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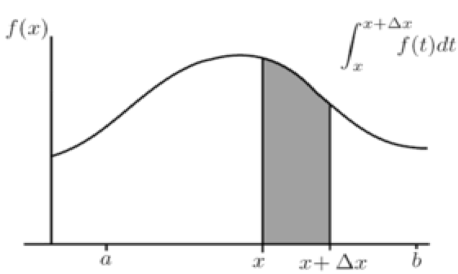
Chapter 4: Calculus

Calculus historically has been called "*the calculus of infinitesimals*". Calculus comes from the Latin (*calculus*) referring to a small stone used for counting.Calculus as a whole, is about the mathematical *analysis of change*. We can think of calculus as the study of change the same way that algebra is the study of operations and their application, and geometry is the study of shapes. Though a rather daunting part of calculus is that there is so much to it. Now one can break up calculus into three main parts, *differential calculus*, *integral calculus*, and the weird stuff that you need not concern yourself with until you decide that you want to have *fun with calculus*. Differential calculus concerns the rate of change and slopes of curves, and integral calculus concerns the area underneath and between those curves. Integral and differential calculus are related under the fundamental theorem of calculus which says that, the area under the curve (the integral part of the left) is equal to the difference in the values of the function (the curve) at two points a and b. This is called the **fundamental theorem of calculus**. Meaning that if you take the value of the function at one point B (which is x +∆x) and subtract that from the value of the function at one point A (which is x) then this tells you the area under the curve between those two points.



$$\boxed{\int_a^b f(x)dx = F(b) - F(a)}.$$

Now the third part of calculus is more of a trick. Because we can take the foundations that we learn from calculus and *use it as a gateway to more advance studies of functions* and limits, a field called mathematical analysis. Other methods, such as theories of computation, take the advance ideas from calculus in their fields such as calculus of variations, lambda calculus, process calculus, etc. Drawing upon the ideas of calculus leads us to study linear algebra and differential equations as *corollaries* to learning the mathematics of change, and how to manipulate that.

Now, many think that calculus is quite daunting, and it is. The prologue for, " Calculus Made Easy Being: a very-simplest introduction to those beautiful methods which are generally called by the terrifying names of the DIFFERENTIAL CALCULUS AND THE INTEGRAL CALCULUS. " Where the author, Silvanus Thompson, amusingly goes along that:

"Considering how many fools can calculate, it is surprising that it should be thought either a difficult or a tedious task for any other fool to learn how to master the same tricks. Some calculus-tricks are quite easy. Some are enormously difficult. The fools who write the textbooks of advanced mathematics—and they are mostly clever fools—seldom take the trouble to show you how easy the easy calculations are. On the contrary, they seem to desire to impress you with their tremendous cleverness by going about it in the most difficult way. Being myself a remarkably stupid fellow, I have had to unteach myself the difficulties, and now beg to present to my fellow fools the parts that are not hard. Master these thoroughly, and the rest will follow. What one fool can do, another can."

For a very fun in depth reading on calculus foundations, please read his book for free on the Gutenburg Project, where it, "is for the use of anyone anywhere at no cost and with almost no restrictions whatsoever. You may copy it, give it away or re-use it under the terms of the Project Gutenberg License included with this eBook or online at www.gutenberg.org."
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Now what scares most people off of calculus is the notation. One *has* to think of mathematics as a language. To learn any language properly you must really know the language; write in it, talk with it, think in it, breathe it, dream in it. You know you've learned the language of mathematics well enough, if you even start to have nightmares in it!

Now what we said before is that there are two variations of calculus, integral and differential. These have dreadful symbols that carry over across everything, so we need to know them.

Differential

(1) d means **“a little bit of”**

Say we say dx, that is a little bit of x, (*or rather an infinitesimally small part of x)* , where du is a little bit of u. These things such as dx, and du, and dy, are called “differentials,” the differential of x, or of u, or of y, as the case may be. [You read them as dee-eks, or dee-you, or dee-wy.] For using differential calculus, for partial derivatives such as

$d/dx,\ d^2x/dt^2,\ d\,\theta/dt,\ d\alpha/dx$

It follows the same logic but with *components*. So to say, dx/dt is a little bit of x *with respect to t*, so the little bit of x that corresponds to the little bit of t. This gets a bit more complicated with higher order derivatives, but it follows the same logic but with a second or third derivative with respect to another quantity.

Integral

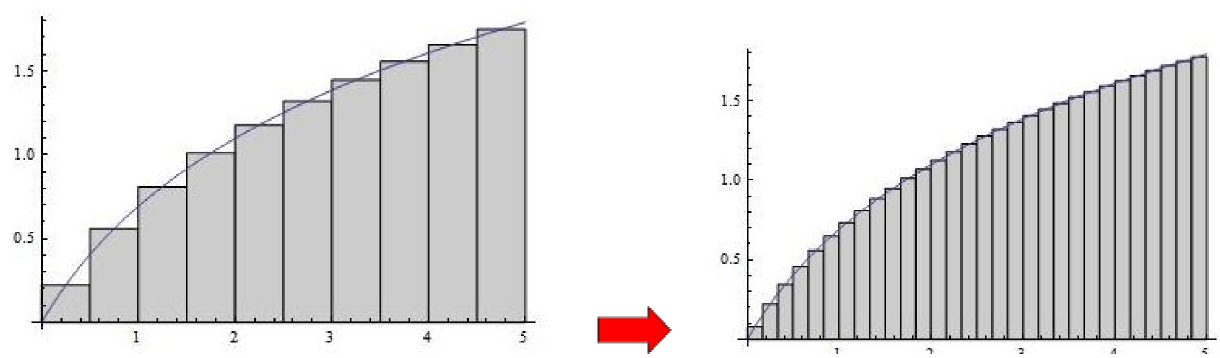
(2) $\int$ means the sum of **the integral**

