Quiz 7 problem bank

1. Verify $\mathbf{x}_1 = \begin{pmatrix} 3t^2 \\ 5t \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} t^{-6} \\ -t^{-7} \end{pmatrix}$ are solutions to $t^2\mathbf{x}' = \begin{pmatrix} -3t & 3t^2 \\ 5 & -2t \end{pmatrix}\mathbf{x}$.

2. Let $\mathbf{x}_1(t) = \begin{pmatrix} t^2 \\ t+3 \end{pmatrix}$ and $\mathbf{x}_2(t) = \begin{pmatrix} t+2 \\ 2 \end{pmatrix}$ be solutions to $\mathbf{x}' = A(t)\mathbf{x}$. Find the solution with $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, and determine the interval on which uniqueness is *guaranteed* by the Wronskian.

3. Let $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{x}_2(t) = e^t \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$, $\mathbf{x}_3(t) = e^t \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ be solutions to $\mathbf{x}' = A\mathbf{x}$. Find the solution with $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 7 \\ -9 \end{pmatrix}$.

4. Find the general form for solutions to the system $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\mathbf{x}$.

5. Find the general form for solutions to $\mathbf{x} = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\mathbf{x}'$. (Hint: $A^{-1}A = I$.)

6. Find the solution to the system $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 6 & 0 \end{pmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$.

7. Given $\mathbf{x}_1 = e^t \begin{pmatrix} \cos(4t) - 3\sin(4t) \\ 2\cos(4t) + 2\sin(4t) \end{pmatrix}$ is *a* solution to $\mathbf{x}' = A\mathbf{x}$, find *the* solution with $\mathbf{x}(0) = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$.

8. Find the solution to the system $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

9. Find the general form for solutions to the system $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\mathbf{x}$.

10. Given $\mathbf{x}_1 = e^{5t}\begin{pmatrix} 1-2t \\ t \end{pmatrix}$ is *a* solution to $\mathbf{x}' = A\mathbf{x}$, find *the* solution with $\mathbf{x}(0) = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$.

The quiz will consist of two randomly chosen problems (with possibly small variations; so make sure to learn processes and not just memorize answers).

Key ideas and processes

- Many of the same ideas translate into linear systems, $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$, e.g. verifying, superposition, $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p$, solving initial conditions, ... The Wronskian is now $w(\mathbf{x}_1, \ldots \mathbf{x}_n) = \left| \mathbf{x}_1 \ \cdots \ \mathbf{x}_n \right|$ (determinant where columns are the functions).
- For $\mathbf{x}' = A\mathbf{x}$, solutions take form $c_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n$, where $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ (e.g. these are eigenvalues and eigenvectors).
- To find eigenvalues find the roots of the polynomial $|A - \lambda I| = 0$ (found taking determinant of matrix A with "$-\lambda$" added to diagonal terms). Note: sum of eigenvalues is sum of diagonal entries; product of eigenvalues is determinant. To find eigenvector for λ_i find non-zero vector with $(A - \lambda_i I)\mathbf{v}_i = \mathbf{0}$. For 2×2 this is done quickly by noting if $A - \lambda I = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, then desired vector is either of $\left(\begin{smallmatrix} \beta \\ -\alpha \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \delta \\ -\gamma \end{smallmatrix}\right)$
- Complex eigenvalues/eigenvectors occur in conjugate pairs. Can use this to rewrite solutions with sine and cosine. Namely if $\lambda = \alpha + \beta i$ and $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ then taking real parts give $\left(e^{\alpha t}\cos(\beta t)\mathbf{a} - e^{\alpha t}\sin(\beta t)\mathbf{b}\right)$ and imaginary parts give $\left(e^{\alpha t}\cos(\beta t)\mathbf{b} + e^{\alpha t}\sin(\beta t)\mathbf{a}\right)$.
- When shortage of eigenvectors we "bump". So if $\mathbf{v}$ is eigenvector (e.g. $(A - \lambda I)\mathbf{v} = 0$), find $\mathbf{w}$ with $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then two solutions are $e^{\lambda t}\mathbf{v}$ and $e^{\lambda t}(t\mathbf{v} + \mathbf{w})$. This can be generalized when even more eigenvectors are missing.

