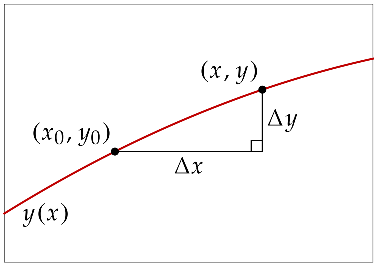
4.3 Euler's Method

Euler's method is a numerical method that can be used to approximate the solutions to explicit first-order equations. It is based on making successive linear approximations to the solution.

Linear Approximation

Suppose we are given a single data point (x_0, y_0) for a function $y(x)$, and suppose we also know the value of the derivative $y'(x_0)$ at this point. In this case, nearby points (x, y) on the graph of the function obey the approximate formula



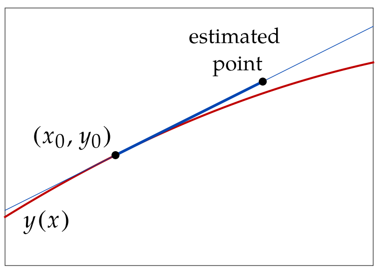
▲ **Figure 1:** The ratio of Δy to Δx is approximately equal to the slope at (x_0, y_0).

$$\frac{\Delta y}{\Delta x} \approx y'(x_0).$$

Here $\Delta x = x - x_0$ is the distance in the x direction, and $\Delta y = y - y_0$ is the distance in the y direction, as shown in Figure 1. This equation is only approximately correct, since it assumes that the average slope between (x_0, y_0) and (x, y) is equal to $y'(x_0)$.

We can use this formula to estimate $y(x)$ for x close to x_0. Specifically, we compute $\Delta x = x - x_0$, and then estimate Δy using the formula

$$\boxed{\Delta y \approx y'(x_0)\,\Delta x}$$



▲ **Figure 2:** The estimated point lies on the tangent line to the graph of $y(x)$ at the point (x_0, y_0).

Adding this Δy to y_0 gives an approximate value for $y(x)$. This is called a **linear approximation**, because the estimated point actually lies on the tangent line to the graph of the function, as shown in Figure 2.

EXAMPLE 1

Suppose $y(x)$ is a differentiable function satisfying $y(1) = 3$ and $y'(1) = 0.5$. Use a linear approximation to estimate the value of $y(1.04)$.

SOLUTION Here we are changing x by the small amount $\Delta x = 0.04$. The corresponding change in y is

$$\Delta y \approx y'(1)\,\Delta x = (0.5)(0.04) = 0.02.$$

Then

$$y(1.04) = y(1) + \Delta y \approx 3 + 0.02 = 3.02.$$

Linear approximation is quite useful in the case of explicit first-order equations, since we have a formula for y' in terms of x and y. Given an initial condition (x_0, y_0), we can plug these coordinates directly into the differential equation to get the value of $y'(x_0)$.

EXAMPLE 2

Let $y(x)$ be the solution to the following initial value problem:

$$y' = 3y^2 + \ln x, \qquad y(1) = 0.5.$$

Estimate $y(1.002)$.

SOLUTION Plugging $x = 1$ and $y = 0.5$ into the differential equation itself gives us the value of $y'(1)$:

$$y'(1) = 3(0.5)^2 + \ln(1) = 0.75.$$

For $\Delta x = 0.002$, the linear approximation gives

$$\Delta y \;\approx\; y'(1)\,\Delta x \;=\; (0.75)(0.002) \;=\; 0.0015.$$

Then

$$y(1.002) \;=\; y(1) + \Delta y \;\approx\; 0.5 + 0.0015 \;=\; 0.5015.$$

The actual value of $y(1.002)$ in this example is about 0.501507, so the linear approximation is fairly accurate.

Euler's Method

The idea of Euler's method is to use repeated linear approximations to estimate a sequence of points that lie on a solution curve. Starting with an initial condition (x_0, y_0), we use a linear approximation to estimate a nearby point on the solution curve. We then use a linear approximation based at the new point to estimate yet another point, and so forth.

EXAMPLE 3

Let $y(x)$ be the solution to the following initial value problem:

$$y' \;=\; \sin y - 3x, \qquad y(0) = 1.$$

Use Euler's method to estimate $y(0.1)$, $y(0.2)$, and $y(0.3)$.

SOLUTION The idea is to use three linear approximations, each with $\Delta x = 0.1$. We will keep track of three decimal places during the process.

