Week 2, Chapter 2.

Exercises 1 thru' 7.

Skeleton solutions.

- 4 to 7 are a bit difficult.

* If you know the derivative of a function, then you know the integral of that derivative plus an unknown constant.

1) Linear if

$$P_0(t)\,u(t) + P_1(t)\,u'(t) + P_5(t)\left(\frac{d^5}{dt^5}u\right) + P_n(t)\left(u^{(n)}(t)\right) = Q(t).$$

All derivatives are power 1.

if not linear $\Rightarrow$ nonlinear.

2) homogeneous if $Q = 0$.

Ex 1.

	linear	non-linear
homogeneous	a c h	e
inhomogeneous	b d f	g ? why ?

Ex2.

$\boxed{\text{Ex2. uses section 2.1}}$

a). solve

$$\frac{du}{dt} = t^2 - 1 \quad \text{with} \quad u(0) = 1.$$

$$\int \frac{du}{\cancel{dt}} \cancel{dt} = \int (t^2 - 1)\, dt$$

$$\int du = \int t^2 dt - \int dt$$

$$\therefore \quad u = \frac{t^3}{3} - t + C$$

$$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1}, n \neq -1.}$$

constant of integration to be found.

Using $u(0) = 1$,

$$1 = 0 - 0 + c \Rightarrow c = 1.$$

complete answer:

$$u = \frac{1}{3}t^3 - t + 1.$$

Ex 2b is the same

Ex 2. c.

Given

$$\frac{dy}{dx} = 1 - y \quad , \quad y(0) = 0.$$

Divide both sides by $1-y$. & integrate w.r.t. x.

$$\underbrace{\int \frac{dy}{(1-y)}}_{\text{apply}} = \int dx$$

apply Fundamental Th. of calculus.

absolute sign

$$-\ln\left(|1-y|\right) = x + C$$

natural log.

$$|1-y| = e^{-c}\, e^{-x} \Rightarrow 1-y = \underbrace{\pm e^{-c}}_{A} e^{-x}$$

A.

$$y = 1 - Ae^{-x}.$$

d is similar.

Ex2, 9

Two methods for 2.9.
M1: you already know properties of ODE.
M2: apply a new method.

$y''(x) = -y(x)$

is eqn. of simple harmonic oscillator.

Try the trigonometric solution

$$y = A\sin x + B\cos x$$

$$\therefore \quad y' = A\cos x - B\sin x$$

$$\& \quad y'' = -A\sin x - B\cos x = -y \Rightarrow \text{ODE}$$

satisfied.
$\therefore$ solution is valid.

Using $y(0) = 0$, $B = 0$.

Using $y'(0) = 1$, $A = 1$.

$\therefore$ solution is

$$y = -\sin x.$$

Method 2.

Try $y = e^{mx}$ and work out m. Hint. m is complex.

Ex 2, 9. method 2.

In ODE,

$$y''(x) = -y(x)$$

try solution

$$y = Ae^{mx}.$$

$$\therefore \; y''(x) = m^2 e^{mx} = m^2 y.$$

Using ODE, ~~you~~ we have

$$m^2 e^{mx} + e^{mx} = 0$$

$$\Rightarrow \quad m = \pm i$$

i is $\sqrt{-1}$, imaginary number.

You have to take each value of m.

A is yet arbitrary.

$$\therefore \; y = A_1 e^{ix} + \cancel{B} A_2 e^{-ix}$$

$$e^{ix} = \cos x + i \sin x \text{ (known)}$$

work out steps.

$$\therefore \; y = A \sin x + B \cos x$$

A & B are combinations of A_1, A_2, and i.

Get A & B from i.c.s.

Ex 2.h

Solve $u''(x) = u'(x) + 1$

by reducing order.

Let $y = u'$.

$\Rightarrow \quad y' = y + 1$ integration happened, write all steps.

$y = Ae^x$. (see soln to ex2,c).

but $y = u'$!

$\Rightarrow \quad u' = Ae^x$.

Integrating w.r.t. x

$$\int u'\,dx = \int A e^x\,dx$$

$\Rightarrow \quad u = Ae^x + C$. (solution.).

Ex 3 is about use of
product rule, chain rule, sec 2.2.

Steps are:

Given ode, write it in form (standard form)

$$u'(t) + p(t)\,u(t) = q(t). \quad \text{— S1.}$$

Get the integral

$$\int p(t)\,dt \quad \text{—— S2.}$$

Then define function $v(t)$

$$v(t) = e^{\int p(t)\,dt} \quad \text{—— S3.}$$

Fourth step is work out

$$X(t) = \int v(t)\,q(t)\,dt \quad \text{—— S4}$$

Your solution is

$$u(t) = \frac{1}{v(t)}\left(X(t) + C\right) \quad \text{— } \cancel{S4}.\ S5$$

if initial conditions given, then use S5 to get C.

Ex 3.a.

$\boxed{\text{try a,b,d,e - all similar.}}$

$$u'(t) + \frac{1}{(t+1)} u(t) = t$$

S1, standard form.

$$p(t) = \frac{1}{t+1}, \quad q(t) = t.$$

find $v(t)$: $\int p(t)\,dt = \int \frac{dt}{1+t} = \ln|t+1|$ S2.

$$v(t) = e^{\ln|t+1|} = t+1.$$

S3.

$$X(t) = \int (t+1)\; t \; dt$$

S4.

$$X(t) = t^3/3 + t^2/2 + c_1$$

you dont have to write this c_1.

$$\therefore u(t) = \frac{1}{(t+1)}\left[\frac{t^3}{3} + \frac{t^2}{2} + c\right]$$

this c is mandatory.