This summation comes in the form that we call an integral, and derives from the limit of infinitesimally small **Riemann Sums**($\sum$). It is better to think of it as $\int dx$ *or* $\int dt$. This means that $\int dx$ is taking the sum of all the little parts of x between two values and same for dt as the little parts of t between two values.

Now we have to think of things as little bits of things that are infinitesimally small. We want them to be infinitesimally small because if they are then our measurements goes from a approximation (which is the limit of the sum of terms, the riemann sum) to an exact solution with calculus. We use these sums to make a bunch of tiny boxes to approximate a curve, with more and more boxes our approximation gets better and better.

We can see this in the Wolfram Demonstration below:

http://demonstrations.wolfram.com/RiemannSums/



All through the calculus we are dealing with quantities that are growing, and with rates of growth. We classify all quantities into two classes: constants and variables. Those which we regard as of *fixed value*, and call constants, we generally denote algebraically by letters from the *beginning of the alphabet*, such as a, b, or c; while those which we consider as capable of growing, or *varying*, we denote by letters from the *end of the alphabet*, such as x, y, z, u, v, w, and sometimes t.

Moreover, we are usually dealing with more than one variable at once, and thinking of the way in which one variable depends on the other: for instance, we think of the way in which the height reached by a projectile depends on the time of attaining that height. Or we are asked to consider a rectangle of given area, and to enquire how any increase in the length of it will compel a corresponding decrease in the breadth of it. Or we think of the way in which any variation in the slope of a ladder will cause the height that it reaches, to vary.

Suppose we have got two such variables that depend one on the other. We know that dx means d times x, for d is not a factor—it means "an element of" or "a bit of" whatever follows. One reads dx thus: "dee-eks." In case the reader has no one to guide him in such matters it may here be simply said that one reads differential coefficients in the following way. The differential coefficient dy/dx is read "dee-wy by dee-eks," or "dee-wy over dee-eks." So also du/dt is read "dee-you by dee-tee." Second differential coefficients will be met with later on. They are like this, d^2y/dx^2, which is read "dee-two-wy over dee-eks-squared," and it means that the operation of differentiating y with respect to x has been (or has to be) performed twice over. Another way of indicating that a function has been differentiated is by putting an accent to the symbol of the function. Thus if $y = F(x)$, which means that y is some unspecified function of x, we may write $F'(x)$ instead of $d\,F(x)/dt$. Similarly, $F''(x)$ will mean that the original function F(x) has been differentiated twice over with respect to x. The difference between using y' and dy/dt is the difference between Leibniz notation and Newton's notation. Newton didn't like Leibniz.

Now that we know the notation of calculus, let's see it's use for differentiation.

Let $y = x^5$.

$$\begin{array}{lll} \text{First differentiation,} & 5x^4. & \\ \text{Second differentiation,} & 5 \times 4x^3 & = 20x^3. \\ \text{Third differentiation,} & 5 \times 4 \times 3x^2 & = 60x^2. \\ \text{Fourth differentiation,} & 5 \times 4 \times 3 \times 2x & = 120x. \\ \text{Fifth differentiation,} & 5 \times 4 \times 3 \times 2 \times 1 & = 120. \\ \text{Sixth differentiation,} & & = 0. \end{array}$$

The corresponding symbol for the differential coefficient is $f'(x)$, which is simpler to write than $\frac{dy}{dx}$. This is called the "derived function" of x.

Suppose we differentiate over again, we shall get the "second derived function" or second differential coefficient, which is denoted by $f''(x)$; and so on.

Now let us generalize.

Let $y = f(x) = x^n$.

$$\begin{array}{ll} \text{First differentiation,} & f'(x) = nx^{n-1}. \\ \text{Second differentiation,} & f''(x) = n(n-1)x^{n-2}. \\ \text{Third differentiation,} & f'''(x) = n(n-1)(n-2)x^{n-3}. \\ \text{Fourth differentiation,} & f''''(x) = n(n-1)(n-2)(n-3)x^{n-4}. \\ & \text{etc., etc.} \end{array}$$

But this is not the only way of indicating successive differentiations. For,

$$\begin{array}{ll} \text{if the original function be} & y = f(x); \\ \text{once differentiating gives} & \dfrac{dy}{dx} = f'(x); \\ \text{twice differentiating gives} & \dfrac{d\left(\dfrac{dy}{dx}\right)}{dx} = f''(x); \end{array}$$

and this is more conveniently written as $\frac{d^2y}{(dx)^2}$, or more usually $\frac{d^2y}{dx^2}$. Similarly, we may write as the result of thrice differentiating, $\frac{d^3y}{dx^3} = f'''(x)$.

Examples.

