Quiz 6 problem bank

1. Two well-mixed tanks are hooked up so that water with 2 pounds of salt per gallon flows into the first tank, well-mixed water flows out of the second tank, water flows from the first tank to the second tank, and water flows from the second tank to the first tank. *All flows are constant*. Determine how much water is in each of the two tanks, given if x is the amount of salt in the first tank and y is the amount of salt in the second tank then $3x' = -x + y + 30$ and $6y' = 2x - 3y$.

2. Solve for x and y given that $x' = 2y$, $y' = 18x$, $x(0) = 5$, and $y(0) = 3$.

3. Write the third-order differential equation $x^{(3)} - 2tx'' + t^2x' - 5x = \arctan(t)$ as a first-order system with three equations.

4. Find A and B given that $x = Ce^{3t} + De^{-2t}$ is the general form for x for the system $x' = 5x - 2y$ and $y' = Ax + By$.

5. Find a differential equation x satisfies given $x' + 2y' = 4x + 5y$ and $2x' - y' = 3x$.

6. Find differential equations for both the functions x and y given $x' = 3x - y + t^2$ and $y' = x + y + e^{3t}$.

7. Find a differential equation y satisfies given $x'' + x' + 2y' - 3y = 0$ and $x' + x + y' - y = 0$.

8. Given that $x = Ae^{2t} + Be^{t} + Ce^{-t} + De^{-2t}$ is the general form for the solution for x for the system $x'' + x + y' - 3y = 0$ and $x' + 3x + y'' - 5y = 0$, find the corresponding solution for y. (Hint: both x and y satisfy the same ODE.)

9. Find $\begin{pmatrix} 1+t & 1-t \\ t & 2-t \end{pmatrix}^{-1}$.

10. Given $A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}$, find $AB^{-1} - BA^{-1}$.

The quiz will consist of two randomly chosen problems (with possibly small variations; so make sure to learn processes and not just memorize answers).

Key ideas and processes

- *Systems* of differential equations involve several functions which can interact with each other. Examples include springs linked in serial, multiple tanks mixing with fluid being pumped in between them, and so on.

- The *order* of the system is the highest derivative involved. There is a trad off between the order of a system and the number of equations involved. Generally speaking, we can have either high order with few functions, or low order with many functions.

  To go from a high-order differential equation to a first-order system, we introduce a series of new variables that are a "ladder" to our high derivative. Namely, if we have an n-th order differential equation in x then we introduce n variables by $x_1 (= x)$, $x_1' = x_2 (= x')$, $x_2' = x_3 (= x'')$, $\ldots$, $x_{n-1}' = x_n (= x^{(n-1)})$, and note that $x_n' = x^{(n)}$ so that all our derivatives can now be replaced.

- To solve a system of (linear = simple) differential equations there are two basic approaches to reduce to problems we have already done:

  - Substitution: Solve for one variable in terms of the other in one equation, and plug into the other equation. Solve the resulting ODE.
  - Elimination: Representing a derivative with a "D" we can represent differential equations using this D-notation by
  $$\begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$$
  where each L_i is a differential *operator* expressed as a polynomial in D. We then have
  $$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} x = \begin{vmatrix} f(t) & L_2 \\ g(t) & L_4 \end{vmatrix} \quad \text{and}$$
  $$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} y = \begin{vmatrix} L_1 & f(t) \\ L_3 & g(t) \end{vmatrix}$$

  When solving a system of equations the constants associated with the pieces might interact between the functions. To figure out the appropriate constants, plug into an equation(s) of the system and determine the relationships needed so that the solution is always valid.

- Linear systems can be represented by matrices ("arrays with benefits"). Matrices have many useful properties, adding, scaling, multiplying (when done correctly), inverses (in some cases), and more.

Comments and (partial) answers on problems

1. 45 gallons in the first tank; 30 gallons in the second
2. $x = 3e^{6t} + 2e^{-6t}$, $y = 9e^{6t} - 6e^{-6t}$
3. $x_1' = x_2$, $x_2' = x_3$, $x_3' = 2tx_3 - t^2x_2 - 5x_1 + \arctan(t)$
4. $A = 7$, $B = -4$
5. $x'' - 4x' + 3x = 0$
6. $x'' - 4x' + 4x = 2t - t^2 - e^{3t}$, $y'' - 4y' + 4y = t^2$
7. $y''' - 2y'' + 3y = 0$
8. $y = 5Ae^{2t} + Be^{t} + \frac{1}{2}Ce^{-t} + De^{-2t}$
9. $\frac{1}{2}\begin{pmatrix} 2-t & t-1 \\ -t & 1+t \end{pmatrix}$
10. $\begin{pmatrix} 16 & 10 \\ 14 & -16 \end{pmatrix}$

