Quiz 5 problem bank

1. A spring-mass-damper system is set up with a spring constant $k = 2$, a mass $m = 3$, and a damping constant $c = 5$. Given that $x(0) = 4$ and $x'(0) = 0$, find x. Also classify it as undamped, under damped, critically damped, or over damped.
2. A spring-mass-damper system is set up with a spring constant $k = 9$ and a mass $m = 49$. Determine what the damping constant, c, on the damper should be set to so that the system will be critically damped. Also, give the general form for motion of the spring in this system.
3. A spring-mass system (with no damper) initially at rest ($x(0) = x'(0) = 0$) has an oscillating external force acting on it and satisfies the differential equation $x'' + x = 2\sin(t)$. Determine $x(t)$.
4. Find the solution for $y'' - y = 16e^{3x}$ with $y(0) = 1$ and $y'(0) = 5$.
5. Find the general form for solutions to the differential equation $y'' + 3y' = 18x + 4e^{-2x}$.
6. Given that $y_p = (5x^3 - 3x^2)e^{2x}$ for the differential equation $y^{(3)} + Ay'' + By' + 12y = 150xe^{2x}$, with A and B not given, then find the general form for solutions.
7. Use *undetermined coefficients* to find the general form for solutions to $y'' - 3y' + 2y = 6e^{3x} + 12e^{-x}$.
8. Use *variation of parameters* to find the general form for solutions to $y'' - 3y' + 2y = 6e^{3x} + 12e^{-x}$.
9. Given $y_1 = \sec(\theta)$ and $y_2 = \tan(\theta)$ are solutions to $y'' - \tan(\theta)y' - \sec^2(\theta)y = 0$, find a particular solution for $y'' - \tan(\theta)y' - \sec^2(\theta)y = \sec(\theta)$.
10. Find the general form for solutions to $y'' + y = \sec^2(x)$.

The quiz will consist of two randomly chosen problems (with possibly small variations; so make sure to learn processes and not just memorize answers).

Key ideas and processes

- Spring-mass-damper systems are of the form $mx'' + cx' + kx = f(t)$; m is mass; c is damping constant; k is spring constant; and $f(t)$ is the external force.
  - *Undamped:* $c = 0$ or roots are pure imaginary; motion looks like a sine wave.
  - *Underdamped:* $c^2 < 4mk$ or complex roots of form $a \pm bi$; motion looks like a decaying sine wave.
  - *Critically damped:* $c^2 = 4mk$ or repeated real roots; motion looks like decaying exponential. ("fastest return to 0")
  - *Overdamped:* $c^2 > 4mk$ or distinct real roots; motion looks like decaying exponential.
- *Undetermined coefficients* – finding y_p for constant-coefficient linear nonhomogeneous ODE where the $f(x)$ has pieces like $x^k e^{\alpha x}\cos(\beta x)$.
  1. Write y_p as linear combination with unknown constants by taking (1) each part of $f(x)$; (2) if needed, add parts with lower powers of x; (3) make sure sine and cosine parts come in pairs.
  2. Bump up pieces of y_p until no overlap with y_c.
  3. Feed y_p into the differential equation. Set coefficients equal and solve for constants.
- *Variation of parameters* – finding y_p for $y'' + p(x)y' + q(x)y = f(x)$.
  1. Find y_1, y_2 for homogeneous ODE.
  2. Find Wronskian $w(y_1, y_2)$.
  3. Use formula

$$y_p = -y_1 \int \frac{y_2 f}{w(y_1,y_2)}\,dx + y_2 \int \frac{y_1 f}{w(y_1,y_2)}\,dx$$

