2 Concavity

In the preceding section, you saw how to use the sign of the derivative $f'(x)$ to determine where $f(x)$ is increasing and decreasing and where its graph has relative extrema. In this section, you will see that the second derivative $f''(x)$ also provides useful information about the graph of $f(x)$. By way of introduction, here is a brief description of a situation from industry that can be analyzed using the second derivative.

The Efficiency of a Worker

The number of units that a factory worker can produce in t hours is often given by a function $Q(t)$ like the one whose graph is shown in Figure 3.11.

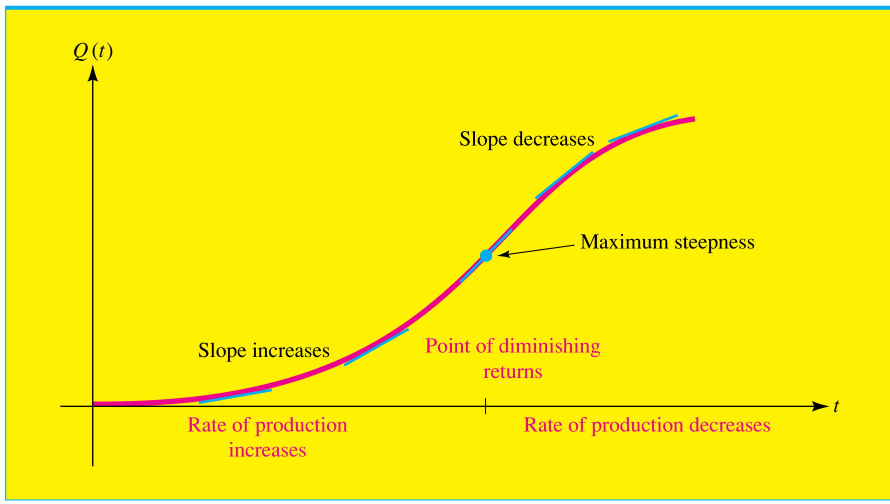


FIGURE 3.11 The output of a factory worker.

Notice that at first the graph is not very steep. The steepness increases, however, until the graph reaches a point of maximum steepness, after which the steepness begins to decrease. This reflects the fact that at first the worker's rate of production is low. The rate of production increases, however, as the worker settles into a routine and continues to increase until the worker is performing at maximum efficiency, after which fatigue sets in and the rate of production begins to decrease. The moment of maximum efficiency is known in economics as the **point of diminishing returns.**

The behavior of the graph of this production function on either side of the point of diminishing returns can be described in terms of its tangent lines. To the left of this point, the slope of the tangent increases as t increases. To the right of this point, the slope of the tangent decreases as t increases. It is this increase and decrease of slopes that we shall examine in this section with the aid of the second derivative. (We shall return to the questions of worker efficiency and diminishing returns in Example 2.4, later in this section.)

CONCAVITY

The following notions of **concavity** are used to describe the increase and decrease of the slope of the tangent to a curve.

Concavity ■ If the function $f(x)$ is differentiable on the interval $a < x < b$, then the graph of f is
concave upward on $a < x < b$ if f' is increasing on the interval
concave downward on $a < x < b$ if f' is decreasing on the interval

In Figure 3.11, for example, the production curve was concave upward to the left of the point of diminishing returns and concave downward to the right of this point. Concavity is illustrated further in Figure 3.12 in which the curve is concave upward to the left of $x = a$ and concave downward to the right. Note that the graph can be described as “spilling water” where it is concave downward and as “holding water” where it is concave upward.

