of liquid markets has brought us to think of them as objects living in the same vector space. In fact we are used to write expressions like $\pi = \sum_i w_i \, A_i$. In presence of liquidity risk, expressions like this will turn out to be meaningless both as relations among formal objects and as relations among their values. Indeed the very concept of 'value of an asset' will turn out to be meaningless and so we will not try (nor need) to define it. An asset does not have any value *per se*. Its value will depend on the portfolio in which we put it and its only objective reality is the list of prices at which we can buy or sell it.

**Definition 1.** An *'Asset' $A$* is a good traded in the market with prices given by a map[4] $m : \mathbb{R}_* \to \mathbb{R}$ called its *'Marginal Supply–Demand Curve'* (MSDC) satisfying

[4] Notation: $\mathbb{R}_* \triangleq \mathbb{R} \setminus \{0\}$

1. $m(x_1) \geq m(x_2)$ if $x_1 < x_2$

2. $m$ is cadlag for $x < 0$ and ladcag for $x > 0$

We call $m^+ := m(0^+)$ the *'best bid'* and $m^- := m(0^-)$ the *'best ask'*. The positive quantity $\delta m := m^- - m^+ \geq 0$ is called the *'bid–ask spread'*. We denote $\mathcal{M}$ the set (actually, a convex cone) of all possible MSDC.

The price $m(x)$ for $x > 0$ ($x < 0$) is the bid (ask) price associated to a quote of size $dx$, expressed in units of the asset. Notice that the only assumption is the no arbitrage necessary requirement that any bid is lower than any price. The decreasing shape of the MSDC is instead not a restrictive hypothesis, but only the result of a rearrangement of all available prices in a single curve with bids (asks) on the right (left) of the origin $x = 0$ and better prices closer to it. In other words an MSDC is decreasing by construction.[5].

MSDCs can be directly observed in real time in the quotes of stock exchanges for listed assets. Realistic cases of MSDC will certainly display a non-zero bid–ask spread and finite size quotes for best bid and offer, namely MSDC locally flat at the left and right of the origin. We do not impose however these conditions for the sake of general modeling purposes.

We will be interested in the stochastic evolution of MSDCs through time, so that the dynamics of assets will be described by stochastic processes with values in the space $\mathcal{M}$ of all MSDCs.

In general, MSDCs can assume positive or negative values. Some assets however may admit natural bounds. We will call a *security* an asset whose MSDC takes positive values only and a *swap* (latu senso) any asset whose MSDC can admit both positive and negative values. Without loss of generality we can always exclude the presence of assets $A$ with MSDC taking only negative values by redefining $A' := -A$ through $m'(x) := -m(-x)$.

We don't define any concept such a mid–price. Our MSDCs are not defined in $x = 0$. The reason of this choice is simply because mid–prices in real finance do not exist.

Notice that assets do not live in a vector space. In fact, there is no notion of addition $A_1 + A_2$ of two assets, because we see no way of giving any useful definition for the corresponding MSDC.

**Definition 2.** The *"euro"* is a special Asset $A_0$ whose MSDC $m_0(x)$ is identically equal to 1 for all $x \in \mathbb{R}_*$. It is the currency which is delivered cash in any trade. The *"market"* is a finite collection of assets $\mathfrak{M} = \{A_i | i = 0, \ldots, N\}$ ($N \geq 1$) containing the euro.

We assume that when trading an asset we hit the most convenient prices available. This action is therefore described by the function

[5] It is only for the sake of simplicity that we assume finite quotations $m(x)$ for all $x \in \mathbb{R}_*$. This will be generalized in [1]

**Definition 3.** The *'Proceeds'* for transaction of $s \in \mathbb{R}_*$ assets are

$$P(s) \triangleq \int_0^s m(x)\,dx$$

Note that $P(s)$ is the total amount we receive when we sell $s > 0$ assets and minus the total amount we pay when we buy $|s|$ assets when $s < 0$.

**Definition 4.** The *"Supply–Demand Curve"* (SDC) for transaction of $s \in \mathbb{R}_*$ assets is[6] :

$$S(s) \triangleq \frac{P(s)}{s}.$$

For $s > 0$, $S(s)$ is the average unit price of the sale of $s$ assets. For $s < 0$, $S(s)$ is minus the average unit price of the purchase of $|s|$ assets.

Consider now a market $\mathfrak{M}$ where $N \geq 1$ (illiquid) assets are traded: we denote $m_i \in \mathcal{M}$ the MSDC of the $i$–th asset.

**Definition 5.** A *"Portfolio"* is a vector $\mathbf{p} = (p_0, \ldots, p_N) \in \mathbb{R}^{N+1}$, where $p_0$ is the *"portfolio liquidity"* and $\vec{p} := (p_1, \ldots, p_n) \in \mathbb{R}^N$ is the *"assets' position"*. We will say that $\mathbf{p}$ is *"long–"*, *"flat–"* or *"short– asset $A_k$"* if $p_k > 0$, $p_k = 0$ or $p_k < 0$ respectively.

We will denote $\mathcal{P} = \mathbb{R}^{N+1}$ the space of all portfolios. We will also sometimes write $\mathbf{p} + a := \mathbf{p} + (a, \vec{0})$ identifying scalars $a$ with portfolios $(a, \vec{0})$ made of cash only.

The case $p_0 < 0$ describes a portfolio which must pay immediately $|p_0|$ euros that it does not possess. This is different from a portfolio financed for example via a loan which in our notation will be instead some illiquid asset $A_k$ $(k > 0)$ endowed with a non–trivial MSDC.

We now describe the action of liquidating a portfolio on the market.