Comments and (partial) answers on problems

1. $x = 12e^{-(2/3)t} - 8e^{-t}$; over-damped
2. $c = 42$; $x = Ae^{-(3/7)t} + Bte^{-3/7}t$
3. $x = \sin(t) - t\cos(t)$
4. $y = 2e^{3x} - e^{x}$
5. $y = A + Be^{-3x} + 3x^2 - 2x - 2e^{-2x}$
6. $y = (5x^3 - 3x^2 + Ax + B)e^{2x} + Ce^{-3x}$
7. $y = Ae^{x} + Be^{2x} + 2e^{-x} + 3e^{3x}$
8. $y = Ae^{x} + Be^{2x} + 2e^{-x} + 3e^{3x}$
9. $y_p = \tan(\theta)\ln\left|\sec(\theta) + \tan(\theta)\right| - \sec(\theta)\ln\left|\sec(\theta)\right|$
10. $y = A\cos(x) + B\sin(x) + \sin(x)\ln\left|\sec(x) + \tan(x)\right| - 1$

1. A spring-mass-damper system is set up with a spring constant $k = 2$, a mass $m = 3$, and a damping constant $c = 5$. Given that $x(0) = 4$ and $x'(0) = 0$, find x. Also classify it as undamped, under damped, critically damped, or over damped.

$Mx'' + cx' + kx = f(t)$

$3x'' + 5x' + 2x = 0$ * assume external force = 0

homogeneous equation

$3r^2 + 5r + 2 = 0$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{25 - 4(3)(2)}}{6} = \frac{-5 \pm \sqrt{1}}{6}$$

$$\frac{-5-1}{6} \text{ \& } \frac{-5+1}{6}$$

$r = -1$ & $-\frac{2}{3}$

Use Elimination

$$x = Ce^{-1t} + De^{-2/3 t}$$

$(2)\ x(0) = 4 = C + D(2) \rightarrow 8 = 2C + 2D$

$8 = -C$ $\boxed{C = -8}$

$4 = -8 + D$ $\boxed{D = 12}$

$$x' = -Ce^{-t} - \frac{2}{3} De^{-2/3 t}$$

$x'(0) = -C - \frac{2}{3} D \rightarrow 0 = -3C - 2D$

$$\boxed{x = -8e^{-t} + 12e^{-2/3 t}}$$

✓ ✓ $\boxed{\text{OVER}}$

C = constant = -5

$$\frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$c^2 - 4mk > 0 \rightarrow$ overdamped

$c^2 - 4mk = 0 \rightarrow$ critically damped, repeated roots

$c^2 - 4mk < 0 \rightarrow$ imaginary roots = under or undamped

2. A spring-mass-damper system is set up with a spring constant $k = 9$ and a mass $m = 49$. Determine what the damping constant, c, on the damper should be set to so that the system will be critically damped. Also, give the general form for motion of the spring in this system.

MCK with a C

$mx'' + cx' + kx = f(t)$

critically damped $= 0$

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$c^2 - 4mk = 0$

$c^2 = 4mk$

$c^2 = 4(49)(9)$

$c = \sqrt{(4)(49)(9)}$

$c = (2)(7)(3) = \boxed{42}$

$$r = \frac{-c \pm \sqrt{0}}{2m} = \frac{-\cancel{42}^{\,6\,3}}{2(\cancel{49})_{7}} = -\frac{3}{7}$$

$r = -\frac{3}{7}, -\frac{3}{7}$

$X = Ae^{-3/7t} + Bte^{-3/7t}$

3. A spring-mass system (with no damper) initially at rest ($x(0) = x'(0) = 0$) has an oscillating external force acting on it and satisfies the differential equation $x'' + x = 2\sin(t)$. Determine $x(t)$.

$C = 0$, no damper

NON HOMOGENEOUS

$x'' + x = 2\sin(t)$

$X = x_p + x_c$

to find x_c, make it homogeneous

$x_c: \ x'' + x = 0$

$r^2 + 1 = 0$

$\boxed{r = 0 \pm i}$

$x_c = Ae^{0t}\sin t + Be^{0t}\cos t$

$\boxed{x_c = A\sin t + B\cos t}$

$x_p = (A_1 \sin t + B_1 \cos t)t$ random constants

bump it up so x_c & x_p aren't the same

$x_p = A_1 t(\sin t) + B_1 t(\cos t)$ use product rule

$x_p' = A_1 t\cos t + A_1 \sin t + B_1 \cos t + B_1 t(-\sin t)$

$x_p'' = A_1 \cos t + A_1 t(-\sin t) + A_1 \cos t - B_1 \sin t + B_1(-\sin t) + B_1 t(-\cos t)$