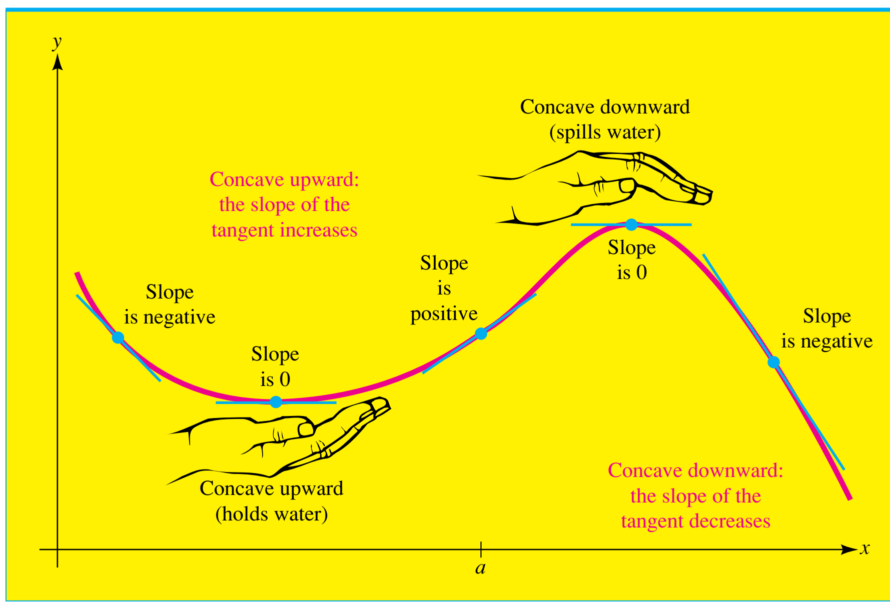


FIGURE 3.12 Concavity and the slope of the tangent.

THE SIGN OF THE SECOND DERIVATIVE

There is a simple characterization of concavity in terms of the sign of the second derivative. It is based on the fact (established in the preceding section) that a quantity increases when its derivative is positive and decreases when its derivative is negative. The second derivative comes into the picture when this fact is applied to the first derivative (or slope of the tangent). Here is the argument.

Suppose the second derivative f'' is positive on an interval I. This implies that the first derivative f' must be increasing on I, so the graph of f is concave upward on I. Similarly, if $f'' < 0$ on an interval, then f' is decreasing there and the graph of f is concave downward. To summarize:

Test for Concavity

If $f''(x) > 0$ on the interval $a < x < b$, then f is concave upward on this interval. If $f''(x) < 0$ on the interval $a < x < b$, then f is concave downward on this interval.

A Word of Advice

Do not confuse the concavity of a curve with its increase or decrease. A curve that is concave upward on an interval may be either increasing or decreasing on that interval. Similarly, a curve that is concave downward may be increasing or decreasing. The four possibilities are illustrated in Figure 3.13.

Explore!

As in the Explore exercise on page 205, store $f(x) = x^3 - x^2 - 4x + 4$ into Y1, $f'(x)$ into Y2 using the bold graphing style and $f''(x)$ into Y3, with a viewing window set at $[-4.7, 4.7]1$ by $[-10, 10]2$. How does the value of $f''(x)$ relate to the concavity of $f(x)$ at $x = 2$? What occurs on the graph of $f(x)$ when $f''(x) = 0$?

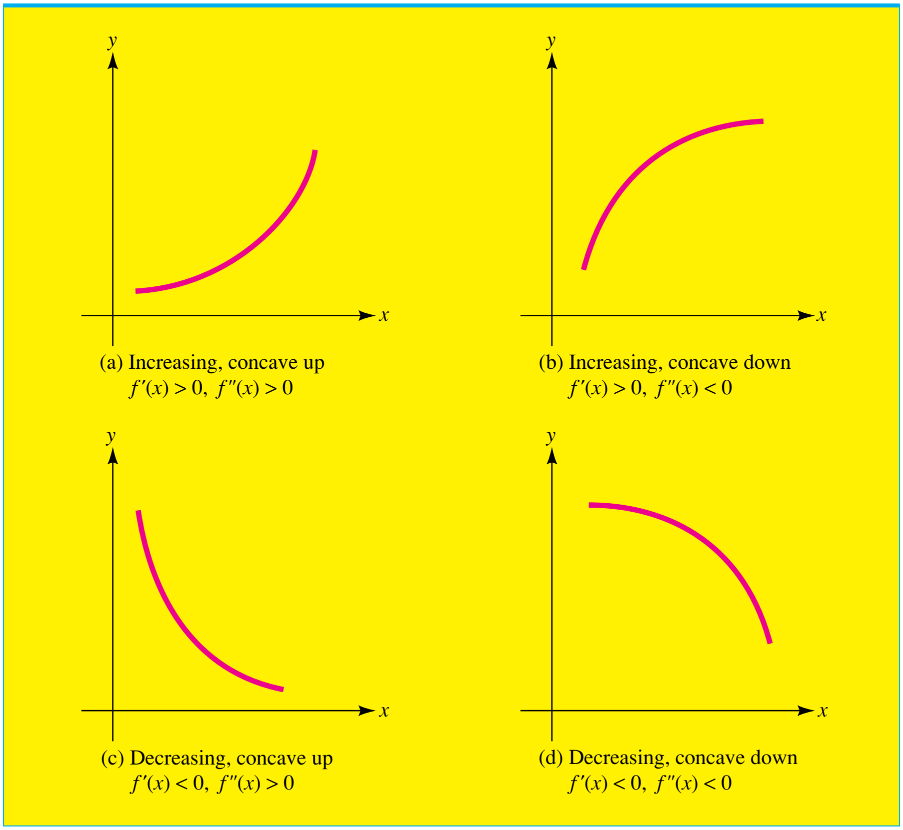


FIGURE 3.13 Possible combinations of increase, decrease, and concavity.