**Definition 6.** The *"Liquidation"* of a portfolio $\mathbf{p} \in \mathcal{P}$ is defined by

$$L(\mathbf{p}) \triangleq \sum_{i=0}^{N} P_i(p_i) = p_0 + \sum_{i=1}^{N} \int_0^{p_i} m_i(x)\,dx$$

The quantity $L(\mathbf{p})$ represents the total proceeds coming from a liquidation of the portfolio. This quantity can be considered an extremely prudent case of MtM policy for a portfolio. Such prudence is necessary only for a portfolio subject to external agents which could force a sudden closure of all positions. The opposite extreme case is

[6] In [4, 9] the supply demand curve is defined in a similar way, apart from stronger regularity conditions and a different convention for the sign of the argument $s$, i.e. $S^{CJP}(s) = S(-s)$. In our formalism MSDCs are more natural structures to adopt than SDCs. Our sign convention help us avoiding a lot of minus signs in the tools we will use more frequently.

**Definition 7.** The *"Uppermost MtM policy"* of a portfolio $\mathbf{p} \in \mathcal{P}$ is defined by

$$U(\mathbf{p}) \triangleq p_0 + \sum_{p_i>0} m_i^+ p_i + \sum_{p_i<0} m_i^- p_i$$

$U(\mathbf{p})$ coincides with the widespread MtM practice of marking all long positions to the best bids and all short positions to the best offers available. Although this practice does take into account bid–offer spreads (as opposed to the practice of marking to mid–prices which we don't ever contemplate), this may often be not a prudent one, because it does not probe at all the depth of bid and offer prices. It is therefore a sufficiently prudent MtM policy only in those cases when one knows for sure that a portfolio will never be subject to liquidations for future payments or else in those cases where the portfolio sizes are smaller than the market depth of corresponding best prices.

It is elementary to check that for any $\mathbf{p} \in \mathcal{P}$, $U(\mathbf{p}) \geq L(\mathbf{p})$. This justifies the following

**Definition 8.** The *liquidation cost* of a portfolio $\mathbf{p} \in \mathcal{P}$ is

$$C(\mathbf{p}) \triangleq U(\mathbf{p}) - L(\mathbf{p}) \in \mathbb{R}_+$$

We need the following useful notions

**Definition 9.** Given two portfolios $\mathbf{p}, \mathbf{q} \in \mathcal{P}$, we say that

1. they are *"concordant"*, and we write $\mathbf{p} \upuparrows \mathbf{q}$, if $p_i q_i \geq 0$ for any $i > 0$
2. they are *"discordant"*, and we write $\mathbf{p} \updownarrows \mathbf{q}$, if $p_i q_i \leq 0$ for any $i > 0$
3. $\mathbf{q}$ is *"attainable"* from $\mathbf{p}$, and we write $\mathbf{q} \in \mathrm{Att}(\mathbf{p}) \subseteq \mathcal{P}$, if $\mathbf{q} = \mathbf{p} - \mathbf{r} + L(\mathbf{r})$ for some $\mathbf{r} \in \mathcal{P}$

The attainable portfolios $\mathbf{q} \in \mathrm{Att}(\mathbf{p})$ are all the portfolios that can be obtained from $\mathbf{p}$ by liquidating at available market prices some portfolio $\mathbf{r} \in \mathcal{P}$.

Useful properties of the functions $L$, $U$ and $C$ are

**Proposition 1.** *The functions $L, U, C$, defined on $\mathcal{P}$ have the following properties*

- *$L : \mathcal{P} \to \mathbb{R}$ is concave and continuous on $\mathcal{P}$, subadditive on concordant portfolios and superadditive on discordant portfolios;*
- *$U : \mathcal{P} \to \mathbb{R}$ is concave, continuous, superadditive on $\mathcal{P}$, additive on concordant portfolios;*
- *$C : \mathcal{P} \to \mathbb{R}_+$ is convex and continuous on $\mathcal{P}$, superadditive on concordant portfolios and subadditive on discordant portfolios.*

*Moreover, if* $\lambda \geq 1$

$$
\begin{aligned}
L(\lambda\mathbf{p}) &\leq \lambda\, L(\mathbf{p}) \\
U(\lambda\mathbf{p}) &= \lambda\, U(\mathbf{p}) \\
C(\lambda\mathbf{p}) &\geq \lambda\, C(\mathbf{p})
\end{aligned}
$$

*Proof.* see [1] $\square$

We will also need the following important

**Lemma 1.** *Let* $\mathbf{q} \in \mathrm{Att}(\mathbf{p})$. *Then* $U(\mathbf{q}) \leq U(\mathbf{p})$.

*Proof.* From prop. 1 we know that $U(\mathbf{q}) = U(\mathbf{p} - \mathbf{r} + L(\mathbf{r})) = U(\mathbf{p} - \mathbf{r}) + L(\mathbf{r})$ is concave in $\mathbf{r}$. To show that it attains a maximum value in $\mathbf{r} = \mathbf{0}$ it is sufficient to compute right and left derivatives in $r_i = 0$ which turn out to be negative and positive respectively from direct computation. $\square$

### 4.2 Liquidity Policy and Mark–to–Market Policy

We want now to generalize the concept of MtM policy. We first need the following

**Definition 10.** A *"liquidity policy"* $\mathcal{L}$ is any closed convex subset $\mathcal{L} \subseteq \mathcal{P}$ of the space of portfolios satisfying

1. $\mathbf{p} \in \mathcal{L} \quad \Rightarrow \quad \mathbf{p} + a \in \mathcal{L}, \quad \forall a > 0$
2. $\mathbf{p} = (p^0, \vec{p}) \in \mathcal{L} \quad \Rightarrow \quad (p^0, \vec{0}) \in \mathcal{L}$

A liquidity policy is therefore a type of constraint on a portfolio $\mathbf{p}$ for which the portfolio liquidity $p_0$ is never too large and the absolute value $|p_k|$ of any illiquid asset $(k > 0)$ is never too small.

An example of liquidity policies is represented by the family of what we will call *cash liquidity policies* defined by

$$
\mathcal{L}(a) \triangleq \{(p_0, \vec{p}) \in \mathcal{P} \,|\, p_0 \geq a\} \qquad a \in \mathbb{R}
$$

The meaning of $\mathcal{L}(a)$ is transparent. It is a minimum requirement $a$ on the cash $p_0$ to be held in portfolio. This is the liquidity policy that an hypothetical ALM department should advise to the Risk Management department if $a$ is the amount of euros supposed to be enough to fulfill expected due payments on the time horizon considered.

