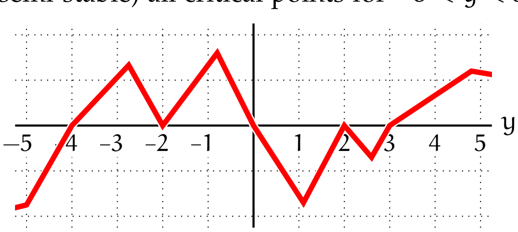
Quiz 3 problem bank

1. Find an implicit solution for $(x^2 - e^{x-y} + 3y^2)y' + 2xy + e^{x-y} + 8x(3x^2+5)^{1/3} = 0$ which contains the point $(x, y) = (1, 1)$.
2. Solve $4xyy' = x^2 + 4y^2$ with $y(1) = 1$.
3. Find the general solution for $(y^2 + 1)y'' = 2y(y')^2$.
4. Find an implicit solution to $2yy' = \dfrac{2xy^2 + 2y^4}{x^2 + 2xy^2}$ by using the substitution $y^2 = xu$.
5. A population of fish, measured in tons, had a stable population of 100 tons. A disease reduced it to 20 tons, at which point the population started recovering (growing) at the rate of 2 tons per year. Assuming the population follows the logistic model, what will be the population after 10 years?
6. In a zombie apocalypse, the number $P(t)$ of zombies follows a logistic model. When $P = 4$ (or 4000 zombies) the rate of growth of the zombie population was measured at $P' = 2$, similarly when $P = 8$ (or 8000 zombies) it was $P' = 3$. Determine the long-term (i.e. as $t \to \infty$) zombie population.
7. Given $y' = \arctan(y)(e^y - 4)$, find all critical points and draw the phase diagram.
8. Find a, b, c so that for $y' = y^3 + ay^2 + by + c$ we have that $y = 1$ is semi-stable and $y = 3$ is a critical point. Moreover, classify the stability of $y = 3$.
9. Given $y' = 2\cos(\frac{1}{3}y) + 1$, find the smallest $a > 0$ so that $y = a$ is a *stable* critical point. Moreover, give the largest interval of the form (c, d) so that any function with initial condition $c < y(0) < d$ will converge to a as $t \to \infty$.
10. Consider $y' = f(y)$ where $f(y)$ is the function shown below. Find and classify (i.e. as stable, unstable, semi-stable) all critical points for $-5 < y < 5$.



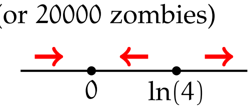
The quiz will consist of two randomly chosen problems (with possibly small variations; so make sure to learn processes and not just memorize answers).

Key ideas and processes

- *Exact ODEs* coming from implicit differentiation of $F(x, y) = 0$; they have the form $M\,dx + N\,dy = 0$ where M and N satisfy certain properties. Find $\int M\,dx$ and $\int N\,dy$, check overlap, and then put together.
- *Substitution* works by introducing a new variable that simplifies the ODE. When looking for substitution find a function on the "inside". *Many* possible substitutions, common substitutions include:
  - (Linear) $z = ax + by + c$
  - (Homogeneous) $z = y/x$ (when ODE can be rewritten so everything looks like y/x)
  - (Bernoulli) $z = y^{1-n}$ (for $y' + P(x)y = Q(x)y^n$)
  - $z(x) = y'$ (ODE has no term with y; so $y'' = z'$)
  - $z(y) = y'$ (ODE has no term with x; so $y'' = zz'$)
- Differential equations can be used to model growth/decay. A common example being populations. Some common models:
  - $P' = kP$ (*Exponential*). Has general solution $P(t) = P_0e^{kt}$. Connected to half-life and doubling-time.
  - $P' = kP(M - P)$ (*Logistic*). Has solution $P(t) = \dfrac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$.

  In many cases, problems do not give the value k but it is derived from additional data points.
- Autonomous differential equations of form $y' = f(y)$. For these constant solutions (critical points) found by solving $f(y) = 0$. Can then find flow to/from critical points by testing sign of y' in each interval, which leads to a *phase diagram* (drawing of real line with critical points marked and intervals oriented according to how solutions move).
- Critical point is *stable* if points "nearby" move closer. Critical point is *unstable* if points "nearby" move away. Critical point is *semi-stable* if on one side it is stable and on the other it is unstable.
- From phase diagram can form solutions; first draw constant solutions then add curves for each interval.