Inflection Points

A point on the graph of a function $f(x)$ where the concavity changes is called an **inflection point.** For example, the point of diminishing returns for the production curve in Figure 3.11 corresponds to an inflection point, and the function in Figure 3.12 has an inflection point at $x = a$.

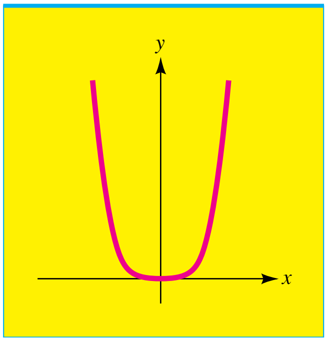


FIGURE 3.14 The graph of $f(x) = x^4$.

At an inflection point $P(c, f(c))$, the graph of $f(x)$ can be neither concave up nor concave down, so $f''(c)$ cannot be positive or negative. Thus, either $f''(c)$ does not exist or $f''(c) = 0$. However, just because $f''(c) = 0$, you cannot automatically conclude that $(c, f(c))$ is an inflection point. For example, if $f(x) = x^4$, then $f''(x) = 12x^2$, and the graph of f is always concave up even though $f''(0) = 0$ (see Figure 3.14). Here is a procedure for determining the concavity of a graph and locating its inflection points.

Procedure for Determining Concavity and Locating Inflection Points of $f(x)$

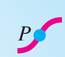
Step 1. Find f'' and determine numbers where $f''(x) = 0$ or $f''(x)$ does not exist.
Step 2. Mark the numbers from step 1 on a number line, dividing the line into a number of subintervals. Evaluate $f''(p)$ for a test number p in each subinterval.
Step 3. The graph is concave up on those intervals where $f''(p) > 0$ and concave down where $f''(p) < 0$. The inflection points are the subdivision points where the concavity changes.

Curve Sketching with the Second Derivative

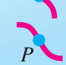
Geometrically, inflection points occur at "twists" on a graph, as indicated in the following.

Behavior of the Graph of $f(x)$ at an Inflection Point $P(c, f(c))$

Graph is rising ($f' > 0$)

Before $P(x < c)$	After $P(x > c)$	Shape of graph at P
$f'' > 0$	$f'' < 0$	
$f'' < 0$	$f'' > 0$	

Graph is falling ($f' < 0$)

Before $P(x < c)$	After $P(x > c)$	Shape of graph at P
$f'' > 0$	$f'' < 0$	
$f'' < 0$	$f'' > 0$	

By adding the criteria for concavity and inflection points to the first derivative methods discussed in Section 1, you can sketch a variety of graphs with considerable detail. Here is an example.

EXAMPLE 2.1

Determine where the function

$$f(x) = 3x^4 - 2x^3 - 12x^2 + 18x + 15$$

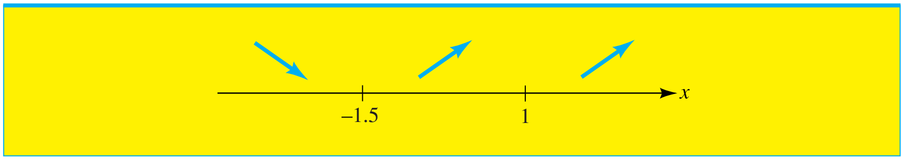
is increasing and decreasing, and where its graph is concave up and concave down. Find all relative extrema and points of inflection, and sketch the graph.

Solution

First, note that since $f(x)$ is a polynomial, it is continuous for all x, as are the derivatives $f'(x)$ and $f''(x)$. The first derivative of $f(x)$ is

$$f'(x) = 12x^3 - 6x^2 - 24x + 18 = 6(x - 1)^2(2x + 3)$$

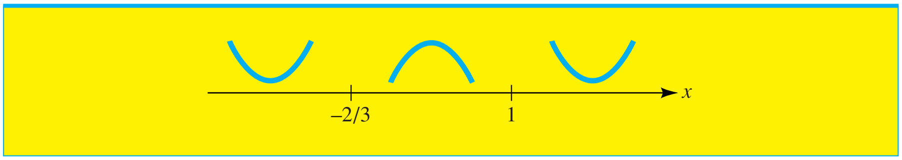
and $f'(x) = 0$ only when $x = 1$ and $x = -1.5$. The sign of $f'(x)$ does not change for $x < -1.5$, nor in the interval $-1.5 < x < 1$, nor for $x > 1$. Evaluating $f'(x)$ at test numbers in each interval (say, at -2, 0, and 3), you obtain the arrow diagram shown. Note that there is a relative minimum at $x = -1.5$ but no extremum at $x = 1$.



