Quiz 2 problem bank

1. Find the general solution for the differential equation $y' = 4te^{2t}$.
2. Find the specific solution for the differential equation $x' = t\sec^2(t^2)$ with $x(0) = 3$.
3. Find the position function $s(t)$ given that $a(t) = 6t - 4$ with $v(1) = 2$ and $s(1) = 6$.
4. Solve for P given $P' = \dfrac{6t\sqrt{P}}{\sqrt{t^2+3}}$ with $P(1) = 25$.
5. Solve $y' = \sin(x)y\ln(y)$ with $y(0) = e$.
6. Find the particular solution for $3xy^2y' = y^3 + 1$ with $y(1) = 2$.
7. Solve $y' + \tan(x)y = \cos^2(x)$ with $y(0) = -1$.
8. Solve the following differential equation: $ty' + (t+2)y = te^{t^3-t}$, with $y(1) = \frac{1}{3}$.
9. Find an implicit solution for $(e^y - 3xy)y' = y^2$. (Hint: rewrite problem in terms of $x' = \dfrac{dx}{dy}$.)
10. A large tank is initially filled with 40 gallons of pure water. At time $t = 0$ valves are opened, one pours brine into the tank at a rate of 8 gallons per minute with 1 pound of salt per gallon, the other empties well-mixed brine out at 4 gallons per minute. How much salt is in the tank at $t = 10$ minutes?

The quiz will consist of two randomly chosen problems (with possibly small variations; so make sure to learn processes and not just memorize answers).

Key ideas and processes

- *Differential equations of the form* $y' = f(x)$ are solved by taking anti-derivatives using the tools from Calculus I and II. This works for general and specific solutions.
- *Physics* problems often involve differential equations. For these it is almost always helpful to draw a picture and translate word problems into mathematical expressions.
- *Separable differential equations* are ones which can be written as $f(y)y' = g(x)$ or $f(y)\,dy = g(x)\,dx$. The process for solving consists of: (1) separate; (2) integrate; (3) uncomplicate (solve for constant and rearrange as needed).
- *Linear first order equations* are ones which can be written as $y' + P(x)y = Q(x)$. For this multiply both sides by an "integrating factor" ($\exp(\int P(x)\,dx)$) making the left hand side a derivative. Now integrate both sides and solve for y.
- *Mixing problems* are $y' = (\text{inflow}) - (\text{outflow})$ where usually $(\text{outflow}) = q(t)y$ where $q(t)$ is proportion of the tank that flows out. These are usually linear first order ODEs.

Comments and (partial) answers on problems

1. $y = (2t-1)e^{2t} + C$
2. $x = \frac{1}{2}\tan(t^2) + 3$
3. $s(t) = t^3 - 2t^2 + 3t + 4$
4. $P = \left(3\sqrt{t^2+3} - 1\right)^2$
5. $y = \exp(\exp(1 - \cos(x)))$
6. $y = (9x-1)^{1/3}$
7. $y = \sin(x)\cos(x) - \cos(x)$
8. $y = \dfrac{e^{t^3-t}}{3t^2}$
9. $y^3x = (y-1)e^y + C$
10. 60 pounds