1st Approximation

We start by making a linear approximation to estimate $y(0.1)$. We have

$$\begin{aligned} & y'(0) \;=\; \sin(1) - 3(0) \;\approx\; 0.841 \\ \text{so} \quad & \Delta y \;\approx\; y'(0)\,\Delta x \;=\; (0.841)(0.1) \;=\; 0.084 \\ \text{so} \quad & y(0.1) \;\approx\; y(0) + \Delta y \;=\; 1 + 0.084 \;=\; 1.084. \end{aligned}$$

2nd Approximation

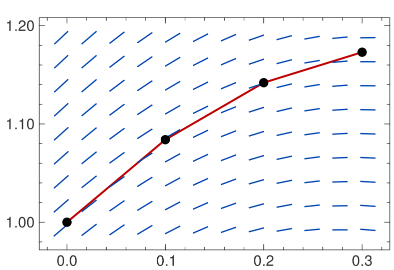
Next we make a linear approximation to estimate $y(0.2)$, using the the point $(0.1, 1.084)$ from the previous approximation as the base point. We have

$$\begin{aligned} & y'(0.1) \;\approx\; \sin(1.084) - 3(0.1) \;\approx\; 0.584 \\ \text{so} \quad & \Delta y \;=\; y'(0.1)\,\Delta x \;\approx\; (0.584)(0.1) \;\approx\; 0.058 \\ \text{so} \quad & y(0.2) \;=\; y(0.1) + \Delta y \;\approx\; 1.084 + 0.058 \;=\; 1.142. \end{aligned}$$

3rd Approximation

Finally we make a linear approximation to estimate $y(0.3)$, using the the point $(0.2, 1.142)$ from the previous approximation as the base point. We have

$$\begin{aligned} & y'(0.2) \;\approx\; \sin(1.142) - 3(0.2) \;\approx\; 0.309 \\ \text{so} \quad & \Delta y \;=\; y'(0.2)\,\Delta x \;\approx\; (0.309)(0.1) \;\approx\; 0.031 \\ \text{so} \quad & y(0.3) \;=\; y(0.2) + \Delta y \;\approx\; 1.142 + 0.031 \;=\; 1.173. \end{aligned}$$



▲ **Figure 3:** The three linear approximations in Example 3.

Figure 3 shows the three linear approximations in this example. By changing slope twice, we manage to follow the slope field much better than we could with a single linear approximation.

When using Euler's method, we typically use the same **step size** Δx for all of the linear approximations. It is common to use a table to keep track of the estimates in each step, as shown in Table 4.1. Each value of y' is computed from the x and y values in the same column using the differential equation, and each value of y is computed from the y and y' values in the previous column using the following linear approximation formula:

x	0.0	0.1	0.2	0.3
y	1.000	1.084	1.142	1.173
y'	0.841	0.584	0.309	—

▲ **Table 4.1:** Table of values for the linear approximations used in Example 1.

$$\boxed{y_{\text{new}} \approx y_{\text{old}} + y'_{\text{old}}\,\Delta x}$$

Note that this formula combines the two parts of the linear approximation (computing Δy and then adding it to y_{old}) into a single equation.

EXAMPLE 4

Let $y(x)$ be the solution to the following initial value problem:

$$y' = (x-1)^2 - y^2, \qquad y(0) = 0.5.$$

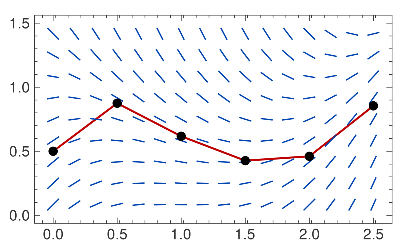
Use Euler's method with step size $\Delta x = 0.5$ to estimate $y(2.5)$, keeping track of four decimal places during the procedure.

SOLUTION Euler's method results in the following table of values:

x	0.0	0.5	1.0	1.5	2.0	2.5
y	0.5	0.8750	0.6172	0.4267	0.4607	0.8546
y'	0.75	−0.5156	−0.3809	0.0679	0.7878	—

Here are the calculations that were used to produce this table:

$$\begin{aligned}
y'(0.0) &= (x-1)^2 - y^2 = (0.0-1)^2 - (0.5)^2 = 0.75 \\
y(0.5) &= y(0.0) + y'(0.0)\,\Delta x \approx (0.5) + (0.75)(0.5) = 0.875 \\
y'(0.5) &= (x-1)^2 - y^2 \approx (0.5-1)^2 - (0.875)^2 \approx -0.5156 \\
y(1.0) &= y(0.5) + y'(0.5)\,\Delta x \approx (0.875) + (-0.5156)(0.5) = 0.6172 \\
y'(1.0) &= (x-1)^2 - y^2 \approx (1.0-1)^2 - (0.6172)^2 \approx -0.3809 \\
y(1.5) &= y(1.0) + y'(1.0)\,\Delta x \approx (0.6172) + (-0.3809)(0.5) \approx 0.4267 \\
y'(1.5) &= (x-1)^2 - y^2 \approx (1.5-1)^2 - (0.4267)^2 \approx 0.0679 \\
y(2.0) &= y(1.5) + y'(1.5)\,\Delta x \approx (0.4267) + (0.0679)(0.5) \approx 0.4607 \\
y'(2.0) &= (x-1)^2 - y^2 \approx (2.0-1)^2 - (0.4607)^2 \approx 0.7878 \\
y(2.5) &= y(2.0) + y'(2.0)\,\Delta x \approx (0.4607) + (0.7878)(0.5) \approx 0.8546
\end{aligned}$$



▲ **Figure 4:** The five steps of Euler's method used in Example 4.

