integrating straight line motion in spherical coordinates

February 18, 2022

1 basic equations

From direct differentiation of the position vector in spherical coordinates, $\boldsymbol{x} = r\boldsymbol{e}_r$, the acceleration is

$$
\begin{aligned}
\boldsymbol{a} &= \boldsymbol{e}_r\ddot{r} + \dot{\boldsymbol{e}}_r\dot{r} + \boldsymbol{e}_\theta r\ddot{\theta} + \boldsymbol{e}_\theta\dot{r}\dot{\theta} + \dot{\boldsymbol{e}}_\theta r\dot{\theta} \\
&+ \boldsymbol{e}_\phi r\sin\theta\ddot{\phi} + \boldsymbol{e}_\phi r\cos\theta\dot{\theta}\dot{\phi} + \boldsymbol{e}_\phi\dot{r}\sin\theta\dot{\phi} + \dot{\boldsymbol{e}}_\phi r\sin\theta\dot{\phi} \\
&= \boldsymbol{e}_r\ddot{r} + \left(\frac{\partial\boldsymbol{e}_r}{\partial\theta}\dot{\theta} + \frac{\partial\boldsymbol{e}_r}{\partial\phi}\dot{\phi}\right)\dot{r} + \boldsymbol{e}_\theta r\ddot{\theta} + \boldsymbol{e}_\theta\dot{r}\dot{\theta} + \left(\frac{\partial\boldsymbol{e}_\theta}{\partial\theta}\dot{\theta} + \frac{\partial\boldsymbol{e}_\theta}{\partial\phi}\dot{\phi}\right)r\dot{\theta} + \boldsymbol{e}_\phi r\sin\theta\ddot{\phi} \\
&+ \boldsymbol{e}_\phi r\cos\theta\dot{\theta}\dot{\phi} + \boldsymbol{e}_\phi\dot{r}\sin\theta\dot{\phi} + \left(\frac{\partial\boldsymbol{e}_\phi}{\partial\theta}\dot{\theta} + \frac{\partial\boldsymbol{e}_\phi}{\partial\phi}\dot{\phi}\right)r\sin\theta\dot{\phi} \\
&= \boldsymbol{e}_r\ddot{r} + \left(\boldsymbol{e}_\theta\dot{\theta} + \sin\theta\boldsymbol{e}_\phi\dot{\phi}\right)\dot{r} + \boldsymbol{e}_\theta r\ddot{\theta} + \boldsymbol{e}_\theta\dot{r}\dot{\theta} + \left(-\boldsymbol{e}_r\dot{\theta} + \cos\theta\boldsymbol{e}_\phi\dot{\phi}\right)r\dot{\theta} + \boldsymbol{e}_\phi r\sin\theta\ddot{\phi} \\
&+ \boldsymbol{e}_\phi r\cos\theta\dot{\theta}\dot{\phi} + \boldsymbol{e}_\phi\dot{r}\sin\theta\dot{\phi} + (-\sin\theta\boldsymbol{e}_r - \cos\theta\boldsymbol{e}_\theta)\, r\sin\theta\dot{\phi}^2. \qquad (1)
\end{aligned}
$$

The components of the acceleration are then collected to be

$$
\begin{aligned}
a_r &= \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 && (2)\\
a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2 && (3)\\
a_\phi &= r\sin\theta\ddot{\phi} + 2\sin\theta\dot{r}\dot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi}. && (4)
\end{aligned}
$$

We see the expected "double dot" terms like $\ddot{r}$, $\ddot{\theta}$ and $\ddot{\phi}$ as well as a host of "inertial forces" similar to the cylindrical expressions.

These formulas can be recovered using the geodesic equation. Using spherical coordinates and a coordinate basis ($g_{rr} = 1, g_{\theta\theta} = r^2, g_{\phi\phi} = r^2\sin^2\theta$), the nonzero connection coefficients are

$$
\begin{aligned}
\Gamma_{ijk} &= \frac{1}{2}\left(g_{ij,k} - g_{jk,i} + g_{ki,j}\right) && (5)\\
\Gamma^{r}{}_{\theta\theta} &= -r && (6)\\
\Gamma^{r}{}_{\phi\phi} &= -r\sin^2\theta && (7)\\
\Gamma^{\theta}{}_{r\theta} &= \Gamma^{\theta}{}_{\theta r} = \frac{1}{r} && (8)\\
\Gamma^{\theta}{}_{\phi\phi} &= -\sin\theta\cos\theta && (9)\\
\Gamma^{\phi}{}_{r\phi} &= \Gamma^{\phi}{}_{\phi r} = \frac{1}{r} && (10)\\
\Gamma^{\phi}{}_{\theta\phi} &= \Gamma^{\phi}{}_{\phi\theta} = \cot\theta. && (11)
\end{aligned}
$$

Plugging into the geodesic equation

$$
a^i = \frac{d^2x^i}{dt^2} + \Gamma^{i}{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0 \tag{12}
$$

we recover the above results.

2 numerical form

Let's write the equations $d\boldsymbol{x}/dt = \boldsymbol{v}$ and $d\boldsymbol{v}/dt = \boldsymbol{a} = 0$ in the standard form

$$
\frac{d\boldsymbol{y}}{dt} = f(\boldsymbol{y}). \tag{13}
$$

Here the independent variable is time t. The dependent variables are

$$
\boldsymbol{y} = \begin{pmatrix} r \\ \theta \\ \phi \\ \dot{r} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} \tag{14}
$$

and the right hand sides are

$$f(\boldsymbol{y}) \quad = \quad \left(\begin{array}{c} \dot{r} \\ \dot{\theta} \\ \phi \\ \ddot{r} \\ \ddot{\theta} \\ \ddot{\phi} \end{array} \right) = \left(\begin{array}{l} \dot{r} \\ \dot{\theta} \\ \phi \\ r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2 \\ -2\frac{\dot{r}\dot{\theta}}{r} + \sin\theta\cos\theta\dot{\phi}^2 \\ -2\frac{\dot{r}\dot{\phi}}{r} - 2\cot\theta\dot{\theta}\dot{\phi} \end{array} \right). \tag{15}$$

Issues with integrating? Coordinate singularities at $r = 0$ and $\theta = 0, \pi$. The stepsize Δt must be very small there in order to take small steps. Near the origin, the angles θ and ϕ change very rapidly due to the $1/r$ in $\ddot{\theta}$ and $\ddot{\phi}$. Also, near $\theta = 0, \pi$ the $\cot\theta = \cos\theta/\sin\theta$ becomes large in $\ddot{\phi}$.