We stress that liquidity policies can be also of other kinds. For instance, an Hedge Fund could be forced to keep its positions within limits imposed by its own risk manager or by the margin protocol of the prime broker. In this case the limits would affect in a complicated way the maximum exposure allowed for its illiquid assets, and the constraints might be expressed in terms of any possible analytics (say notionals, sensitivities, ...) of the portfolio.

The notion of liquidity policy is *not* supposed to be a constraint that a portfolio *need* to satisfy all the time. It is instead a constraint that a portfolio should be *prepared* to satisfy at some random future instant. The value of a portfolio subject to this kind of constraint needs to be a MtM policy which accounts for this possibility and does not drop if this random instant occurs.

We can now state the central definition of our formalism, namely the value function $V^{\mathcal{L}}(\mathbf{p})$ of a portfolio $\mathbf{p}$ under a liquidity policy $\mathcal{L}$.

**Definition 11.** The *"MtM policy"* or simply the *"Value"* of a portfolio $\mathbf{p}$ under the liquidity policy $\mathcal{L}$, is a map $V^{\mathcal{L}} : \mathcal{P} \to \mathbb{R}$ defined by

$$V^{\mathcal{L}}(\mathbf{p}) \triangleq \sup \{U(\mathbf{q}) \,|\, \mathbf{q} \in \mathrm{Att}(\mathbf{p}) \cap \mathcal{L}\} \tag{1}$$

This definition clearly provides a non–linear map on the space of portfolios, as we anticipated in section 2. This notion of value function is new and represents a fundamental ingredient for portfolio risk management in the case of illiquid markets.

A very important property of $V^{\mathcal{L}}$ is the convexity of the optimization problem it defines.

**Proposition 2.** *The optimization problem (1) in* $\mathbf{q}$ *is equivalent to the following convex optimization problem in* $\mathbf{r}$

$$V^{\mathcal{L}}(\mathbf{p}) = \sup \{U(\mathbf{p} - \mathbf{r}) + L(\mathbf{r}) \,|\, \mathbf{r} \in \mathcal{C}_{\mathcal{L}}(\mathbf{p})\} \tag{2}$$

*where*

$$\mathcal{C}_{\mathcal{L}}(\mathbf{p}) \triangleq \{\mathbf{r} \in \mathcal{P} | \mathbf{p} - \mathbf{r} + L(\mathbf{r}) \in \mathcal{L}\}$$

*If* $\mathcal{C}_{\mathcal{L}}(\mathbf{p}) = \varnothing$ *then* $V^{\mathcal{L}}(\mathbf{p}) = -\infty$*, else the infimum is attained and* $V^{\mathcal{L}}(\mathbf{p}) \in \mathbb{R}$

*Proof.* Equivalence comes from changing variable $\mathbf{r} = \mathbf{p} - \mathbf{q}$. $U(\mathbf{p} - \mathbf{r}) + L(\mathbf{r})$ is a concave function in $\mathbf{r}$ thanks to prop. 1. The set $\mathcal{C}_{\mathcal{L}}(\mathbf{p}) \subseteq \mathcal{P}$ is easily shown to be convex thanks to the concavity of $L$ and the properties of a liquidity policy. □

**Remark 1.** Notice that the cash component of $\mathbf{r}$ plays no role in (2), because $U(\mathbf{p} - \mathbf{r}) + L(\mathbf{r})$ and $\mathbf{p} - \mathbf{r} + L(\mathbf{r})$ do not depend on $r_0$. Therefore we can always put the additional constraint $r_0 = 0$.

**Remark 2.** The convexity property of the optimization problem (2) implicit in the definition of MtM policies is a crucial ingredient for future concrete industrial implementations of this framework in the field of financial risk management. Had it been not convex in general, this method would have had no chance of being ever considered for implementation by financial practitioners. Being convex, we can fortunately say that this optimization, though never trivial, is at the same time never challenging for standard convex optimization techniques. A deeper characterization of this optimization problem will be given in [1]. In this paper we limit ourselves to give an illustrative example in section 6 where an explicit solution is displayed for a wide class of stylized MSDCs.

We can easily recognize that the maps $L(\mathbf{p})$ and $U(\mathbf{p})$ are nothing but two extreme cases of MtM policies $V^{\mathcal{L}}(\mathbf{p})$, the *"be prepared to liquidate everything"* and *"don't be prepared to liquidate anything"* policies formally described by

$$\begin{aligned}
&\mathcal{L}^L \triangleq \{(p_0, \vec{0}) \in \mathcal{P} | p_0 \in \mathbb{R}\} \\
&\mathcal{L}^U \triangleq \mathcal{P}
\end{aligned}$$

We call $U$ the uppermost MtM policy because of the following

**Proposition 3.** *For any liquidity policy $\mathcal{L}$ we have*

$$V^{\mathcal{L}}(\mathbf{p}) \leq U(\mathbf{p})$$

*Proof.* From (1) and lemma 1. □

We can now check that if it turns out to be necessary to attain a portfolio which satisfies $\mathcal{L}$, the value of the function $V^{\mathcal{L}}$ does not drop.

