Quiz 8 problem bank

1. Given that $\mathbf{x}' = A\mathbf{x}$ has the general form for solution $\mathbf{x} = c_1 e^{3t}\begin{pmatrix}4\\3\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}3\\1\end{pmatrix}$, then find the matrix A.
2. Use a fundamental matrix approach to solve $\mathbf{x}' = \begin{pmatrix}2 & -1\\-4 & 2\end{pmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix}-7\\6\end{pmatrix}$.
3. Find e^{At} for $A = \begin{pmatrix}-2 & 20\\-1 & 7\end{pmatrix}$.
4. Find e^{At} for $A = \begin{pmatrix}1 & 1\\-2 & 3\end{pmatrix}$.
5. Find e^{At} for $A = \begin{pmatrix}7 & -2\\8 & -1\end{pmatrix}$.
6. Find $\mathbf{x}_p$ for $\mathbf{x}' = \begin{pmatrix}3 & 2\\5 & 4\end{pmatrix}\mathbf{x} + \begin{pmatrix}t-2\\t+2\end{pmatrix}$.
7. Verify that $e^t\begin{pmatrix}3t^2+t\\-6t^2+t\end{pmatrix}$ is a solution to $\mathbf{x}' = \begin{pmatrix}5 & 2\\-8 & -3\end{pmatrix}\mathbf{x} + e^t\begin{pmatrix}1\\1\end{pmatrix}$. Also, give the general form for solutions.
8. Find the general form for $\mathbf{x}' = \begin{pmatrix}2 & -1\\5 & -4\end{pmatrix}\mathbf{x} + \begin{pmatrix}8e^{4t}\\e^{4t}\end{pmatrix}$.
9. Find $\mathbf{x}_p$ for $\mathbf{x}' = \begin{pmatrix}3 & -2\\2 & -1\end{pmatrix}\mathbf{x} + \begin{pmatrix}0\\15\sqrt{t}e^t\end{pmatrix}$. (Hint: the eigenvalues are $1, 1$.)
10. Find $\mathbf{x}_p$ for $\mathbf{x}' = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}0\\\sec(t)\end{pmatrix}$. You may use $e^{At} = \begin{pmatrix}\cos(t) & \sin(t)\\-\sin(t) & \cos(t)\end{pmatrix}$.

The quiz will consist of two randomly chosen problems (with possibly small variations; so make sure to learn processes and not just memorize answers).

Key ideas and processes

- Given a full set of solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ to a homogeneous differential equation we form a *fundamental matrix* $\Phi(t) = \begin{pmatrix}\mathbf{x}_1 & \cdots & \mathbf{x}_n\end{pmatrix}$.

  Give $\mathbf{x}' = A\mathbf{x}$ and initial conditions $\mathbf{x}(0)$, then *the* solution is given by $\mathbf{x} = \Phi(t)\big(\Phi(0)\big)^{-1}\mathbf{x}(0)$.

- Given a matrix A we can compute e^{At} in two ways:

  1. By $\Phi(t)\big(\Phi(0)\big)^{-1}$ where the columns of $\Phi(t)$ come from the solutions to $\mathbf{x}' = A\mathbf{x}$.
  2. By $e^{At} = I + tA + \frac{1}{2!}t^2A^2 + \cdots$. This is the best option when all eigenvalues are the same. In particular if the eigenvalues of a 2×2 matrix A are λ, λ, then $\boxed{e^{At} = e^{\lambda t}\big(I + t(A - \lambda I)\big)}$.

- We have $(e^{At})^{-1} = e^{-At}$ and $\frac{d}{dt}\big(e^{At}\big) = Ae^{At}$.

- *"Method of undetermined coefficients"* For $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ if parts making up $\mathbf{f}(t)$ are what comes out of constant coefficient ODE, then we work the same procedure as with normal undetermined coefficients with two small changes: (1) instead of undetermined coefficients we use undetermined vectors; (2) we still need to initially bump but now take powers all the way down (we might not always use the bump).

  Once we have the "form" for $\mathbf{x}_p$ we plug it into both sides of the equation and then solve for what the vectors need to be in order for it to be a solution.