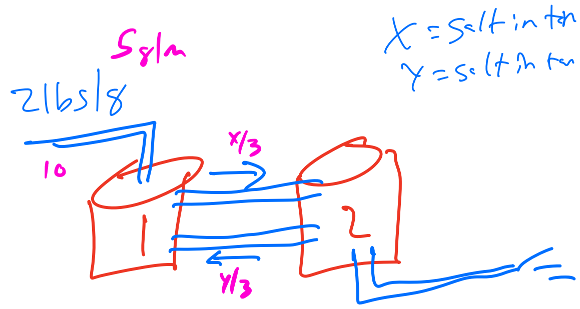
P1) $3x' = -x + y + 30$

$6y' = 2x - 3y$

All flows constant

x = salt in tank 1
y = salt in tank 2

$\boxed{x' \,\&\, y' \text{ represent change } (+\text{ in}) \quad (-\text{out})}$ $\rightarrow$



$x' = \dfrac{-x + y + 30}{3}$ $\qquad$ $y' = \dfrac{2x - 3y}{6}$

$x' = \dfrac{-x}{3} + \dfrac{y}{3} + 10$ $\qquad$ $y' = \dfrac{x}{3} - \dfrac{y}{2}$

$x' = (\text{rate})(\text{concentration}) - (\text{rate})(\text{concentration}) = (\text{proportion})(\text{amt}) - (\text{proportion})(\text{amt})$

T_1 = amt liquid in tank 1

T_2 : amt liquid in tank 2

$\boxed{y' = \frac{x}{3} - \frac{y}{3} - \frac{y}{6} \quad (\text{combined})}$

don't need this

$3\left(-\frac{1}{3}T_1 + \frac{1}{3}T_2 + 5 = 0\right)3$

$-T_1 + T_2 + 15 = 0$

$-2T_1 + 2T_2 + 30 = 0$

$2T_1 - 3T_2$

$-T_2 + 30 = 0$

$T_2 = 30\,g$

$\boxed{\text{Tank 2 has 30 gallons}}$

$\left(\frac{1}{3}T_1 - \frac{1}{2}T_2 = 0\right)6$

$2T_1 - 3T_2 = 0$

$2T_1 - 3(30) = 0$

$2T_1 - 90 = 0$

$T_1 = 45\,g$

$\boxed{\text{Tank 1 has 45 gallons}}$

P2) Solve $x' = 2y \mid y' = 18x$ $\quad x(0)=5$ & $y(0)=3$

$x' = 2y$

$\boxed{y = \frac{x'}{2}}$

$y' = \frac{x''}{2}$

$\frac{x''}{2} = 18x$

$x'' = 36x$

homogeneous $x'' - 36x = 0$

$r^2 - 36 = 0$

$r: 6, -6$

$x = Ae^{6t} + Be^{-6t}$

$x(0) = A + B = 5$

$$\boxed{\begin{aligned} x &= 3e^{6t} + 2e^{-6t} \\ y &= 9e^{6t} \end{aligned}}$$

can do same process for y or use info already have

$x' = 6Ae^{6t} - 6Be^{-6t}$

$y = \frac{x'}{2} = \frac{6Ae^{6t} - 6Be^{-6t}}{2}$

$= 3Ae^{6t} - 3Be^{-6t}$

$y(0) = 3 = 3A - 3B$

$1 = A - B$

$5 = A + B$

$6 = 2A$ $\quad \boxed{A = 3}$

$1 = 3 - B$ $\quad \boxed{B = 2}$

P3) third order, write as first order $x^{(3)} - 2t x'' + t^2 x' - 5x = \arctan(t)$

1 $x_1 = x$ → no derivative

2 $x_2 = x_1' = x'$

3 $x_3 = x_2' = x_1'' = x''$

4 $\cancel{x_4} = x_3' = x_2'' = x_1''' = x'''$

$$x_3' - 2t\,x_3 + t^2 x_2 - 5x_1 = \arctan(t)$$

$$\boxed{\begin{aligned} x_3' &= \arctan(t) + 2t\,x_3 - t^2 x_2 + 5x_1 \\ x_2 &= x_1' \\ x_3 &= x_2' \end{aligned}}$$

