PHYS 3740 HW # 6 solutions Wietfeldt

① a) We assume the trajectory $y(t) = at^2 + bt + c$

initial condition : $y(t=0) = 0$ so $\boxed{c = 0}$

final condition : $y(t=\tau) = 0$ so $a\tau^2 + b\tau = 0 \rightarrow \boxed{b = -a\tau}$

b) $T = \frac{1}{2}m\dot{y}^2 = \frac{1}{2}m(2at+b)^2 = \frac{1}{2}ma^2(2t-\tau)^2$ using above

$U = mgy = mg(at^2 - a\tau t) = mga(t^2 - \tau t)$

$\mathcal{L} = T - U = \boxed{\frac{1}{2}ma^2(2t-\tau)^2 - mga(t^2 - \tau t)}$

d) action $\mathcal{A} = \int_{t=0}^{\tau} \mathcal{L}\, dt = \int_0^{\tau}\left[\frac{1}{2}ma^2(4t^2 - 4\tau t + \tau^2) - mgat^2 + mga\tau t\right]dt$

$= \frac{1}{2}ma^2\left(\frac{4}{3}\tau^3 - 2\tau^3 + \tau^3\right) - \frac{1}{3}mga\tau^3 + \frac{1}{2}mga\tau^3$

$= m\tau^3\left[\left(\frac{2}{3} - 1 + \frac{1}{2}\right)a^2 + \frac{1}{6}ag\right] = \boxed{\frac{m\tau^3}{6}(a^2 + ag)}$

e) $\frac{d\mathcal{A}}{da} = \frac{m\tau^3}{6}(2a + g) = 0 \rightarrow \boxed{a = -\frac{1}{2}g,\ b = \frac{1}{2}g\tau,\ c = 0}$

f) so $y(t) = -\frac{1}{2}gt^2 + \frac{1}{2}g\tau t$ $\quad \dot{y} = -gt + \frac{1}{2}g\tau \rightarrow \boxed{V_0 = \frac{1}{2}g\tau}$

$\boxed{\ddot{y} = -g}$

g) This is equivalent to a projectile fired upward with initial velocity V_0 at $y=0$ that falls back to $y=0$ at $t=\tau$.

Newton's 2nd Law : $m\ddot{y} = F_y = -\frac{dU}{dy} = -mg \rightarrow \boxed{\ddot{y} = -g}$

The projectile reaches maximum height at $t = \frac{1}{2}\tau$

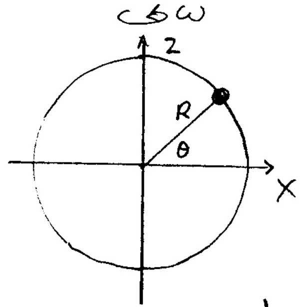
so $\dot{y} = \int \ddot{y}\, dt = -gt + V_0 = 0$ at $t = \frac{1}{2}\tau \rightarrow -\frac{1}{2}g\tau + V_0 = 0$

and $\boxed{V_0 = \frac{1}{2}g\tau}$ <u>Same result</u>

② a) i) 1 ii) 2 iii) $f_1 = z - B\sin kx - A = 0$, $f_2 = y = 0$

b) i) 2 ii) azimuthal angle ϕ and height y iii) $f_1 = y - b(x^2+z^2) = 0$

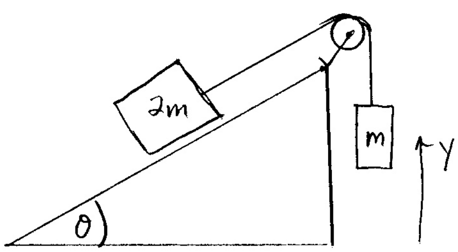
c)



i) 1 ii) θ = the angle in the rotating plane as shown

iii) The bead will always lie on the surface of a sphere of radius R, but the displacement perpendicular to the plane of the hoop must be zero. So two holonomic constraints:
$$f_1 = x^2 + y^2 + z^2 - R^2 = 0$$
$$f_2 = y\cos\omega t - x\sin\omega t = 0$$
(with the hoop in the xz plane at $t=0$)

