Problem Set 3 (PHYS512) - Matias Castro Tapia

In [195]:

```
import numpy as np
import matplotlib.pyplot as plt
```

Problem 1

I defined the routine rk4_step to use the Runge Kutta mthod for integrating diferential equations. Thus:

$$f(x,y) = \frac{dy}{dx}$$
$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + h/2, y_n + k_1/2)$$
$$k_3 = hf(x_n + h/2, y + k_2/2)$$
$$k_4 = hf(x_n + h, y_n + k_3)$$

And then:

$$y_{n+1} = y_n + k_1/6 + k_2/3 + k_3/3 + k_4/6$$

or

$$y(x+h) = y(x) + k_1/6 + k_2/3 + k_3/3 + k_4/6$$

In [2]:

```
def rk4_step(fun,x,y,h):
    k1=h*fun(x,y)
    k2=h*fun(x+(h/2),y+(k1/2))
    k3=h*fun(x+(h/2),y+(k2/2))
    k4=h*fun(x+h,y+k3)
    dy=(k1/6)+(k2/3)+(k3/3)+(k4/6)
    return y+dy
```

I used rk4_step to solve $\frac{dy}{dx} = \frac{y}{1+x^2}$. I defined $h = \frac{20-(-20)}{200}$ because we want to use 200 steps from the initial value $x = -20$ to the last $x = 20$ to solve the ODE. Also, $y(-20) = 1$ for the boundary condition.

In [237]:

```
xx=np.linspace(-20,20,201)
```

In [238]:

```
def dydx(x,y):
    return y/(1+(x**2))
```

In [239]:

```
y0=1
yy=np.ones(len(xx))
h=(xx[-1]-xx[0])/200
yy[0]=y0
for i in range(len(xx)-1):
    yy[i+1]=rk4_step(dydx,xx[i],yy[i],h)
```

The analic solution is $y = c0\exp(\arctan(x))$. For this case the constant must be $c0 = 1/\exp(\arctan(-20))$ for our boundary condition.

In [244]:

```
c0=1/np.exp(np.arctan(-20))
```

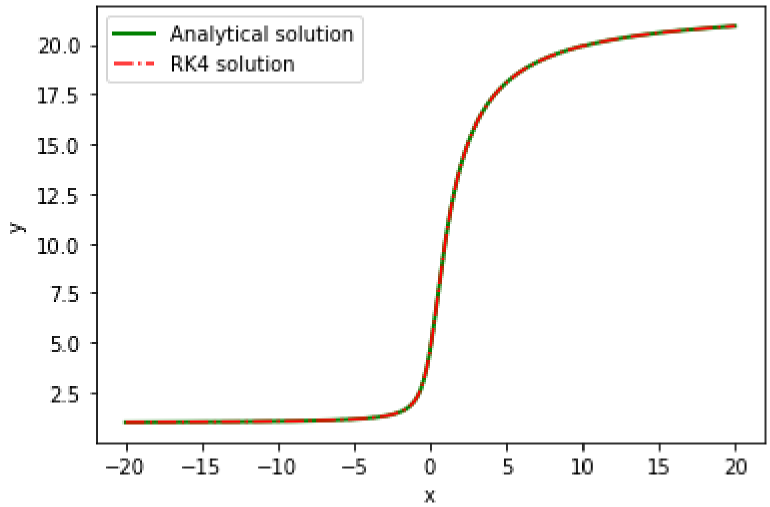
In [245]:

```
yan=c0*np.exp(np.arctan(xx))
```

Now comparing the RK4 solution and the analytical:

In [252]:

```
plt.plot(xx,yan,color='green',label='Analytical solution',linewidth=2)
plt.plot(xx,yy,'r-.', label='RK4 solution')
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.show()
```



I defined the routine rk4_stepd, which uses the rk4_step routine to compute first $y(x+h/2)$, then uses this to compute the following y with another step of $h/2$, then $y(x+h) = y(x+h/2) + RK4_{terms} = y_2$. Rigorously, this value should be closer to $y(x+h) = y_2 + 2(\frac{h}{2})^5\psi + O(h^6)$ with $\psi = y^{(5)}/5!$ since RK4 method accuracy is trucated to ignore terms of order h^5 or higher.