Comments and (partial) answers on problems

1. (plug it in and verify)
2. $\begin{pmatrix} 4 + 2t - t^2 \\ 1 - t \end{pmatrix}$; $-1 < t < 6$
3. $\begin{pmatrix} 3 + 2e^t \\ 3 + 4e^t \\ -3 - 6e^t \end{pmatrix}$
4. $\mathbf{x} = Ae^t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{5t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$
5. $\mathbf{x} = Ae^{\frac{1}{2}t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + Be^t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
6. $\mathbf{x} = \begin{pmatrix} 3e^{-2t} + 4e^{3t} \\ -9e^{-2t} + 8e^{3t} \end{pmatrix}$
7. $\mathbf{x} = e^t\begin{pmatrix} 7\cos(4t) - \sin(4t) \\ -2\cos(4t) + 6\sin(4t) \end{pmatrix}$
8. $\mathbf{x} = \begin{pmatrix} 2\cos(t) + 3\sin(t) \\ \cos(t) - 5\sin(t) \end{pmatrix}$
9. $\mathbf{x} = Ae^{2t}\begin{pmatrix} t \\ t+1 \end{pmatrix} + Be^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
10. $\mathbf{x} = e^{5t}\begin{pmatrix} 8 - 4t \\ -3 + 2t \end{pmatrix}$

1. Verify $\mathbf{x}_1 = \begin{pmatrix} 3t^2 \\ 5t \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} t^{-6} \\ -t^{-7} \end{pmatrix}$ are solutions to $t^2\mathbf{x}' = \begin{pmatrix} -3t & 3t^2 \\ 5 & -2t \end{pmatrix}\mathbf{x}$.

plug it in

$$x_1: \text{LHS}: t^2 x_1' = t^2 \begin{pmatrix} 6t \\ 5 \end{pmatrix} = \begin{pmatrix} 6t^3 \\ 5t^2 \end{pmatrix}$$

$$\text{RHS}: \begin{pmatrix} -3t & 3t^2 \\ 5 & -2t \end{pmatrix} \begin{pmatrix} 3t^2 \\ 5t \end{pmatrix} = \begin{pmatrix} -9t^3 + 15t^3 \\ 15t^2 - 10t^2 \end{pmatrix} = \begin{pmatrix} 6t^3 \\ 5t^2 \end{pmatrix}$$

$$x_2: \text{LHS}: t^2 x_2' = t^2 \begin{pmatrix} -6t^{-7} \\ 7t^{-8} \end{pmatrix} = \begin{pmatrix} -6t^{-5} \\ 7t^{-6} \end{pmatrix}$$

$$\text{RHS}: \begin{pmatrix} -3t & 3t^2 \\ 5 & -2t \end{pmatrix} \begin{pmatrix} t^{-6} \\ -t^{-7} \end{pmatrix} = \begin{pmatrix} -3t^{-5} - 3t^{-5} \\ 5t^{-6} + 2t^{-6} \end{pmatrix} = \begin{pmatrix} -6t^{-5} \\ 7t^{-6} \end{pmatrix}$$

DONE

2. Let $\mathbf{x}_1(t) = \begin{pmatrix} t^2 \\ t+3 \end{pmatrix}$ and $\mathbf{x}_2(t) = \begin{pmatrix} t+2 \\ 2 \end{pmatrix}$ be solutions to $\mathbf{x}' = A(t)\mathbf{x}$. Find the solution with $\mathbf{x}(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, and determine the interval on which uniqueness is *guaranteed* by the Wronskian.

$X = Ax_1 + Bx_2$ ← this is superposition

$$X = A\begin{pmatrix} t^2 \\ t+3 \end{pmatrix} + B\begin{pmatrix} t+2 \\ 2 \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix} = A\begin{pmatrix} 0 \\ 3 \end{pmatrix} + B\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3A \end{pmatrix} + \begin{pmatrix} 2B \\ 2B \end{pmatrix} = \begin{pmatrix} 2B \\ 3A+2B \end{pmatrix}$$

