5.2-1. Consider the following problem.

$$\text{Maximize} \quad Z = 8x_1 + 4x_2 + 6x_3 + 3x_4 + 9x_5,$$

subject to

$$\begin{aligned}
x_1 + 2x_2 + 3x_3 + 3x_4 \quad\quad\quad &\le 180 \qquad \text{(resource 1)}\\
4x_1 + 3x_2 + 2x_3 + \ x_4 + \ x_5 &\le 270 \qquad \text{(resource 2)}\\
x_1 + 3x_2 \quad\quad\quad + \ x_4 + 3x_5 &\le 180 \qquad \text{(resource 3)}
\end{aligned}$$

and

$$x_j \ge 0, \qquad j = 1, \ldots, 5.$$

You are given the facts that the basic variables in the optimal solution are x_3, x_1, and x_5 and that

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}^{-1} = \frac{1}{27}\begin{bmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{bmatrix}.$$

(a) Use the given information to identify the optimal solution.

$$X_B = B^{-1} \cdot b \qquad b = \begin{bmatrix} 180 \\ 270 \\ 180 \end{bmatrix}$$

$$= \frac{1}{27} \cdot \begin{bmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{bmatrix} \cdot \begin{bmatrix} 180 \\ 270 \\ 180 \end{bmatrix}$$

$$= \begin{array}{c} \begin{bmatrix} 50 \\ -30 \\ 50 \end{bmatrix} \end{array} \begin{array}{l} x_3 \\ x_1 \\ x_5 \end{array}$$

$x_1 = 30$
$x_2 = 0$
$x_3 = 50$
$x_4 = 0$
$x_5 = 50$

$$Z = 8 \cdot 30 + 0 \cdot 4 + 6 \cdot 50 + 3 \cdot 0 + 9 \cdot 50$$

$$Z = 990$$

I **5.2-2.*** Work through the matrix form of the simplex method step by step to solve the following problem.

Maximize $\quad Z = 5x_1 + 8x_2 + 7x_3 + 4x_4 + 6x_5,$

subject to

$$2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 \leq 20$$
$$3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 \leq 30$$

Maximize : $Z = 5x_1 + 8x_2 + 7x_3 + 4x_4 + 6x_5 + 0S_1 + 0S_2$

St : $2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 + S_1 = 20$

$3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 + S_2 = 30$

$x_1, x_2, x_3, x_4, x_5, S_1, S_2 \geq 0$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ S_1 \\ S_2 \end{bmatrix} \quad C = \begin{bmatrix} 5 \\ 8 \\ 7 \\ 4 \\ 6 \\ 0 \\ 0 \end{bmatrix} \quad A = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & S_1 & S_2 \end{array} \\ \begin{bmatrix} 2 & 3 & 3 & 2 & 2 & 1 & 0 \\ 3 & 5 & 4 & 2 & 4 & 0 & 1 \end{bmatrix} \end{array} \quad b = \begin{bmatrix} 20 \\ 30 \end{bmatrix} \quad X_B = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \quad X_N = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \quad C_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad C_N = \begin{bmatrix} 5 \\ 8 \\ 7 \\ 4 \\ 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 2 & 3 & 3 & 2 & 2 \\ 3 & 5 & 4 & 2 & 4 \end{bmatrix}$$

Optimality Check:

$$C_N^T - C_B^T \cdot B^{-1} \cdot N$$

$$[5\,8\,7\,4\,6] - [0\;0] \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 3 & 2 & 2 \\ 3 & 5 & 4 & 2 & 4 \end{bmatrix} \Rightarrow [5\,8\,7\,4\,6]$$

enters x_2

Ratio Test : $\dfrac{b}{x_2} \Rightarrow \dfrac{\begin{bmatrix} 20 \\ 30 \end{bmatrix}}{\begin{bmatrix} 3 \\ 5 \end{bmatrix}} \Rightarrow \begin{bmatrix} 6\frac{1}{3} \\ 6 \end{bmatrix} \begin{matrix} S_1 \\ S_2 \end{matrix}$

