Linear Algebra

Definitions and facts

- The dot product of two orthogonal vectors is 0
- Rank: The number of linearly independent vectors in a matrix
- Orthogonal: $U^T U = I$ (Independence and orthogonality can be thought of as the same thing)
- Orthonormal:
- Singular Value: Singular values are the square roots of eigenvalues: ($\sigma_i = \sqrt{\lambda_i}$), as can be seen by comparing SVD and the Spectral Theorem
- Eigenvalue:
- Eigenvector:
- Null Space: $Ax = 0$
- Invertible: A matrix is invertible if it is full rank and all eigenvalues are non zero. Prop of inverse: $AA^{-1} = I$
- Trace: Defined as the of the sum of the main diagonal of a matrix. Is also the sum of the eigenvalues of the matrix
- Determinant: Represents the volume enclosed in the vectors. Is also the product of the eigenvalues of the matrix

Projection

For projection of a line $\bar{x}$ onto another line, the

$$\min_w ||x - x_0 - wu||$$

Norms

l1-norm

$$\sum_i x_i$$

- Least absolute deviation
- Robust

l2-norm

$$(\sum_i x_i^2)^{1/2}$$

- Least Squares
- Not very robust (because susceptible to outliers)

lp-norm

$$(\sum_i x_i^p)^{1/p}$$

- Generalized form of the lp-norm

Fundamental Theorem of Linear Algebra

You can decompose any vector into two orthogonal ones, one in the null space and the second in the range of the transpose

Least Squares

$$||Ax - y||_2^2$$

Positive Semi-definte PSD

$$x^T Ax \geq 0 : \forall x$$

All eigenvalues are positive and non-zero

QR Decomposition

Decompose a matrix into an orthogonal matrix Q and an upper triangular matrix R
Can be used to solve linear least squares

$$A = QR$$

Principle Component Analysis (PCA)

The goal is to maximize variance The eigenvectors that correspond to the largest

$$\lambda_{max} = \max_u u^T \Sigma u$$

Spectral Theorem/ Singular Eigenvalue Decomposition

$$A = \sum_i^n \lambda_i u_i u_i^T = U \Lambda U^T$$

U is an orthnormal basis of the eigenvectors of A Λ is a diagonal matrix of the eigenvalues of A
Only for symmetric, square matrices
There are exactly n (possibly non distinct) real eigenvalues that are associated with an eigenvector from the orthonormal basis U

Singular Value Decomposition

Generalization of Spectral Theorem to non-symmetric and rectangular matrices Express any matrix as the sum of rank 1 matrices

$$A = \sum_i^n \sigma_i u_i v_i^T = USV^T$$

$$\text{where } \tilde{S} = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$$

where all v_i's are orthogonal and all u_i's are orthogonal
U is the orthonormal basis of eigenvectors of A
σ_i's are singular values
All the singular values are unique
S is a matrix with σ_i's down the diagonal in decreasing order

Cauchy Schwarz

$$x^T y = |xy| \leq ||x||_2 ||y||_2$$