$2B = 4$

$\boxed{B = 2}$

$3A + 2B = 1$

$3A + 2(2) = 1$

$\boxed{A = -1}$

$$\boxed{X = -\begin{pmatrix} t^2 \\ t+3 \end{pmatrix} + 2\begin{pmatrix} t+2 \\ 2 \end{pmatrix}} \text{ or } \begin{pmatrix} -t^2 \\ -t-3 \end{pmatrix} + \begin{pmatrix} 2t+4 \\ 4 \end{pmatrix} = \boxed{\begin{pmatrix} -t^2+2t+4 \\ -t+1 \end{pmatrix}}$$

$$W = \begin{vmatrix} t^2 & t+2 \\ t+3 & 2 \end{vmatrix} = 2t^2 - (t+3)(t+2)$$

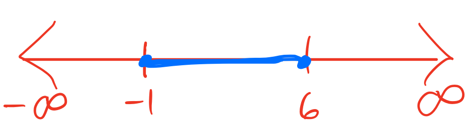
$$= 2t^2 - t^2 - 3t - 2t - 6$$

$$= t^2 - 5t - 6$$

$0 = t^2 - 5t - 6$

$0 = (t-6)(t+1)$

$\boxed{t = -1, 6}$



$\boxed{\text{Interval} = -1, 6}$

3. Let $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{x}_2(t) = e^t \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$, $\mathbf{x}_3(t) = e^t \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ be solutions to $\mathbf{x}' = A\mathbf{x}$. Find the solution with $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 7 \\ -9 \end{pmatrix}$.

initial conditions

use super position

$$x = Ax_1 + Bx_2 + Cx_3$$

$$x = A\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + B\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + C\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

easier to combine into one matrix

$$x = \begin{pmatrix} A + Be^t - Ce^t \\ A + 0 + 2Ce^t \\ -A - 2Be^t + Ce^t \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 5 \\ 7 \\ -9 \end{pmatrix} = \begin{pmatrix} A + B - C \\ A + 2C \\ -A - 2B + C \end{pmatrix}$$

start here add these together

$$\begin{aligned} 5 &= A + B - C \\ -9 &= -A - 2B + C \\ \hline -4 &= -B \end{aligned} \qquad \boxed{B = 4}$$

$$A + 4 - C = 5$$

$$\boxed{A - C = 1}$$

$$A - 2 = 1$$

$$\boxed{A = 3}$$

$$\begin{aligned} &(A - C = 1) \\ - &(A + 2C = 7) \\ \hline &-3 = -6 \end{aligned} \qquad \boxed{C = 2}$$

$$x = \begin{pmatrix} 3 + 4e^t - 2e^t \\ 3 + 4e^t \\ -3 - 8e^t + 2e^t \end{pmatrix}$$

or

$$x = \begin{pmatrix} 3 + 2e^t \\ 3 + 4e^t \\ -3 - 6e^t \end{pmatrix}$$ ✓

4. Find the general form for solutions to the system
$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \mathbf{x}.$

$x = e^{rt}$

$x = e^{\lambda t}$

$x = v \cdot e^{\lambda t}$

$\lambda I = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$\underbrace{|A - \lambda I| = 0}$ this must be true to find eigenvalues

$$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{pmatrix} = 0$$

Find eigenvalue

$(4-\lambda)(2-\lambda) - 3 = 0$

$8 - 2\lambda - 4\lambda + \lambda^2 - 3 = 0$

$\lambda^2 - 6\lambda + 5 = 0$

$(\lambda - 5)(\lambda - 1) = 0$

$\boxed{\lambda = 5, 1}$

Find eigenvector

$(A - \lambda I)(v) = 0 \quad \lambda = 5$

$$\begin{pmatrix} 2-5 & 1 \\ 3 & 4-5 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ +3 \end{pmatrix}}_{\text{eigenvector for } \lambda = 5} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(A - \lambda I)(v) = 0$

$$\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\text{eigenvector for } \lambda = 1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{x = Ae^{5t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + Be^{t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

✓ ✓

5. Find the general form for solutions to $\mathbf{x} = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} \mathbf{x}'$. (Hint: $A^{-1}A = I$.)