✓✓

P4)

$x' - 5x + 2y = 0$

$Dx - 5x + 2y = 0$

$(D-5)x + 2y = 0$

$y' - Ax - By = 0$

$Dy - Ax - By = 0$

$-Ax + (D-B)y = 0$

$D =$ derivative of

$D = \frac{d}{dx}\ \frac{d}{dy}\ \frac{d}{dt}$

$-Ax + (D-B)y = 0$

$$\begin{pmatrix} D-5 & 2 \\ -A & D-B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

coefficients

$$\begin{vmatrix} D-5 & 2 \\ -A & D-B \end{vmatrix} x = \begin{vmatrix} 0 & 2 \\ 0 & D-B \end{vmatrix}$$

Kramers' rule

$[(D-5)(D-B) - (-A)(2)]x = 0 - 0 = 0$

roots: 3, −2 from original question.

$[D^2 - 5D - DB + 5B + 2A]x = 0$

$(r-3)(r+2) = 0$

$x'' - 5x' - Bx' + 5Bx + 2Ax = 0$

$r^2 - r + 6 = 0$

$x'' (-5-B)x' + (5B+2A)x = 0$ homogeneous

$r^2 + (-5-B)r + (5B+2A)x = 0$

$=$

$r^2 + (-5+B)r + (5B+2A)$

$-1 = -5 - B$

$\boxed{B = -4}$

$-6 = 5B + 2A$

$-6 = 5(-4) + 2A$

$\boxed{A = 7}$

P5) $x' + 2y' = 4x + 5y$ & $2x' - y' = 3x$

$x' - 2y' - 4x - 5y = 0$	$2x' - y' - 3x = 0$
$Dx + 2Dy - 4x - 5y = 0$	$2Dx - Dy - 3x = 0$
$(D-4)x + (2D-5)y = 0$	$(2D-3)x - Dy = 0$

$$\begin{pmatrix} D-4 & 2D-5 \\ 2D-3 & -D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Kramer's rule

$$\begin{vmatrix} D-4 & 2D-5 \\ 2D-3 & -D \end{vmatrix} x = \begin{vmatrix} 0 & 2D-5 \\ 0 & -D \end{vmatrix}$$

$$(-D)(D-4) - (2D-3)(2D-5)\, x = 0 - 0 = 0$$

$$[-D^2 + 4D - 4D^2 + 10D + 6D - 15]\, x = 0$$

$$[-5D^2 + 20D - 15]\, x = 0$$

$$[-5x'' + 20x' - 15x = 0] \quad \frac{-1}{5}$$

$$x'' - 4x' + 3x = 0$$

P6) $x' = 3x - y + t^2$ & $y' = x + y + e^{3t}$

$x' - 3x + y = t^2$

$Dx - 3x + y = t^2$

$y' - x - y = e^{3t}$

$Dy - x - y = e^{3t}$

$(D-3)x + y = t^2$

$(D-1)y - x = e^{3t}$

$$\begin{pmatrix} D-3 & 1 \\ -1 & D-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t^2 \\ e^{3t} \end{pmatrix}$$

$$\begin{vmatrix} D-3 & 1 \\ -1 & D-1 \end{vmatrix} x = \begin{vmatrix} t^2 & 1 \\ e^{3t} & D-1 \end{vmatrix}$$

$[(D-3)(D-1) - (-1)]x = (D-1)t^2 - e^{3t}$

$(D^2 - 3D - D + 3 + 1)x = (D-1)t^2 - e^{3t}$

$D^2 - 4D + 4x = (D-1)t^2 - e^{3t}$

$\boxed{x'' - 4x' + 4x = 2t - t^2 - e^{3t}}$ ✓ ✓ x

$$\begin{vmatrix} D-3 & -1 \\ -1 & D-1 \end{vmatrix} y = \begin{vmatrix} D-3 & t^2 \\ -1 & e^{3t} \end{vmatrix}$$