$= 2A_1 \cos t - \cancel{A_1 t\sin t} - 2B_1 \sin t - \cancel{B_1 t\cos t} + \cancel{A_1 t\sin t} + \cancel{B_1 t\cos t} = 2\sin t$

$2A_1 \cos t - 2B_1 \sin t = 2\sin t + 0\cos t$

$-2B_1 = 2$ $\quad$ $2A_1$ must $= 0$

$B_1 = -1$ $\quad$ $A_1 = 0$ $\quad$ $\boxed{x_p = -t\cos t}$

$x = A\sin t + \cancel{B\cos t} - t\cos t$

$x(0) = 0 = 0 + B - 0$

$\boxed{B = 0}$

$x = A\sin t - t\cos t$

$x' = A\cos t - \cos t - (t(\sin t))$

$= A\cos t - \cos t + t\sin t$

$x'(0) = 0 = A - 1 + 0$ $\quad$ $\boxed{A = 1}$

$$\boxed{x = \sin t - t\cos t}$$

4. Find the solution for $y'' - y = 16e^{3x}$ with $y(0) = 1$ and $y'(0) = 5$.

non homogeneous $y = y_p + y_c$

☆ you have to start with y_c

y_c: $y'' - y = 0$

$r^2 - 1 = 0$

$\boxed{r = \pm 1}$ or $1, -1$

$y_c = Ae^{x} + Be^{-x}$

no similarities

$y_p = A_1 e^{3x}$

$y_p' = 3A_1 e^{3x}$

$y_p'' = 9A_1 e^{3x}$

$9A_1 e^{3x} - A_1 e^{3x} = 16e^{3x}$

$8A_1 e^{3x} = 16e^{3x}$

$8A_1 = 16$ $\boxed{A_1 = 2}$

$\boxed{y_p = 2e^{3x}}$

$y = Ae^{x} + Be^{-x} + 2e^{3x}$

$y(0) = 1 + A + B + 2 \rightsquigarrow \boxed{A + B = -1}$

$e^0 = 1$

$y' = Ae^{x} - Be^{-x} + 6e^{3x}$

$y'(0) = 5 = A - B + 6$ $\boxed{A - B = -1}$

$$\begin{aligned} A + B &= -1 \\ + \quad A - B &= -1 \\ \hline 2A &= -2 \end{aligned}$$

$\boxed{A = -1}$

$\boxed{B = 0}$

$$\boxed{y = -e^{x} + 2e^{3x}}$$

5. Find the general form for solutions to the differential equation $y'' + 3y' = 18x + 4e^{-2x}$.

$y = y_p + y_c$

Use undetermined coefficients

$y_c: \; y'' + 3y' = 0$

$r^2 + 3r = 0$

$r: 0, -3$

$y = Ae^{0x} + Be^{-3x} = A + Be^{-3x}$

$y_p = A_1x^1 + B_1x^0 + C_1e^{-2x}$ $\quad A_1x^2 + B_1x + C_1e^{-2x}$

$= (A_1x + B_1)x$ -BUMP it up

$= A_1x^2 + B_1x$

$y_p' = 2A_1x + B_1 - 2C_1e^{-2x}$

$y_p'' = 2A_1 + [-2C_1][-2]e^{-2x} = 2A_1 + 4C_1e^{-2x}$

$2A_1 + 4C_1e^{-2x} + 3[2A_1x + B_1 - 2C_1e^{-2x}] = 18x + 4e^{-2x}$

$2A_1 + 4C_1e^{-2x} + 6A_1x + 3B_1 - 6C_1e^{-2x} = 18x + 4e^{-2x}$

$x[6A_1] + e^{-2x}[4c_1 - 6c_1] + (2A_1 + 3B_1) = 18x + 4e^{-2x}$

$6A_1 = 18 \quad \boxed{A_1 = 3}$

$4c_1 - 6c_1 = 4 \quad -2c_1 = 4 \quad \boxed{c_1 = -2}$

$2A_1 + 3B = 0 \rightarrow 2(3) + 3B_1 = 0 \quad \boxed{B_1 = -2}$

$y_p = 3x^2 - 2x - 2e^{-2x}$ $\quad \boxed{y = 3x^2 - 2x - 2e^{-2x} + A + Be^{-3x}}$

6. Given that $y_p = (5x^3 - 3x^2)e^{2x}$ for the differential equation $y^{(3)} + Ay'' + By' + 12y = 150xe^{2x}$, with A and B not given, then find the general form for solutions.