**Proposition 4.** *Let $\mathbf{q}^\star \in \mathrm{Att}(\mathbf{p}) \cap \mathcal{L}$ be a solution of optimization problem (1), $U(\mathbf{q}^\star) = V^{\mathcal{L}}(\mathbf{p})$. Then we have*

$$V^{\mathcal{L}}(\mathbf{q}^\star) = V^{\mathcal{L}}(\mathbf{p})$$

*Proof.* Since $\mathbf{q}^\star \in \mathcal{L}$, lemma 1 and eq. (1) immediately give $V^{\mathcal{L}}(\mathbf{q}^\star) = U(\mathbf{q}^\star)$. □

We can now state an important strong characterization of the value map under any liquidity policy $\mathcal{L}$

**Theorem 1.** *Let $\mathcal{L}$ be any liquidity policy. Then the map $V^{\mathcal{L}} : \mathcal{P} \to \mathbb{R}$*

1. *is concave*

$$V^{\mathcal{L}}(\theta \mathbf{p}_1 + (1-\theta)\mathbf{p}_2) \geq \theta\, V^{\mathcal{L}}(\mathbf{p}_1) + (1-\theta)\, V^{\mathcal{L}}(\mathbf{p}_2), \qquad \forall \theta \in [0,1]$$

2. *is translationally supervariant*

$$V^{\mathcal{L}}(\mathbf{p} + e) \geq V^{\mathcal{L}}(\mathbf{p}) + e, \qquad \forall e \geq 0$$

*Proof. 1.* Let $\mathbf{p}_\theta = \theta\mathbf{p}_1 + (1-\theta)\mathbf{p}_2$. Let $\mathbf{r}_i$ $(i = 1, 2)$ be the solution of the optimal problem (2) for $V^{\mathcal{L}}(\mathbf{p}_i)$. Then, defining $\mathbf{r}_\theta = \theta\mathbf{r}_1+(1-\theta)\mathbf{r}_2$, it is easy to show (using also the concavity of $L$, prop. 1) that $\mathbf{r}_\theta \in \mathcal{C}_{\mathcal{L}}(\mathbf{p}_\theta)$. Using again (2) we have $V^{\mathcal{L}}(\mathbf{p}_\theta) \geq U(\mathbf{p} - \mathbf{r}_\theta) + L(\mathbf{r}_\theta) \geq \theta(U(\mathbf{p} - \mathbf{r}_1) + L(\mathbf{r}_1)) + (1-\theta)(U(\mathbf{p} - \mathbf{r}_2) + L(\mathbf{r}_2)) = \theta V^{\mathcal{L}}(\mathbf{p}_1) + (1-\theta)V^{\mathcal{L}}(\mathbf{p}_2)$, where we have used the concavity of $U$ and $L$ (prop. 1).

*2.* From (2) applied to $V^{\mathcal{L}}(\mathbf{p} + e)$, noting that $U(\mathbf{p} + e - \mathbf{r}) = U(\mathbf{p} - \mathbf{r}) + e$ and that $e \geq 0$ implies $\mathcal{C}_{\mathcal{L}}(\mathbf{p} + e) \supseteq \mathcal{C}_{\mathcal{L}}(\mathbf{p})$ by definition of liquidity policy (def. 10). □

Both the results of this theorem are extremely important and deserve interpretation.

First of all, concavity of $V^{\mathcal{L}}$ tells us, even before introducing risk measures, that already at the level of value functions, for liquidity risk we have a *diversification principle*. Blending two portfolios generates additional value to the blend of their two values. This confirms the commonsense rule that to protect oneself against liquidity risk it is better reduce the *granularity* of a portfolio.

Secondly, from supervariance we see that adding a positive cash portfolio to a given portfolio, also generates additional value with respect to the sum of values of the two constituent portfolios. The point is that when adding a positive amount in euros to a portfolio, we are not only adding nominal value, but also improving the liquidity properties of the original portfolio.

**Remark 3.** All section 4.2 could be rewritten entirely replacing the role of operator $U$ with another operator $U'$ based on the concept of marking–to–mid market,

$$U'(\mathbf{p}) \triangleq \sum_{i=0}^{N} m_i(0)\, p_i$$

All the results of this paper would still hold, because $U'$ has all the properties of $U$ (and also more because it is strictly linear and not only concave). We insist that $U$ plays a more fundamental role than $U'$ because it is based on market data and not on market abstractions. We will never follow this approach.

# 5 Coherent Measures of Risk, revisited

Having discussed the no more linear relation that exists between portfolios and portfolio values in an illiquid context, we can turn to our initial project of discussing whether the set of coherent axioms is still valid or needs to be modified.

Our plan should be clear by now, but we stress it again. *We choose to keep coherency axioms exactly the way they are.* Instead of trying to modify the set of axioms, we just take them seriously, according to their only orthodox interpretation of being axioms written on portfolio values[7] We do not agree on the typical arguments against coherent axioms coming from liquidity risk arguments, as they are flawed by the implicit assumption of linearity for the value map on the portfolio space. And in section 4.2 we have shown that this is not only an unnecessary hypotheses, but also an inadequate one for illiquid markets. We also stress that leaving aside these faulty arguments, we don't see any other good reason why we should give up the solid structure of coherency axioms. Having said this, we nevertheless keep well in mind that in choosing axioms one is simply laying the foundations for a deductive science, and as such, every different set of axioms is neither right nor wrong, but is just the starting point for a different theory or school of thought if you prefer. It is on the ground of their descriptive power that we can compare two approaches based on different axioms, and here we are just saying that this is our choice and not that this is the only or the best choice.

## 5.1 Coherent Portfolio Risk Measures

We have now the opportunity of discussing both measures of risk $\rho(X)$ defined as functions on portfolio values r.v.'s $X$ and measures of risk defined on the space of portfolios $\rho^{\mathcal{L}}(\mathbf{p})$ by defining the latter explicitly through a given measure of risk $\rho$ and a chosen MtM policy $V^{\mathcal{L}}$.

Let a probability space $(\Omega, \mathcal{F}, P)$ be given, where $\mathcal{F}$ is the $\sigma$-algebra describing the information available at the time horizon $T > 0$. It is now possible to give a natural stochastic counterpart of the notions given in the previous sections. Referring to [1] for a thorough and precise development of this aspect, for our present purposes it is sufficient to say that a *random MSDC* is a random variable defined on $(\Omega, \mathcal{F}, P)$ and taking values in $\mathcal{M}$, the set of all MSDC (see Definition 1), endowed with the natural $\sigma$-algebra generated by the cylinders. This means that $V^{\mathcal{L}}(\mathbf{p})$, the Value of a portfolio, is now a random variable and no more a number: we don't know the exact shape of the future (joint) MSDC's, and we can only make a probabilistic model for them.