NBV Constant : $B^{-1}N = \begin{bmatrix} 2 & 3 & 3 & 2 & 2 \\ 3 & 5 & 4 & 2 & 4 \end{bmatrix}$

entering column

$$x_B = \begin{bmatrix} S_1 \\ x_2 \end{bmatrix} \quad x_N = \begin{bmatrix} x_1 \\ S_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \quad B = \begin{array}{c} \begin{array}{cc} S_1 & x_2 \end{array} \\ \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix} \end{array} \quad N = \begin{array}{c} \begin{array}{ccccc} x_1 & S_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 2 & 0 & 3 & 2 & 2 \\ 3 & 1 & 4 & 2 & 4 \end{bmatrix} \end{array} \quad C_B = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \quad C_N = \begin{bmatrix} 5 \\ 0 \\ 7 \\ 4 \\ 6 \end{bmatrix}$$

Optimality Check:

$$C_N^T - C_B^T \cdot B^{-1} \cdot N$$

$$[5\,0\,7\,4\,6] - [0\;8] \begin{bmatrix} 1 & -3/5 \\ 0 & 1/5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 3 & 2 & 2 \\ 3 & 1 & 4 & 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{5} & -\frac{8}{5} & \frac{3}{5} & \frac{4}{5} & -\frac{2}{5} \end{bmatrix}$$

enters x_4

$$Z = \begin{bmatrix} 0 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3/5 \\ 0 & 1/5 \end{bmatrix} \cdot \begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

$$= 48$$

$\to s_1$ enters

Ratio test: $\frac{b}{x_4} = \frac{\begin{bmatrix} 20 \\ 30 \end{bmatrix}}{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \Rightarrow \begin{bmatrix} 10 \\ 15 \end{bmatrix}$

$x_B = \begin{bmatrix} x_4 \\ x_2 \end{bmatrix}$ $x_N = \begin{bmatrix} x_1 \\ s_2 \\ x_3 \\ s_1 \\ x_5 \end{bmatrix}$ $B = \begin{array}{c} \begin{matrix} x_4 & x_2 \end{matrix} \\ \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix} \end{array}$ $N = \begin{array}{c} \begin{matrix} x_1 & s_2 & x_3 & s_1 & x_5 \end{matrix} \\ \begin{bmatrix} 2 & 0 & 3 & 1 & 2 \\ 3 & 1 & 4 & 0 & 4 \end{bmatrix} \end{array}$ $C_B = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ $C_N = \begin{bmatrix} 5 \\ 0 \\ 7 \\ 0 \\ 6 \end{bmatrix}$

Optimality Check:

$$C_N^T - C_B^T \cdot B^{-1} \cdot N$$

$$\begin{bmatrix} 5 & 0 & 7 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 8 \end{bmatrix} \cdot \begin{bmatrix} 5/4 & -3/4 \\ -1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 3 & 1 & 2 \\ 3 & 1 & 4 & 0 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

Optimal!

$$Z = \begin{bmatrix} 4 & 8 \end{bmatrix} \cdot \begin{bmatrix} 5/4 & -3/4 \\ -1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

$$\boxed{Z = 50}$$

$x_2 = 5$

$x_4 = 5/2$

$x_1 = x_3$

$x_5 = 0$

6.1-3. For each of the following linear programming models, give your recommendation on which is the more efficient way (probably) to obtain an optimal solution: by applying the simplex method directly to this primal problem or by applying the simplex method directly to the dual problem instead. Explain.

(a) Maximize $Z = 10x_1 - 4x_2 + 7x_3$,

subject to

$$\begin{aligned} 3x_1 - x_2 + 2x_3 &\le 25 \\ x_1 - 2x_2 + 3x_3 &\le 25 \\ 5x_1 + x_2 + 2x_3 &\le 40 \\ x_1 + x_2 + x_3 &\le 90 \\ 2x_1 - x_2 + x_3 &\le 20 \end{aligned}$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

(b) Maximize $Z = 2x_1 + 5x_2 + 3x_3 + 4x_4 + x_5$,

subject to

$$\begin{aligned} x_1 + 3x_2 + 2x_3 + 3x_4 + x_5 &\le 6 \\ 4x_1 + 6x_2 + 5x_3 + 7x_4 + x_5 &\le 15 \end{aligned}$$

and

$$x_j \ge 0, \quad \text{for } j = 1, 2, 3, 4, 5.$$

With the First problem, once making the dual, there will only be 3 Functional Constraints. This will require the Simplex method to Solve. This will require less math and be easier.