Figure 4 shows the five linear approximations used in this example. Note that each linear segment has the correct slope at its left endpoint, but the slope gradually becomes wrong over the course of each step.

Euler's method may remind you of using a Riemann sum to approximate a definite integral. Indeed, in the special case where the differential equation has the form

$$y' = f(x),$$

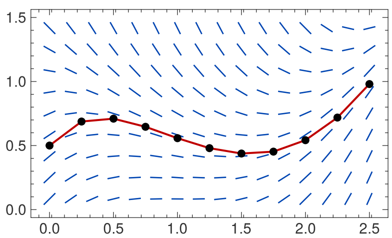
solving the differential equation is the same as integrating f, and Euler's method gives the same result as the left endpoint rule for a Riemann sum.

A Closer Look Numerical Methods

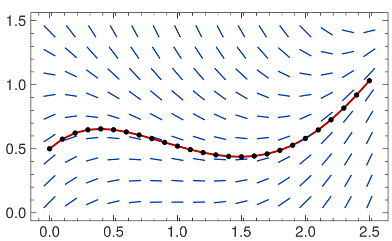
Euler's method is only the simplest numerical method for solving differential equations. Finding better ways of approximating solutions to differential equations is an ongoing subject of research, and is one of the primary goals of the field of mathematics known as **numerical analysis**.

One basic improvement is to estimate the slope of each straight segment using a combination of y' values corresponding to different values of x. This idea leads to a set of possible methods known collectively as **Runge-Kutta methods**. Another possible improvement, often combined with Runge-Kutta, is to change the step size Δx adaptively, using small steps when the value of y' seems to be changing quickly, and using large steps when y' is roughly constant.

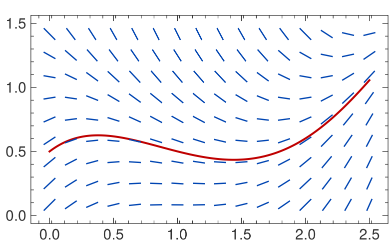
Modern computer algebra systems such as *Mathematica*, *Sage*, and *Matlab* have a variety of different numerical methods built into the system. Though you can choose a method explicitly to solve a given problem, most computer algebra systems also have a built-in algorithm to choose an appropriate method based on the properties of the given equation.



▲ **Figure 5:** Euler's method for the initial-value problem in Example 4 with step size $\Delta x = 0.25$.



▲ **Figure 6:** Euler's method for the initial-value problem in Example 4 with step size $\Delta x = 0.1$.



▲ **Figure 7:** Euler's method for the initial-value problem in Example 4 with step size $\Delta x = 0.01$.

As with a Riemann sum, Euler's method becomes more precise as the step size becomes smaller, since the slope is being adjusted more often. For example, we can improve the estimated values in Example 4 by decreasing the step size, as shown in Figures 5, 6, and 7. The last of these three graphs (with step size 0.01) follows the actual solution curve to within 0.004, which is about a third of the width of the red line.

With a computer, it is possible to implement Euler's method using very small steps (e.g. $\Delta x = 0.000001$), which leads to very accurate numerical approximations for the solution curves. Combining this with other numerical methods (see the *Numerical Methods* box above) can increase the speed and accuracy further, making computers an indispensable tool for scientific modeling.

EXERCISES

1. Consider the following initial value problem:

$$y' = x^2 - y^3, \qquad y(0) = 1.$$

Use Euler's method with a step size of 0.2 to estimate $y(1)$, keeping track of three decimal places during the calculation.

2. Consider the following initial value problem:

$$y' = xy - 2, \qquad y(2) = 1.$$

Use Euler's method with a step size of 0.5 to estimate $y(4)$, keeping track of three decimal places during the calculation.

3. A ball is dropped from the top of a tall building. If air resistance is taken into account, the downward velocity v of the ball is modeled by the differential equation

$$\frac{dv}{dt} = g - kv^2$$

where $g = 9.8\ \text{m/sec}^2$ is the acceleration due to gravity, and $k = 0.1/\text{m}$ is the **drag coefficient**. Assuming the initial velocity of the ball is zero, use Euler's method with a step size of 0.25 sec to estimate the velocity of the ball after one second. Keep track of three decimal places during the calculation.