On the other hand, the routine also compute $y(x+h) = y(x) + RK4_{terms} = y_1$, i.e., using a single step of h. For this case we will have $y(x+h) = y_1 + h^5\psi + O(h^6)$. Then, if we substract both expressions for $y(x+h)$ we will have:

$$0 = y_1 - y_2 + h^5\frac{15}{15}$$

$$(y_2 - y_1)\frac{16}{15} = h^5\psi$$

and replacing in the first expression for $y(x+h)$:

$$y(x+h) = y_2 + \frac{y_2 - y_1}{15}$$

The routine rk4_stepd returns this final expression. We can note that rk4_step evaluate $f(x, y)$ 4 times to obtain the RK4 terms, while rk4_stepd evaluate $f(x, y)$ 12 times because it uses rk4_step 3 times. Therefore, if we want to use almost the same number of function evaluations in both routines we must use about 66 steps in rk4_stepd.

In [253]:

```
def rk4_stepd(fun,x,y,h):
    yhalf=rk4_step(fun,x,y,h/2)
    y2=rk4_step(fun,x,yhalf,h/2)
    y1=rk4_step(fun,x,y,h)
    return  y2 + (y2-y1)/15
    #h2=h/2
    #k12=h2*fun(x,y,h2)
    #k22=h2*fun(x+(h2/2),y+(k12/2))
    #k32=h2*fun(x+(h2/2),y+(k22/2))
    #k42=h2*fun(x+h2,y+k32)
    #dy2=(k12/6)+(k22/3)+(k32/3)+(k42/6)
    #yhalf=y+dy2
```

In [272]:

```
xx2=np.linspace(-20,20,67)
```

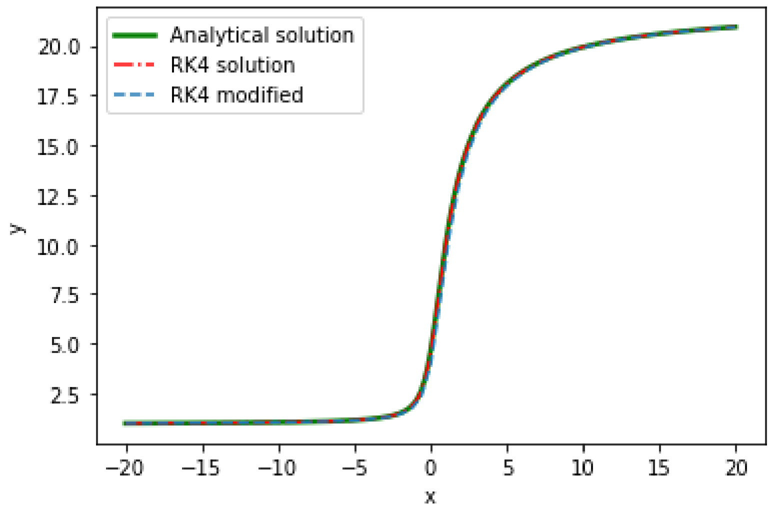
In [273]:

```
yyy=np.ones(len(xx2))
h2=(xx2[-1]-xx2[0])/66
yyy[0]=y0
for i in range(len(xx2)-1):
    yyy[i+1]=rk4_stepd(dydx,xx2[i],yyy[i],h2)
```

Comparing both mehods:

In [274]:

```
plt.plot(xx,yan,color='green',label='Analytical solution',linewidth=2.5)
plt.plot(xx,yy,'r-.', label='RK4 solution')
plt.plot(xx2,yyy,'--',label='RK4 modified')
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.show()
```



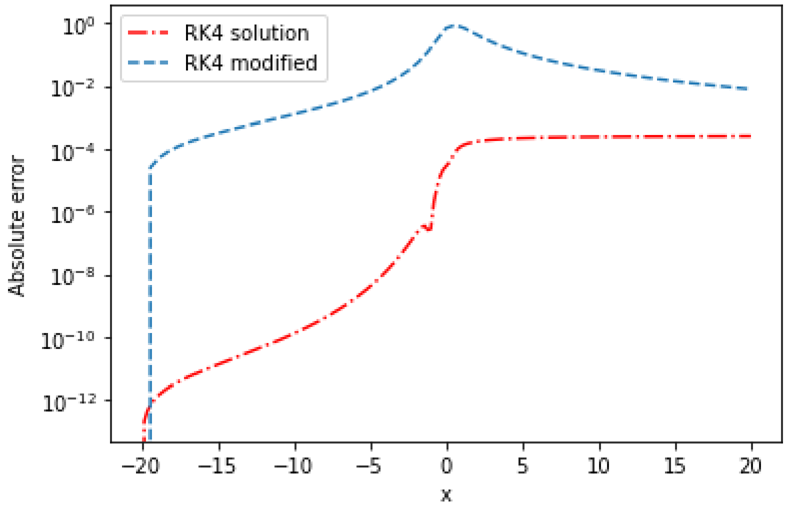
In [275]:

```
yan2=c0*np.exp(np.arctan(xx2))
```

Let's compare the errors too.

