Delta-Sigma Attribution: Understanding Differences in Risk

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Investors face the challenge of understanding changes in risk. Did a recent increase in risk come from turnover into more aggressive positions, a spike in market volatility, or a loss of diversification? Which of the investor's positions drove the change? Which parts of the market became more risky?

A related issue is understanding differences among risk models. Do such differences indicate a weakness of one of the models, or do they provide insight into the evolving structure of the markets?

The delta-sigma attribution[1] framework addresses these issues by relating a change in risk to the underlying portfolio and market variables driving the change.

Understanding Differences in Risk

Delta-sigma attribution takes this basic form:

$$\text{Risk change} = \sum_{\text{Return Sources}} \text{Exposure change} + \text{Standalone risk change} + \text{Correlation change}$$

A change in risk can be directly related to its sources: changes in the portfolio exposures, the standalone risk of portfolio components, and the correlation structure of the market. Because the decomposition is additive, different contributions can be aggregated to zoom in and out between high-level and low-level contributions.

Figure 1 shows a delta-sigma decomposition of the cumulative changes in risk of a sample portfolio of US corporate bonds. Starting at about 6% volatility in 1998, the risk went to 4% in 2002 before rapidly rising to 8% in 2008. The delta-sigma analysis shows that much of the initial decline in this portfolio's risk was not due to changes in the markets, but to changes in the portfolio itself. The curve labeled Cumulative ΔX shows a total risk reduction of about 2% over time due to a gradual decline in the duration of the portfolio, which began stabilizing after 2002.

In contrast, changes in risk after 2002 were primarily driven by changes in the standalone volatility of the underlying risk factors, declining until 2007 and then rising sharply in the crisis. An additional

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[1] Portions of this note first appeared in Reference 1.

contribution during the crisis came from a loss of diversification, a demonstration that the crisis was a correlation crisis as well as a volatility crisis. However, not all correlations went to 1, and this long-only portfolio benefited from offsetting moves of interest rates and corporate spreads.

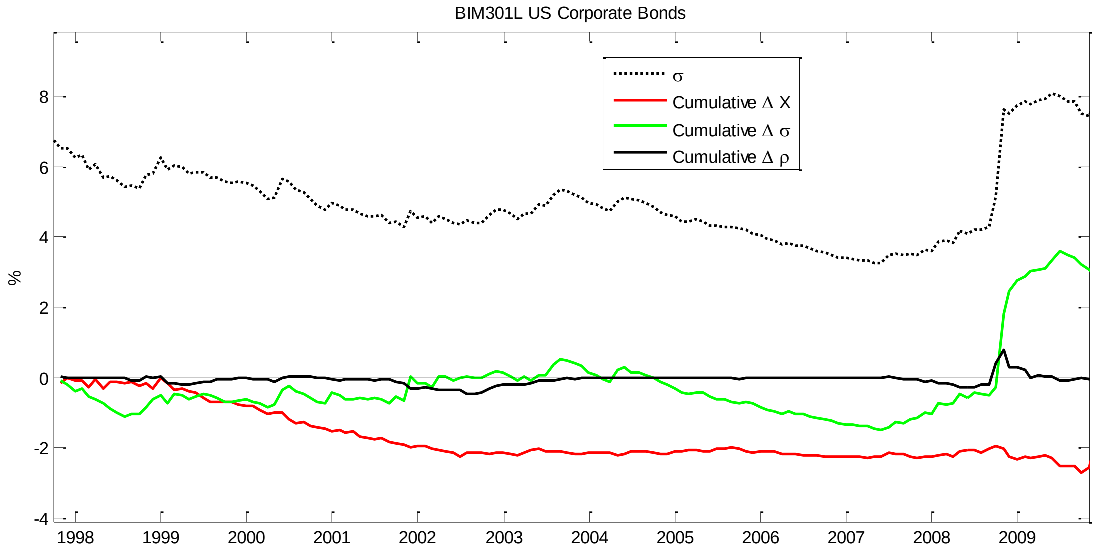

Figure 1. A delta-sigma attribution of the changes in risk forecasts for a portfolio of US corporate bonds. The curve labeled ΔX shows that the initial decline from about 6% in 1998 to 4% in 2002 was a result of the portfolio become inherently less risky, due to a declining duration, rather than a reflection of the markets. During the crisis of 2008, the risk doubled from 4% to 8%, largely driven by an increase in market volatility, $\Delta\sigma$. A loss of diversification, $\Delta\rho$, also contributed a 1% rise in risk, a demonstration that the crisis was a correlation crisis as well as a volatility crisis. However, as explored further in Table 1, not all correlations went to 1, and this long-only portfolio benefited from offsetting moves of interest rates and corporate spreads.