NON HOMOGENEOUS

$Y = Y_p + Y_c$

Y_c: $y''' + Ay'' + By' + 12y = 0$

$r^3 + Ar^2 + Br + 12 = 0$ two unknowns

(our) $Y_p = Axe^{2x} + Bx^0e^{2x}$ this is what we would normally do,

$= \left[A_1xe^{2x} + B_1e^{2x}\right]x^2$ but look at the actual Y_p given.

(given) $Y_p = (5x^3 - 3x^2)e^{2x} = 5x^3e^{2x} - 3x^2e^{2x}$

bump Y_p 2 times to make Y_c look like Y_p

$Y_c = A_2xe^{2x} + B_2e^{2x} + (\ldots\ldots)$

$r: 2, 2$

$(r-2)(r-2)(r-\lambda) = r^3 + Ar^2 + Br + 12$

$(r^2 - 4r + 4)(r - \lambda) = r^3 + Ar^2 + Br + 12$

$r^3 - \lambda r^2 - 4r^2 + 4r\lambda + 4r - 4\lambda = r^3 + Ar^2 + Br + 12$

$r^3 + (-\lambda - 4)r^2 + (4\lambda + 4)r - 4\lambda = r^3 + Ar^2 + Br + 12$

$\boxed{-4\lambda = 12}$ $\boxed{\lambda = -3}$ $r: 2, 2, -3$

$Y_c = A_3xe^{2x} + Be^{2x} + Ce^{-3x}$

$Y = A_3xe^{2x} + Be^{2x} + Ce^{-3x} + (5x^3 - 3x^2)e^{2x}$

7. Use *undetermined coefficients* to find the general form for solutions to $y'' - 3y' + 2y = 6e^{3x} + 12e^{-x}$.

$$Y = Y_p + Y_c$$

Y_c: $y'' - 3y' + 2y = 0$

$$r^2 - 3r + 2 = 0$$

$$(r-2)(r-1) = 0$$

$$\boxed{r: 2, 1}$$

$$Y_c = Ae^{2x} + Be^{x}$$

$$Y_p = A_1 e^{3x} + B_1 e^{-x}$$

$$Y_p' = 3A_1 e^{3x} - B_1 e^{-x}$$

$$Y_p'' = 9A_1 e^{3x} + B_1 e^{-x}$$

$$(9A_1 e^{3x} + B_1 e^{-x}) - 3(3A_1 e^{3x} - B_1 e^{-x}) + 2(A_1 e^{3x} + B_1 e^{-x}) = 6e^{3x} + 12e^{-x}$$

$$\cancel{9A_1 e^{3x}} + B_1 e^{-x} - \cancel{9A_1 e^{3x}} + 3B_1 e^{-x} + 2A_1 e^{3x} + 2B_1 e^{-x} = 6e^{3x} + 12e^{-x}$$

$$e^{3x}[2A_1] + e^{-x}[B_1 + 3B_1 + 2B_1] = 6e^{3x} + 12e^{-x}$$

$$2A_1 = 6 \qquad \boxed{A_1 = 3}$$

$$6B_1 = 12 \qquad \boxed{B_1 = 2}$$

$$Y_p = 3e^{3x} + 2e^{-x}$$

$$\boxed{Y = 3e^{3x} + 2e^{-x} + Ae^{2x} + Be^{x}}$$

8. Use *variation of parameters* to find the general form for solutions to $y'' - 3y' + 2y = 6e^{3x} + 12e^{-x}$. $f(x)$

(Same equation as #7)

$$\boxed{y_p = -y_1 \int \frac{y_2 f}{w(y_1, y_2)}\,dx + y_2 \int \frac{y_1 f}{w(y_1, y_2)}\,dx}$$

y_c : $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$ $\boxed{r : 2, 1}$

$$y = A\underbrace{e^{2x}}_{y_1} + B\underbrace{e^{x}}_{y_2}$$

$$w = \begin{vmatrix} e^{2x} & e^{x} \\ 2e^{2x} & e^{x} \end{vmatrix} = e^{2x}e^{x} - 2e^{2x}e^{x} = e^{3x} - 2e^{3x}$$