The Second problem, if dual is used, the problem can be solved graphially which is easier than Simplex.

6.1-4. Consider the following problem.

Maximize $\quad Z = -x_1 - 2x_2 - x_3,$

subject to

$$x_1 + x_2 + 2x_3 \leq 12$$
$$x_1 + x_2 - x_3 \leq 1$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

(a) Construct the dual problem.

(b) Use duality theory to show that the optimal solution for the primal problem has $Z \leq 0$.

a) Minimize $W = 12y_1 + y_2$

$y_1 + y_2 \geq -1$

$y_1 + y_2 \geq -2$

$2y_1 - y_2 \geq -1$

$y_1, y_2 \geq 0$

b) Maximize $c^T \cdot x$

St: $Dx \leq a$

$x \geq 0$

Convert Problem

Minimize $a^T \cdot y$

St: $D^T \cdot y \geq c$

$y \geq c$

Decision Variables = Dual Variables

Since all x terms in primal maximize are >0 the optimal solution $Z \leq 0$

6.1-5. Consider the following problem.

Maximize $Z = 2x_1 + 6x_2 + 9x_3,$

subject to

$$x_1 + x_3 \leq 3 \quad \text{(resource 1)}$$
$$x_2 + 2x_3 \leq 5 \quad \text{(resource 2)}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

(a) Construct the dual problem for this primal problem.

I **(b)** Solve the dual problem graphically. Use this solution to identify the shadow prices for the resources in the primal problem.

a) minimize $\omega = 3y_1 + 5y_2$

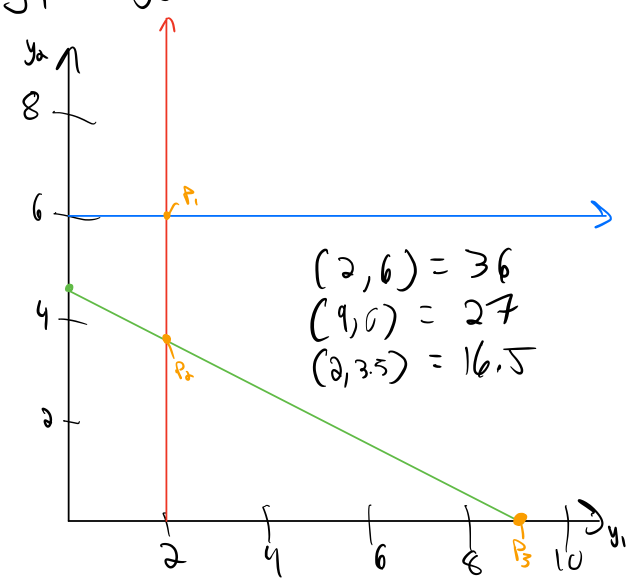
$y_1 \geq 2$

$y_2 \geq 6$

$y_1 + 2y_2 \geq 9$

$y_1, y_2 \geq 0$

b)



$(2,6) = 36$

$(9,0) = 27$

$(2,3.5) = 16.5$

4.9-4. Consider the following problem.

$$\text{Maximize} \quad Z = x_1 - 7x_2 + 3x_3,$$

subject to

$$
\begin{aligned}
2x_1 + x_2 - x_3 &\le 4 \qquad \text{(resource 1)} \\
4x_1 - 3x_2 \quad &\le 2 \qquad \text{(resource 2)} \\
-3x_1 + 2x_2 + x_3 &\le 3 \qquad \text{(resource 3)}
\end{aligned}
$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

D,I **(a)** Work through the simplex method step by step to solve the problem.

(b) Identify the shadow prices for the three resources and describe their significance.

C **(c)** Use a software package based on the simplex method to solve the problem and then to generate sensitivity information. Use this information to identify the shadow price for each resource, the allowable range for each objective function coefficient, and the allowable range for each right-hand side.