**Definition 12.** Let a random MSDC be fixed. Given a coherent measure of risk $\rho : \mathcal{V} \to \mathbb{R}$, where $\mathcal{V}$ is a subspace of random variables, and a liquidity policy $\mathcal{L}$ we will call *coherent portfolio risk measure* (CPRM) induced by $\mathcal{L}$ on $\rho$ the map $\rho^{\mathcal{L}} : \mathcal{P} \to \mathbb{R}$ defined by

$$\rho^{\mathcal{L}}(\mathbf{p}) \triangleq \rho(V^{\mathcal{L}}(\mathbf{p}))$$

**Remark 4.** Plainly, $\rho^{\mathcal{L}}$ is well defined only on those portfolios $\mathbf{p}$ such that $V^{\mathcal{L}}(\mathbf{p}) \in \mathcal{V}$. Moreover, as we have seen before, it may well happen that $V^{\mathcal{L}}(\mathbf{p}) = -\infty$: we can extend

[7] A similar attitude is taken in [9], where the authors adopt coherent measures on market–liquidity–sensitive asset distributions. Their formalism is different from ours in many respects, as they are more focused on no–arbitrage pricing theory for single asset markets, while we are exclusively focused (for the moment) on portfolio risk–theory for multi–assets illiquid markets. However, we definitely agree with them in the interpretation of coherent axioms they give.

$\rho^{\mathcal{L}}$ to this case by formally putting $\rho(-\infty) = +\infty$. For a more precise treatment of these mathematical issues we refer to [1].

We immediately observe that, exactly as it happens for portfolio "values", also portfolio "risks" in an illiquid context can be discussed only after having fixed a liquidity policy $\mathcal{L}$. The following step is to see how the axioms of coherency translate into conditions on the portfolio measures $\rho^{\mathcal{L}}$

We notice that coherency axiom (M) remains essentially unchanged when we look at it from the point of view of CPRMs. It is just rephrased in a different notation

**Proposition 5.** *Let $\rho^{\mathcal{L}}$ be any CPRM. Then, for all $\mathbf{p}, \mathbf{q} \in \mathcal{P}$*

$$\rho^{\mathcal{L}}(\mathbf{p}) \leq \rho^{\mathcal{L}}(\mathbf{q}) \qquad \textit{if} \qquad V^{\mathcal{L}}(\mathbf{p}) \geq V^{\mathcal{L}}(\mathbf{q})$$

Extremely more interesting is the effect produced by theorem 1 on the coherent axioms.

**Theorem 2.** *Let $\rho^{\mathcal{L}}$ be any CPRM. Then*

1. *$\rho^{\mathcal{L}}$ is convex*

$$\rho^{\mathcal{L}}(\theta\mathbf{p}_1 + (1-\theta)\mathbf{p}_2) \leq \theta\, \rho^{\mathcal{L}}(\mathbf{p}_1) + (1-\theta)\, \rho^{\mathcal{L}}(\mathbf{p}_2), \qquad \forall\theta \in [0,1],\ \ \forall\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{P}$$

2. *$\rho^{\mathcal{L}}$ is translationally subvariant*

$$\rho^{\mathcal{L}}(\mathbf{p} + e) \leq \rho^{\mathcal{L}}(\mathbf{p}) + e, \qquad \forall e \geq 0,\ \ \forall\mathbf{p} \in \mathcal{P}$$

*Proof. 1.*

$$\begin{aligned} \rho^{\mathcal{L}}(\theta\mathbf{p}_1 + (1-\theta)\mathbf{p}_2) &= \rho(V^{\mathcal{L}}(\theta\mathbf{p}_1 + (1-\theta)\mathbf{p}_2)) \\ &\overset{\text{th. 1 and (M)}}{\leq} \rho(\theta V^{\mathcal{L}}(\mathbf{p}_1) + (1-\theta)V^{\mathcal{L}}(\mathbf{p}_2)) \\ &\overset{\text{(S) and (PH)}}{\leq} \theta\rho(V^{\mathcal{L}}(\mathbf{p}_1)) + (1-\theta)\rho(V^{\mathcal{L}}(\mathbf{p}_2)) \\ &= \theta\rho^{\mathcal{L}}(\mathbf{p}_1) + (1-\theta)\rho^{\mathcal{L}}(\mathbf{p}_2) \end{aligned}$$

*2.* From supervariance (theorem 1), (M) and (TC). □

This is one of the main results of the paper. Convexity of risk measures, which was proposed as an axiom (C) to weaken the couple (S) and (PH), is found here instead as a *result* of our formalism when we study the properties of a CPRM. The result is completely independent on the chosen liquidity policy $\mathcal{L}$. This means that the diversification principle is still working on risk measures also when markets are illiquid under any possible circumstance. We see therefore that in a hypothetical risk–reward plane, the diversification principle in illiquid markets affects both axes. This is a completely new result to us and bear a lot of important consequences in the field of asset allocation that will deserve separate analysis.

Translational subvariance of CPRMs is the immediate reflex of translational supervariance of MtM policies. The same comments made for theorem 1 apply also here. Adding euros to a portfolio not only adds to it nominal value but increases its level of liquidity. This is reflected in the CPRM as an additional benefit. This is absolutely natural from a financial point of view and we stress that to the best of our knowledge, nobody had ever criticized the coherency axioms for not taking into account this aspect (reading them as axioms on portfolio functions) in the presence of illiquidity. Criticisms had focused on (PH) and (S) only. This is also another reason why we believe that "convex measures of risk" are inadequate for describing liquidity risk. They contain the (TC) axiom and they assume linearity between value and portfolios, so they cannot account for this benefit.

The property of translational subvariance under the addition of cash first appeared in [5] in a different context.

A natural question is wondering whether (PH) and (S) still hold in full generality or not, and if not what has become of them. The answer is that they both do not hold anymore in general for CPRMs, nor it is easy to provide a general deformation of these properties for all possible portfolios and liquidity policies. We justify this conclusion in the following proposition which collects and compares the results for three particular cases of liquidity policies, namely $L$, $U$ and cash liquidity policies $V^{\mathcal{L}(a)}$.