1. Find the general solution for the differential equation $y' = 4te^{2t}$.

General Solution = Find y

$$\int y' = \int 4te^{2t}$$

use integration by parts

$$\int u \cdot dv = u \cdot v - \int v\, du$$

$$y$$

differentiate

$$u = 4t \quad dv = e^{2t}\,dt$$

integrate

$$du = 4dt \quad v = \frac{e^{2t}}{2}$$

$$y = \frac{4te^{2t}}{2} - \int \frac{e^{2t}}{2} \cdot 4dt$$

$$y = 2te^{2t} - \int 2e^{2t}\,dt$$

$$y = 2te^{2t} - \frac{2e^{2t}}{2} + C$$

$$\boxed{y = 2te^{2t} - e^{2t} + c}$$

or

$$\boxed{y = e^{2t}[2t-1] + c}$$

2. Find the specific solution for the differential equation $x' = t\sec^2(t^2)$ with $x(0) = 3$.

$$\int x' = \int t \sec^2(t^2)\, dt$$

use u substitution

x

$u = t^2$

$du = 2t\, dt$

$\int \sec^2 u$

$= \tan u$

$$x = \int \frac{1}{2} \cdot \sec^2(u)\, du$$

$$x = \frac{1}{2} \tan u + C$$

plug t^2 in for u

$$x = \frac{1}{2} \tan(t^2) + C$$

now plug in initial conditions

$x(0) = 3$

$$3 = \frac{1}{2} \tan(0) + C$$

$\tan(0) = 0$

$$\boxed{3 = C}$$

$$x = \frac{1}{2} \tan(t^2) + 3$$

3. Find the position function $s(t)$ given that $a(t) = 6t - 4$ with $v(1) = 2$ and $s(1) = 6$.

physics problem

$$\int a(t)\,dt = v(t) = \int (6t-4)\,dt$$

simplify. $$v(t) = \frac{6t^2}{2} - 4t + c$$

$$v(t) = 3t^2 - 4t + c$$

when $t=1$, $v=2$

$$2 = 3(1)^2 - 4(1) + c$$

$$2 = 3 - 4 + c$$

$$\boxed{3 = c}$$

$$\boxed{v(t) = 3t^2 - 4t + 3}$$

$$\int v(t)\,dt = s(t) = \int (3t^2 - 4t + 3)\,dt$$

$$s(t) = t^3 - \frac{4t^2}{2} + 3t + D$$

Simplify. $$s(t) = t^3 - 2t^2 + 3t + D$$

position is 6 when time is 1

$$(6) = (1)^3 - 2(1)^2 + 3(1) + D$$

$$6 = 1 - 2 + 3 + D$$

$$\boxed{4 = D}$$

$$\boxed{s(t) = t^3 - 2t^2 + 3t + 4}$$

4. Solve for P given $P' = \dfrac{6t\sqrt{P}}{\sqrt{t^2+3}}$ with $P(1) = 25$.

Separate variables first

$$\frac{P'}{\sqrt{P}} = \frac{6t}{\sqrt{t^2+3}}$$

$$\frac{dP}{\sqrt{P}\,dt} = \frac{6t}{\sqrt{t^2+3}}$$

$$\int \frac{dP}{\sqrt{P}} = \int \frac{6t\,dt}{\sqrt{t^2+3}}$$

$$\int P^{-1/2}\,dP = \int 6t\,(t^2+3)^{-1/2}\,dt$$

use u substitution

$u = t^2+3$

$du = 2t\,dt$

$$\frac{P^{-1/2+1}}{-1/2+1} = \frac{P^{1/2}}{1/2} = \int 3(u)^{-1/2}\,du$$

$$\frac{P^{1/2}}{1/2} = \int 3(u)^{-1/2}\,du = \frac{3u^{1/2}}{1/2} + C$$

$$= \frac{1}{2}\left(\frac{P^{1/2}}{1/2} = \frac{3(t^2+3)^{1/2}}{1/2} + C\right)\frac{1}{2}$$

$$P^{1/2} = 3(t^2+3)^{1/2} + C$$

$$(25)^{1/2} = 3(1^2+3)^{1/2} + C$$

$$5 = 3(2) + C$$

$$-1 = C$$

$$\left(P^{1/2}\right)^2 = \left(3(t^2+3)^{1/2} - 1\right)^2$$

$$\boxed{P = (3(t^2+3)^{1/2} - 1)^2}$$

5. Solve $y' = \sin(x) y \ln(y)$ with $y(0) = e$.

Separate variables first

$$\frac{dy}{dx} = \sin(x)\, y \ln(y)$$

use u substitution

$$\int \frac{dy}{y \ln(y)} = \int \sin(x)\, dx$$

$u = \ln(y)$

$du = \frac{1}{y} dy$

$$\int \frac{du}{u} = \int \sin(x)\, dx$$

$$\ln(u) = -\cos(x) + c$$ ← plug y back in.

$$\ln(\ln(y)) = -\cos(x) + c$$ ← Find c.

$\ln(1) = 0$

$$\ln(\ln(e)) = -\cos(0) + c$$

$$\ln(1) = -1 + c \rightarrow 0 = -1 + c$$

$$\boxed{1 = c}$$

$$e^{[\ln(\ln(y))]} = e^{[-\cos(x) + 1]}$$

$$\ln(y) = e^{-\cos(x)} \cdot e$$

$$e^{\ln(y)} = e^{e^{-\cos(x)} \cdot e}$$

$$y = e^{e^{(-\cos x)} \cdot e^1} = e^{e^{(-\cos x + 1)}}$$

6. Find the particular solution for $3xy^2y' = (y^3+1)$ with $y(1) = 2$.

combine y^3+1

$$\frac{3xy^2y'}{(y^3+1)} = 1$$

$$\frac{y^2}{(y^3+1)}\frac{dy}{dx} = \frac{1}{3x}$$

$$\int \frac{y^2}{(y^3+1)}dy = \int \frac{1}{3x}dx$$

Use u substitution

$u = y^3+1$

$du = 3y^2\,dy$

$$\int \frac{1}{3}\cdot\frac{1}{u}du = \int \frac{1}{3x}dx$$

Cancel out $\frac{1}{3}$

$$\int \frac{1}{u}du = \frac{1}{x}dx$$

$\ln(u) = \ln(x) + C$

plug y back in for u.