Comments and (partial) answers on problems

1. $x^2y + e^{x-y} + y^3 + (3x^2+5)^{4/3} = 19$
2. $y = x\sqrt{\frac{1}{2}\ln(x) + 1}$
3. $y = \tan(Cx + D)$
4. $ye^{(y^2/x)} = Cx$
5. $100/(1 + 4e^{-5/4})$ tons
6. (This is solving for M); $P = 20$ (or 20000 zombies)
7. Critical points: 0 and $\ln(4)$ 

8. $a = -5, b = 7, c = -3$; $y = 3$ will be unstable
9. $a = 2\pi$; $(c, d) = (-2\pi, 4\pi)$
10. $-4, 3$ = unstable; $-2, 2$ = semi-stable; 0 = stable

1. $(x^2 - e^{x-y} + 3y^2)y' + 2xy + e^{x-y} + 8x(3x^2+5)^{1/3} = 0$

which contains the point $(x,y) = (1,1)$

- this is an exact problem

Exact: $F_y\,dy + F_x\,dx = 0$

GOAL: $F = \text{constant}$

$$\underbrace{(x^2 - e^{x-y} + 3y^2)y'}_{} + \left(2xy + e^{x-y} + 8x(3x^2+5)^{1/3}\right) = 0$$

- these are similar

$$(x^2 - e^{x-y} + 3y^2)dy + \left(2xy + e^{x-y} + 8x(3x^2+5)^{1/3}\right)dx = 0$$

$$\int (x^2 - e^{x-y} + 3y^2)\,dy = \boxed{\boxed{x^2y + e^{x-y}} + y^3 + C(x)}$$

$$\int \left(2xy + e^{x-y} + 8x(3x^2+5)^{1/3}\right)dx = \boxed{x^2y + e^{x-y}}$$

this confirms EXACT

$$\int 8x\underbrace{(3x^2+5)}^{1/3}dx = \int 8 \cdot \frac{1}{6} u^{1/3}\,du$$

$$= \frac{4}{3} \cdot \frac{3}{4} \cdot u^{4/3} = \underbrace{(3x^2+5)}^{4/3}$$

$u = 3x^2+5$

$\frac{du}{6} = \frac{6x\,dx}{6} = \frac{1}{6}du = x\,dx$

$$x^2y + e^{x-y} + y^3 + (3x^2+5)^{4/3} = E$$

plug in $(1,1)$ for x & y

$$1 + 1 + 1 + (8)^{4/3}$$

note: always do cube root first,
so $8^{1/3} = 2$ and $2^4 = \boxed{16}$

$$3 + 16 = \boxed{19}$$

$$\boxed{x^2y + e^{x-y} + y^3 + (3x^2+5)^{4/3} = 19}$$

② Solve $4xyy' = x^2 + 4y^2$ with $y(1) = 1$

need (1)y', so divide

$$y' = \frac{x^2}{4xy} + \frac{4y^2}{4xy} = \frac{1}{4}x(y)^{-1} + \frac{1}{x}(y)$$

$$y' - \frac{1}{x}y = \frac{1}{4}x\boxed{(y)^{-1}}$$

Bernoulli, this would be first order without this.

$$yy' - \frac{1}{x}\boxed{y^2} = \frac{1}{4}x \rightarrow \frac{1}{2}z' - \frac{1}{x}z = \frac{1}{4}x$$

sub

$z = y^2$
$z' = 2yy'$

Protip: If you see something, try it

$z = y^2 \quad z' = 2yy'$

$2xz' = x^2 + 4z$

$$z' - \frac{2}{x}z = \frac{1}{2}x$$

$$e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln(x^{-2})} = x^{-2} = \boxed{\frac{1}{x^2}}$$

$$\underbrace{\frac{1}{x^2}z' - \frac{2}{x^3}z}_{} = \frac{1}{2}\cdot\frac{1}{x}$$

$$\int \frac{d}{dx}\left(\frac{1}{x^2}z\right)dx = \int \frac{1}{2}\cdot\frac{1}{x}dx$$

$$= \frac{1}{x^2}z = \frac{1}{2}\ln x + C \longrightarrow \frac{1}{x^2}y^2 = \frac{1}{2}\ln x + C$$