Now let us try $y = f(x) = 7x^4 + 3.5x^3 - \frac{1}{2}x^2 + x - 2$.

$$\begin{aligned}\frac{dy}{dx} &= f'(x) = 28x^3 + 10.5x^2 - x + 1,\\ \frac{d^2y}{dx^2} &= f''(x) = 84x^2 + 21x - 1,\\ \frac{d^3y}{dx^3} &= f'''(x) = 168x + 21,\\ \frac{d^4y}{dx^4} &= f''''(x) = 168,\\ \frac{d^5y}{dx^5} &= f'''''(x) = 0.\end{aligned}$$

In a similar manner if $y = \phi(x) = 3x(x^2 - 4)$,

$$\begin{aligned}\phi'(x) &= \frac{dy}{dx} = 3\big[x \times 2x + (x^2 - 4) \times 1\big] = 3(3x^2 - 4),\\ \phi''(x) &= \frac{d^2y}{dx^2} = 3 \times 6x = 18x,\\ \phi'''(x) &= \frac{d^3y}{dx^3} = 18,\\ \phi''''(x) &= \frac{d^4y}{dx^4} = 0.\end{aligned}$$

Now the most important problems of calculus are those where time is independent, and it becomes a **rate** instead. These rates of changes are what we use in physics to describe a system.

$$v = \frac{dy}{dt}.$$

Now let's imagine that we are on a train, and this train is moving across some distance from a point to another, and represent this element of distance as dy . For a particular time there is an element of that time dt . The speed of that journey will be the velocity (distance) * time (seconds) which is a speed (distance*seconds --> m/s), so the magnitude (number without a unit) of velocity, given by $dy\ dt$. The rate at which one quantity (say distance) is changing in relation to another quantity (say time) , is said to be stating that *one differential coefficient of one is with respect to another* . A velocity, scientifically expressed, is the rate at which a very small distance in any given direction is being passed over in a given instant of time.

But if the velocity, v, *is not uniform*, then it must be either increasing or else decreasing. The rate at which a velocity is increasing is called the **acceleration**, if a moving body is, at any particular instant, gaining an additional velocity dv in an element of time dt, then the acceleration a at that instant may be written.

$$a = \frac{dv}{dt};$$

but dv is itself $d\left(\frac{dy}{dt}\right)$. Hence we may put

$$a = \frac{d\left(\frac{dy}{dt}\right)}{dt};$$

and this is usually written $a = \frac{d^2y}{dt^2}$;

or the *acceleration is the second differential coefficient of the distance,* with respect to time. Acceleration is expressed as a *change of velocity in unit time,* for instance, as being so many meters per second per second or meters per second squared (m/s^2).

To *accelerate a mass m requires the continuous application of force*. The **force** necessary to accelerate a mass is proportional to the mass, and it is also proportional to the acceleration which is being imparted. Hence we may write for the force f, the expression

$$f = ma;$$

$$f = m\frac{dv}{dt};$$

$$f = m\frac{d^2y}{dt^2}.$$

The *product of a mass by the speed* at which it is going is called its **momentum**, and is in symbols mv. If we differentiate momentum with respect to time we shall get $d(mv)/dt$ for the rate of change of momentum. But, since m is a constant quantity, this may be written dv/dt, which we see above is the same as f. That is to say, force may be expressed either as mass times acceleration, or as rate of change of momentum. Again, if a force is employed to move something (*against an equal and opposite counter-force*), it does **work**; and the amount of work done is measured by the product of the force into the distance (*in its own direction*) through which its point of application moves forward. So if a force f moves forward through a length y, the work done (*which we may call* w) will be

$$w = f \times y;$$

Where we take f as a constant force. If the force varies at different parts of the range y, then we must find an expression for its value from when time varies from point to point. If f be the force along the small element of length dy, the amount of work done will be $f\,x\,dy$. But as dy is only an element of length, only an element of work will be done. If we write w for work, then an element of work will be dw; and we have

$$dw = f \times dy;$$

which may be written

$$dw = ma \cdot dy;$$

or

$$dw = m\frac{d^2y}{dt^2} \cdot dy;$$

or

$$dw = m\frac{dv}{dt} \cdot dy.$$

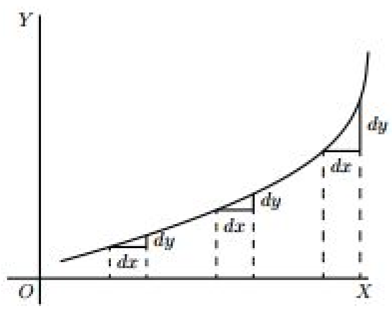
Further, we may transpose the expression and write

$$\frac{dw}{dy} = f.$$

Newton, who was (*along with Leibnitz*) an inventor of the methods of the calculus, regarded all quantities that were varying as flowing; and the ratio which we nowadays call the *differential coefficient he regarded as the rate of flowing*, or the **fluxion** (*a word that you'll never have to remember but it's fun)* of the quantity. He did not use the notation of the dy and dx, and dt (*this was due to Leibnitz*), but had instead a notation of his own. If y was a quantity that varied, or "flowed," then his symbol for its rate of variation (or "fluxion") was *y with a dot on top.* If x was the variable, then its fluxion was *x with a dot on top.*. The dot over the letter indicated that it had been differentiated. But this notation does not tell us what is the independent variable with respect to which the differentiation has been effected. When we see dy/dt we know that y is to be differentiated with respect to t. If we see dy/dx we know that y is to be differentiated with respect to x. But if we see merely y with a dot on top, we cannot tell without looking at the context whether this is to mean dy/dx, dy/dt, dy/du, or what is the other variable. So, therefore, this fluxional notation is less informing than the differential notation. Adopting this fluxional notation we may write the mechanical equations considered in the paragraphs above, as follows

distance	x,
velocity	$v = \dot{x}$,
acceleration	$a = \dot{v} = \ddot{x}$,
force	$f = m\dot{v} = m\ddot{x}$,
work	$w = x \times m\ddot{x}$.