The second derivative is

$$f''(x) = 36x^2 - 12x - 24 = 12(x - 1)(3x + 2)$$

and $f''(x) = 0$ only when $x = 1$ and $x = \frac{-2}{3}$. The sign of $f''(x)$ does not change for $x < \frac{-2}{3}$, nor in the interval $\frac{-2}{3} < x < 1$, nor for $x > 1$. Evaluating $f''(x)$ at test numbers in each interval, you obtain the concavity diagram shown.



The patterns in these two diagrams suggest that there is a relative minimum at $x = -1.5$ and inflection points at $x = \frac{-2}{3}$ and $x = 1$ (since the concavity changes at both points).

To sketch the graph, you find that

$$f(-1.5) = -17.06 \quad f\left(\frac{-2}{3}\right) = -1.15 \quad f(1) = 22$$

and plot a "cup" ◡ at (−1.5, −17.06) to mark the relative minimum located there. Likewise, plot twists ∫ at $\left(\frac{-2}{3}, -1.15\right)$ and ⌐ at (1, 22) to mark the shape of the graph near the inflection points. Using the arrow and concavity diagrams, you get the preliminary diagram shown in Figure 3.15a. Finally, complete the sketch as shown in Figure 3.15b by passing a smooth curve through the minimum point, the inflection points, and the *y* intercept (0, 15).

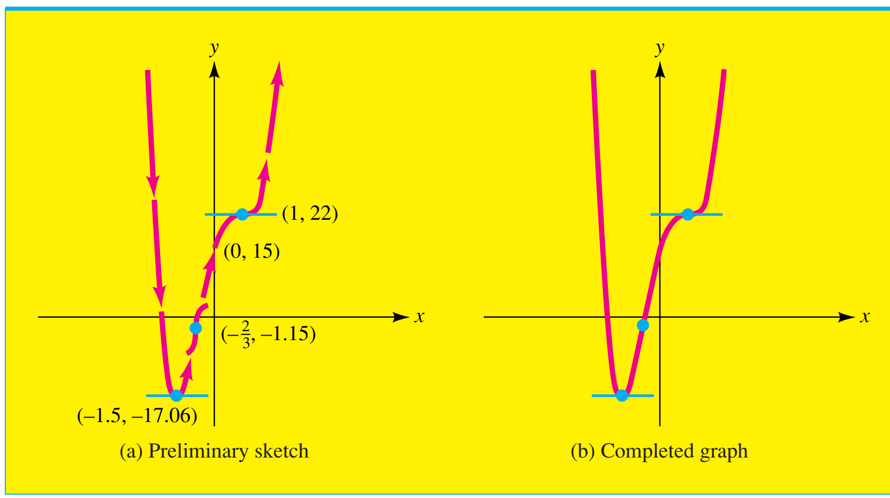


FIGURE 3.15 The graph of $f(x) = 3x^4 - 2x^3 - 12x^2 + 18x + 15$.

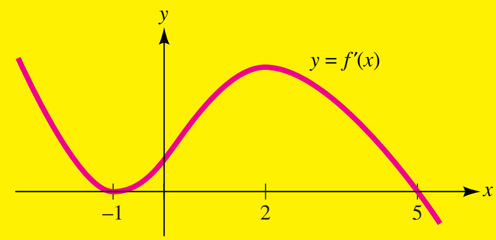
Explore!

Graphically confirm the results of Example 2.1 by first graphing $f(x)$ in bold against $f'(x)$ and observing the location of relative extrema. Then graph $f(x)$ in bold against $f''(x)$ and observe how the concavity of $f(x)$ changes.

Sometimes you are given the graph of a derivative function $f'(x)$ and asked to analyze the graph of $f(x)$ itself. For instance, it would be quite reasonable for a manufacturer who knows the marginal cost $C'(x)$ associated with producing x units of a particular commodity to want to know as much as possible about the total cost $C(x)$. The following example illustrates a procedure for carrying out this kind of analysis.

EXAMPLE 2.2

The graph of the derivative $f'(x)$ of a function $f(x)$ is shown in the figure. Find intervals of increase and decrease and concavity for $f(x)$ and locate all relative extrema and inflection points. Then sketch a curve that has all these features.