In [277]:

```
plt.plot(xx,np.abs(yy-yan),'r-.',label='RK4 solution')
plt.plot(xx2,np.abs(yyy-yan2),'--',label='RK4 modified')
plt.yscale('log')
plt.xlabel('x')
plt.ylabel('Absolute error')
plt.legend()
plt.show()
```



It seems like the modified method does not make the solution better. However, the errors seem to converge to an error of about 10^{-1}.

Problem 3

(Problem 2 is after Problem 3)

I loaded the file with the "dish zenith" data. We can expand the expression for the paraboloid that we want to fit:

$$z - z_0 = a((x - x_0)^2 + (y - y_0)^2)$$
$$z = a(x^2 + y^2) - 2ax_0x - 2ay_0y + a(x_0^2 + y_0^2) + z_0$$

Then, we can define new variables $\alpha = a$, $\beta = -2ax_0$, $\gamma = -2ay_0$, and $\delta = a(x_0^2 + y_0^2) + z_0$:

$$z = \alpha(x^2 + y^2) + \beta x + \gamma y + \delta$$

For solving the best-fit parameters we should the best find $m = (\alpha, \beta, \gamma, \delta)$, where the matrix for the model would be (using the row element A_i): $A_i = ((x_i^2 + y_i^2)\ x_i\ y_i\ 1)$. And the least squares solutions should be:

$$m = (A^T A)^{-1} A^T z$$

In [281]:

```
dat=np.loadtxt('dish_zenith.txt')
```

In [282]:

```
xd=np.array([i[0] for i in dat])
yd=np.array([i[1] for i in dat])
zd=np.array([i[2] for i in dat])
```

In [283]:

```
mat=np.ones([len(xd),4])
for i in range(len(xd)):
    mat[i]=([xd[i]**2+yd[i]**2,xd[i],yd[i],1])
```

$$m = (A^T A)^{-1} A^T z$$

In [284]:

```
m=np.linalg.inv(mat.T@mat)@mat.T@zd
```

In [35]:

```
m
```

Out[35]:

```
array([ 1.66704455e-04,  4.53599028e-04, -1.94115589e-02, -1.51231182e+03])
```

Now, we have to obtain the old parameters a, x_0, y_0, and z_0

In [285]:

```
a=m[0]
x0=-m[1]/(2*m[0])
y0=-m[2]/(2*m[0])
z0=m[3]-(x0**2+y0**2)*a
```

In [286]:

```
a,x0,y0,z0
```

Out[286]:

```
(0.00016670445477740132,
 -1.3604886221978416,
 58.22147608157977,
 -1512.877210036787)
```

Due to (x,y,z) in the data are in mm we can estimate the focal length in meters as $\frac{(1/4a)}{1000}$

In [287]:

```
f=(1/(4*m[0]))/1000 #in meters
```

In [289]:

```
f
```

Out[289]:

```
1.4996599841252194
```

We are going to define the error in the data comparing the fit with the data $\sigma_i = |z_i - (Am)_i|$

In [290]:

```
sig2=np.abs(zd-(mat@m))**2
```

I defined the noise matrix N using σ_i^2 as elements of a diagonal matrix

In [291]:

```
Nnn=np.zeros([len(zd),len(zd)])
for i in range(len(zd)):
    Nnn[i][i]=sig2[i]
```

Now, the square errors in the best-fit parameters would be the diagonal of $(A^T N^{-1} A)^{-1}$

In [297]:

```
errors2=np.diag(np.linalg.inv(mat.T@np.linalg.pinv(Nnn)@mat))
```

In [298]:

```
errors2
```

Out[298]:

```
array([4.08048563e-18, 1.18646669e-11, 1.57044668e-11, 3.10090531e-05])
```

In [299]:

```
errorm=np.sqrt(errors2)
```

In [300]:

```
errorm
```

Out[300]:

```
array([2.02002120e-09, 3.44451258e-06, 3.96288618e-06, 5.56857729e-03])
```

And the error in the focal length should be $\Delta f = (|\frac{df}{da}|\Delta a)$ with $\Delta a = \sigma_a$ and $\Delta f = \sigma_f$. Considering $|\frac{df}{da}| = \frac{1}{4a^2}$

In [304]:

```
errorf=errorm[0]/(4*(a**2))
```

we have to multiply by (1/1000) to have the result in meters.