**Proposition 6.**

- *Let $L$ be the Liquidation operator. Then*
  1. $$\rho^{\mathcal{L}^L}(\lambda \mathbf{p}) \geq \lambda\, \rho^{\mathcal{L}^L}(\mathbf{p}) \qquad \lambda \geq 1,\ \mathbf{p} \in \mathcal{P}$$
  2. *$\rho^{\mathcal{L}^L}$ is subadditive on discordant portfolios*
  $$\rho^{\mathcal{L}^L}(\mathbf{p}_1 + \mathbf{p}_2) \leq \rho^{\mathcal{L}^L}(\mathbf{p}_1) + \rho^{\mathcal{L}^L}(\mathbf{p}_2) \qquad \mathbf{p}_1 \updownarrows \mathbf{p}_2$$
- *Let $U$ be the Uppermost MtM policy. Then*
  1. *$\rho^{\mathcal{L}^U}$ is positive homogeneous*
  $$\rho^{\mathcal{L}^U}(\lambda \mathbf{p}) = \lambda\, \rho^{\mathcal{L}^U}(\mathbf{p}) \qquad \lambda \geq 0,\ \mathbf{p} \in \mathcal{P}$$
  2. *$\rho^{\mathcal{L}^U}$ is subadditive*
  $$\rho^{\mathcal{L}^U}(\mathbf{p}_1 + \mathbf{p}_2) \leq \rho^{\mathcal{L}^U}(\mathbf{p}_1) + \rho^{\mathcal{L}^U}(\mathbf{p}_2) \qquad \mathbf{p}_1, \mathbf{p}_2 \in \mathcal{P}$$

3. $\rho^{\mathcal{L}^U}$ *is translationally covariant*

$$\rho^{\mathcal{L}^U}(\mathbf{p}_1 + a) = \rho^{\mathcal{L}^U}(\mathbf{p}_1) - a \qquad a \in \mathbb{R}$$

- *Let* $V^{\mathcal{L}(b)}$ *be a Cash MtM policy,* $b \in \mathbb{R}$*. Then*

  1. $$\rho^{\mathcal{L}(b)}(\lambda \mathbf{p}) \leq \lambda\, \rho^{\mathcal{L}(b)}(\mathbf{p}) \qquad \lambda \geq 1,\ \mathbf{p} \in \mathcal{P}$$

  2. $\rho^{\mathcal{L}(b)}$ *is subadditive on concordant portfolios*

  $$\rho^{\mathcal{L}(b)}(\mathbf{p}_1 + \mathbf{p}_2) \leq \rho^{\mathcal{L}(b)}(\mathbf{p}_1) + \rho^{\mathcal{L}(b)}(\mathbf{p}_2) \qquad \mathbf{p}_1 \upuparrows \mathbf{p}_2$$

*Proof.* See [1]. □

These results show that axioms (PH) and (S) take different routes when converted to CPRMs, depending on the liquidity policy chosen.

Under dilatation $\mathbf{p} \to \lambda \mathbf{p}$ ($\lambda \geq 1$) of a portfolio, for instance, CPRMs $\rho^{\mathcal{L}^L}$ and $\rho^{\mathcal{L}(b)}$ display violations of positive homogeneity of opposite kind. Risk scales *more* than proportionally in the case of $\mathcal{L}^L$ and *less* than proportionally in the case $\mathcal{L}(b)$. If the first result is probably expected, the second may surprise at first sight. But if we think of the meaning of a cash liquidity policy, which requires a portfolio to generate a fixed (not scaling with $\lambda$) amount $b$ of euros, we can easily see that this result is natural. In fact, in such a situation, adding any other asset to a portfolio, can never deteriorate the liquidity properties of the portfolio because the latter already contains assets enough to liquidate $b$. For $L$ the situation is opposite because when you add assets to a portfolio you are asked to prepare yourself to liquidate them as well.

Under addition of portfolios, again $\rho^{\mathcal{L}^L}$ and $\rho^{\mathcal{L}(b)}$ behave in an opposite way, for essentially the same reason just explained. Subadditivity is generally lost in these cases and it holds for $\rho^{\mathcal{L}^L}$ if the added portfolios are discordant while it holds for $\rho^{\mathcal{L}(b)}$ if the added portfolios are concordant.

We can condense these observations by saying that although $\rho^{\mathcal{L}^L}$ and $\rho^{\mathcal{L}(b)}$ both encourage diversification in the sense of *blending*, because they are both convex, the presence of liquidity risk induces in them an opposite behaviour in the case of *addition* of portfolios. The liquidity risk effect of $\rho^{\mathcal{L}^L}$ discourages the addition of concordant portfolios while that of $\rho^{\mathcal{L}(b)}$ encourages it and vice versa for discordant portfolios. A little thought on the meaning of these liquidity policies tells us that our formalism produces results that we should expect from financial intuition.

As for the $\rho^{\mathcal{L}^U}$, we see that formally, it satisfies exactly all the axioms of coherency. This fact is extremely important, because the liquidity policy $\mathcal{L}^U$ clearly represents one of the ways in which we can explore the limit for *"vanishing liquidity risk"* of our setup. And in this limit we planned to recover exactly the coherent risk axioms, including the double interpretation (portfolio vs. portfolio value) for (TC), (PH) and (S). Notice that these properties hold formally for $U$ even in the presence of bid–offer spreads, to which our definition of $U$ is sensitive. We can therefore conclude that had we modeled liquidity risk by the only introduction of bid ask spreads as many do, we would have ended up with a framework completely identical to coherent measures of risk, without any additional richness for describing liquidity risk.