2×2

$$\boxed{w = -e^{3x}}$$

$$y_p = -e^{2x}\int \frac{e^{x}[6e^{3x} + 12e^{-x}]}{-e^{3x}}\,dx + e^{x}\int \frac{e^{2x}[6e^{3x} + 12e^{-x}]}{-e^{3x}}\,dx$$

$$= e^{2x}\int e^{-2x}[6e^{3x} + 12e^{-x}]\,dx - e^{x}\int e^{-x}[6e^{3x} + 12e^{-x}]\,dx$$

$$= e^{2x}\int (6e^{x} + 12e^{-3x})\,dx - e^{-x}[6e^{2x} + 12e^{-2x}]\,dx$$

$$= e^{2x}[6e^{x} - 4e^{-3x} + C]\,dx - e^{x}[3e^{2x} - 6e^{-2x} + D]\,dx$$

Constants don't matter

$$= 6e^{3x} - 4e^{-x} + Ce^{2x} - 3e^{3x} + 6e^{-x} + De^{x}$$

$$y = \underbrace{3e^{3x} + 2e^{-x}}_{y_p} + \underbrace{Ce^{2x} - De^{x}}_{y_c}$$

$$\boxed{y = 3e^{3x} + 2e^{x}}$$ ✓ ✓

9. Given $\underline{y_1 = \sec(\theta)}$ and $y_2 = \tan(\theta)$ are solutions to $y'' - \tan(\theta)y' - \sec^2(\theta)y = 0$, find a particular solution for $y'' - \tan(\theta)y' - \sec^2(\theta)y = \underline{\sec(\theta)}$. $\to f(x)$

$$y = y_p + y_c$$

$$W = \begin{vmatrix} \sec\theta & \tan\theta \\ \sec\theta\tan\theta & \sec^2\theta \end{vmatrix} = \sec^3\theta - \sec\theta\tan^2\theta$$

$$= \sec\theta\,[\underbrace{\sec^2\theta - \tan^2\theta}_{=1}] \boxed{= \sec\theta}$$

$$y_p = -\sec\theta \int \frac{\tan\theta\,\cancel{\sec\theta}}{\cancel{\sec\theta}}\,d\theta + \tan\theta \int \frac{\sec\theta\,\cancel{\sec\theta}}{\cancel{\sec\theta}}\,d\theta$$

$$y_p = -\sec\theta \int \tan\theta\,d\theta + \tan\theta \int \sec\theta\,d\theta$$

$$y_p = -\sec\theta \ln|\sec\theta| + \ln|\tan\theta|$$

10. Find the general form for solutions to
$y'' + y = \sec^2(x)$.

$$Y = Y_p + Y_c$$

$$y'' + y = 0 \qquad y = A\underbrace{\cos x}_{Y_2} + B\underbrace{\sin x}_{Y_1}$$

$$r^2 + 1 = 0$$

$$\boxed{r = \pm i}$$

$$W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = \boxed{-1}$$

$$Y_p = -\sin x \int \frac{\cos x \sec^2 x}{-1}\,dx + \cos x \int \frac{\sin x \sec^2 x}{-1}\,dx$$

$$Y_p = \sin x \int \sec x \,dx - \cos x \int \tan x \sec x \,dx$$

$$Y_p = \sin x \ln|\sec x + \tan x| - \cos x \sec x$$

$$Y_p = \sin x \ln|\sec x + \tan x| - 1$$

$$\boxed{Y = \sin x \ln|\sec x + \tan x| - 1 + A\cos x + B\sin x}$$