$[(D-1)(D-3) - (-1)]y = (D-3)e^{3t} - (-t^2)$

$(D^2 - 4D + 4)y = (D-3)e^{3t} + t^2$

$y'' - 4y' + 4y = 3e^{3t} - 3e^{3t} + t^2$

$\boxed{y'' - 4y' + 4y = t^2}$ ✓ ✓

P7) $x'' + x' + 2y' - 3y = 0$ & $x' + x + y' - y = 0$

$D^2x + Dx + 2Dy - 3y = 0$

$Dx + x + Dy - y = 0$

$(D^2 + D)x + (2D - 3)y = 0$

$(D+1)x + (D-1)y = 0$

$(D+1)x + (D-1)y = 0$

Solve for y per question

$$\begin{vmatrix} D^2+D & 2D-3 \\ D+1 & D-1 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} D^2+D & 2D-3 \\ D+1 & D-1 \end{vmatrix} y = \begin{vmatrix} D^2+D & 0 \\ D+1 & 0 \end{vmatrix}$$

$$\left[(D^2+D)(D-1) - (D+1)(2D-3)\right] y = 0 - 0 = 0$$

$$\left[D^3 + \cancel{D^2} - \cancel{D^2} - \cancel{D} - 2D^2 - \cancel{2D} + \cancel{3D} + 3\right] y = 0$$

$$\left[D^3 - 2D^2 + 3\right] y = 0$$

$$\boxed{y''' - 2y'' + 3y = 0}$$

P8)

$y'''' - 5y'' + 4y = 0$ $\quad$ $x'''' - 5x'' + 5x = 0$

$r^4 - 5r^2 + 4 = 0$

$r^4 - 5r^2 + 4 = 0$

$r: 2, 1, -1, 2$

$y = Ee^{2t} + Fe^{t} + Ge^{-t} + He^{-2t}$

$y' = 2Ee^{2t} + Fe^{t} - Ge^{-t} + -2He^{-2t}$

$(4Ae^{2t} + Be^{t} + Ce^{-t} + 4be^{-2t}) + (Ae^{2t} + Be^{t} + Ce^{-t} + De^{-2t})$

x''

$+ (2Ee^{2t} + Fe^{t} - Ge^{-t} - 2He^{-2t}) - 3(Ee^{2t} + Fe^{t} + Ge^{-t} + 2He^{-2t}) = 0$

$e^{2t}(4A + A + 2E - 3E) + e^{t}(B + B + F - 3F) + e^{-t}(C + C - G - 3G) + e^{-2t}(4D + D - 2H - 3H) = 0$

$4A + A + 2E - 3E = 0$

$5A - E = 0$ $\quad$ $\boxed{E = 5A}$

$B + B + F - 3F = 0$

$2B - 2F = 0$

$\boxed{B = F}$

$C + C - G - 3G = 0$

$2C - 4G = 0$

$\boxed{C = 2G}$

$4D + D - 2H - 3H = 0$

$5D - 5H = 0$

$\boxed{D = H}$

$y = 5Ae^{2t} + Be^{t} + \frac{C}{2}e^{-t} + De^{-2t}$ ✓

P9) $\begin{vmatrix} 1+t & 1-t \\ t & 2-t \end{vmatrix}^{-1}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

swap + negate

$$= \frac{1}{(2-t)(1+t) - (t)(1-t)} \begin{pmatrix} 2-t & -(1-t) \\ -t & 1+t \end{pmatrix}$$

$$\frac{1}{2 - \cancel{t} + \cancel{2t} - \cancel{t^2} - \cancel{t} + \cancel{t^2}} \begin{pmatrix} 2-t & t-1 \\ -t & 1+t \end{pmatrix}$$

$$\boxed{\frac{1}{2} \begin{pmatrix} 2-t & t-1 \\ -t & 1+t \end{pmatrix}}$$ ✓✓

P10) $A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}$

Find $AB^{-1} - BA^{-1}$

$$A^{-1} = \frac{1}{(1)(3)-(2)(1)} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = \frac{1}{3-2} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$B^{-1} = \frac{1}{(1)(4)-(-3)(-1)} \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} = \frac{1}{4-3} \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$$

$$AB^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} = \begin{bmatrix} (3)(4)+(2)(3) & 3(1)+2(1) \\ (1)(4)+(1)(3) & (1)(1)+(1)(1) \end{bmatrix} = \begin{bmatrix} 18 & 5 \\ 7 & 2 \end{bmatrix}$$

$$BA^{-1} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = \begin{bmatrix} 1+1 & -2+-3 \\ -3-4 & 6+12 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -7 & 18 \end{bmatrix}$$

$$AB^{-1} - BA^{-1} = \begin{bmatrix} 18 & 5 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -5 \\ -7 & 18 \end{bmatrix} = \begin{bmatrix} 16 & 10 \\ 14 & -16 \end{bmatrix}$$