Another possible way of taking the limit of vanishing liquidity risk is assuming that portfolios are extremely small with respect to the market depth. In our formalism this amounts to consider MSDCs which are essentially flat at the right and at the left of the origin separately. But in this situation it is easy to recognize that $V^{\mathcal{L}} \to U$ for any liquidity policy $\mathcal{L}$. Therefore in this limit we recover coherent measures formally for all CPRMs $\rho^{\mathcal{L}}$.

# 6 An analytically solvable class of $V^{\mathcal{L}}$

An interesting class of optimization problems which allow for an analytical solution is represented by cash liquidity policies $\mathcal{L}(a)$ applied when the market is characterized by continuous and strictly decreasing maps $m : \mathbb{R} \to \mathbb{R}$.

The solution is provided by the following

**Proposition 7.** *Let $\mathcal{L}(a)$ be a cash liquidity policy, and suppose that the MSDCs $m_i(s)$ are continuous on $\mathbb{R}$ and strictly decreasing for all $i = 1, \dots, n$. Then the solution $\mathbf{r}^a = (0, \vec{r}^a)$ of the problem*

$$V^{\mathcal{L}(a)}(\mathbf{p}) = \sup \left\{ U(\mathbf{p} - \mathbf{r}) + L(\mathbf{r}) \,|\, \mathbf{r} \in \mathcal{C}_{\mathcal{L}(a)}(\mathbf{p}) \right\}$$

*is unique and given by*

$$\begin{aligned} {r^a}_i &= \zeta_i \left( \frac{m_i(0)}{1+\lambda} \right) & \qquad p_0 < a \\ {r^a}_i &= 0 & \qquad p_0 \geq a \end{aligned} \tag{3}$$

*where $\zeta_i$ is the inverse of the function $m_i$, and the Lagrange multiplier $\lambda$, representing the* marginal cost of liquidation per euro liquidated*, is determined by*

$$L(\mathbf{r}^a) = a - p_0 \tag{4}$$

*Proof.* An easy lagrangian exercise. In view of remark 1, we look for a solution of the type $\mathbf{r} = (0, \vec{r})$. The case $p_0 \geq a$ is trivial because then $\mathbf{p} \in \mathcal{L}(a)$, so $V^{\mathcal{L}(a)}(\mathbf{p}) = U(\mathbf{p})$. Consider the case $p_0 < a$. Writing a lagrangian

$$\mathfrak{L}(\vec{r}, \lambda) := -U(\mathbf{p} - \mathbf{r}) - L(\mathbf{r}) - \lambda(L(\mathbf{r}) - a + p_0)$$

we obtain the equations

$$
\begin{aligned}
m_i(0) &= (1+\lambda)\, m_i(r_i) \\
L(\mathbf{r}^a) &= a - p_0
\end{aligned}
\tag{5}
$$

Equations (3) and (4) then follow immediately. The lagrangian multiplier $\lambda$ plays clearly the role of marginal cost of liquidation per euro liquidated, as it satisfies

$$dC = \lambda dL$$

$\square$

This proposition allows us to solve the optimization by means of a fast root search of $\lambda$, eq. (4).

**Example 1.** Consider a market of $N$ illiquid securities where all MSDCs are exponential functions $m_i(x) = A_i\, e^{-k_i\, x}$ with $A_i, k_i \geq 0$ for $i = 1, \ldots, N$. We have for this case

$$L(\mathbf{q}) = q_0 + \sum_{i=1}^{N} \frac{A_i}{k_i} \,(1 - e^{-k_i\, q_i})$$

Let's compute the cash MtM policy $V^{\mathcal{L}(a)}(\mathbf{p})$ of a portfolio $\mathbf{p}$ using proposition 7. Let's suppose $p_0 < a$, otherwise the problem is trivial. Equations (3) and (4) read

$$
\begin{aligned}
&{r^a}_i = \frac{1}{k_i} \log(1+\lambda) \qquad\qquad \forall i = 1, \ldots, N \\
&\sum_{i=1}^{N} \frac{A_i}{k_i} \,(1 - e^{-k_i\, r_i^a(\lambda)}) = a - p_0
\end{aligned}
$$

In general, the second equation requires numerical methods for determining $\lambda$. In this case the solution is analytical

$$
\begin{aligned}
\lambda &= \frac{a}{\sum_{i=1}^{N} A_i/k_i - a} \\
r_i^a &= \frac{1}{k_i} \log(1+\lambda)
\end{aligned}
$$

Finally, we have the value of the portfolio

$$V^{\mathcal{L}(a)}(\mathbf{p}) = U(\mathbf{p} - \mathbf{r}^a) + L(\mathbf{r}^a) = \sum_{i=1}^{N} A_i(p_i - r_i^a) + a$$

We have used this model to perform a Montecarlo simulation of the portfolio under different assumptions on the state vector $(\vec{A}, \vec{k})$. Essentially we can interpret $\vec{A}$ as the market risk drivers and $\vec{k}$ as the liquidity risk drivers.

1. $\vec{A}$ joint normal, $\vec{k} = 0$. It is a classical gaussian portfolio with no liquidity risk.

2. $\vec{A}$ joint normal, $\vec{k} \neq 0$, fixed. It is a liquidity risk model with one–factor nontrivial MSDCs. The factor can be identified with the bid price.

3. $(\vec{A}, \vec{k})$ joint normal, with $\vec{k}$ independent from $\vec{A}$. It is a liquidity risk model with two–factor MSDCs. Liquidity risk factors are independent from market factors.

4. $(\vec{A}, \vec{k})$ joint normal, $\vec{k}$ and $\vec{A}$ negatively correlated. It is a liquidity risk model with two–factor MSDCs. Liquidity risk factors are dependent on market factors.

The resulting probability distributions obtained by the simulation are plotted in figure 1. Notice how adding liquidity risk, even in the simplest case 2, produces a significant dislocation of the distribution toward left. Adding a more realistic dependence structure between liquidity and market risk drivers, as in case 4, deforms the gaussian into a more dispersed distribution.

We stress that this is just an example chosen for its tractability. The formalism lends itself very seldom to analytical examples which are necessarily very stylized. In concrete risk management situations, numerical procedures are always needed, which require accurate modeling but never present challenging computational obstacles.