③:



a) $T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}(2m)\dot{y}^2 = \frac{3}{2}m\dot{y}^2$

$U = U_0 + mgy - 2mgy\sin\theta$

so $\mathcal{L} = T - U = \boxed{\frac{3}{2}m\dot{y}^2 - U_0 + mgy(2\sin\theta - 1)}$

b) $\frac{\partial\mathcal{L}}{\partial y} = mg(2\sin\theta - 1)$ $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}} = \frac{d}{dt}(3m\dot{y}) = 3m\ddot{y}$

So $\cancel{m}g(2\sin\theta - 1) - 3\cancel{m}\ddot{y} = 0 \Rightarrow \boxed{\ddot{y} = \frac{1}{3}g(2\sin\theta - 1)}$ Same as we found using Newton's 2nd Law

④ a) $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ and $y = -\frac{b}{x} \Rightarrow x = -\frac{b}{y} \Rightarrow \dot{x} = \frac{b}{y^2}\dot{y}$

so $T = \frac{1}{2}m\dot{y}^2\left(1 + \frac{b^2}{y^4}\right)$ $U = mgy$ (let $U=0$ at $x \to \infty$)

and so $\mathcal{L} = T - U = \boxed{\frac{1}{2}m\dot{y}^2\left(1 + \frac{b^2}{y^4}\right) - mgy}$

b) $\frac{\partial\mathcal{L}}{\partial y} = \frac{1}{2}m\dot{y}^2\left(-4\frac{b^2}{y^5}\right) - mg = -2m\dot{y}^2\frac{b^2}{y^5} - mg$

$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}} = \frac{d}{dt}\left(m\dot{y}\left(1 + \frac{b^2}{y^4}\right)\right) = m\ddot{y}\left(1 + \frac{b^2}{y^4}\right) + m\dot{y}\left(-4\frac{b^2}{y^5}\right)\dot{y}$

$= m\ddot{y}\left(1 + \frac{b^2}{y^4}\right) - 4m\dot{y}^2\frac{b^2}{y^5}$

so $-2\cancel{m}\dot{y}^2\frac{b^2}{y^5} - \cancel{m}g - \cancel{m}\ddot{y}\left(1 + \frac{b^2}{y^4}\right) + 4\cancel{m}\dot{y}^2\frac{b^2}{y^5} = 0$

and we have: $\boxed{\ddot{y} = \left[\frac{2b^2\dot{y}^2}{y^5} - g\right]\left[1 + \frac{b^2}{y^4}\right]^{-1}}$ This would have been hard to find using Newton's 2nd Law.

(5) We will use polar coordinates r, θ. The velocity components are $V_r = \dot{r}$ and $V_\theta = \omega r = \dot{\theta} r$ so $T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$

The force is $F = -Ar^{\alpha-1}$ so $U(r) = -\int F\,dr = \frac{A}{\alpha}r^\alpha$

Then the Lagrangian is : $\mathcal{L} = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{A}{\alpha}r^\alpha$

<u>Lagrange eqn for r:</u> $\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 - Ar^{\alpha-1}$

$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\ddot{r}$ so $mr\dot{\theta}^2 - Ar^{\alpha-1} - m\ddot{r} = 0 \Rightarrow \boxed{\ddot{r} = r\dot{\theta}^2 - \frac{A}{m}r^{\alpha-1}}$

<u>Lagrange eqn for θ:</u> $\frac{\partial \mathcal{L}}{\partial \theta} = 0 \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{d}{dt}\left(mr^2\dot{\theta}\right)$

so $\frac{d}{dt}\left(mr^2\dot{\theta}\right) = m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) = 0 \Rightarrow \boxed{\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r}}$