In [306]:

```
errorf/1000
```

Out[306]:

```
1.8171949618123915e-05
```

In [308]:

```
np.abs(f-1.5)
```

Out[308]:

```
0.00034001587478060813
```

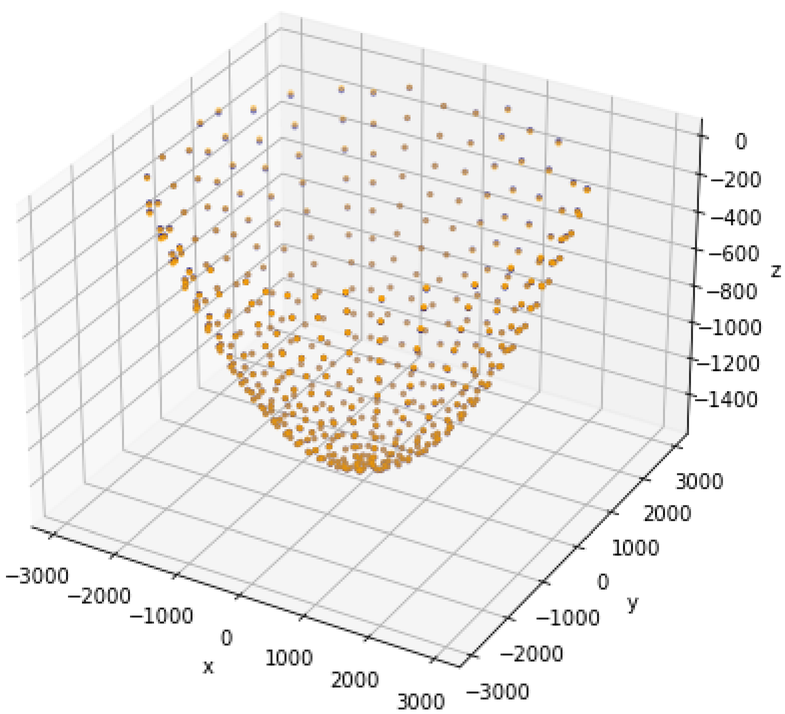
Comparing what I obtained 1.4996599841252194 meters with the expected value 1.5 meters we have a difference of about 10^{-4} which is one order of magnitude larger than the error we estimated.

In [328]:

```
fig = plt.figure(figsize=(7,7))
ax = fig.add_subplot(projection='3d')
ax.set_xlabel('x')
ax.set_ylabel('y')
ax.set_zlabel('z')
#ax.plot_surface(xd,yd,mat@m,marker='.',cmap='Oranges')
ax.scatter(xd,yd,mat@m,marker='.',color='orange')
ax.scatter(xd, yd, zd, marker='.',color='navy')
```

Out[328]:

```
<mpl_toolkits.mplot3d.art3d.Path3DCollection at 0x258e25c3e20>
```



Problem 2

I defined an array with all the half-lifes=th=$t_{1/2}$ in the order of Pb_{206} to U_{238}. They are in years.

In [331]:

```
th=np.array([138.376*0.00274,5.015,22.3,164.3*(3.171e-14),19.9*(1.903e-6),
             26.8*(1.903e-6),3.10*(1.903e-6),3.82535*0.00274,1600,75380,
             245500,6.70*(0.000114),24.10*0.00274,4.468*(1e9)])
```

In [332]:

```
th
```

Out[332]:

```
array([3.7915024e-01, 5.0150000e+00, 2.2300000e+01, 5.2099530e-12,
       3.7869700e-05, 5.1000400e-05, 5.8993000e-06, 1.0481459e-02,
       1.6000000e+03, 7.5380000e+04, 2.4550000e+05, 7.6380000e-04,
       6.6034000e-02, 4.4680000e+09])
```

Then, ll=λ are the decay constants and are related to the half-lifes as $\lambda = \ln(2)/t_{1/2}$