To understand the changing covariance structure during the crisis, we can use the delta-sigma framework to look at a different kind of change in risk: the difference between models. Rather than look at risk forecasts across time, the same framework can be used at a single point in time to compare models with different time horizons.

Table 1 looks at the risk of the same portfolio of US corporate bonds on September 30, 2008, just before the crisis reached a climax. The risk differences are between the long-horizon Barra Integrated Model (BIM301L) and its short-horizon counterpart (BIM301S). Unsurprisingly, risk was significantly higher at a short-horizon during this crisis period.

A delta-sigma attribution of the difference between the long- and short-horizon risk forecasts provides insight into changes underway in the markets. At an aggregate level, most of the difference is due to greater short-term volatility, with just 7 basis points from changing correlations.

However, when the correlation differences are viewed in greater detail, we see significant cancellations at work. A loss of diversification from increased correlation among credit spreads is offset by a 77 basis point reduction from the US Shift factor: A flight to quality created a negative correlation between credit spreads and government interest rates.

Comparing two models with different horizons side-by-side provides a view into the changing structure of the markets by identifying the elements that are in flux. A delta-sigma decomposition cannot predict what will happen next, but it puts a spotlight on the market dynamics the risk manager should notice.

Table 1. A delta-sigma attribution of the difference between long- and short-horizon risk forecasts for a portfolio of US corporate bonds just before the climax of the financial crisis of 2008. The short-horizon Barra Integrated Model (BIM301S) forecasts risk about 1% greater than the long-horizon BIM301L. Of the 97 basis point difference, most (91 bps) is due to greater short-term volatilities, with just 7 bps from changing correlations. When viewed from a more granular level, the correlation effect is seen to include a 77 bps benefit from the US Shift factor correlations, reflecting the drop in interest rates offsetting widening credit spreads. The large difference in correlation at different horizons indicates a market relationship in flux.

US Corporate Bonds 9/30/2008	BIM301S σ_L (bps)	BIM301L σ_S (bps)	σ_S-σ_L (bps)	ΔX_k	$\Delta\sigma_k$	$\Delta\rho_k$
Total	506	603	**97**	0	**91**	**7**
US Shift	290	219	-70	0	6	**-77**
US Swap Shift	49	83	34	0	19	15
US Financials/Banks A	26	48	22	0	12	10
US Financials/Cap Mkts A	18	33	15	0	8	8
US Swap Twist	-12	-27	-15	0	-6	-9
US Financials BBB	13	24	11	0	6	6

As a final example, Table 2. shows how delta-sigma attribution can be used to monitor the risk of a portfolio run by multiple managers. Between period 1 and period 2, the total risk increases 2.5%, with the largest contribution from Manager 3. However, the delta-sigma analysis suggests that Manager 1's activity may be more important. Large transactions were made after a spike in the risk of the initial holdings, with a rebalancing contribution of -6.6% mostly offsetting the 7.1% increase in market volatility.

Table 2. A delta-sigma decomposition of the risk contributions of three managers. The change in Manager 1's total risk contribution of .4% is the result of large contributions from increased market volatility (7.1%) offset by turnover (-6.6%) into less risky holdings.

	σ_2	σ_1	σ_2-σ_1	ΔX_k	$\Delta\sigma_k$	$\Delta\rho_k$
Total	34.7	32.2	2.5	-7.9	10.2	0.2
Manager 1	14.9	14.5	0.4	-6.6	7.1	-0.1
Manager 2	9.9	10.0	-0.1	-1.4	1.1	0.2
Manager 3	9.8	7.6	2.2	0.1	2.0	0.1

The Nuts and Bolts

As described in Reference 2, the starting point for understanding differences between two risk forecasts is the x-sigma-rho risk decomposition:

$$\sigma(R) = \sum_k X_k \sigma_k \rho_k . \tag{1}$$

Intuitively, the risk from return source k grows with the portfolio exposure X_k and the standalone risk σ_k, and may increase or decrease depending on its correlation with the portfolio $\rho_k = corr(R, r_k)$. This framework applies not just to standard deviation and tracking error, but to any homogenous[2] risk measure $\sigma(R)$, such as value at risk or expected shortfall, in which case ρ_k is a generalized correlation. Furthermore, the return sources to which risk is attributed can be anything – assets, factors, sectors, managers, etc. – as long as the portfolio return can be written as $R = \sum_k X_k r_k$.

The delta-sigma framework takes x-sigma-rho one step further, to understand differences in risk, $\Delta\sigma$, in terms of the changes in the underlying variables, X_k, σ_k and ρ_k.

- Was a change in risk due to the turnover or to the markets?
- What parts of the market drove risk higher?
- Which managers and which transactions contributed most to the change in risk?
- What market dynamics are in flux, giving different contributions to long- and short-horizon risk forecasts?