Using $u(0) = 1 \Rightarrow c = 1.$

$$\therefore u = \frac{1}{t+1}\left[\frac{t^3}{3} + \frac{t^2}{2} + 1\right]$$

Write final solution explicitly.

Ex 3, c

Simpler solution without use of product rule.

$$y' = e^x y + e^x \quad , \quad y = y(x).$$

$$y' = e^x (y+1)$$

$$\frac{y'}{y+1} = e^x$$

Integrate w.r.t. x,

$$\int \frac{dy}{y+1} = \int e^x \, dx$$

$$\ln|y+1| = e^x + C$$

$$y+1 = A e^{e^x}$$

$$y = A e^{e^x} - 1.$$

Do this with product rule.

Ex 4.

'if and only if' means

assume one & show other is true

AND

assume other & show one is true.

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n^{th} order linear homogenous ODE :

$$p_1 u + p_2 u' + p_3 u'' + \dots p_{n+1} u^n = 0. \quad : P_1$$

if u_1 & u_2 satisfy ODE P_1, then

$$p_1 u_1 + p_2 u_1' + p_3 u_2'' + \dots + p_{n+1} u_1^n = 0 \quad - E_1$$

&

$$p_1 u_2 + p_2 u_2' + \dots + p_{n+1} u_2^n = 0. - E_2.$$

Multiply eq. E_1 by A and eq_2 by B, and add them up. $A, B =$ any numbers.

$$A\left[\boxed{p_1 u_1} + \boxed{p_2 u_1'} + \boxed{p_3 u_1''} + \dots + p_{n+1} u_1^n\right]$$
$$+ B\left[\boxed{p_1 u_2} + \boxed{p_2 u_2'} + \boxed{p_3 u_2''} + \dots + p_{n+1} u_2^n\right] = 0.$$

collect all 'similar' terms together.

$$p_1 A u_1 + p_1 B u_2$$

$$+ p_2 A u_1' + p_2 B u_2'$$

$$+ p_3 A u_1'' + p_3 B u_2''$$

$$+ \dots$$

$$+ p_{n+1} A u_1^n + p_{n+1} B u_2^n = 0 .$$

$$\Rightarrow p_1 (\underbrace{A u_1 + B u_2})$$

$$+ p_2 (\underbrace{A u_1 + B u_2})'$$

$$+ p_3 (\underbrace{A u_1 + B u_2})''$$

$$+ \dots$$

$$+ p_{n+1} (\underbrace{A u_1 + B u_2})^n = 0$$

LHS is ~~a~~ the same homogeneous ODE ($\text{same } p_i\text{ s}$), and it works out ~~to~~ 0.

HALF WORK DONE - DO REST OF IT.

2×5 same as

2×6.

Ex6. This is an important concept.

To show that

$$u(t) = \int_a^t g\big(u(s)\big)\,ds \quad \text{— eq.1.}$$

constant ← a

is a solution of

$$\frac{du(t)}{dt} = g\big(u(t)\big) \quad \text{— eq.2.}$$

Let h be the integral of g.
Then eq.1. can be written as

$$u(t) = \int_a^t g\big(u(s)\big)\,ds = h\big(u(t)\big) - h\big(u(a)\big)$$

↳ constant, since a is constant.

Take derivative w.r.t. ~~time~~ t on both sides

$$u'(t) = \frac{d}{dt}\Big[h\big(u(t)\big)\Big] = g\big(u(t)\big)$$

since $h = \int g$! (fundamental th. of calculus).
$\Rightarrow g = h'$.

Ex 7. Use of the fundamental theorem of calculus.

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Show that

$$u(t) = \frac{1}{\vartheta(t)}\left[u(a) + \int_a^t \vartheta(s) q(s)\, ds\right] \quad \text{- eq. 1}$$

solves

$$u' + p \cdot u = q. \quad \text{- eq. 2.}$$

(the '(t)' is not written for clarity).

Re-arrange 1,

$$\vartheta \cdot u = u(a) + \int_a^t \vartheta(s) q(s)\, ds.$$

LHS RHS.

Time derivative of RHS.

$$\frac{d}{dt}\left[\int_a^t \vartheta(s) q(s)\, ds\right]$$

for this to work, the integral must be definite.

$$= \vartheta(t)\, q(t).$$

due to the theorem.

Time derivative of LHS.

$$\frac{d}{dt}\left[v \cdot u \right] =$$

$$v \frac{du}{dt} + u \frac{dv}{dt}$$

ps tells us

$$\frac{dv}{dt} = v \cdot p \quad \left(v(t) \cdot p(t) \right)$$

v is special, $v = e^{\int p\,dt}$.

$$\therefore \frac{d}{dt}\left[v \cdot u \right] = v \frac{du}{dt} + u\, v\, p$$

$$= v \underbrace{\left[\frac{du}{dt} + p(t)\, u \right]}_{\text{left side of std. form.}}$$

$$= v \cdot q = \text{derivative of RHS.}$$

$\therefore$ u is a solution.