purificazione fissata

$\rho \rightsquigarrow |\psi\rangle_{SA} = (1_S \otimes U_A)\, |\psi_0\rangle_{SA}$ $\qquad |1'\rangle_{SA} = \sum_j |j\rangle_S |j\rangle_A$ ← non normalizzato, non è stato di $\mathcal{H}_{SA}$

Schmidt dec. $\quad |\psi_0\rangle_{SA} = \sum_j \sqrt{\lambda_j}\, |j\rangle_S |j\rangle_A = (\sqrt{\rho_S} \otimes 1_A)\, |1'\rangle_{SA}$

dipende da ρ_S per la scelta di $|j\rangle_S$ (voglio eliminare questa dip.)

$\Rightarrow |\psi\rangle_{SA} = (\sqrt{\rho_S} \otimes U_A)\, |1'\rangle_{SA}$

Introduco base canonica $\quad |1\rangle_{SA} = \sum_K |K\rangle_S |K_A\rangle \quad$ non dipende da ρ_S

$|1'\rangle_{SA} = (V_S \otimes V_A)\, |1\rangle_{SA}$

$\Rightarrow |\psi\rangle_{SA} = (\sqrt{\rho_S}\, V_S \otimes U_A V_A)\, |1\rangle_{SA}$

$$\begin{cases} |\psi_1\rangle_{SA} = (\sqrt{\rho_1}\, V_S^{(1)} \otimes U_A^{(1)} V_A^{(1)})\, |1\rangle_{SA} \\ |\psi_2\rangle_{SA} = (\sqrt{\rho_2}\, V_S^{(2)} \otimes U_A^{(2)} V_A^{(2)})\, |1\rangle_{SA} \end{cases}$$

Proviamo a massimizzare il prodotto scalare

$$|\langle\psi_1|\psi_2\rangle_{SA}|^2 = \left| \langle 1| \left(V_S^{+(1)} \sqrt{\rho_1}\sqrt{\rho_2}\, V_S^{(2)} \otimes V_A^{+(1)} U_A^{+(1)} U_A^{(2)} V_A^{(2)} \right) |1\rangle_{SA} \right|^2$$

$$\max_{\text{pur.}} |\langle\psi_1|\psi_2\rangle_{SA}|^2 = \max_V \left| {}_{SA}\langle 1| \left(V_S^{+(1)} \sqrt{\rho_1}\sqrt{\rho_2}\, V_S^{(2)} \otimes V \right) |1\rangle_{SA} \right|^2 \qquad V_S^{+(1)} \sqrt{\rho_1}\sqrt{\rho_2}\, V_S^{(2)} \equiv \theta$$

$$= \max_V \left| \sum_{K,K'} \theta_{KK'} V_{KK'} \right|^2 = \max_V |\mathrm{tr}(\theta V^T)|^2 \qquad V^T = U \quad U \text{ unitaria}$$

$$= \max_U |\mathrm{tr}(U\theta)|^2 = \max_U \left| \mathrm{tr}\left(U V_S^{+(1)} \sqrt{\rho_1}\sqrt{\rho_2}\, V_S^{(2)} \right) \right|^2 \qquad V_S^{(2)} U V_S^{+(1)} = U' \text{ unitaria}$$

$$= \max_{U'} |\mathrm{tr}(U' \sqrt{\rho_1}\sqrt{\rho_2})|^2 \overset{\text{POLAR}}{=} \max_{U'} |\mathrm{tr}(U' U'' |\sqrt{\rho_1}\sqrt{\rho_2}|)|^2 = \max_U |\mathrm{tr}(U |\sqrt{\rho_1}\sqrt{\rho_2}|)|^2$$

$$\left| \mathrm{tr}\left(\underset{V^+V}{U} \overset{\geq 0}{\sqrt{\theta^+\theta}} \right) \right| = \left| \mathrm{tr}\left(\underbrace{V U V^+}_{U'} \sqrt{D} \right) \right| = \left| \sum_i \sqrt{\lambda_i}\, U'_{ii} \right| \leq \sum_i |\sqrt{\lambda_i}| = \mathrm{tr}|\theta|$$

$U_{ik} (U^+)_{kj} = \delta_{ij}$

$U_{ik} U^*_{jk} = \delta_{ij}$

$\sum_K U_{iK} U^*_{iK} = 1$

$\sum_K |U_{iK}|^2 = 1$

$\therefore |U_{iK}|^2 \leq 1 \qquad |U_{iK}| \leq 1$

$$\Rightarrow |\langle\psi_1|\psi_2\rangle_{SA}|^2 \leq \left(\mathrm{tr}|\sqrt{\rho_1}\sqrt{\rho_2}| \right)^2 = F(\rho_1, \rho_2)$$