To answer these questions, the delta-sigma methodology extends x-sigma-rho, attributing differences in risk forecasts to changes in the underlying exposures, volatilities, and correlations. It is based on the simple observation that the change in a product of variables A and B can be written as the sum of contributions proportional to the changes ΔA and ΔB:

$$\Delta\left(AB\right) = \bar{A}\,\Delta B + \bar{B}\,\Delta A, \tag{2}$$

where $\bar{A}$ and $\bar{B}$ are the averages of A and B respectively. Despite a resemblance to a Taylor expansion, the relationship is exact; it is not a first order approximation. The use of averages in the coefficients eliminates the second order $\Delta A \Delta B$ term, allowing an additive decomposition of differences in risk forecasts directly proportional to changes in the underlying variables. Figure 2 shows a geometric representation of this identity, with a derivation in the Appendix.

Applying Equation 2 to the x-sigma-rho decomposition (Equation 1) yields the central equation of delta-sigma attribution:

$$\Delta\sigma = \sum_k \left(\Delta X_k \overline{\sigma_k \rho_k} + \Delta\sigma_k \bar{X}_k \bar{\rho}_k + \Delta\rho_k \bar{X}_k \bar{\sigma}_k \right). \tag{3}$$

The first term, proportional to the change in exposure ΔX_k, is the effect of changes in the portfolio exposures, the variables the manager can control. The coefficient $\overline{\sigma_k \rho_k}$ gives the portfolio sensitivity to changing exposure to return source k, while summing $\Delta X_k \overline{\sigma_k \rho_k}$ over all k gives the total contribution

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[2] A homogenous risk measure is one for which the total risk increases proportional to the size of the investment, $\sigma(\lambda R) = \lambda\sigma(R)$ for any $\lambda > 0$. Common risk measures such as standard deviation, value at risk, and expected shortfall are all homogenous risk measures. Examples of risk measures that are not homogenous can be found among measures of liquidity risk, for which doubling the portfolio more than doubles the risk: $\sigma(\lambda R) > \lambda\sigma(R)$ for $\lambda > 1$.

from turnover. Over long time scales, individual asset-level exposures may change, and it may be useful to further decompose the change into direct contributions from holdings Δh_i and changing asset-level exposures $\Delta X_{i,k}$:

$$\Delta X_k = \sum_i \left(\Delta X_{i,k} \bar{h}_k + \Delta h_i \bar{X}_{i,k} \right). \tag{4}$$

The other terms in Equation 3 are the effect of changes in the markets: the standalone risk of the return sources and the changing correlations with the total portfolio.