Geometric meaning of differentiation, is that we find the rate of change of the function. Meaning we take the slope at a certain time and use that to find out about the nature of the graph. By taking different derivatives we discover more about the graph, how fast it is changing over a time, it's concavity, whether it's increasing or decreasing, its minima and maxima. Examine the following graphs for these properties.



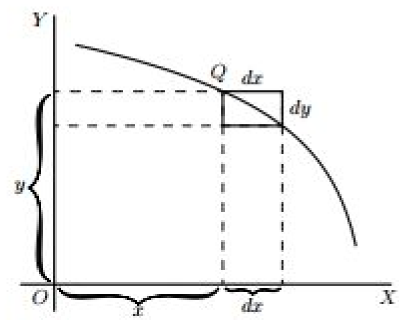


FIG. 11.

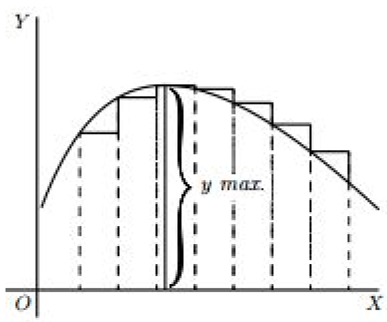
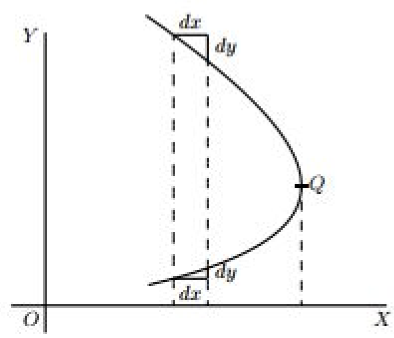


FIG. 16.



Now integrating (*mathematically speaking*) is the reverse of differentiating. Let's see this in the example that is provided.

DIFFERENTIATING is the process by which when y is given us (as a function of x), we can find $\frac{dy}{dx}$.

Like every other mathematical operation, the process of differentiation may be reversed; thus, if differentiating $y = x^4$ gives us $\frac{dy}{dx} = 4x^3$; if one begins with $\frac{dy}{dx} = 4x^3$ one would say that reversing the process would yield $y = x^4$. But here comes in a curious point. We should get $\frac{dy}{dx} = 4x^3$ if we had begun with *any* of the following: x^4, or $x^4 + a$, or $x^4 + c$, or x^4 with *any* added constant. So it is clear that in working backwards from $\frac{dy}{dx}$ to y, one must make provision for the possibility of there being an added constant, the value of which will be undetermined until ascertained in some other way. So, if differentiating x^n yields nx^{n-1}, going backwards from $\frac{dy}{dx} = nx^{n-1}$ will give us $y = x^n + C$; where C stands for the yet undetermined possible constant.

Clearly, in dealing with powers of x, the rule for working backwards will be: Increase the power by 1, then divide by that increased power, and add the undetermined constant.

So, in the case where

$$\frac{dy}{dx} = x^n,$$

working backwards, we get

$$y = \frac{1}{n+1}x^{n+1} + C.$$

If differentiating the equation $y = ax^n$ gives us

$$\frac{dy}{dx} = anx^{n-1},$$

it is a matter of common sense that beginning with

$$\frac{dy}{dx} = anx^{n-1},$$

and reversing the process, will give us

$$y = ax^n.$$

So, when we are dealing with a multiplying constant, we must simply put the constant as a multiplier of the result of the integration.

Thus, if $\frac{dy}{dx} = 4x^2$, the reverse process gives us $y = \frac{4}{3}x^3$.

But this is incomplete. For we must remember that if we had started with

$$y = ax^n + C,$$

where C is any constant quantity whatever, we should equally have found

$$\frac{dy}{dx} = anx^{n-1}.$$

Now let's see an easier example of this done with integration and differentiation.

If we begin with a simple case,

$$\frac{dy}{dx} = x^2.$$

We may write this, if we like, as

$$dy = x^2\,dx.$$

Now this is a "differential equation" which informs us that an element of y is equal to the corresponding element of x multiplied by x^2. Now, what we want is the integral; therefore, write down with the proper symbol the instructions to integrate both sides, thus:

$$\int dy = \int x^2\,dx.$$

[Note as to reading integrals: the above would be read thus:

"*Integral dee-wy* equals *integral eks-squared dee-eks.*"]

We haven't yet integrated: we have only written down instructions to integrate—if we can. Let us try. Plenty of other fools can do it—why not we also? The left-hand side is simplicity itself. The sum of all the bits of y is the same thing as y itself. So we may at once put:

$$y = \int x^2\,dx.$$

Typically the general rule is that

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}.$$

Now there is a matter of **double integration and triple integration**, which is *used to find the area and volume of something in a certain plan*e.