In [333]:

```
ll=(np.log(2)*np.ones(len(th))/th)
```

In [334]:

```
ll
```

Out[334]:

```
array([1.82815968e+00, 1.38214792e-01, 3.10828332e-02, 1.33042886e+11,
       1.83034769e+04, 1.35910146e+04, 1.17496513e+05, 6.61307916e+01,
       4.33216988e-04, 9.19537252e-06, 2.82341010e-06, 9.07498273e+02,
       1.04968226e+01, 1.55135895e-10])
```

In [113]:

```
ll[::-1]
```

Out[113]:

```
array([1.55135895e-10, 1.04968226e+01, 9.07498273e+02, 2.82341010e-06,
       9.19537252e-06, 4.33216988e-04, 6.61307916e+01, 1.17496513e+05,
       1.35910146e+04, 1.83034769e+04, 1.33042886e+11, 3.10828332e-02,
       1.38214792e-01, 1.82815968e+00])
```

I defined the routine ff to introduce the ODEs related to the decay of the elements from U_{238} to Pb_{206}. Then, every dndt in the routine are similar to the intermediate elements and just decay for U_{238} and just increase in the stable Pb_{206}:

$$\frac{dN_{U_{238}}}{dt} = -\lambda_{U_{238}} N_{U_{238}}$$
$$\frac{dN_{Th_{234}}}{dt} = -\lambda_{Th_{234}} N_{Th_{234}} + \lambda_{U_{238}} N_{U_{238}}$$
$$\frac{dN_{Pa_{234}}}{dt} = -\lambda_{Pa_{234}} N_{Pa_{234}} + \lambda_{Th_{234}} N_{Th_{234}}$$
$$\dots$$

$$\frac{dN_{Pb_{206}}}{dt} = \lambda_{Po_{210}} N_{Po_{210}}$$

In the input of the routine I fixed l=ll[::-1] because the half-lifes and the constant were in the order from Pb_{206} to U_{238}, while the equations were fixed as described above.

In [335]:

```
def ff(t,n,l=ll[::-1]):
    dndt=np.zeros(len(l)+1)
    dndt[0]=-l[0]*n[0]
    for i in range(1,len(l)):
        dndt[i]=-l[i]*n[i]+l[i-1]*n[i-1]
    dndt[len(l)]=l[len(l)-1]*n[len(l)-1]
    return dndt
    #dndt[1]=-l[1]n[1]+l[0]n[0]
```

We must define that the fraction of U_{238} should be 1 for the initial time and 0 for the other elements.

In [336]:

```
n0=np.array([1]+[0]*len(ll))
```

In [337]:

```
n0
```

Out[337]:

```
array([1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0])
```

t0 is the array with the integration time from 0 to 10^{10}.

In [230]:

```
t0=[0,1e10]
```

I used the implicit method of integration of integrate.solve_ivp from scipy since the time interval of integration is very large.

In [338]:

```
from scipy import integrate
```

In [340]:

```
ans_stiff=integrate.solve_ivp(ff,t0,n0,method='Radau')
```

ans_stiff.t are the time steps used in the integration, while ans_stiff.y are the solution for every N.

In [341]:

```
ans_stiff.t
```

Out[341]:

```
array([0.00000000e+00, 2.02507037e+00, 2.22757741e+01, 2.24782811e+02,
       2.24985318e+03, 2.25005569e+04, 2.25007594e+05, 6.51844652e+05,
       1.48408887e+06, 2.75857217e+06, 5.54151454e+06, 1.59917738e+07,
       8.64570005e+07, 7.91109268e+08, 3.88581261e+09, 8.58372893e+09,
       1.00000000e+10])
```