D) Calculate the objective function value assuming that resource 2 amount is 10 instead of 2.
E) Calculate the objective function value assuming that the coefficient of x1 is 7.5 instead of 1
F) Would the current optimal solution change if the coefficient of x3 is 28 instead of 3.

a) Max $Z = x_1 - 7x_2 + 3x_3 = 0 = Z - x_1 + 7x_2 - 3x_3$

st $2x_1 + x_2 - x_3 + S_1 = 4$
$4x_1 - 3x_2 + S_2 = 2$
$-3x_1 + 2x_2 + x_3 + S_3 = 3$

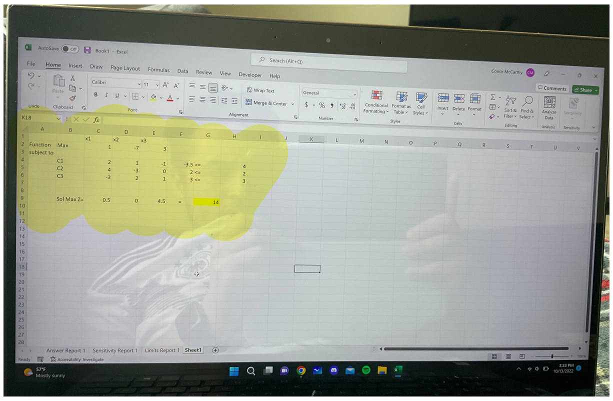
$x_1, x_2, x_3, S_1, S_2, S_3 \ge 0$

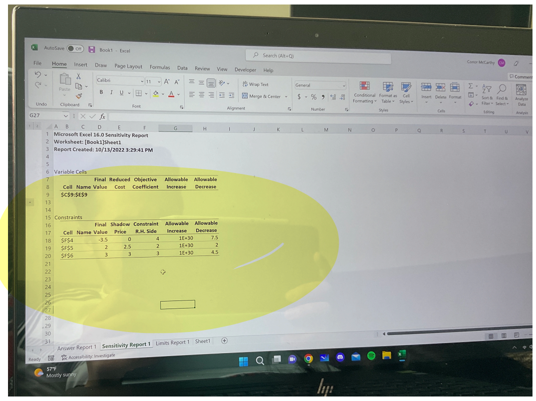
	x_1	x_2	x_3	S_1	S_2	S_3		
Z	-1	7	-3	0	0	0		
S_1	2	1	-1	1	0	0	4	4/-1
S_2	4	-3	0	0	1	0	2	2/0
S_3	-3	2	1	0	0	1	3	3/1
Z	-10	13	0	0	0	3		
S_1	-1	3	0	1	0	1	7	
S_2	4	-3	0	0	1	0	2	
x_3	-3	2	1	0	0	1	3	
Z	0	11/2	0	0	5/2	3		
S_1	0	9/4	0	1	1/4	1	15/2	
x_1	1	-3/4	0	0	1/4	0	1/2	
x_3	0	-1/4	1	0	3/4	1	9/2	

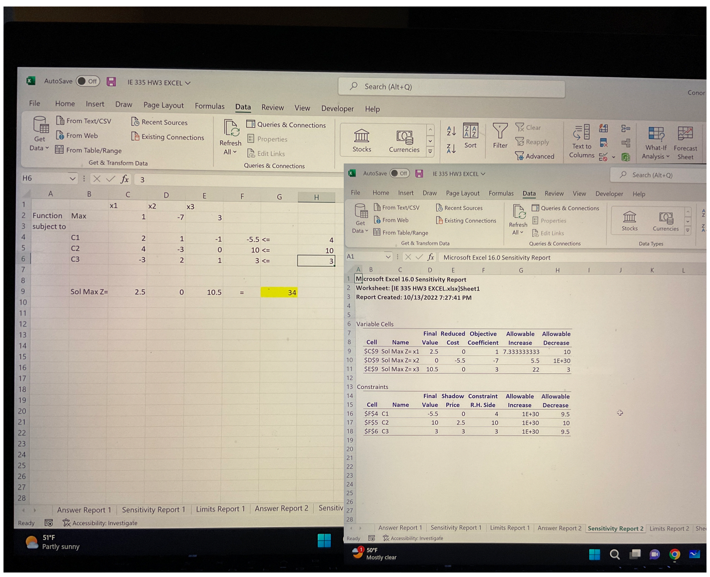
$x_1 = 0.5$ $Z = 14$
$x_3 = 4.5$
$x_2 = 0$