$\ln(y^3+1) = \ln(x) + C$

$e^{\{\ln(y^3+1)\}} = e^{\{\ln(x)+C\}}$

$y^3+1 = e^{\ln(x)}\cdot e^C$

$y^3+1 = C\cdot x$

$2^3+1 = C\cdot 1$

$\boxed{9 = C}$

plug in 9 for C

$y^3+1 = 9x$

$y^3 = 9x-1$

$\boxed{y = (9x-1)^{1/3}}$

7. Solve $y' + \underbrace{\tan(x)}_{P(x)}y = \underbrace{\cos^2(x)}_{Q(x)}$ with $y(0) = -1$.

$$1 \cdot y' + P(x) \cdot y = Qx$$

$$I.F = e^{\int P(x)dx} = P(x)$$

$$I.F = P(x) = e^{\int \tan(x)\,dx} \Rightarrow$$

$$= e^{\ln(\sec x)} = \sec(x)$$

$$\sec(x)y' + \sec(x)\tan(x)y = \sec(x)\cos^2(x)$$

product rule

$$\sec(x)y' + \sec'(x)y = \sec(x)\cos^2(x)$$

$$\int \frac{d}{dx}[\sec(x)y] = \int 1 \cdot \cos(x)$$

$$\sec(x) \cdot y = \int 1 \cdot \cos(x)$$

$$\sec(x)y = \sin(x) + c$$

$$y = \frac{\sin(x) + c}{\sec(x)}$$

Solve for c

$$-1 = \frac{\sin(0) + c}{\sec(0)}$$

$$-1 = \frac{0 + c}{1}$$

$$\boxed{c = -1}$$

$$\boxed{y = \frac{\sin(x) - 1}{\sec(x)}}$$ ✓

or

$$\boxed{y = \cos(x)[\sin(x) - 1]}$$ ✓✓

or

$$\boxed{y = \cos(x)\sin(x) - \cos(x)}$$ ✓✓✓

8. Solve the following differential equation:
$ty' + (t+2)y = te^{t^3-t}$, with $y(1) = \frac{1}{3}$.

$1 \cdot y' + P(t)\, y = Q(t)$

$y' + \boxed{\frac{t+2}{t}}\, y = \frac{\cancel{t} e^{t^3-t}}{\cancel{t}}$

P(t)

$I.F = e^{\int P(t)\,dt} = e^{\int \frac{t+2}{t} dt} = e^{\int 1+\frac{2}{t} dt} = e^{t+2\ln t}$

$= e^{t} \cdot e^{2\ln t} = e^{t} \cdot e^{\ln t^2} = \underline{e^{t} \cdot t^2}$

$e^{t} \cdot t^2 \cdot y' + \left(1+\frac{2}{t}\right)\left(e^{t} \cdot t^2\right) y = \left(e^{t^3-t}\right)\left(e^{t} \cdot t^2\right)$

We need to use product rule

$\left(e^{t} \cdot t^2\right) y' + \left(e^{t} \cdot t^2 + 2te^{t}\right) y = \left(e^{t^3-t}\right)\left(e^{t} \cdot t^2\right)$

$\frac{d}{dt}\left[e^{t} t^2 y\right] = e^{t^3} \cdot \cancel{e^{-t}} \cdot \cancel{e^{t}} \cdot t^2$

$u = t^3$
$du = 3t^2\, dt$

$\int \frac{d}{dt}\left[e^{t} t^2 y\right] = \int e^{t^3} \cdot t^2\, dt$

$e^{t} t^2 y = \frac{1}{3}\int e^{u}\, du$

$e^{t} t^2 y = \frac{1}{3} e^{u} + C$

$e^{t} t^2 y = \frac{1}{3} e^{(t^3)} + C$

$\left[e^{1} \cdot (1)^2 \cdot \frac{1}{3} = \frac{1}{3} e^{1} + C\right] = \left[\frac{e}{3} = \frac{e}{3} + C\right]$ $\boxed{C = 0}$