In many cases it is necessary to integrate some expression for two or more variables contained in it; and in that case the sign of integration appears more than once. Thus,

$$\iint f(x, y,)\,dx\,dy$$

Similarly in the case of solids, where we deal with three dimensions. Consider any element of volume, the small cube whose dimensions are $dx\ dy\ dz$. If the figure of the solid be expressed by the function $f(x, y, z)$, then the whole solid will have the *volume-integral*,

$$\text{volume} = \iiint f(x, y, z) \cdot dx \cdot dy \cdot dz.$$

Let us go back to an example of an integral between two points P and Q for a function to find the area of one strip. If we continue this process, then we get the area under the curve.

ONE use of the integral calculus is to enable us to ascertain the values of areas bounded by curves.

Let us try to get at the subject bit by bit.

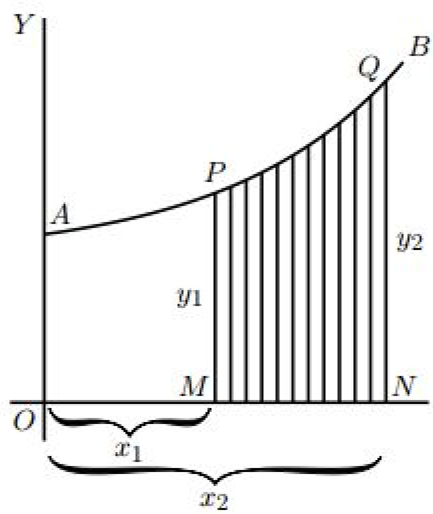
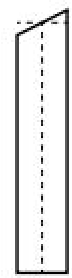


FIG. 52.

Let AB (Fig. 52) be a curve, the equation to which is known. That is, y in this curve is some known function of x. Think of a piece of the curve from the point P to the point Q.

The secret of solving this problem is to conceive the area as being divided up into a lot of narrow strips, each of them being of the width dx. The smaller we take dx, the more of them there will be between x_1 and x_2. Now, the whole area is clearly equal to the sum of the areas of all such strips. Our business will then be to discover an expression for the area of any one narrow strip, and to integrate it so as to add together all the strips. Now think of any one of the strips. It will be like this: being bounded between two vertical sides, with a flat bottom dx, and with a slightly curved sloping top. Suppose we take its *average* height as being y; then, as its width is dx, its area will be $y\,dx$. And seeing that we may take the width as narrow as we please, if we only take it narrow enough its average height will be the same as the height at the middle of it. Now let us call the unknown value of the whole area S, meaning surface. The area of one strip will be simply a bit of the whole area, and may therefore be called dS. So we may write



$$\text{area of 1 strip} = dS = y \cdot dx.$$

If then we add up all the strips, we get

$$\text{total area } S = \int dS = \int y\,dx.$$

This is as far as we need for simple differentiation and integration. We use the mathematical foundations put down by Newton (*and Leibnitz because Newton always gets all of the credit*) to use this for physical applications.

On the Nature of Motion

According to Aristotle, continuation of motion depends on continued action of a force. The motion of a hurled body, therefore, requires elucidation. Aristotle maintained that the air of the atmosphere was responsible for the continuation of such motion. John

Philoponos of the 6th century rejected this Aristotelian view. He claimed that the hurled body acquires a motive power or an inclination for forced movement from the agent producing the initial motion and that this power or condition and not the ambient medium secures the continuation of such motion. According to Philoponos this impressed virtue was temporary. It was a self-expending inclination, and thus the violent motion thus produced comes to an end and changes into natural motion. Ibn Sina adopted this idea in its rough outline, but the violent inclination as he conceived it was a non-self-consuming one. It was a permanent force whose effect got dissipated only as a result of external agents such as air resistance. He is apparently the first to conceive such a permanent type of impressed virtue for non-natural motion. [...] Indeed, self-motion of the type conceived by Ibn Sina is almost the opposite of the Aristotelian conception of violent motion of the projectile type, and it is rather reminiscent of the principle of inertia, i.e., Newton's first law of motion.

Newton, as opposed to Aydin Sayili who is quoted above, had more methodical viewpoint on the nature of motion. Newton employed that, the principles which cause motion in a physical way are two, of which one is not physical, as it has no principle of motion in itself. Of this kind is whatever causes movement, not being itself moved, such as (1) that which is completely unchangeable, the primary reality, and (2) the essence of that which is coming to be, i.e. the form; for this is the end or 'that for the sake of which'. Hence since nature is for the sake of something, we must know this cause also. We must explain the 'why' in all the senses of the term, namely, (1) that from this that will necessarily result ('from this' either without qualification or in most cases); (2) that 'this must be so if that is to be so' (as the conclusion presupposes the premisses); (3) that this was the essence of the thing; and (4) because it is better thus (not without qualification, but with reference to the essential nature in each case). We must explain then (1) that Nature belongs to the class of causes which act for the sake of something; (2) about the necessary and its place in physical problems, for all writers ascribe things to this cause, arguing that since the hot and the cold, &c., are of such and such a kind, therefore certain things necessarily are and come to be — and if they mention any other cause (one his 'friendship and strife', another his 'mind'), it is only to touch on it, and then good-bye to it.