Solve for C

$1 = 0 + C \quad \boxed{C = 1}$

$\frac{1}{x^2} y^2 = \frac{1}{2} \ln x + 1$

$y^2 = \frac{1}{2} x^2 \ln x + x^2$

$\boxed{y = \sqrt{\frac{1}{2} x^2 \ln x + x^2}}$

note: Cannot be negative because parameters are $y(1) = 1$ positive

---

3 Find general solution for $(y^2+1) y'' = 2y(y')^2$

$y' = v = v(y)$
$y'' = v'v$

$(y^2+1) v'\cancel{v} = 2y(v)^{\cancel{2}}$

note: separable problem

$(y^2+1) \frac{dv}{dy} = 2yv$

$\int \frac{1}{v} dv = \int \frac{2y}{y^2+1} dy$

$e^{\ln y' = \ln v} = e^{\ln(y^2+1) + c}$ get rid of logs

$\frac{dy}{dx} \quad y' = C(y^2+1)$

note: another separable problem

$$\int \frac{1}{y^2+1}\,dy = \int C\,dx$$

$$\arctan y = Cx + K$$

take tan both sides

$$\boxed{y = \tan(Cx+K)}$$

---

(4) $2yy' = \dfrac{2xy^2 + 2y^4}{x^2 + 2xy^2}$ by using susbstitution $\boxed{y^2 = xu}$

$2yy' = u + xu'$

$$u + xu' = \frac{2x(xu) + 2(xu)^2}{x^2 + 2x(xu)}$$

note: get u' by itself

$$= \frac{2x^2u + 2x^2u^2}{x^2 + 2x^2u}$$

$$= \frac{(2u + 2u^2)}{(1+2u)}$$

$$xu' = \frac{2u+2u^2}{1+2u} - \frac{u}{1} \cdot \frac{(1+2u)}{(1+2u)}$$

$$= \frac{(2u+2u^2) - (u+2u^2)}{1+2u} = \frac{u}{1+2u}$$

$$x\frac{du}{dx} = \frac{u}{1+2u}$$

$$\int\left(\frac{1}{u}+2\right)du = \ln u + 2u$$

$$\int\left(\frac{1+2u}{u}\right)du = \int\frac{1}{x}\,dx$$

$$\ln u + 2u = \ln x + C$$

$$\ln\left(\frac{y^2}{x}\right) + 2\left(\frac{y^2}{x}\right) = \ln x + C$$

---

(5) Fish problem

Logistic model: $P' = kP(M-P)$

note: this is on the equation sheet

$$P = \frac{MP_0}{P_0 + (M-P_0)e^{-Mkt}}$$

Stable population = 100, so $M = 100$

P' @ $t=0$ = 2 tons/yr

$\boxed{\text{GOAL: Find } P(10)}$

rate = P', we want P however

$2 = k\cdot 20(100-20) \leftarrow$ @ $t=0$

$$k = \frac{1}{800} \qquad P = \frac{100\cdot 20}{20+80e^{-t/8}} = \frac{100}{1+4e^{-t/8}}$$