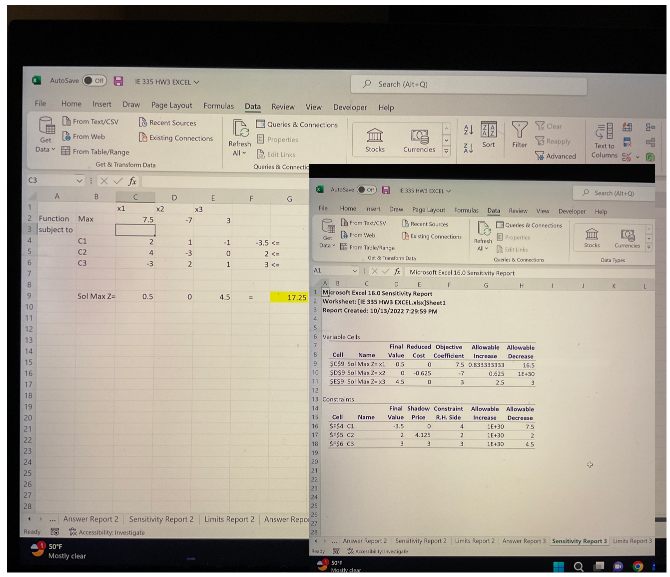
b) Shadow price is marginal value of a given amount which Z will increase given 1 unit more of resource.
Here the shadow prices are:
$b_1 = 0$ $b_2 = 2.5$ $b_3 = 3$.

c) Screen shots next page









D)

Max $Z = x_1 - 7x_2 + 3x_3$

$= 0 = Z - x_1 + 7x_2 - 3x_3$

st

$2x_1 + x_2 - x_3 + s_1 = 4$

$4x_1 - 3x_2 + s_2 = 10$

$-3x_1 + 2x_2 + x_3 + s_3 = 3$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

	x_1	x_2	x_3	s_1	s_2	s_3		
Z	-1	7	-3	0	0	0		
s_1	2	1	-1	1	0	0	4	x
s_2	4	-3	0	0	1	0	10	x
← s_3	-3	2	1	0	0	1	3	3/1
Z	-10	13	0	0	0	3		
s_1	-1	3	0	1	0	1	7	
← s_2	4	-3	0	0	1	0	10	10/4
x_3	-3	2	1	0	0	1	3	
Z	0	11/2	0	0	5/2	3		
s_1	0	5/4	0	1	1/4	1	19/2	
x_1	1	-3/4	0	0	1/4	0	5/2	
x_3	0	1/4	1	0	3/4	1	21/2	

$x_1 = 2.5$
$x_3 = 10.5$
$Z = 34$

E)

Max $Z = 7.5x_1 - 7x_2 + 3x_3$

$= 0 = Z - 7.5x_1 + 7x_2 - 3x_3$

st

$2x_1 + x_2 - x_3 + s_1 = 4$

$4x_1 - 3x_2 + s_2 = 2$

$-3x_1 + 2x_2 + x_3 + s_3 = 3$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

	x_1	x_2	x_3	s_1	s_2	s_3	
Z	-7.5	7	-3	0	0	0	
s_1	2	1	-1	1	0	0	4
← s_2	4	-3	0	0	1	0	2
s_3	-3	2	1	0	0	1	3
Z	0	1.375	-3	0	1.875	0	
s_1	0	5/2	-1	1	-1/2	0	3
x_1	1	-3/4	0	0	1/4	0	0.5
← s_3	0	-1/4	1	0	3/4	1	4.5
Z	0	0.625	0	0	4.125	3	
s_1	0	2.25	0	1	1/4	1	7.5
x_1	1	-3/4	0	0	1/4	0	0.5
x_3	0	-1/4	1	0	3/4	0	4.5

$x_1 = 1/2$
$x_3 = 9/2$
$Z = 17.25$

F) Yes the optimal solution would change. The increased weight/value of x_3 would change the optimal solution. BUT!! The value of x_3 does not change it will remain at 4.5 because the constraining equations don't change.