Implicit v. Explicit Updates

The equation: $$\frac{\partial(\rho X)}{\partial t} = \left(\Gamma n_H - \alpha n_p^2\right) m_p$$

$$X = \frac{n_p}{n_p + n_H} = \frac{n_p}{n} = \frac{n_p m_p}{\rho}$$

n: constant. $\qquad n = \rho / m_p$

Present time: X, n_p, n_H. $\qquad n_p = n - n_H$.

Future time: X^+, n_p^+, n_H^+. $\qquad n_p^+ = n - n_H^+$

Explicit formulation:

$$\frac{\Delta(\rho X)}{\Delta t} = \left(\Gamma n_H - \alpha n_p^2\right) m_p \,; \quad \Delta(\rho X) = \Delta(n_p m_p) = \ldots$$

$$\ldots = m_p \cdot \Delta n_p = m_p \cdot \left(n_p^+ - n_p\right)$$

$$\downarrow$$

$$\frac{m_p\left(n_p^+ - n_p\right)}{\Delta t} = \left(\Gamma_{n_H} - \alpha n_p^2\right) m_p$$

$$\boxed{n_p^+ = n_p + \Delta t\left(\Gamma_{n_H} - \alpha n_p^2\right)}$$

EXPLICIT SOURCE TERM UPDATE

Implicit formulation:

The difference: we solve for ionization equilibrium using the NEXT timestep.

$$\frac{\Delta(\rho X)}{\Delta t} = \left(\Gamma n_H^+ - \alpha n_p^{+2}\right) m_p$$

$$\frac{n_p^+ - n_p}{\Delta t} = \cancel{\Gamma n_H^+ - \alpha\left(n - n_H^+\right)^2 \quad \text{using } n_p^{+2} = \left(n - n_H^+\right)^2}$$

$$\Gamma\left(n - n_p^+\right) - \alpha n_p^{+2} \qquad \text{since } n_H^+ = n - n_p^+$$

Quadratic for n_p^{+2}!

$$\alpha n_p^{+2} - \Gamma(n - n_p^+) + \frac{n_p^+ - n_p}{\Delta t} = 0$$

Let $n_c \equiv \Gamma/\alpha$, $n_r \equiv 1/\alpha\Delta t$.

$$n_p^{+2} - n_c(n - n_p^+) + n_r(n_p^+ - n_p) = 0$$

$$n_p^+ = -\frac{n_c + n_r}{2} \pm \frac{1}{2}\left[n_c^2 + n_r^2 + 4(n_c n + n_r n_p) + 2n_c n_r\right]^{1/2}$$

$$\boxed{n_p^+ = -\frac{(n_c + n_r)}{2} \pm \frac{1}{2}\left[(n_c + n_r)^2 + 4(n_c n + n_r n_p)\right]^{1/2}}$$

As $\Delta t \to \infty$, we approach equilibrium. $n_r = \frac{1}{\alpha\Delta t} \to 0$ gives

$$n_p^+ = -\frac{n_c}{2} \pm \frac{1}{2}\left[n_c^2 + 4n_c n\right]^{1/2} = -\frac{n_c}{2} \pm \frac{n_c}{2}\left[1 + \frac{4n}{n_c}\right]^{1/2}$$

Large n: $n_p^+ \approx 0$? Actually $(n_c n)^{1/2}$

Small n:

$$= -\frac{n_c}{2} \pm \frac{n_c}{2}\left[1 + \frac{4n}{n_c}\right]^{1/2}$$

$$\approx -\frac{n_c}{2} \pm \frac{n_c}{2}\left[1 + \frac{2n}{n_c}\right] \approx n$$