In [342]:

```
ans_stiff.y
```

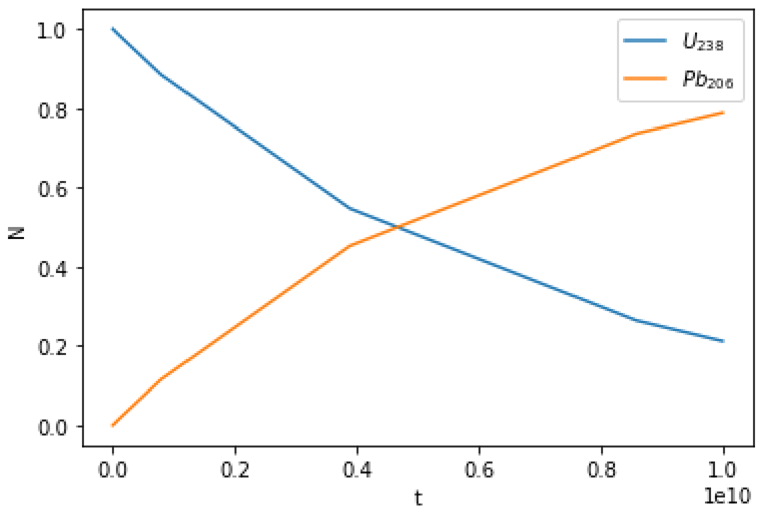
Out[342]:

```
array([[1.00000000e+00, 1.00000000e+00, 9.99999997e-01, 9.99999965e-01,
        9.99999651e-01, 9.99996509e-01, 9.99965094e-01, 9.99898881e-01,
        9.99769791e-01, 9.99572138e-01, 9.99140682e-01, 9.97522177e-01,
        9.86676964e-01, 8.84502937e-01, 5.47262098e-01, 2.64048082e-01,
        2.11964064e-01],
       [0.00000000e+00, 1.38562853e-11, 1.47672959e-11, 1.47793023e-11,
        1.47793144e-11, 1.47792680e-11, 1.47788037e-11, 1.47778251e-11,
        1.47759173e-11, 1.47729961e-11, 1.47666195e-11, 1.47426991e-11,
        1.45824142e-11, 1.30723516e-11, 8.08816146e-12, 3.90245099e-12,
        3.13268464e-12],
       [0.00000000e+00, 1.60150743e-13, 1.70808268e-13, 1.70948768e-13,
        1.70948911e-13, 1.70948374e-13, 1.70943003e-13, 1.70931684e-13,
        1.70909616e-13, 1.70875828e-13, 1.70802071e-13, 1.70525389e-13,
        1.68671411e-13, 1.51204867e-13, 9.35538923e-14, 4.51387478e-14,
        3.62350385e-14],
       [0.00000000e+00, 3.00143853e-10, 3.44072632e-09, 3.48458779e-08,
        3.47911842e-07, 3.38205783e-06, 2.58360678e-05, 4.62172758e-05,
        5.40874176e-05, 5.48957319e-05, 5.49008544e-05, 5.48130795e-05,
        5.42172114e-05, 4.86028208e-05, 3.00716714e-05, 1.45092583e-05,
        1.16472778e-05],
       [0.00000000e+00, 8.16409001e-16, 1.07727223e-13, 1.10463598e-11,
        1.09854306e-09, 1.01413251e-07, 4.92142242e-06, 1.30323766e-05,
        1.64932454e-05, 1.68518206e-05, 1.68572549e-05, 1.68304576e-05,
        1.66475043e-05, 1.49235945e-05, 9.23356754e-06, 4.45509712e-06,
        3.57632021e-06],
       [0.00000000e+00, 4.84941063e-21, 7.30645149e-18, 7.42760525e-15,
        6.04881011e-12, 1.76817675e-09, 1.03101850e-07, 2.76097309e-07,
        3.50030404e-07, 3.57691497e-07, 3.57808595e-07, 3.57239871e-07,
        3.53356547e-07, 3.16765186e-07, 1.95989830e-07, 9.45629869e-08,
        7.59102465e-08],
       [0.00000000e+00, 3.10361915e-26, 4.77662385e-23, 4.86478454e-20,
        3.96244809e-17, 1.15831535e-14, 6.75411076e-13, 1.80868912e-12,
        2.29301833e-12, 2.34320548e-12, 2.34397258e-12, 2.34024691e-12,
        2.31480760e-12, 2.07510082e-12, 1.28391210e-12, 6.19473794e-13,
        4.97281335e-13],
       [0.00000000e+00, 1.74679283e-29, 2.68843342e-26, 2.73805586e-23,
        2.23019236e-20, 6.51936889e-18, 3.80142933e-16, 1.01798802e-15,
        1.29058397e-15, 1.31883091e-15, 1.31926265e-15, 1.31716573e-15,
        1.30284767e-15, 1.16793304e-15, 7.22626751e-16, 3.48659643e-16,
        2.79885823e-16],
       [0.00000000e+00, 1.50995732e-28, 2.32417092e-25, 2.36709114e-22,
        1.92803709e-19, 5.63609951e-17, 3.28639697e-15, 8.80067065e-15,
        1.11573066e-14, 1.14015059e-14, 1.14052384e-14, 1.13871102e-14,
        1.12633283e-14, 1.00969695e-14, 6.24722482e-15, 3.01421885e-15,
        2.41965808e-15],
       [0.00000000e+00, 1.12110413e-28, 1.72577088e-25, 1.75765223e-22,
        1.43163938e-19, 4.18501417e-17, 2.44027237e-15, 6.53482634e-15,
        8.28471648e-15, 8.46604356e-15, 8.46881510e-15, 8.45535422e-15,
        8.36344151e-15, 7.49737659e-15, 4.63879753e-15, 2.23816996e-15,
        1.79668641e-15],
       [0.00000000e+00, 1.54236759e-35, 2.37424252e-32, 2.41810352e-29,
        1.96958885e-26, 5.75756532e-24, 3.35722342e-22, 8.99033742e-22,
        1.13977622e-21, 1.16472243e-21, 1.16510373e-21, 1.16325184e-21,
        1.15060687e-21, 1.03145733e-21, 6.38186126e-22, 3.07917948e-22,
        2.47180510e-22],
```

```
       [0.00000000e+00, 9.76611372e-25, 1.53581717e-20, 7.01395596e-17,
        8.10307666e-14, 2.45693592e-11, 1.43671787e-09, 3.84800418e-09,
        4.87853870e-09, 4.98532492e-09, 4.98695731e-09, 4.97903073e-09,
        4.92490689e-09, 4.41491480e-09, 2.73160826e-09, 1.31797164e-09,
        1.05799907e-09],
       [0.00000000e+00, 1.11739397e-26, 1.39354237e-21, 1.42040759e-17,
        1.80582367e-14, 5.52158128e-12, 3.23087100e-10, 8.65364389e-10,
        1.09712378e-09, 1.12113920e-09, 1.12150632e-09, 1.11972373e-09,
        1.10755193e-09, 9.92860885e-10, 6.14305626e-10, 2.96395864e-10,
        2.37931180e-10],
       [0.00000000e+00, 3.26067306e-28, 9.45193554e-23, 1.06495409e-18,
        1.36432073e-15, 4.17427876e-13, 2.44263547e-11, 6.54243206e-11,
        8.29461074e-11, 8.47617537e-11, 8.47895095e-11, 8.46547398e-11,
        8.37345127e-11, 7.50634980e-11, 4.64434946e-11, 2.24084871e-11,
        1.79883677e-11],
       [0.00000000e+00, 2.26820542e-28, 6.66195609e-22, 9.61897502e-17,
        1.44308413e-12, 5.35321808e-09, 4.04375344e-06, 4.15888455e-05,
        1.59272185e-04, 3.55750552e-04, 7.87196266e-04, 2.40581629e-03,
        1.32518121e-02, 1.15433214e-01, 4.52698397e-01, 7.35932858e-01,
        7.88020635e-01]])
```