- *"Variation of parameters"* For $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ we run this much like a first-order linear equation and get

  $$\boxed{\mathbf{x} = e^{At}\int e^{-At}\mathbf{f}(t)\,dt}.$$

  Recall we integrate vectors entrywise. If we choose constants to be zero, this finds $\mathbf{x}_p$; if we choose constants to be arbitrary this finds general form.

Comments and (partial) answers on problems

1. (find e^{At}, take derivative at $t=0$) $A = \begin{pmatrix}-6 & 12\\-3 & 7\end{pmatrix}$
2. $\mathbf{x} = \begin{pmatrix}-2-5e^{4t}\\-4+10e^{4t}\end{pmatrix}$
3. $e^{At} = \begin{pmatrix}5e^{2t}-4e^{3t} & -20e^{2t}+20e^{3t}\\e^{2t}-e^{3t} & -4e^{2t}+5e^{3t}\end{pmatrix}$
4. $e^{At} = e^{2t}\begin{pmatrix}\cos(t)-\sin(t) & \sin(t)\\-2\sin(t) & \cos(t)+\sin(t)\end{pmatrix}$
5. $e^{At} = e^{3t}\begin{pmatrix}1+4t & -2t\\8t & 1-4t\end{pmatrix}$
6. $\mathbf{x}_p = \begin{pmatrix}-t+3\\t-4\end{pmatrix}$
7. (verification by plugging in); $\mathbf{x} = e^t\begin{pmatrix}3t^2+t\\-6t^2+t\end{pmatrix} + c_1e^t\begin{pmatrix}1+4t\\-8t\end{pmatrix} + c_2e^t\begin{pmatrix}2t\\1-4t\end{pmatrix}$
8. $\mathbf{x} = e^{4t}\begin{pmatrix}3\\2\end{pmatrix} + c_1e^t\begin{pmatrix}1\\1\end{pmatrix} + c_2e^{-3t}\begin{pmatrix}1\\5\end{pmatrix}$
9. $\mathbf{x}_p = e^t\begin{pmatrix}-8t^{5/2}\\10t^{3/2}-8t^{5/2}\end{pmatrix}$
10. $\mathbf{x}_p = \begin{pmatrix}\cos(t)\ln\left|\cos(t)\right| + t\sin(t)\\\sin(t)\ln\left|\sec(t)\right| + t\cos(t)\end{pmatrix}$

1. Given that $\mathbf{x}' = A\mathbf{x}$ has the general form for solution $\mathbf{x} = c_1 e^{3t}\begin{pmatrix}4\\3\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}3\\1\end{pmatrix}$, then find the matrix A.

eigenvalues

$$\phi t = [x_1\ x_2 \dots]$$

$$e^{At} = \phi t\,[\phi(0)]^{-1}$$

$$\phi(t) = \begin{pmatrix} 4e^{3t} & 3e^{-2t} \\ 3e^{3t} & e^{-2t} \end{pmatrix}$$

* $\phi(0)$ should be eigenvectors

$$\phi(0) = \begin{pmatrix} 4 & 3 \\ 3 & 1 \end{pmatrix} \quad [\phi(0)]^{-1} = \frac{1}{4-9}\begin{pmatrix} 1 & -3 \\ -3 & 4 \end{pmatrix} = \frac{1}{-5}\begin{pmatrix} 1 & -3 \\ -3 & 4 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 4e^{3t} & 3e^{-2t} \\ 3e^{3t} & e^{-2t} \end{pmatrix}\frac{1}{-5}\begin{pmatrix} 1 & -3 \\ -3 & 4 \end{pmatrix} = \frac{1}{-5}\begin{pmatrix} 4e^{3t} & 3e^{-2t} \\ 3e^{3t} & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4e^{3t} - 9e^{-2t} & -12e^{3t} + 12e^{-2t} \\ 3e^{3t} - 3e^{-2t} & -9e^{3t} + 4e^{-2t} \end{pmatrix}$$