# 7 Conclusions

We have described a new formalism for liquidity risk which is fully compatible with the axioms of coherency. This formalism reduces exactly to the paradigm of CRMs when liquidity risk is assumed to be negligible, a limit that can be taken either by assuming portfolios to be much smaller than market depth or by assuming a reckless attitude in marking–to–market, associated to a MtM policy $\mathcal{L} = U$. In the presence of liquidity risk, the formalism provides an appropriate description of this additional dimension of financial risk which allows to capture correctly the interplay between the liquidity of markets and the liquidity needs of a portfolio. This is achieved by the introduction of general liquidity policies $\mathcal{L} \neq U$. This formalism accounts for the dialogue that should always exist between an ALM department (which is supposed to monitor the adequacy of the cash reserves of a bank in relation to future payment needs) and the Accounting department (which is in charge for defining an appropriate MtM policy for the banks' assets, consistent with current liquidity of the market).

Our formalism naturally leads to a redefinition of the concept of value of a portfolio. An analysis of the microstructure of illiquid markets suggests us to define the value of a portfolio through an optimization problem conditional to market conditions and liquidity needs of the portfolio. This value turns out to be a non-linear but always concave map in

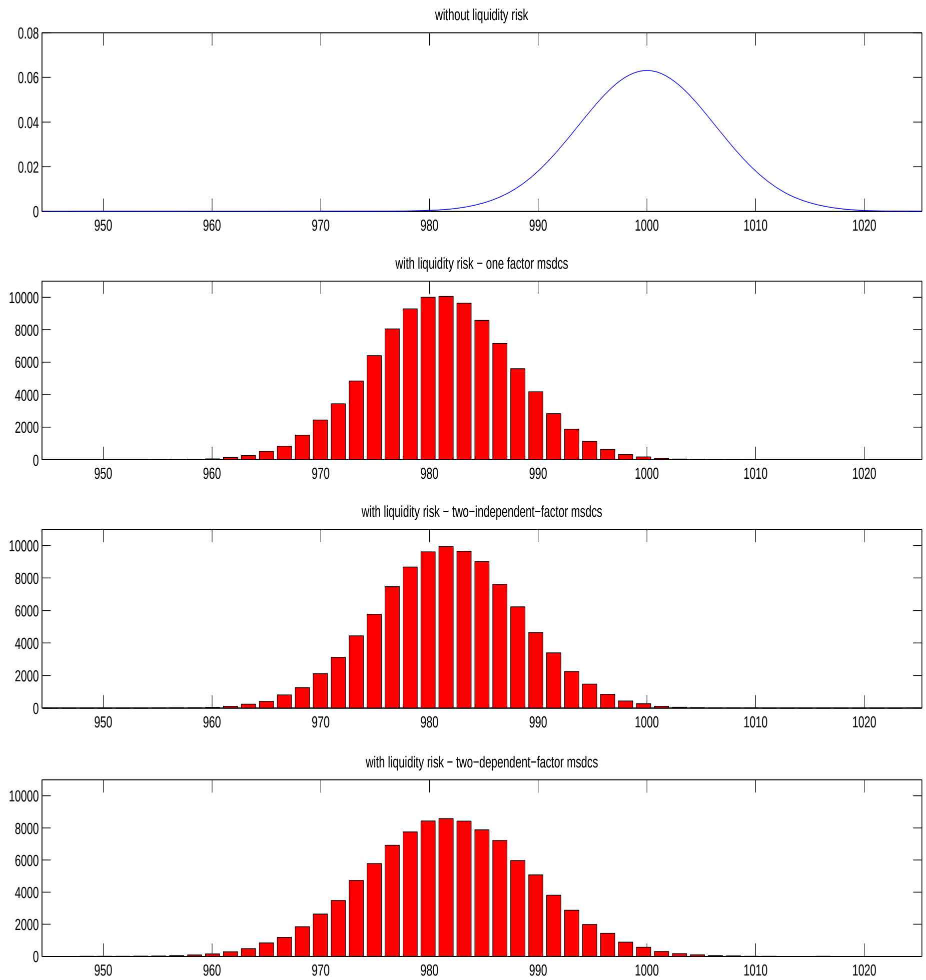


Figure 1: The distributions obtained in Example 1

the space of portfolios, explaining why coherency axioms need not be redefined to account for violations of positive homogeneity. The most important consequence of this fact is that, when defined on the space of portfolios (we call them CPRMs in this case), CMRs turn out to be generally convex, as the literature of "convex risk measures" had imposed axiomatically.

We observe also that CPRMs have general translational subvariance properties under the addition of cash portfolios, a property which is very natural and that had not been investigated in the literature of convex measures of risk. Other properties as subadditivity and positive homogeneity are generally lost for CPRMs and we analyze specific cases of illiquid markets, in which it is in fact reasonable to observe different phenomena for a CPRM under addition of portfolios or portfolio scaling.

Our formalism captures also a purely liquidity–theoretical version of the diversification principle which can directly be seen in the concavity of MtM policies. This aspect, which accounts for commonsense risk management rule of keeping a portfolio granularity as small as possible is probably first seen in a consistent formalism for illiquid markets.

This work seeds a lot of future research. In particular.

- the stochastic modeling of markets described by general MSDCs. The selection of suitably tractable models for basic industrial implementation
- the characterization of the optimization problem induced by MtM policies
- the redefinition of concept of arbitrage in illiquid markets. The no–arbitrage pricing theory in this regime. It is clear that a market opportunity in this context will turn out to be an arbitrage only for those market agents which can afford to be less prudent in their MtM policy. This means that the efficiency of the market in our formalism is expected to be a function of the circulating liquidity of the market. That is why we anticipated that this formalism is a promising one to capture also the link to what we called facet 3.
- the impact of asset allocation methods deriving from the concavity of $V^{\mathcal{L}}$. The $y$–axis of the Markowitz plane is so to say no more linear. Diversification works also at a pure liquidity–theoretical level

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