Also note that $mr^2\dot{\theta} = mr^2\omega =$ angular momentum L

so $\boxed{\frac{dL}{dt} = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \text{ angular momentum is conserved.}}$

From the $\ddot{r}$ equation above : $\ddot{r} - r\dot{\theta}^2 + \frac{A}{m}r^{\alpha-1} = 0$ multiply by $m\dot{r}$

$m\dot{r}\ddot{r} - mr\dot{r}\dot{\theta}^2 + A\dot{r}r^{\alpha-1} = 0$ Now $m\dot{r}\ddot{r} = \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right)$

and $mr\dot{r}\dot{\theta}^2 = \frac{(mr^2\dot{\theta})^2\dot{r}}{mr^3} = \frac{\ell^2\dot{r}}{mr^3} = \frac{d}{dt}\left(-\frac{\ell^2}{2mr^2}\right) = \frac{d}{dt}\left(-\frac{(mr^2\dot{\theta})^2}{2mr^2}\right)$

$= \frac{d}{dt}\left(-\frac{1}{2}mr^2\dot{\theta}^2\right)$

Also $A\dot{r}r^{\alpha-1} = \frac{d}{dt}\left(\frac{A}{\alpha}r^\alpha\right)$ Putting this all together :

$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{A}{\alpha}r^\alpha\right) = \frac{d}{dt}(T+U) = \boxed{\frac{dE_{mech}}{dt} = 0}$

$\boxed{\text{mechanical energy is conserved}}$

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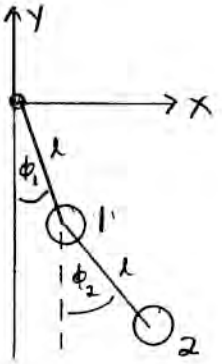

$$T = \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \tfrac{1}{2}m\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

and $x_1 = l\sin\phi_1 \quad y_1 = -l\cos\phi_1$

$x_2 = x_1 + l\sin\phi_2 \quad y_2 = y_1 - l\cos\phi_2$

$$\text{so } T = \tfrac{1}{2}m\Big[\left(l\cos\phi_1\dot{\phi}_1\right)^2 + \left(l\sin\phi_1\dot{\phi}_1\right)^2 + \left(l\cos\phi_1\dot{\phi}_1 + l\cos\phi_2\dot{\phi}_2\right)^2 + \left(l\sin\phi_1\dot{\phi}_1 + l\sin\phi_2\dot{\phi}_2\right)^2\Big]$$

$$\text{Therefore } T = \tfrac{1}{2}ml^2\left[\dot{\phi}_1^2 + \dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\cos\phi_1\cos\phi_2\dot{\phi}_1\dot{\phi}_2 + 2\sin\phi_1\sin\phi_2\dot{\phi}_1\dot{\phi}_2\right]$$

$$= \underline{\tfrac{1}{2}ml^2\left[2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)\right]}$$

$$U = mgy_1 + mgy_2 = -mg\left(l\cos\phi_1 + l\cos\phi_1 + l\cos\phi_2\right) = \underline{-mgl\left(2\cos\phi_1 + \cos\phi_2\right)}$$

$$\text{and } \boxed{\mathcal{L} = T - U = \tfrac{1}{2}ml^2\left[2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2)\right] + mgl\left[2\cos\phi_1 + \cos\phi_2\right]}$$