$x' = Ax$

identity matrix ex. $Ix = x$

$x = Ax'$

$x = Ax'$

$A^{-1}x = A^{-1}Ax'$

$A^{-1}x = Ix' = x'$

$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} x = x'$

$\frac{1}{4(-1)-(-3)2} = \frac{1}{-4+6} \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix} x = x'$

$\frac{1}{2}\begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix} x = x'$

$\begin{pmatrix} -1/2 & 3/2 \\ -1 & 2 \end{pmatrix} x = x'$

$|A - \lambda I| = 0$

$= \begin{vmatrix} -1/2 - \lambda & 3/2 \\ -1 & 2-\lambda \end{vmatrix}$

$0 = \left(-\frac{1}{2} - \lambda\right)(2-\lambda) - \left(-3/2\right)$

$0 = -1 - 2\lambda + \frac{\lambda}{2} + \lambda^2 + \frac{3}{2}$

$0 = -2 - 4\lambda + \lambda + 2\lambda^2 + 3$

$2\lambda^2 - 3\lambda + 1 = 0$

$(2\lambda - 1)(\lambda - 1) = 0$

$\boxed{\lambda = \left(\frac{1}{2}, 1\right)}$

@ $\lambda = 1/2$

$\begin{pmatrix} -1 & 3/2 \\ -1 & 3/2 \end{pmatrix} \underbrace{\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}}_{\text{eigenvector @ } 1/2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

@ $\lambda = 1$

$\begin{pmatrix} -3/2 & 3/2 \\ -1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{eigenvector @ } 1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x = Ae^{1/2 t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + Be^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✓✓

6. Find the solution to the system $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 6 & 0 \end{pmatrix} \mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$.

$$|A - \lambda I| = 0 = \begin{vmatrix} 1-\lambda & 1 \\ 6 & 0-\lambda \end{vmatrix} = (1-\lambda)(0-\lambda) - 6$$
$$= -\lambda + \lambda^2 - 6 = 0$$
$$= (\lambda - 3)(\lambda + 2) = 0$$
$$\boxed{\lambda = 3, 2}$$

@ $\lambda = -2$

$$\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} -1 \\ 3 \end{pmatrix}}_{\text{eigenvector}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

@ $\lambda = 3$

$$\begin{pmatrix} -2 & 1 \\ 6 & -3 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\text{eigenvector}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

note: use values that make initial matrix equal to 0

$$\boxed{X = Ae^{-2t}\begin{pmatrix} -1 \\ 3 \end{pmatrix} + Be^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

$$X = \begin{pmatrix} -Ae^{-2t} + Be^{3t} \\ 3Ae^{-2t} + 2Be^{3t} \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 7 \\ -1 \end{pmatrix} = \begin{pmatrix} -A + B \\ 3A + 2B \end{pmatrix}$$

$(-A + B = 7) \times 3$

$3A + 2B = -1$
$-3A + 3B = 21$

$5B = 20 \quad \boxed{B = 4}$

$-A + 4 = 7$
$\boxed{A = -3}$

$$\boxed{X = \begin{pmatrix} 3e^{-2t} + 4e^{3t} \\ -9e^{-2t} + 8e^{3t} \end{pmatrix}}$$

✓ ✓

7. Given $\mathbf{x}_1 = e^t \begin{pmatrix} \cos(4t) - 3\sin(4t) \\ 2\cos(4t) + 2\sin(4t) \end{pmatrix}$ is *a* solution to $\mathbf{x}' = A\mathbf{x}$, find *the* solution with $\mathbf{x}(0) = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$.

Complex eigenvalue $\lambda = a \pm bi$ $\quad v = (\alpha + \beta i)$

alpha beta

$$\left(e^{at}\cos bt + ie^{at}\sin bt\right)(\alpha + \beta i)$$

$$X = \left(A\left(e^{at}\cos bt \cdot \alpha - e^{at}\sin bt \cdot \beta\right) + B\left(e^{at}\cos bt \cdot \beta + e^{at}\sin bt \cdot \alpha\right)\right.$$

$$x_1 = \begin{pmatrix} e^t\cos(4t) - 3e^t\sin(4t) \\ 2e^t\cos 4t + 2e^t\sin(4t) \end{pmatrix}$$

$$= \begin{pmatrix} e^t\cos 4t \\ 2e^t\cos 4t \end{pmatrix} + \begin{pmatrix} -3e^t\sin 4t \\ 2e^t\sin 4t \end{pmatrix}$$

simplify this

$$= e^t\cos 4t\begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^t\sin 4t\begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$a = 1, \; b = 4$$

$$\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \alpha = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$x = A\left(e^t\cos 4t \cdot \begin{pmatrix} -3 \\ 2 \end{pmatrix} - e^t\cancel{\sin 4t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) + B\left(e^t\cos 4t\begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^t\cancel{\sin 4t}\begin{pmatrix} -3 \\ 2 \end{pmatrix}\right)$$

cos 1 = 1 $\quad$ sin 0 = 0 $\quad$ cos 1 = 1 $\quad$ sin 0 = 0

$$x(0) = \begin{pmatrix} 7 \\ -2 \end{pmatrix} = A\begin{pmatrix} -3 \\ 2 \end{pmatrix} + B\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 7 \\ -2 \end{pmatrix} = \begin{pmatrix} -3A + B \\ 2A + 2B \end{pmatrix}$$

