Cross Product and Triple Product

1. For this problem, let $\vec{v} = \langle 1, 2, 1\rangle$ and $\vec{w} = \langle 0, -1, 3\rangle$.

   (a) *Compute $\vec{v} \times \vec{w}$.*

   Solution. $\langle 7, -3, -1\rangle$.

   (b) *Compute $\vec{w} \times \vec{v}$.*

   Solution. $\langle -7, 3, 1\rangle$.

   (c) *Let $\vec{u} = \vec{v} \times \vec{w}$, the vector you found in (a). What is the angle between $\vec{u}$ and $\vec{v}$? $\vec{u}$ and $\vec{w}$?*

   Solution. To find the angle between two vectors, we use the dot product.

   $\vec{u} \cdot \vec{v} = \langle 7, -3, -1\rangle \cdot \langle 1, 2, 1\rangle = (7)(1) + (-3)(2) + (-1)(1) = 0$, so $\vec{u}$ is orthogonal to $\vec{v}$.

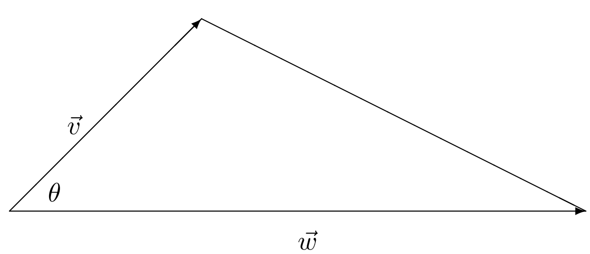
   $\vec{u} \cdot \vec{w} = \langle 7, -3, -1\rangle \cdot \langle 0, -1, 3\rangle = (7)(0) + (-3)(-1) + (-1)(3) = 0$, so $\vec{u}$ is also orthogonal to $\vec{w}$.

2. *In general, what is the relationship between $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$?*

   Solution. From the definition of $\vec{v} \times \vec{w}$, you can compute directly that $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

   Here's another way to get the same conclusion. Let θ be the angle between $\vec{v}$ and $\vec{w}$. Then, we know that the length of $\vec{v} \times \vec{w}$ is $|\vec{v}||\vec{w}| \sin\theta$, and the length of $\vec{w} \times \vec{v}$ is $|\vec{w}||\vec{v}| \sin\theta$. So, $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ have the same length. Also, $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ are both orthogonal to both $\vec{v}$ and $\vec{w}$, so they are either the same vector or negatives of each other. The right-hand rule tells us they must point in opposite directions, so $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

3. *Any two vectors $\vec{v}$ and $\vec{w}$ which are not parallel determine a triangle, as shown. What is the relationship between the area of the triangle and $\vec{v} \times \vec{w}$?*

   Solution. The base of the triangle has length $|\vec{w}|$, and the height of the triangle is $|\vec{v}| \sin\theta$, so the area of the triangle is $\frac{1}{2}|\vec{v}||\vec{w}| \sin\theta$, which is equal to $\frac{1}{2}|\vec{v} \times \vec{w}|$.

   (Note that it is NOT correct to say that the area of the triangle is half of the **vector** $\vec{v} \times \vec{w}$; after all, area is a scalar, not a vector. Rather, the area of the triangle is half of the **length** of the vector $\vec{v} \times \vec{w}$).

4. *If $\vec{v}$ and $\vec{w}$ are parallel, what is $\vec{v} \times \vec{w}$?*

   Solution. If $\vec{v}$ and $\vec{w}$ are parallel, then the angle θ between them is either 0 or π. In either case, $\sin\theta = 0$, so $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta = 0$. This means that $\vec{v} \times \vec{w}$ must be the zero vector $\vec{0}$.

5. *If the scalar triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$ is equal to 0, what can you say about the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$?*

Solution. The fact that $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$ means that $\vec{u}$ is orthogonal to $\vec{v} \times \vec{w}$. We also know that $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$, so this means that $\vec{u}, \vec{v}, \vec{w}$ are *all* orthogonal to the vector $\vec{v} \times \vec{w}$. In particular, if we stick the tails of $\vec{u}$, $\vec{v}$, and $\vec{w}$ at the same point, then $\vec{u}$, $\vec{v}$, and $\vec{w}$ all lie in the same plane. (We say that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are coplanar.)

Note: Some students pointed out a special case: $\vec{u} \cdot (\vec{v} \times \vec{w})$ is equal to 0 if $\vec{v}$ and $\vec{w}$ are parallel (since then $\vec{v} \times \vec{w} = \vec{0}$, by #4). In this case, $\vec{u}$, $\vec{v}$, and $\vec{w}$ are still coplanar. To visualize this, imagine the plane that contains $\vec{u}$ and $\vec{v}$: $\vec{w}$ will automatically be in this plane because it's parallel to $\vec{v}$.

6. *Find an equation for the plane which passes through the points* $(1, 0, 1)$, $(0, 2, 0)$, *and* $(2, 1, 0)$.

   Solution. Let's give the three points names, say $P = (1, 0, 1)$, $Q = (0, 2, 0)$, and $R = (2, 1, 0)$. A point $S = (x, y, z)$ is in the plane if (and only if) the three vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ are coplanar. As we saw in #5, this is the same as saying that $\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR}) = 0$.

   So now we compute some things: $\overrightarrow{PQ} = \langle -1, 2, -1 \rangle$ and $\overrightarrow{PR} = \langle 1, 1, -1 \rangle$, so $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1, -2, -3 \rangle$. $\overrightarrow{PS} = \langle x - 1, y, z - 1 \rangle$, so $\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR}) = 0$ can be rewritten as $-1(x - 1) - 2y - 3(z - 1) = 0$, or $\boxed{x + 2y + 3z = 4}$.

   Note that it is very easy to check that this answer is correct — the three points given in the problem all satisfy this equation.

7. *True or false: If* $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, *then* $\vec{v} = \vec{w}$.

   Solution. False. There are lots of examples where this is not true. A simple one is to suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are all parallel but not equal to each other. Then, $\vec{u} \times \vec{v}$ and $\vec{u} \times \vec{w}$ are both $\vec{0}$, but $\vec{v} \neq \vec{w}$.

8. *True or false: If* $\vec{v} \times \vec{w} = \vec{0}$ *and* $\vec{v} \cdot \vec{w} = 0$, *then at least one of* $\vec{v}$ *and* $\vec{w}$ *must be* $\vec{0}$.

   Solution. True.

   Here's one way to think about it: $\vec{v} \times \vec{w} = \vec{0}$ means $\vec{v}$ and $\vec{w}$ are parallel. $\vec{v} \cdot \vec{w} = 0$ means $\vec{v}$ and $\vec{w}$ are perpendicular. The only way to have a pair of vectors that are both parallel to and perpendicular to each other is if at least one of them is the zero vector $\vec{0}$.

   If you prefer to write out equations, here's another way to think about the problem. Since $\vec{v} \times \vec{w} = \vec{0}$, $|\vec{v} \times \vec{w}| = 0$. If θ is the angle between $\vec{v}$ and $\vec{w}$, then this says that $|\vec{v}||\vec{w}| \sin\theta = 0$. On the other hand, $0 = \vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$. It's not possible for both $\sin\theta$ and $\cos\theta$ to be 0, so it must be the case that $|\vec{v}||\vec{w}| = 0$. That is, one of the vectors $\vec{v}$ and $\vec{w}$ must have length 0, and the only vector with 0 length is $\vec{0}$.