@ $t=10$: $\boxed{P(10) = \dfrac{100}{1+4e^{-5/4}}}$

6 $\boxed{\text{Zombie problem}}$

☆ logistic model

Note: Find M

$P' = kP(M-P)$

☆ cancel something out

$$\frac{2 = k \cdot 4 \cdot (M-4)}{3 = k \cdot 8 \, (M-8)}$$

divide

$$= \frac{2}{3} = \frac{(M-4)}{2(M-8)}$$

$$= 4(M-8) = 3(M-4)$$

$$= 4M - 32 = 3M - 12$$

$\boxed{M = 20 \quad \text{or } 20\text{k zombies}}$

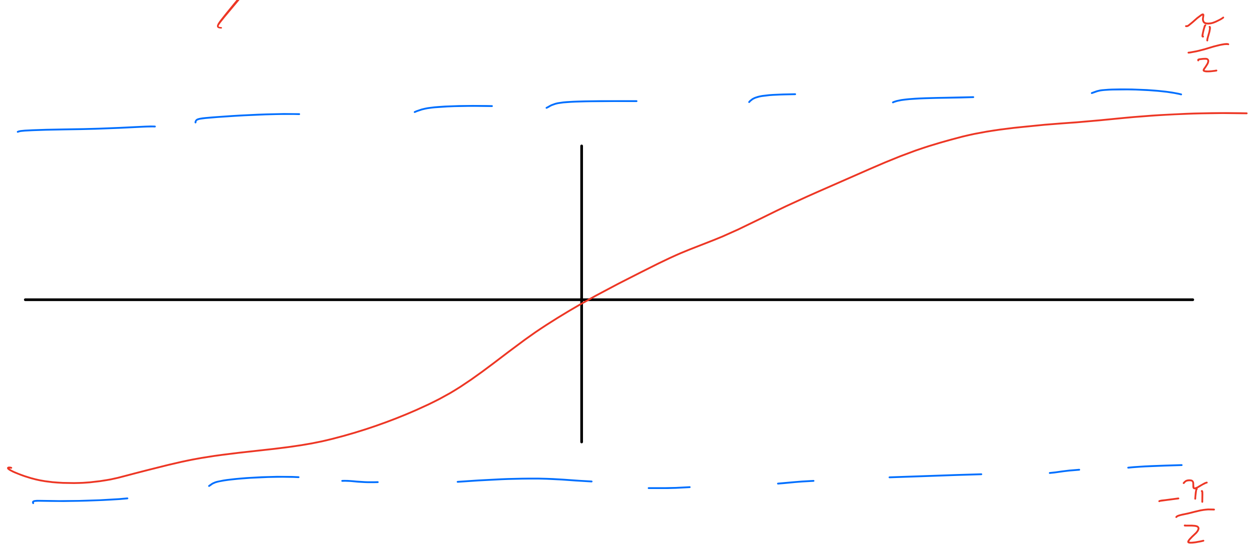
7 Understand the problem

$$0 = \arctan(y)\,(e^{y} - 4)$$

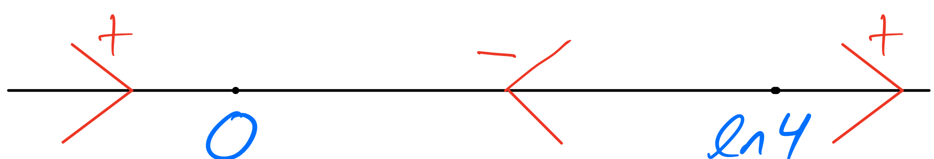
$\{$

$y = \ln 4$

$y = 0$



critical points $0, \ln 4$

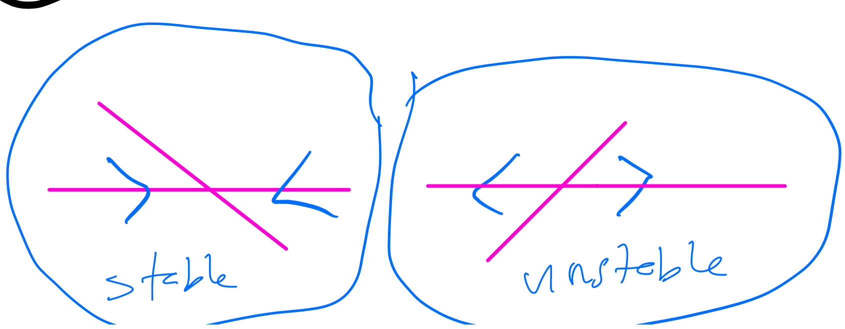


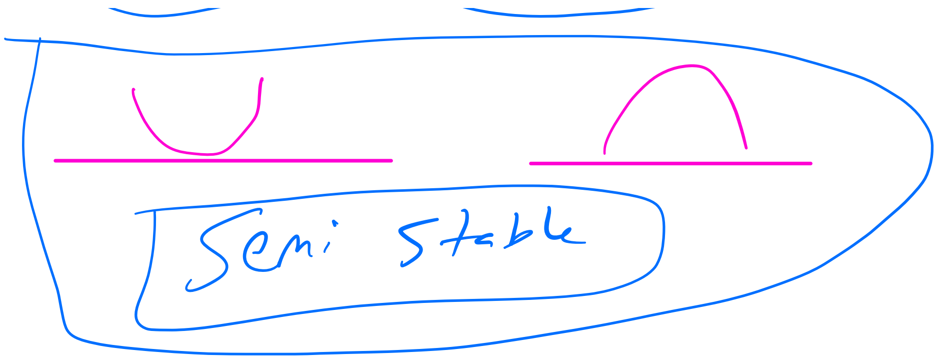
☆ pick a number, ex. -1, plug it into equation