$-3A + B = 7$

$2A + 2B = -2$

$6A - 2B = -14$

$8A = -16 \quad \boxed{A = -2}$

$2(-2) + 2B = -2$

$\boxed{B = 1}$

$$\boxed{\begin{aligned} x &= -2\left(e^t\cos 4t\begin{pmatrix} 3 \\ -2 \end{pmatrix} - e^t\sin 4t\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) \\ &\quad + \left(e^t\cos 4t\begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^t\sin 4t\begin{pmatrix} -3 \\ 2 \end{pmatrix}\right) \end{aligned}}$$

8. Find the solution to the system $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$$|A-\lambda I| = 0 = \begin{vmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} = (-1-\lambda)(1-\lambda) - (-2)$$

$$= -1 \cancel{-\lambda} + \cancel{\lambda} + \lambda^2 + 2 = 0$$

$$\lambda^2 + 1 = 0 \quad \boxed{\lambda = 0 \pm i}$$

@ $\lambda = 0 + i$

$$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ i-1 \end{pmatrix}}_{\text{Swap + negate}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(e^{0t}\cos t + ie^{0t}\sin t\right)\begin{pmatrix} 1 \\ i-1 \end{pmatrix} = \left(e^{0t}\cos t + ie^{0t}\sin t\right)\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$X = A\left[\cos t\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] + B\left[\cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

$$X = \begin{pmatrix} A\cos t + B\sin t \\ -A\cos t - A\sin t + B\cos t - B\sin t \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} A \\ -A + B \end{pmatrix} \quad \boxed{A = 2} \quad -A + B = 1 \quad \boxed{B = 3}$$

$$X = \begin{pmatrix} 2\cos t + 3\sin t \\ -2\cos t - 2\sin t + 3\cos t - 3\sin t \end{pmatrix} = \begin{pmatrix} 2\cos t + 3\sin t \\ \cos t - 5\sin t \end{pmatrix}$$

9. Find the general form for solutions to the system $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \mathbf{x}$.

$$|A-\lambda I| = 0 = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - (-1)$$
$$= 3 - 3\lambda - \lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$
$$(\lambda - 2)^2 = 0 \qquad \boxed{\lambda = 2, 2}$$

@ $\lambda = 2$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Swap + negate

⚠ cannot use same eigenvector for same eigenvalue, need to bump one up.

@ $\lambda = 2$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{w} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{v}$$

$\boxed{\text{What}}$ times this = this ?

w correction vector

or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\boxed{x = Ae^{2t} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{v} + Be^{2t}\left(t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{v} + \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{w}\right)}$$

✓✓

10. Given $\mathbf{x}_1 = e^{5t}\begin{pmatrix}1-2t\\ t\end{pmatrix}$ is *a* solution to $\mathbf{x}' = A\mathbf{x}$, find *the* solution with $\mathbf{x}(0) = \begin{pmatrix}8\\ -3\end{pmatrix}$.

W = correction vector

bump up eigenvector to get correction vector

$\lambda = 5, 5$

$$x_1 = e^{5t}\Big(t\underbrace{\begin{pmatrix}-2\\1\end{pmatrix}}_{v} + \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{w}\Big)$$

$$X = Ae^{5t}\begin{pmatrix}-2\\1\end{pmatrix} + Be^{5t}\Big(t\begin{pmatrix}-2\\1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}\Big)$$

$$X(0) = A\begin{pmatrix}-2\\1\end{pmatrix} + B\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-2A+B\\ A+0\end{pmatrix} = \begin{pmatrix}8\\-3\end{pmatrix}$$

$\boxed{A = -3}$

$-2(-3) + B = 8$

$-6 + B = 8$

$\boxed{B = 2}$

$$X = -3e^{5t}\begin{pmatrix}-2\\1\end{pmatrix} + 2e^{5t}\Big(t\begin{pmatrix}-2\\1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}\Big)$$