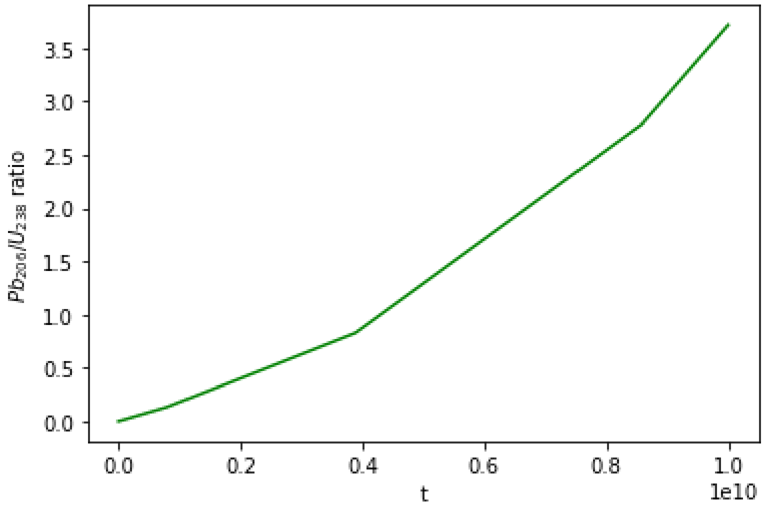
In [347]:

```python
plt.plot(ans_stiff.t,ans_stiff.y[0],label='$U_{238}$')
plt.plot(ans_stiff.t,ans_stiff.y[-1],label='$Pb_{206}$')
plt.xlabel('t')
plt.ylabel('N')
plt.legend()
plt.show()
```