$$Ae^{At} = \frac{-1}{5}\begin{pmatrix} 12e^{3t} + 18e^{-2t} & -36e^{3t} - 24e^{-2t} \\ 9e^{3t} + 6e^{-2t} & -27e^{3t} - 8e^{-2t} \end{pmatrix}$$

plug in $t = 0$

$$A = \frac{-1}{5}\begin{pmatrix} 12+18 & -36-24 \\ 9+6 & -27-8 \end{pmatrix} = \frac{-1}{5}\begin{pmatrix} 30 & -60 \\ 15 & 35 \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} -6 & 12 \\ 3 & 7 \end{pmatrix}}$$

extract data & plug in 0 & use derivatives

2. Use a fundamental matrix approach to solve
$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} -7 \\ 6 \end{pmatrix}$.

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ -4 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - (4)$$

$$\cancel{4} - 4\lambda + \lambda^2 - \cancel{4} = 0, \quad \lambda^2 - 4\lambda = 0 \quad \boxed{\lambda = (0, 4)}$$

@ $\lambda = 0$

$A - \lambda I$

$$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

@ $\lambda = 4$

$$\begin{pmatrix} -2 & -1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x = c_1 e^{\boxed{0t \begin{pmatrix} 1 \\ 2 \end{pmatrix}}} + c_2 \boxed{e^{4t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}}$$

$$\Phi(t) = \begin{pmatrix} 1 & -e^{4t} \\ 2 & 2e^{4t} \end{pmatrix} \quad \Big| \quad \Phi(0) = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \quad \Big| \quad [\Phi(0)]^{-1} = \frac{1}{2-(-2)} \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & -e^{4t} \\ 2 & 2e^{4t} \end{pmatrix} \frac{1}{4} \underbrace{\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -7 \\ 6 \end{pmatrix}}_{\text{multiply these}}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & -e^{4t} \\ 2 & 2e^{4t} \end{pmatrix}\begin{pmatrix} -14 + 6 \\ 14 + 6 \end{pmatrix} = \begin{pmatrix} -8 \\ 20 \end{pmatrix} \frac{1}{4}\begin{pmatrix} 1 & -e^{4t} \\ 2 & 2e^{4t} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -8 & -20e^{4t} \\ -16 & +40e^{4t} \end{pmatrix} = \boxed{\begin{pmatrix} -2 & -5e^{4t} \\ -4 & 10e^{4t} \end{pmatrix}}$$

3. Find e^{At} for $A = \begin{pmatrix} -2 & 20 \\ -1 & 7 \end{pmatrix}$.

$$\begin{vmatrix} -2-\lambda & 20 \\ -1 & 7-\lambda \end{vmatrix} = (7-\lambda)(-2-\lambda) - (-20) = -14 - 5\lambda + \lambda^2 + 20 = \lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-2)(\lambda-3) = 0$$

$$\boxed{\lambda = 2, 3}$$

@ $\lambda = 2$

$$\begin{pmatrix} -4 & 20 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

@ $\lambda = 3$

$$\begin{pmatrix} -5 & 20 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$X = A_1 \underbrace{\boxed{e^{2t}\begin{pmatrix} 5 \\ 1 \end{pmatrix}}}_{X_2} + B_1 \underbrace{\boxed{e^{3t}\begin{pmatrix} 4 \\ 1 \end{pmatrix}}}_{X_1}$$

$$\phi(t) = \begin{pmatrix} 4e^{3t} & 5e^{2t} \\ e^{3t} & e^{2t} \end{pmatrix}$$

$$\phi(0) = \begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix}$$

$$[\phi(0)]^{-1} = \frac{1}{4-5}\begin{pmatrix} 1 & -5 \\ -1 & 4 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 4e^{3t} & 5e^{2t} \\ e^{3t} & e^{2t} \end{pmatrix}\begin{pmatrix} -1 & 5 \\ 1 & -4 \end{pmatrix} = \boxed{\begin{pmatrix} -4e^{3t} + 5e^{2t} & 20e^{3t} - 20e^{2t} \\ -e^{3t} + e^{2t} & 5e^{3t} - 4e^{2t} \end{pmatrix}}$$

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4. Find e^{At} for $A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$.

