$E_{1,2} \geq 0 \qquad E_1 + E_2 = 1$

$$P_{ERROR} = p_1 \, tr\left((1-E_1)\rho_1\right) + p_2 \, tr\left(E_1 \rho_2\right) = p_1 + tr\left(E_1(p_2\rho_2 - p_1\rho_1)\right)$$

$$= p_2 + tr\left(E_2(p_1\rho_1 - p_2\rho_2)\right)$$

$$= \frac{1}{2}\left(1 + tr\left(E_1(p_2\rho_2 - p_1\rho_1) - E_2(p_2\rho_2 - p_1\rho_1)\right)\right)$$

$$= \frac{1}{2}\left(1 - tr\left((E_1 - E_2)(p_1\rho_1 - p_2\rho_2)\right)\right)$$

$p_1\rho_1 - p_2\rho_2 = \hat{\theta} = \hat{\theta}^+$ real eigenvalues $\lambda_j \Rightarrow$ separate $\lambda_j > 0$ and $\lambda_j < 0$

$$\hat{\theta} \sim \left(\begin{array}{c|c} A \geq 0 & 0 \\ \hline 0 & -B < 0 \end{array}\right) = A - B \qquad A, B \geq 0 \qquad A \cdot B = BA = 0$$

$$P_{error} = \frac{1}{2}\left(1 - tr\left((E_1 - E_2)(A-B)\right)\right) = \frac{1}{2}\left(1 - \overset{\leq 0}{tr(E_1 A)} - \overset{\leq 0}{tr(E_2 B)} + \overset{\geq 0}{tr(E_1 B)} + \overset{\geq 0}{tr(E_2 A)}\right)$$

$$tr\left(\overset{\geq 0}{E_1} \overset{\geq 0}{A}\right) \overset{?}{\geq} 0$$

$$= \sum_{j,k} \langle j | E_1 | k \rangle \langle k | A | j \rangle = \sum_{jk} \lambda_j \lambda_k |\langle j | k \rangle|^2 \geq 0$$

$E_1 |j\rangle = \overset{\geq 0}{\lambda_j} |j\rangle \qquad A|k\rangle = \overset{\geq 0}{\lambda_k} |k\rangle \qquad E_{1,2} \leq 1$

$$P_{ERROR} \geq \frac{1}{2}\left(1 - tr(E_1 A + E_2 B)\right) \geq \frac{1}{2}\left(1 - tr(A+B)\right) \qquad \text{(no POVM dependency)}$$

$\theta = A - B \qquad \theta^2 = \theta^+ \theta \underset{\uparrow AB=0}{=} A^2 + B^2 \qquad \sqrt{\theta^+\theta} = \sqrt{A^2 + B^2} = A + B$

$$\min P_{error} = \frac{1}{2}\left(1 - \|p_1\rho_1 - p_2\rho_2\|_1\right) \qquad \text{Helstrom theorem}$$

In order to obtain the minimum P_{error}, we made the following assumptions

$$\left.\begin{array}{l} tr(E_2 A) = tr(E_1 B) = 0 \\ tr(E_1 A) = tr(A) \\ tr(E_2 B) = tr(B) \end{array}\right\} \begin{array}{l} \exists \text{ such POVM? } \quad E_{1,2} = \Pi_{A,B} \qquad \Pi_A + \Pi_B = 1 \\ \text{(proj. measure only for 2 outputs)} \end{array}$$