---

8 $0 = y^3 + ay^2 + by + c$

☆ Note: we are looking for roots





$y=1 \leftarrow$ double root

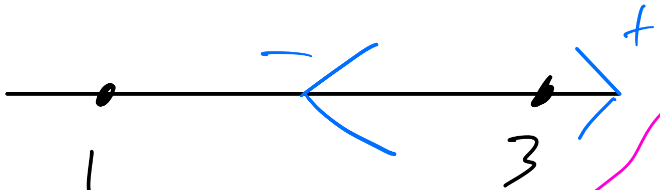
$y=3 \leftarrow$ single root

$$0 = y^3 + ay^2 + by + c = (y-1)^2(y-3)$$

$$= (y^2-2y+1)(y-3)$$

$$= (y^3 - 3y^2 - 2y^2 + 6y + y - 3)$$

$$= y^3 \underbrace{-5}_{a}y^2 + \underbrace{7}_{b}y \underbrace{-3}_{c}$$



3 is unstable

$a=-5$

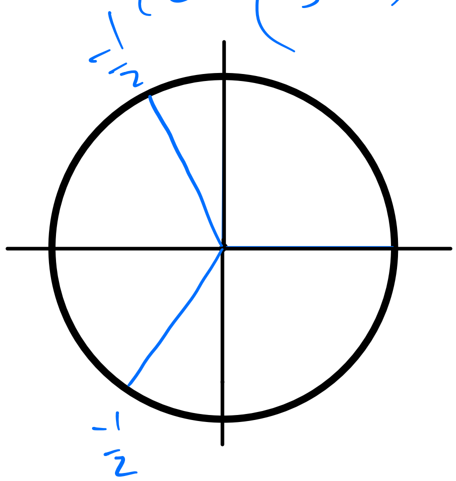
$b=7$

$c=-3$

9

$$0 = 2\cos\left(\tfrac{1}{3}y\right) + 1$$

$$\cos\left(\tfrac{1}{3}y\right) = -\tfrac{1}{2}$$



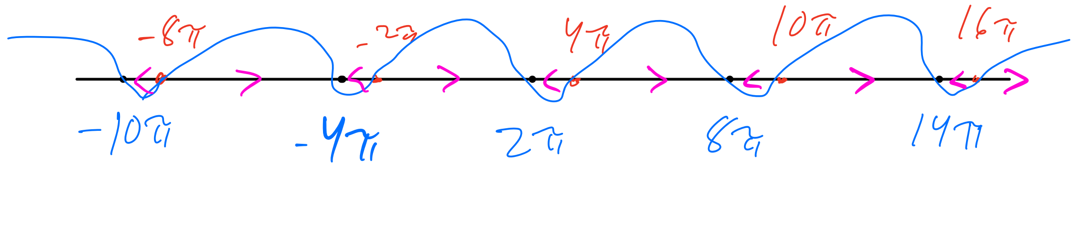
$$\frac{2\pi}{3} + 2\pi k$$

$$-\frac{2\pi}{3} + 2\pi k$$

$$\frac{1}{3}y = \frac{2\pi}{3} + 2\pi k$$

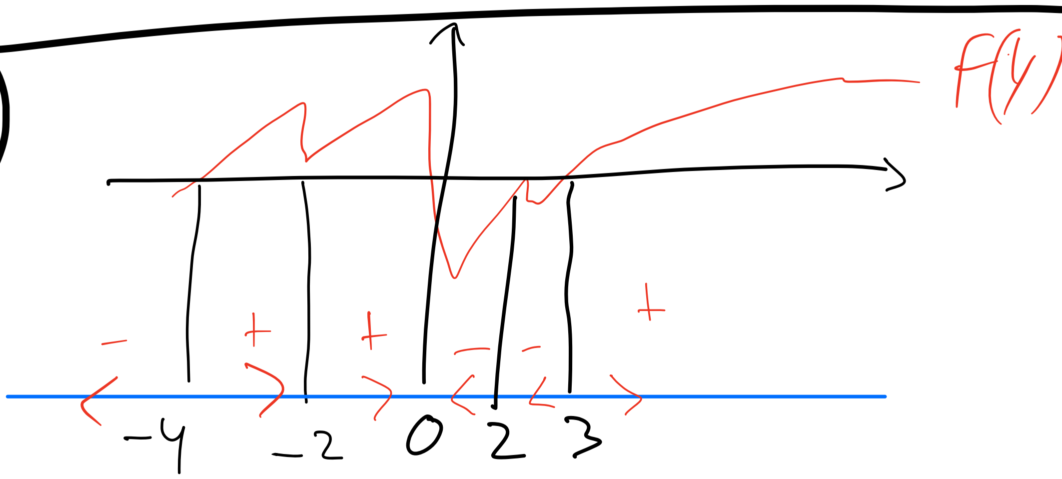
$$\frac{1}{3}y = -\frac{2\pi}{3} + 2\pi k$$

$$y = 2\pi + 6\pi k$$
$$y = -2\pi + 6\pi k$$



$$\boxed{\begin{array}{l} a = 2\pi, \text{ stable} \\ (c,d) = (-2\pi, 4\pi) \end{array}}$$

10



-4 : unstable

-2 : semistable

0 : stable

2 : semistable

3 : unstable