Math 3607: Homework 2

6PM, Friday, January 27, 2023

- Problems marked with ✏ are to be done by hand; those marked with 💻 are to be solved using a computer.
- **Important note.** Do not use *Symbolic Math Toolbox*. Any work done using `sym` or `syms` will receive NO credit.

1. (Leap year) 💻 A year is a *leap year* if it is a multiple of 4, except for years divisible by 100 but not by 400. In simpler terms, a non-century year is a leap year if it is divisible by 4; a century year is a leap year if it is divisible by 400.

   For example,

   - 2020 was a leap year. (non-century year; divisible by 4)
   - 1900 was not a leap year. (century year; not divisible by 400)
   - 2000 was a leap year. (century year; divisible by 400)

   (a) Write a function ($_{\text{at}}$ the end of the live script_) which determines whether a given year is a leap year or not.

   (b) Use the function to confirm each of the examples above.

   (c) (Optional) Use (or modify if you like) the function to find all leap years between 1900 and 2023.

2. (3-D coordinate conversion) Recall from calculus that Cartesian coordinates (x, y, z) in $\mathbb{R}^3$ are related to spherical coordinates (ρ, ϕ, θ) by

   $$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi,$$

   where $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi)$.

   (a) ✏ Write down the expressions for ρ, ϕ, θ in terms of x, y, z. Show your work.

   (b) 💻 Write a function ($_{\text{at}}$ the end of the live script_) which takes Cartesian coordinates as inputs and converts them to their spherical counterparts. (Convention: for points on the z-axis, set $\theta = 0$.)

   (c) 💻 Then run your function with the following inputs:

   - $(x, y, z) = (1, 2, -2)$
   - $(x, y, z) = (0, -3, 4)$
   - $(x, y, z) = (-5, -12, 1)$
   - $(x, y, z) = (0, 0, -7)$

3. (Taylor series for exponential function) One way to compute the exponential function e^x is to use its Taylor series expansion around $x = 0$. Unfortunately, many terms are required if $|x|$ is large. A property of the exponential, namely $e^{2x} = (e^x)^2$, can be helpful and lead to the following algorithm:

    Step 1. Divide x by 2 until $|x| < 1/2$.

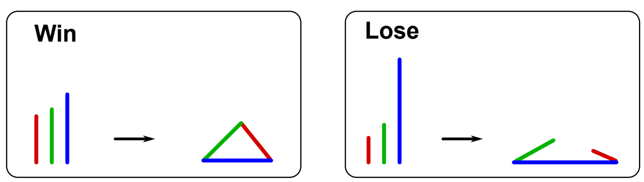
    Step 2. Use a Taylor polynomial (16 terms should be more than enough) to approximate e^x where x is the one obtained in the previous step.

    Step 3. Square the result of Step 2 repeatedly.

    (a) Write a function `myexp` (at the end of the live script_) that does this. The function `polyval` can help with evaluating the Taylor polynomial.

    (b) Test your function on the values of $x = -30, -3, 3, 30$.

4. (Game of 3-stick) In the game of 3-Stick, you pick three sticks each having a random length between 0 and 1. You win if you can form a triangle using three sticks; otherwise, you lose.



    Simulate one million games. Use the results to estimate the probability of winning a game.

    Note. For this problem, you do not need to write a function. Simply write your program in a single code block.

    Note. The only output must be the summary of the million simulations. Use `disp` or `fprintf`.

5. (Gap of 10) Simulate the tossing of a *biased* coin whose tails is twice more likely to be showing than its heads, until the gap between the number of heads and that of tails reaches 10. Print out the number of tosses needed. Then use a for-loop to repeat the simulation ten times. Your output should look like

```
Simulation  1 done in 18 tosses.
Simulation  2 done in 30 tosses.
Simulation  3 done in 18 tosses.
Simulation  4 done in 12 tosses.
Simulation  5 done in 50 tosses.
Simulation  6 done in 30 tosses.
Simulation  7 done in 16 tosses.
Simulation  8 done in 34 tosses.
Simulation  9 done in 10 tosses.
Simulation 10 done in 22 tosses.
```

Note. As in the previous problem, you do not need to write a function for this. Simply write your code in a single code block.

6. (Sequences converging to π) Each of the following sequences converges to π:

$$a_n = \frac{6}{\sqrt{3}} \sum_{k=0}^{n} \frac{(-1)^k}{3^k(2k+1)},$$

$$b_n = 16 \sum_{k=0}^{n} \frac{(-1)^k}{5^{2k+1}(2k+1)} - 4 \sum_{k=0}^{n} \frac{(-1)^k}{239^{2k+1}(2k+1)}.$$

Write a program that prints $a_0, \ldots, a_{n_a}$, where n_a is the smallest integer so that $|a_{n_a} - \pi| \leqslant 10^{-6}$ and prints $b_0, \ldots, b_{n_b}$, where n_b is the smallest integer so that $|b_{n_b} - \pi| \leqslant 10^{-6}$.

Note. Write your program in a single code block for this problem as well.