I like to think that Newton's idea that motion was caused by (1) the mathematical principles that he came up with to define his laws of motion (2) it happens because it happens. I say it happens because it happens, from a story that my professor once told me. He said that a teacher posed him the question, why does this pencil fall to the ground? And he went on a tangent describing the different lagrangians, and hamiltonians, and newton's laws, etc. *The professor just told him, that the pencil falls because it does*. Good to also keep this in mind while studying mechanics. Now for Newton's laws ...

Newton placed the first law of motion to establish **frames of reference** for which the laws are applicable. The first law of motion *postulates (thinks about)* the existence of at least one **frame of reference** called a Newtonian or **inertial reference frame**. Newton's first law is often referred to as the **law of inertia**. Thus, a condition necessary for the uniform motion of a particle relative to an inertial reference frame is that the **total net force** acting on it is zero. Newton's laws are *valid only in an inertial reference frame*.

In every material universe, the motion of a particle in a preferential reference frame Φ *is determined by the action of forces whose total vanished for all times when and only when the velocity of the particle is constant in* Φ*. That is, a particle initially at rest or in uniform motion in the preferential frame* Φ *continues in that state unless compelled by forces to change it.*

The second law states that the net force on an object is equal to the rate of change (that is, the *derivative*) of its linear momentum **p** in an inertial reference frame:

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\mathbf{v})}{\mathrm{d}t}.$$

The second law can also be stated in terms of an object's acceleration. Since Newton's second law is only valid for constant-mass systems,mass can be taken outside the differentiation operator by the constant factor rule in differentiation. Thus, as we have seen before.

$$\mathbf{F} = m\,\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\mathbf{a},$$

Where **F** is the net force applied, *m* is the mass of the body, and **a** is the body's acceleration. Thus, *the net force applied to a body produces a proportional acceleration. In other words, if a body is accelerating,* then there is a force on it.

Consistent with the first law, *the time derivative of the momentum is non-zero when the momentum changes direction*, even if there is no change in its magnitude; such is the case with uniform circular motion. The relationship also implies the conservation of momentum: when the net force on the body is zero, the momentum of the body is constant. Any net force is equal to the rate of change of the momentum. Any mass that is gained or lost by the system will cause a change in momentum that is not the result of an external force. A different equation is necessary for variable-mass systems (*like the rocket-equation)*. Newton's second law requires modification if the effects of special relativity are to be taken into account, because at high speeds the approximation that momentum is the product of rest mass and velocity is not accurate.

An **impulse** **J** occurs when a force **F** acts over an interval of time Δt, and it is given by

$$\mathbf{J} = \int_{\Delta t} \mathbf{F}\, \mathrm{d}t.$$

Since force is the time derivative of momentum, it follows that

$$\mathbf{J} = \Delta \mathbf{p} = m\Delta \mathbf{v}.$$

This relation between impulse and momentum is closer to Newton's wording of the second law. Impulse is a concept frequently used in the analysis of collisions and impact. Variable-mass systems, like a rocket burning fuel and ejecting spent gases, are not closed and cannot be directly treated by making mass a function of time in the second law;that is, the following formula is wrong:

$$\mathbf{F}_{\mathrm{net}} = \frac{\mathrm{d}}{\mathrm{d}t}\big[m(t)\mathbf{v}(t)\big] = m(t)\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{v}(t)\frac{\mathrm{d}m}{\mathrm{d}t}. \qquad \text{(wrong)}$$

The falsehood of this formula can be seen by noting that it does not respect Galilean invariance: a variable-mass object with **F** = 0 in one frame will be seen to have **F** ≠ 0 in another frame.[16] The correct equation of motion for a body whose mass *m* varies with time by either ejecting or accreting mass is obtained by applying the second law to the entire, constant-mass system consisting of the body and its ejected/accreted mass; the result is

$$\mathbf{F} + \mathbf{u}\frac{\mathrm{d}m}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$

where **u** is the velocity of the escaping or incoming mass relative to the body. From this equation one can derive the equation of motion for a varying mass system, for example, the **Tsiolkovsky rocket equation**. Under some conventions, the quantity **u** d*m*/d*t* on the left-hand side, which represents the advection of momentum, is defined as a force (the force exerted on the body by the changing mass, such as rocket exhaust) and is included in the quantity **F**. Then, by substituting the definition of acceleration, the equation becomes **F** =*m***a**.

Lagrangian mechanics is a reformulation of classical mechanics, introduced by the Italian-French mathematician and astronomer Joseph-Louis Lagrange in 1788. In Lagrangian mechanics, the trajectory of a system of particles is derived by solving the Lagrange equations in one of two forms, either the *Lagrange equations of the first kind*,which treat constraints explicitly as extra equations, often using Lagrange multipliers; or the *Lagrange equations of the second kind*, which incorporate the constraints directly by judicious choice of generalized coordinates. In each case, a mathematical function called the **Lagrangian** is a function of the generalized coordinates, their time derivatives, and time, and contains the information about the dynamics of the system.