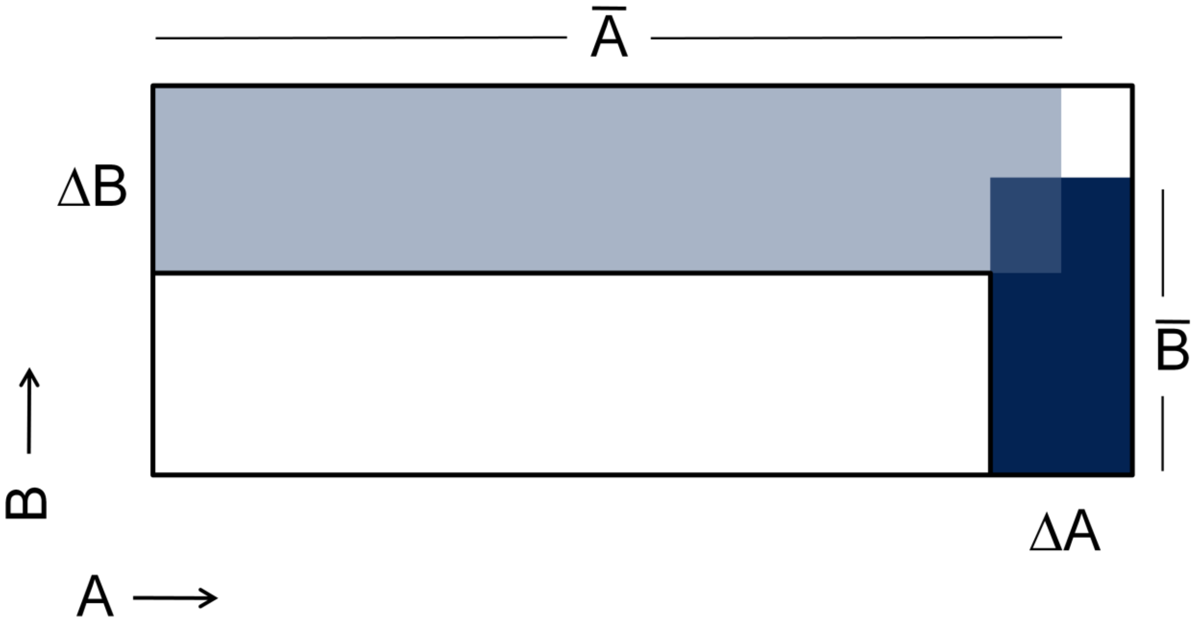

Figure 2. A geometric representation of the $\Delta(\boldsymbol{AB}) = \bar{\boldsymbol{A}}\Delta\boldsymbol{B} + \bar{\boldsymbol{B}}\Delta\boldsymbol{A}$ relationship. The areas of the two shaded regions add up to the difference between the outer and inner rectangles.

Conclusion

By relating differences in risk to the underlying causes, the delta-sigma framework provides answers to many of the questions facing risk managers. For any two risk forecasts, the framework decomposes the difference $\Delta\sigma$ into additive contributions proportional to changes in the underlying exposures and market variables, providing greater insight into the evolution of risk.

Appendix A: Mathematical Derivation

For any quantities A and B, define the differences and averages as follows:

$$\Delta A = A_2 - A_1,$$
$$\Delta B = B_2 - B_1,$$
$$\bar{A} = \left(A_1 + A_2\right)/2,$$
$$\bar{B} = \left(B_1 + B_2\right)/2.$$

These definitions may be inverted to give

$$A_1 = \bar{A} - \Delta A/2, \tag{5}$$

$$A_2 = \bar{A} + \Delta A/2, \tag{6}$$

$$B_1 = \bar{B} - \Delta B/2, \tag{7}$$

$$B_2 = \bar{B} + \Delta B/2. \tag{8}$$

The difference in the product of A and B is

$$\Delta\left(AB\right) = A_2 B_2 - A_1 B_1.$$

Plugging in Equations 5-8 yields

$$\Delta\left(AB\right) = (\bar{A} + \Delta A/2)(\bar{B} + \Delta B/2) - (\bar{A} - \Delta A/2)(\bar{B} - \Delta B/2).$$

Expanding the products,

$$\Delta\left(AB\right) = (\bar{A}\bar{B} + \Delta A \Delta B/4 + \bar{A}\Delta B/2 + \bar{B}\Delta A/2) - (\bar{A}\bar{B} + \Delta A \Delta B/4 - \bar{A}\Delta B/2 - \bar{B}\Delta A/2),$$

the $\bar{A}\bar{B}$ and $\Delta A \Delta B$ terms cancel out, while the cross terms survive to give the desired result:

$$\Delta\left(AB\right) = \bar{A}\Delta B + \bar{B}\Delta A.$$

This identity is linear,

$$\Delta\left(\sum_k A_k B_k\right) = \sum_k \left(\bar{A}_k \Delta B_k + \bar{B}_k \Delta A_k\right),$$

and it may be nested[3]

$$\begin{aligned}\Delta\left(\left(AB\right)C\right) &= \overline{AB}\Delta C + \bar{C}\Delta\left(AB\right) \\ &= \overline{AB}\Delta C + \bar{C}\left(\bar{A}\Delta B + \bar{B}\Delta A\right).\end{aligned}$$

The delta-sigma framework is generated by the application of these properties to the additive x-sigma-rho risk decomposition, Equation 1:

$$\begin{aligned}\Delta\sigma &= \sum_k \left(\overline{\sigma_k \rho_k}\Delta X_k + \bar{X}_k \Delta\left(\sigma_k \rho_k\right)\right), \\ &= \sum_k \left(\overline{\sigma_k \rho_k}\Delta X_k + \bar{X}_k \left(\bar{\rho}_k \Delta\sigma_k + \bar{\sigma}_k \Delta\rho_k\right)\right),\end{aligned}$$

[3] Note that $\overline{AB} = \bar{A}\bar{B} + \Delta A \Delta B/4$, so the order of nesting matters.

$$= \sum_k \left(\overline{\sigma_k \rho_k} \Delta X_k + \overline{X}_k \overline{\rho}_k \Delta\sigma_k + \overline{X}_k \overline{\sigma}_k \Delta\rho_k \right),$$

Note that the coefficient of ΔX_k is $\overline{\sigma_k \rho_k}$, the average of the product $\sigma_k \rho_k$, not the product of the averages as in the coefficients of $\Delta\sigma_k$ and $\Delta\rho_k$, a result of the order of nesting.

References

1. Shepard, P. “The Barra Integrated Model (BIM301),” MSCI Model Insights, February 2011.
2. Menchero, J. and Davis, B. 2011. “Risk Contribution is Exposure times Volatility times Correlation,” MSCI Model Insights, January 2010.

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