Chapter 18
Linear, First-Order Difference Equations

In this chapter we will learn how to solve autonomous and non-autonomous linear, first-order difference equations. The linear autonomous first-order difference equation has following general form:

$$y_{t+1} = ay_t + b, \ t = 0, 1, 2, .. \tag{18.1}$$

where a and b are known constants. (18.1) can be solved using **iterative method** and **guess and verify method**. In the iterative method we start from the first period ($t = 0$) and solve forward.

The solution is

$$y_t = a^t y_0 + b\left(\frac{1-a^t}{1-a}\right), \text{ if } a \neq 1 \tag{18.2}$$

and

$$y_t = y_0 + bt \text{ if } a = 1 \tag{18.3}$$

(18.2) can also be written as

$$y_t = a^t \left(y_0 - \frac{b}{1-a}\right) + \frac{b}{1-a} \tag{18.4}$$

where as we will see below that $\frac{b}{1-a}$ is **steady state value** of y. Also steady state exists only in the case $a \neq 1$.

If we know the initial condition (y_0) we can use iterative method. In the case, we don't know the initial condition we can still solve the difference equation using **guess and**

verify method. The general approach is very much identical to the one we used in solving first order linear autonomous differential equation.

We divide the difference equation in two parts: **steady state part** and **homogeneous part**. The steady state solution $\overline{y}$ is given by

$$\overline{y} = a\overline{y} + b$$

which implies

$$\overline{y} = \frac{b}{1-a} \tag{18.5}$$

Notice that steady-state solution exists only when $a \neq 1$.

The homogeneous part is given by

$$y_{t+1} = ay_t. \tag{18.6}$$

Now assume that the solution of homogeneous part has following form:

$$y_h = Cd^t \tag{18.7}$$

where C and d are two undetermined coefficients. Basically we have to find C and d which satisfy the differential equation (18.1). To do this, put (18.7) in (18.6), we get

$$Cd^{t+1} = aCd^t$$

which implies

$$d = a. \tag{18.8}$$

Thus the general solution for the homogeneous part is given by

$$y_h = Ca^t. \tag{18.9}$$

The complete solution for the differential equation (18.1) is given by

$$y = y_h + \overline{y}$$

which implies

$$y_t = Ca^t + \frac{b}{1-a} \text{ if } a \neq 1. \qquad (18.10)$$

Important Remark: The form of (18.10) is slightly different from what is given in the text book (Theorem 18.2). In the text book, the solution is of the form

$$y_t = C_1 a^t + b\left(\frac{1-a^t}{1-a}\right) \qquad (18.11)$$

where C_1 is some constant. (18.11) can be rewritten as

$$y_t = C_1\left(1 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}. \qquad (18.12)$$

Thus

$$C = C_1\left(1 - \frac{b}{1-a}\right). \qquad (18.13)$$

If we know, the initial value y_0 then we can pin down C which turns out to be equal to

$$C = y_0 - \frac{b}{1-a}. \tag{18.14}$$

From (18.10) it is immediately clear that the solution will converge to steady state if and only if $|a| < 1$.

In the case, $a = 1$ steady-state does not exist and one has to modify the guess. The homogeneous solution is still given by $y_h = Ca^t$. But with $a = 1$,

$$y_h = C. \tag{18.15}$$

In order to find solution for non-homogeneous part assume that

$$y_{nh} = kt. \tag{18.16}$$

Putting (18.16) in (18.1), we get

$$k(t+1) = kt + b$$

which implies

$$k = b.$$

Thus, the solution for non-homogeneous part is

$$y_{nh} = bt. \tag{18.17}$$

The complete solution in the case $a = 1$ is given by, $y_t = y_h + y_{nh}$ which implies

$$y_t = C + bt. \tag{18.18}$$

Non-autonomous Equation

The general form of linear, non-autonomous, first-order difference equation is

$$y_{t+1} = a_t y_t + b_t,\ t = 0, 1, 2, .. \qquad (18.19)$$

By using **iterative method** one can show that the solution is given by

$$y_t = \prod_{i=0}^{t-1} a_i y_0 + \sum_{k=0}^{t-1} b_k \prod_{i=k}^{t-1} \frac{a_i}{a_k}. \qquad (18.20)$$