No new physics is introduced in Lagrangian mechanics compared to Newtonian mechanics. Newton's laws can include non-conservative forces like friction, however they must include constraint forces explicitly and are best suited to Cartesian coordinates. Lagrangian mechanics is ideal for systems with conservative forces and for bypassing constraint forces in any coordinate system. Dissipative and driven forces can be accounted for by splitting the external forces into a sum of potential and non-potential forces, leading to a set of modified **Euler-Lagrange equations**.Generalized coordinates can be chosen by convenience, to exploit symmetries in the system or the geometry of the constraints, which may simplify solving for the motion of the system. Lagrangian mechanics also reveals conserved quantities and their symmetries in a direct way, as a special case of Noether's theorem.

Lagrangian mechanics is important not just for its broad applications, but also for its role in advancing deep understanding of physics. Although Lagrange only sought to describe classical mechanics in his treatise *Mécanique analytique*, **Hamilton's principle** that can be used to derive the Lagrange equation was later recognized to be applicable to much of fundamental theoretical physics as well, particularly quantum mechanics and the theory of relativity.

Conservative force

A particle of mass *m* moves under the influence of a conservative force derived from the gradient ∇ of the a scalar potential,

$$\mathbf{F} = -\nabla V(\mathbf{r}) \,.$$

If there are more particles, in accordance with the above results, the total kinetic energy is a sum over all the particle kinetic energies, and the potential is a function of all the coordinates.

Cartesian coordinates

The Lagrangian of the particle can be written

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \,.$$

The equations of motion for the particle are found by applying the Euler–Lagrange equation, for the *x* coordinate

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \,,$$

with derivatives

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} \,, \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \,, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \,,$$

hence

$$m\ddot{x} = -\frac{\partial V}{\partial x} \,,$$

and similarly for the *y* and *z* coordinates. Collecting the equations in vector form we find

$$m\ddot{\mathbf{r}} = -\nabla V$$

which is Newton's second law of motion for a particle subject to a conservative force.

Polar coordinates in 2d and 3d

The Lagrangian for the above problem in spherical coordinates is

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2) - V(r)\,,$$

so the Euler–Lagrange equations are

$$m\ddot{r} - mr(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) + \frac{\partial V}{\partial r} = 0\,,$$

$$\frac{d}{dt}(mr^2\dot{\theta}) - mr^2\sin\theta\cos\theta\,\dot{\varphi}^2 = 0\,,$$

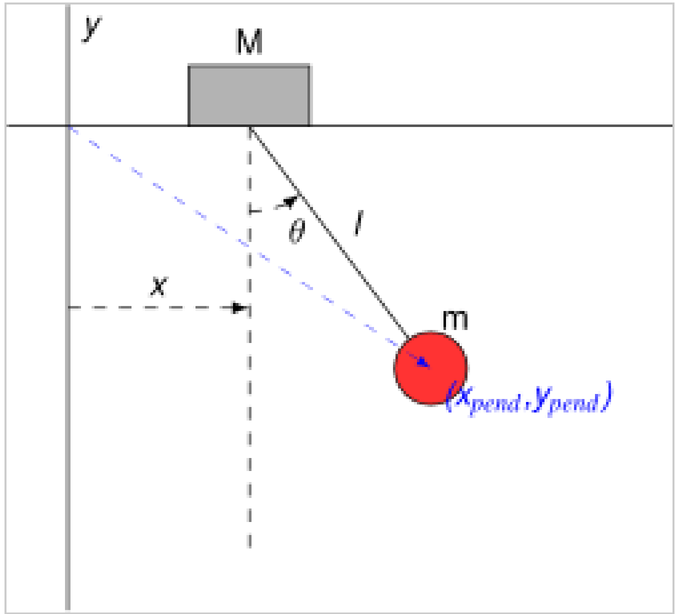
$$\frac{d}{dt}(mr^2\sin^2\theta\,\dot{\varphi}) = 0\,.$$

The ϕ coordinate is cyclic since it does not appear in the Lagrangian, so the conserved momentum in the system is the angular momentum

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\sin^2\theta\dot{\varphi}\,,$$

in which r, θ and $d\phi/dt$ can all vary with time, but only in such a way that p_ϕ is constant.

Pendulum on a movable support



Consider a pendulum of mass *m* and length ℓ, which is attached to a support with mass *M*, which can move along a line in the *x*-direction. Let *x* be the coordinate along the line of the support, and let us denote the position of the pendulum by the angle θ from the vertical. The coordinates and velocity components of the pendulum bob are

$$\begin{aligned} x_{\text{pend}} &= x + \ell \sin\theta \qquad &\Rightarrow \qquad \dot{x}_{\text{pend}} &= \dot{x} + \ell\dot{\theta}\cos\theta \\ y_{\text{pend}} &= -\ell\cos\theta \qquad &\Rightarrow \qquad \dot{y}_{\text{pend}} &= \ell\dot{\theta}\sin\theta \end{aligned}$$

The generalized coordinates can be taken to be *x* and θ. The kinetic energy of the system is then

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}_{\text{pend}}^2 + \dot{y}_{\text{pend}}^2\right)$$

and the potential energy is

$$V = mgy_{\text{pend}}$$

giving the Lagrangian

$$
\begin{aligned}
L &= T - V \\
&= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left[\left(\dot{x} + \ell\dot{\theta}\cos\theta\right)^2 + \left(\ell\dot{\theta}\sin\theta\right)^2\right] + mg\ell\cos\theta \\
&= \tfrac{1}{2}\left(M+m\right)\dot{x}^2 + m\dot{x}\ell\dot{\theta}\cos\theta + \tfrac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta
\end{aligned}
$$