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) - (-2) = 3-4\lambda+\lambda^2+2 = \underline{\lambda^2-4\lambda+5=0}$$

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = \lambda = \frac{4 \pm \sqrt{-4}}{2}$$

$$\boxed{\lambda = 2 \pm i} \quad \lambda = \frac{4 \pm 2i}{2}$$

@ $\lambda = 2+i$

$$\begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix}\begin{pmatrix} 1 \\ i+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad e^{2t}\cos t + ie^{2t}\sin t \begin{pmatrix} 1 \\ i+1 \end{pmatrix} =$$

$$= \left(e^{2t}\cos t + ie^{2t}\sin t\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right)$$

$$x = A_1\left(e^{2t}\cos t\begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{2}\sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + B_1\left(e^{2t}\sin t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{2t}\cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$\phi(t) = \begin{pmatrix} e^{2t}\cos t & e^{2t}\sin t \\ e^{2}\cos t - e^{2t}\sin t & e^{2t}\sin t + e^{2t}\cos t \end{pmatrix}$$

$$\phi(0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \Big| \quad [\phi(0)]^{-1} = \frac{1}{1-0}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{2t}\cos t & e^{2t}\sin t \\ e^{2t}\cos t - e^{2t}\sin t & e^{2t}\sin t + e^{2t}\cos t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} e^{2t}\cos t - e^{2t}\sin t & e^{2t}\sin t \\ -2e^{2t}\sin t & e^{2t}\sin t + e^{2t}\cos t \end{pmatrix}}$$

5. Find e^{At} for $A = \begin{pmatrix} 7 & -2 \\ 8 & -1 \end{pmatrix}$.

$$\begin{vmatrix} 7-\lambda & -2 \\ 8 & -1-\lambda \end{vmatrix} = (-1-\lambda)(7-\lambda) - (-16)$$

$$= -7 - 6\lambda + \lambda^2 + 16 = \lambda^2 - 6\lambda + 9 = 0$$

$$\boxed{\lambda = 3, 3}$$

$$e^{At} = e^{\lambda t}\{I + (A - \lambda I)t\}$$

$$= e^{3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} t\right]$$

$$\boxed{e^{At} = e^{3t}\begin{bmatrix} 1+4t & -2t \\ 8t & 1-4t \end{bmatrix}}$$

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6. Find $\mathbf{x}_p$ for $\mathbf{x}' = \underbrace{\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} \mathbf{x}}_{x_c} + \begin{pmatrix} t-2 \\ t+2 \end{pmatrix}$.

this extra means non-homogeneous

$X = X_c + X_p$

$X_p = t\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}$ ← constant

$X_c = \begin{vmatrix} 3-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 10 = 12 - 7\lambda + \lambda^2 - 10 = \lambda^2 - 7\lambda + 2 = 0$

$\lambda =$

$X_p' = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} at + c \\ bt + d \end{pmatrix} + \begin{pmatrix} t-2 \\ t+2 \end{pmatrix}$

$\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} at+c \\ bt+d \end{pmatrix} = \begin{pmatrix} 3at + 3c + 2bt + 2d + t - 2 \\ 5at + 5c + 4bt + 4d + t + 2 \end{pmatrix}$

extract $\begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} t(3a+2b+1) + (3c+2d-2) \\ t(5a+4b+1) + (5c+4d+2) \end{pmatrix}$

$(3a + 2b + 1 = 0) \cdot -2$

$5a + 4b + 1 = 0$

$-6a - 4b - 2 = 0$

$-a - 1 = 0$ $\boxed{a = -1}$

$3(-1) + 2b + 1 = 0$

$\boxed{b = 1}$

$3c + 2d - 2 = a$

$5c + 4d + 2 = b$

$(3c + 2d - 2 - 2 = -1) \cdot -2$

$+ 5c + 4d + 2 = 1$

$-6c - 4d + 4 = 2$

$-c + 6 = 3$

$\boxed{c = 3}$

$5(3) + 4d + 2 = 1$

$\boxed{d = -4}$

$$\boxed{X_p = t\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \end{pmatrix}}$$