<u>Lagrange eqn for ϕ_1:</u> $\dfrac{\partial\mathcal{L}}{\partial\phi_1} = -ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1 - \phi_2) - 2mgl\sin\phi_1$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} = \frac{d}{dt}\left(2ml^2\dot{\phi}_1 + ml^2\dot{\phi}_2\cos(\phi_1 - \phi_2)\right) = ml^2\left(2\ddot{\phi}_1 + \ddot{\phi}_2\cos(\phi_1 - \phi_2) - \dot{\phi}_2\sin(\phi_1 - \phi_2)(\dot{\phi}_1 - \dot{\phi}_2)\right)$$

$$\text{so } -ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1 - \phi_2) - 2mgl\sin\phi_1 - ml^2\left(2\ddot{\phi}_1 + \ddot{\phi}_2\cos(\phi_1 - \phi_2) - \dot{\phi}_2\sin(\phi_1 - \phi_2)(\dot{\phi}_1 - \dot{\phi}_2)\right) = 0$$

$$\text{or: } \boxed{2\ddot{\phi}_1 + \ddot{\phi}_2\cos(\phi_1 - \phi_2) + \dot{\phi}_2^2\sin(\phi_1 - \phi_2) + \frac{2g}{l}\sin\phi_1 = 0}$$

<u>Lagrange eqn for ϕ_2:</u> $\dfrac{\partial\mathcal{L}}{\partial\phi_2} = ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1 - \phi_2) - mgl\sin\phi_2$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_2} = \frac{d}{dt}\left(ml^2\dot{\phi}_2 + ml^2\dot{\phi}_1\cos(\phi_1 - \phi_2)\right) = ml^2\left(\ddot{\phi}_2 + \ddot{\phi}_1\cos(\phi_1 - \phi_2) - \dot{\phi}_1\sin(\phi_1 - \phi_2)(\dot{\phi}_1 - \dot{\phi}_2)\right)$$

$$\text{giving similarly: } \boxed{\ddot{\phi}_2 + \ddot{\phi}_1\cos(\phi_1 - \phi_2) - \dot{\phi}_1^2\sin(\phi_1 - \phi_2) + \frac{g}{l}\sin\phi_2 = 0}$$

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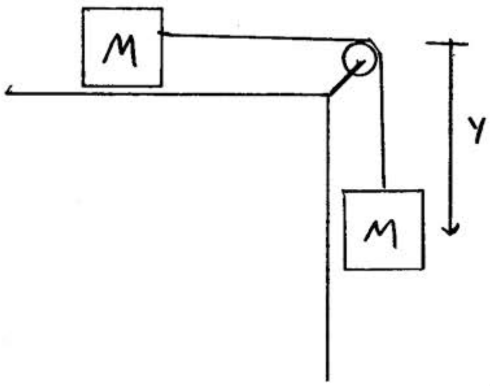

a) With a massless string:

$T = \frac{1}{2} M\dot{y}^2 + \frac{1}{2} M\dot{y}^2 = M\dot{y}^2$

$U = -Mgy$ (let $U=0$ at $y=0$)

so $\mathcal{L} = T - U = M\dot{y}^2 + Mgy$

$\frac{\partial \mathcal{L}}{\partial y} = Mg \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 2M\ddot{y}$

so $Mg - 2M\ddot{y} = 0 \Rightarrow \boxed{\ddot{y} = \frac{g}{2}}$

b) The string now has kinetic energy $\frac{1}{2}m\dot{y}^2$ so $T = \left(M + \frac{m}{2}\right)\dot{y}^2$

The potential energy of the string is $U_{string} = -\left(\frac{m}{\ell}y\right)g\left(\frac{y}{2}\right) = -\frac{mgy^2}{2\ell}$

so $U = -Mgy - \frac{mgy^2}{2\ell}$ and $\mathcal{L} = \left(M + \frac{m}{2}\right)\dot{y}^2 + Mgy + \frac{mgy^2}{2\ell}$

$\frac{\partial \mathcal{L}}{\partial y} = Mg + \frac{mg}{\ell}y \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 2\left(M + \frac{m}{2}\right)\ddot{y}$

so $Mg + \frac{mg}{\ell}y - (2M+m)\ddot{y} = 0 \Rightarrow \boxed{\ddot{y} = \left(\frac{g}{2M+m}\right)\left(M + \frac{my}{\ell}\right)}$