Since *x* is absent from the Lagrangian, it is a cyclic coordinate. The conserved momentum is

$$p_x = \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + m\ell\dot{\theta}\cos\theta\,.$$

and the Lagrange equation for the support coordinate *x* is

$$(M+m)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta = 0$$

The Lagrange equation for the angle θ is

$$\frac{d}{dt}\left[m(\dot{x}\ell\cos\theta + \ell^2\dot{\theta})\right] + m\ell(\dot{x}\dot{\theta} + g)\sin\theta = 0;$$

and simplifying

$$\ddot{\theta} + \frac{\ddot{x}}{\ell}\cos\theta + \frac{g}{\ell}\sin\theta = 0.$$

These equations may look quite complicated, but finding them with Newton's laws would have required carefully identifying all forces, which would have been much more laborious and prone to errors. By considering limit cases, the correctness of this system can be verified: For example, $\ddot{x} \to 0$ should give the equations of motion for a simple pendulum that is at rest in some inertial frame, while $\ddot{\theta} \to 0$ should give the equations for a pendulum in a constantly accelerating system, etc. Furthermore, it is trivial to obtain the results numerically, given suitable starting conditions and a chosen time step, by stepping through the results iteratively.

Hamiltonian mechanics is a theory developed as a reformulation of classical mechanics and predicts the same outcomes as non-Hamiltonian classical mechanics. It uses a different mathematical formalism, providing a more abstract understanding of the theory. Historically, it was an important reformulation of classical mechanics, which later contributed to the formulation of statistical mechanics and quantum mechanics.Hamiltonian mechanics was first formulated by William Rowan Hamilton in 1833, starting from Lagrangian mechanics, a previous reformulation of classical mechanics introduced by Joseph Louis Lagrange in 1788.

In Hamiltonian mechanics, a classical physical system is described by a set of canonical coordinates $\boldsymbol{r} = (\boldsymbol{q}, \boldsymbol{p})$, where each component of the coordinate q_i, p_i is indexed to the frame of reference of the system.

The time evolution of the system is uniquely defined by Hamilton's equations:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}$$
$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = +\frac{\partial \mathcal{H}}{\partial \boldsymbol{p}}$$

where $\mathcal{H} = \mathcal{H}(\boldsymbol{q}, \boldsymbol{p}, t)$ is the Hamiltonian, which often corresponds to the total energy of the system.For a closed system, it is the sum of the kinetic and potential energy in the system.In Newtonian mechanics, the time evolution is obtained by computing the total force being exerted on each particle of the system, and from Newton's second law, the time-evolutions of both position and velocity are computed. In contrast, in Hamiltonian mechanics, the time evolution is obtained by computing the Hamiltonian of the system in the generalized coordinates and inserting it in the Hamiltonian equations. This approach is equivalent to the one used in Lagrangian mechanics. In fact, as is shown below, the Hamiltonian is the **Legendre transform** of the Lagrangian when holding q and t fixed and denoting p as the dual variable, and thus both approaches give the same equations for the same generalized momentum. The main motivation to use Hamiltonian mechanics instead of Lagrangian mechanics comes from the symplectic structure of Hamiltonian systems.

While Hamiltonian mechanics can be used to describe simple systems such as a bouncing ball, a pendulum or an oscillating spring in which energy changes from kinetic to potential and back again over time, its strength is shown in more complex dynamic systems, such as planetary orbits in celestial mechanics. The more degrees of freedom the system has, the more complicated it’s time evolution is and, in most cases, it becomes chaotic.

Basic physical interpretation

A simple interpretation of the Hamilton mechanics comes from its application on a one-dimensional system consisting of one particle of mass *m*. The Hamiltonian represents the **total energy** of the system, which is the sum of **kinetic and potential energy,** traditionally denoted *T* and *V*, respectively. Here *q* is the space coordinate and *p* is the momentum, *mv*.Then

$$\mathcal{H} = T + V, \quad T = \frac{p^2}{2m}, \quad V = V(q).$$

Note that *T* is a function of *p* alone, while *V* is a function of *q* alone (i.e., *T* and *V* are scleronomic).

In this example, the time-derivative of the momentum *p* equals the *Newtonian force*, and so the first Hamilton equation means that the force equals the negative gradient of potential energy. The time-derivative of *q* is the velocity, and so the second Hamilton equation means that the particle’s velocity equals the derivative of its kinetic energy with respect to its momentum.

Calculating a Hamiltonian from a Lagrangian

Given a Lagrangian in terms of the generalized coordinates q_i and generalized velocities $\dot{q}_i$ and time,

1. The momenta are calculated by differentiating the Lagrangian with respect to the (generalized) velocities: $p_i(q_i, \dot{q}_i, t) = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$.
2. The velocities $\dot{q}_i$ are expressed in terms of the momenta p_i by inverting the expressions in the previous step.
3. The Hamiltonian is calculated using the usual definition of $\mathcal{H}$ as the Legendre transformation of $\mathcal{L}$:

$$\mathcal{H} = \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} = \sum_i \dot{q}_i p_i - \mathcal{L} \, .$$

Then the velocities are substituted for through the above results.