7. Verify that $e^t\begin{pmatrix}3t^2+t\\-6t^2+t\end{pmatrix}$ is a solution to $\mathbf{x}'=\begin{pmatrix}5&2\\-8&-3\end{pmatrix}\mathbf{x}+e^t\begin{pmatrix}1\\1\end{pmatrix}$. Also, give the general form for solutions.

$$\text{LHS: } x' = e^t\begin{pmatrix}3t^2+t\\-6t^2+t\end{pmatrix}+e^t\begin{pmatrix}6t+1\\-12t+1\end{pmatrix}=\boxed{e^t\begin{pmatrix}3t^2+7t+1\\-6t^2-11t+1\end{pmatrix}}$$

$$\text{RHS: } \begin{pmatrix}5&2\\-8&3\end{pmatrix}e^t\begin{pmatrix}3t^2+t\\-6t^2+t\end{pmatrix}+e^t\begin{pmatrix}1\\1\end{pmatrix}=e^t\begin{pmatrix}15t^2+5t-12t^2+2t\\-24t^2+18t^2-8t-3t\end{pmatrix}+e^t\begin{pmatrix}1\\1\end{pmatrix}$$

$$\boxed{=e^t\begin{pmatrix}3t^2+7t+1\\-6t^2-11t+1\end{pmatrix}}\ \checkmark\checkmark$$

$$X_c=e^{At}=e^{\lambda t}[I+(A-\lambda I)t]$$

$$e^{At}=e^t\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}4&2\\-8&-4\end{pmatrix}t\right]$$

$$e^{At}=e^t\begin{pmatrix}1+4t&2t\\-8t&1-4t\end{pmatrix}$$

$$X_c=A_1e^t\begin{pmatrix}1+4t\\-8t\end{pmatrix}+B_1e^t\begin{pmatrix}2t\\1-4t\end{pmatrix}$$

$$\boxed{X=A_1e^t\begin{pmatrix}1+4t\\-8t\end{pmatrix}+B_1e^t\begin{pmatrix}2t\\1-4t\end{pmatrix}+e^t\begin{pmatrix}3t^2+t\\-6t^2+t\end{pmatrix}}\ \checkmark\checkmark$$

8. Find the general form for $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 8e^{4t} \\ e^{4t} \end{pmatrix}$.

x_c: $\begin{vmatrix} 2-\lambda & -1 \\ 5 & -4-\lambda \end{vmatrix} = (-4-\lambda)(2-\lambda) - (-5)$
$= -8 + 2\lambda + \lambda^2 + 5 = \lambda^2 + 2\lambda - 3 = 0$
$(\lambda+3)(\lambda-1) = 0$
$\boxed{\lambda = -3, 1}$

@ $\lambda = -3$
$$\begin{pmatrix} 5 & -1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ✓$$

@ $\lambda = 1$
$$\begin{pmatrix} 1 & -1 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ✓$$

$$x_c = A_1 e^{-3t}\begin{pmatrix} 1 \\ 5 \end{pmatrix} + B_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x_p = e^{4t}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$x_p' = 4e^{4t}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix} e^{4t}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 8e^{4t} \\ e^{4t} \end{pmatrix}$$

$$4\cancel{e^{4t}}\begin{pmatrix} a \\ b \end{pmatrix} = \cancel{e^{4t}}\begin{pmatrix} 2a - b \\ 5a - 4b \end{pmatrix} + \cancel{e^{4t}}\begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 4a \\ 4b \end{pmatrix} = \begin{pmatrix} 2a - b + 8 \\ 5a - 4b + 1 \end{pmatrix}$$

$2a - b + 8 = 4a \rightarrow (-2a - b + 8 = 0)(-8)$
$5a - 4b + 1 = 4b \rightarrow + \; 5a - 8b + 1 = 0$

$16a + 8b - 64 = 0$
$21a - 63 = 0$ $\boxed{a = 3}$

$2(3) - b + 8 = 0$
$\boxed{b = 2}$

$$\boxed{x_p = A_1 e^{-3t}\begin{pmatrix} 1 \\ 5 \end{pmatrix} + B_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{4t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}}$$