$e^{t} \cdot t^2 \cdot y = \frac{1}{3} e^{t^3}$

$\boxed{y = \frac{1}{3} e^{t^3} \cdot \frac{1}{e^{t} \cdot t^2}}$

9. Find an implicit solution for $(e^y - 3xy)y' = y^2$.
(Hint: rewrite problem in terms of $x' = \frac{dx}{dy}$.)

$$y' = \frac{dy}{dx} \quad x' = \frac{dx}{dy}$$

$$y' = \frac{1}{x'}$$

$$(e^y - 3xy)\left(\frac{1}{x'}\right) = y^2$$

$$e^y - 3xy = y^2 x'$$

$$y^2 x' + 3xy = e^y$$

$$x' + \frac{3x}{y} = \frac{e^y}{y^2}$$

$$I.F = e^{\int P(y)dy} = e^{\int \frac{3}{y} dy} = e^{3\ln y} = e^{\ln y^3} = y^3$$

$$y^3 x' + 3y^2 x = e^y \cdot y$$

$$\frac{d}{dy}\left[y^3 \cdot x\right] = e^y \cdot y$$

$$\int \frac{d}{dy}\left[y^3 \cdot x\right] = \int e^y \cdot y \, dy$$

$$y^3 \cdot x = \int e^y \cdot y \, dy$$

integration by parts

$u = y \quad dv = e^y dy$

$du = dy \quad v = e^y$

$$y^3 \cdot x = y e^y - \int e^y dy$$

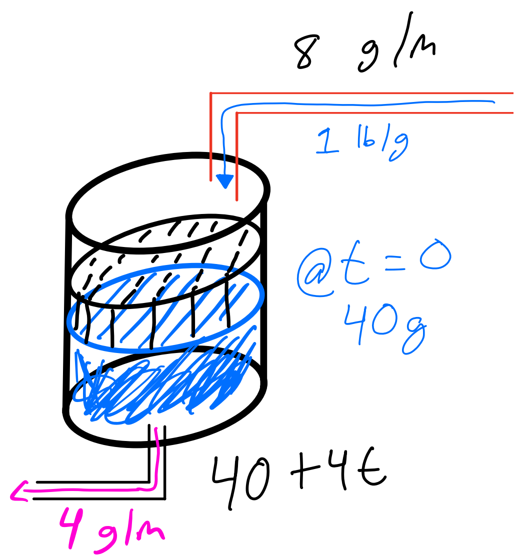
$$y^3 \cdot x = y \cdot e^y - e^y + C$$

$$\boxed{y^3 \cdot x = e^y(y-1) + C}$$

done, no conditions and implicit was asked for

10. A large tank is initially filled with 40 gallons of pure water. At time t = 0 valves are opened, one pours brine into the tank at a rate of 8 gallons per minute with 1 pound of salt per gallon, the other empties well-mixed brine out at 4 gallons per minute. How much salt is in the tank at t = 10 minutes?

$\rightarrow s(0) = 0$



$s(t)$ = amount of salt in tank

$s'(t)$ = change of salt in tank

change = (amt in) − (amt out)

$\underbrace{\text{(amt in)}}_{\text{(rate)(concentration)}} - \underbrace{\text{(amt out)}}_{\text{(rate)(concententration)}}$

$$s'(t) = 8 \cdot 1 - 4\left[\frac{s(t)}{40+4t}\right]$$

$$s'(t) = \frac{4s(t)}{40+4t} = 8 \rightarrow s'(t) + \frac{s(t)}{10+t} = 8$$

$$I.F = e^{\int P(t)\,dt} = e^{\int \frac{1}{10+t}dt} = e^{\ln(10+t)} = 10+t$$

$$(10+t)s'(t) + 1 \cdot s(t) = 8(10+t)$$

$$\frac{d}{dt}$$

$$\int \frac{d}{dt}\left[(10+t)\,s(t)\right] = \int [80+8t]\,dt$$

$$[10+t\cdot s(t)] = 80t + \frac{8t^2}{2} + C$$

$$(10+0)\cdot 0 = 80(0) + 4(0)^2 + C$$

$$\boxed{0 = C}$$

$$(10+t)\,s(t) = 80t + 4t^2$$

$$s(t) = \frac{80t+4t^2}{10+t}$$

$$s(10) = \frac{80(10)+4(10)^2}{10+10} = 60$$

$$\boxed{s(10) = 60 \text{ lb}}$$