In [349]:

```
plt.plot(ans_stiff.t,ans_stiff.y[-1]/ans_stiff.y[0],'g')
plt.xlabel('t')
plt.ylabel('$Pb_{206}/U_{238}$ ratio')
plt.show()
```

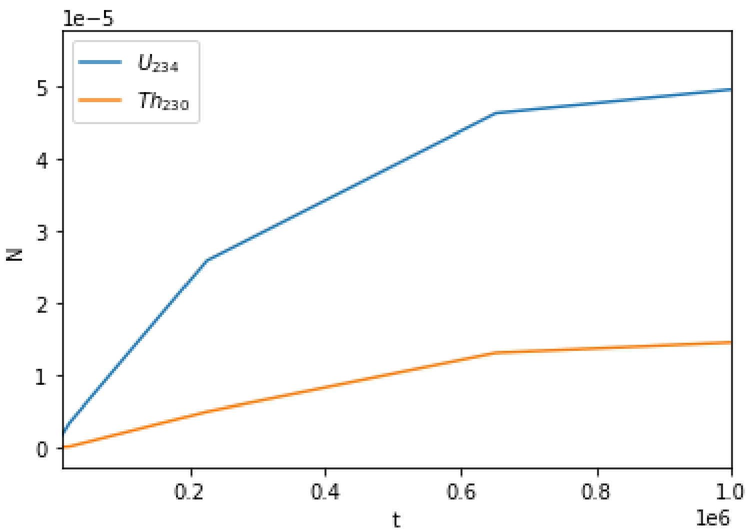


It makes sense that the Pb_{206} increases almost at the same rate of the U_{238} decay because all the intermediate elements decay quicker than U_{238}. We can see in the plot of Pb_{206}/U_{238} ratio that almost at 0.5×10^{10} years the amount of Pb_{206} have been increased until the same value of U_{238}, i.e, $Pb_{206}/U_{238} \sim 1$; this also makes sense since the half-life of U_{238} is about $t_{1/2} \sim 0.5 \times 10^{10}$

Now we can see the ratio of U_{234} and Th_{230}.

In [353]:

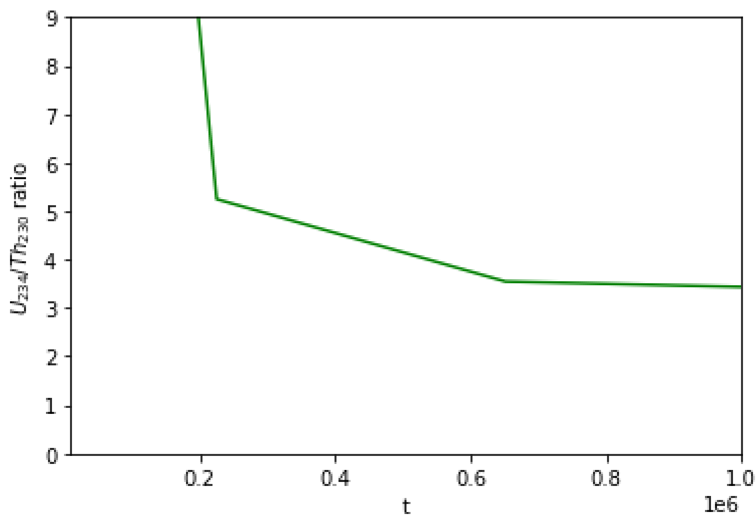
```
plt.plot(ans_stiff.t,ans_stiff.y[3],label='$U_{234}$')
plt.plot(ans_stiff.t,ans_stiff.y[4],label='$Th_{230}$')
plt.xlabel('t')
plt.ylabel('N')
#plt.yscale('log')
plt.xlim(10**4,10**6)
plt.legend()
plt.show()
```



In [358]:

```
plt.plot(ans_stiff.t,ans_stiff.y[3]/ans_stiff.y[4],'g')
plt.xlabel('t')
plt.ylabel('$U_{234}/Th_{230}$ ratio')
plt.xlim(10**4,10**6)
plt.ylim(0,9)
plt.show()
```

```
C:\Users\Odette\AppData\Local\Temp\ipykernel_5516\1670163967.py:1: RuntimeWa
rning: invalid value encountered in true_divide
  plt.plot(ans_stiff.t,ans_stiff.y[3]/ans_stiff.y[4],'g')
```



We can see that the U_{234} is reduced while the Th_{230} increases. Then for smaller ratios of U_{234}/Th_{230} the rocks will be older.