✓

9. Find $\mathbf{x}_p$ for $\mathbf{x}' = \begin{pmatrix}3 & -2\\ 2 & -1\end{pmatrix}\mathbf{x} + \begin{pmatrix}0\\ 15\sqrt{t}e^{t}\end{pmatrix}$. (Hint: the eigenvalues are $1, 1$.)

$$X = e^{At}\int e^{-At} f(t)\,dt$$

$$e^{At} = e^{\lambda t}\left[I + (A+\lambda I)t\right]$$

$$e^{At} = e^{t}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}2 & -2\\ 2 & -2\end{pmatrix}t\right]$$

$$e^{At} = e^{t}\left\{\begin{pmatrix}1+2t & -2t\\ 2t & 1-2t\end{pmatrix}\right\}$$

$$e^{-At} = e^{-t}\left\{\begin{pmatrix}1-2t & 2t\\ -2t & 1+2t\end{pmatrix}\right\}$$

$$e^{-At}f(t) = e^{-t}\begin{pmatrix}1-2t & 2t\\ -2t & 1+2t\end{pmatrix}\begin{pmatrix}0\\ 15\sqrt{t}e^{t}\end{pmatrix}$$

$$= e^{-t}\begin{pmatrix}30t^{3/2}e^{t}\\ 15t^{1/2}e^{t} + 30t^{3/2}e^{t}\end{pmatrix}$$

$$= \begin{pmatrix}30t^{3/2}e^{t}\\ 15t^{1/2}e^{t} + 30t^{3/2}e^{t}\end{pmatrix}$$

$$\int e^{-At}f(t)\,dt = \int\begin{pmatrix}30t^{3/2}\\ 15t^{1/2} + 30t^{3/2}\end{pmatrix}dt$$

$$= \begin{pmatrix}30\cdot\frac{2}{5}\cdot t^{5/2}\\ 15\cdot\frac{2}{3}t^{3/2} + 30\cdot\frac{2}{5}\cdot t^{5/2}\end{pmatrix}$$

$$= \begin{pmatrix}12t^{5/2}\\ 10t^{3/2} + 12t^{5/2}\end{pmatrix}$$

$$X = e^{t}\begin{pmatrix}1+2t & -2t\\ 2t & 1-2t\end{pmatrix}\begin{pmatrix}12t^{5/2}\\ 10t^{3/2} + 12t^{5/2}\end{pmatrix}$$

$$X = e^{t}\begin{pmatrix}12t^{5/2} \cancel{+24t^{7/2}} - 20t^{5/2} \cancel{-24t^{7/2}}\\ \cancel{24t^{7/2}} + 10t^{3/2} + 12t^{5/2} - 20t^{5/2} \cancel{-24t^{7/2}}\end{pmatrix}$$

$$\boxed{X = e^{t}\begin{pmatrix}-8t^{5/2}\\ 10t^{3/2} - 8t^{5/2}\end{pmatrix}}$$

10. Find $\mathbf{x}_p$ for $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ \sec(t) \end{pmatrix}$. You may use $e^{At} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$.

non homogeneous

$$X = e^{At} \int e^{-At} f(t)\,dt$$

$$e^{-At} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$$e^{-At} \cdot f(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 0 \\ \sec t \end{pmatrix} = \begin{pmatrix} -\sin t \sec t \\ \cos t \sec t \end{pmatrix} = \begin{pmatrix} -\tan t \\ 1 \end{pmatrix}$$

$$\int e^{-At} f(t)\,dt = \int \begin{pmatrix} -\tan t \\ 1 \end{pmatrix} dt = \begin{pmatrix} -\ln|\sec t| \\ t \end{pmatrix} + \underbrace{\begin{pmatrix} c \\ d \end{pmatrix}}$$

this will become x_c

$$e^{At} \int e^{-At} f(t) \cdot dt = \begin{pmatrix} \cos t & \sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} -\ln|\sec t| \\ t \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} -\cos t \ln|\sec t| + (\sin t) \\ \ln|\sec t| \cdot \sin t + (-\cos t) \end{pmatrix}}$$

✓ ✓