Solution

First, note that for $x < -1$, the graph of $f'(x)$ is above the x axis, so $f'(x) > 0$ and the graph of $f(x)$ is rising. The graph of $f'(x)$ is also falling for $x < -1$, which means that $f''(x) < 0$ and the graph of $f(x)$ is concave down. Other intervals can be analyzed in a similar fashion, and the results are summarized in the table.

x	Feature of $y = f'(x)$	Feature of $y = f(x)$
$x < -1$	Positive; decreasing	Increasing; concave down
$x = -1$	x intercept; horizontal tangent	Horizontal tangent; possible inflection point ($f'' = 0$)
$-1 < x < 2$	Positive; increasing	Increasing; concave up
$x = 2$	Horizontal tangent	Possible inflection point
$2 < x < 5$	Positive; decreasing	Increasing; concave down
$x = 5$	x intercept	Horizontal tangent
$x > 5$	Negative; decreasing	Decreasing; concave down

Since the concavity changes at $x = -1$ (down to up), an inflection point occurs there, along with a horizontal tangent. At $x = 2$, there is also an inflection point (concavity changes from up to down), but no horizontal tangent. The graph of $f(x)$ is rising to the left of $x = 5$ and falling to the right, so there must be a relative maximum at $x = 5$.

One possible graph that has all the features required for $y = f(x)$ is shown in Figure 3.16. Note, however, that since you are not given the values of $f(-1)$, $f(2)$, and $f(5)$, many other graphs will also satisfy the requirements.

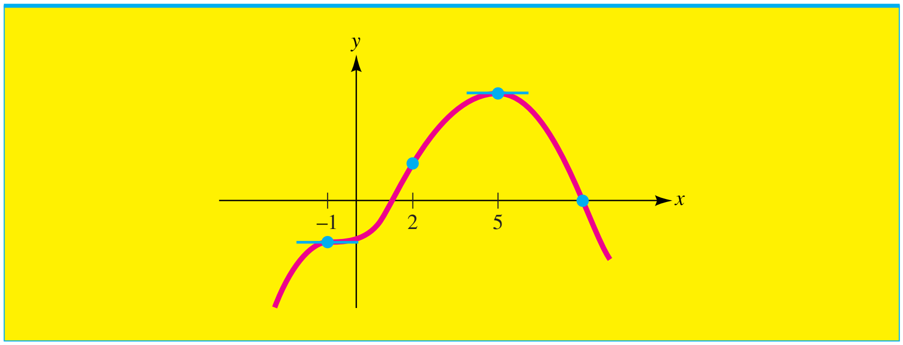


FIGURE 3.16 A possible graph of $f(x)$.

THE SECOND DERIVATIVE TEST

The second derivative can be used to classify the first-order critical points of a function as relative maxima or relative minima. Here is a statement of the procedure, which is known as the **second derivative test.**

The Second Derivative Test ■ Suppose $f'(a) = 0$.

If $f''(a) > 0$, then f has a relative minimum at $x = a$.
If $f''(a) < 0$, then f has a relative maximum at $x = a$.

However, if $f''(a) = 0$ or if $f''(a)$ does not exist, the test is inconclusive and f may have a relative maximum, a relative minimum, or no relative extremum at all at $x = a$.

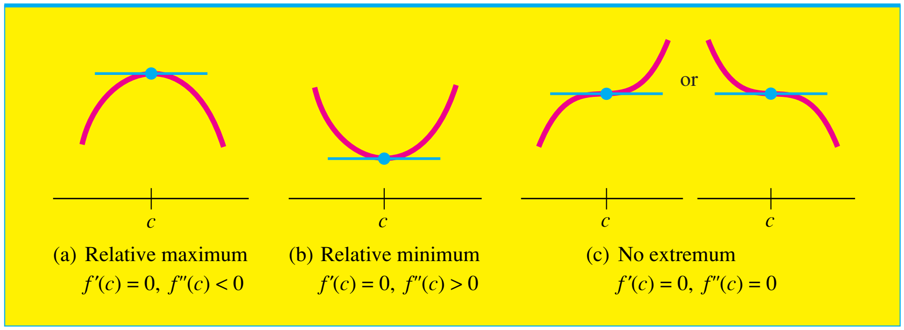


FIGURE 3.17 The second derivative test.

To see why the second derivative test works, look at Figure 3.17, which shows the four possibilities that can occur when $f'(c) = 0$. Figure 3.17a suggests that at a relative maximum, the graph of f must be concave downward, so $f''(c) < 0$. Likewise, at a relative minimum (Figure 3.17b), the graph of f must be concave upward, so $f''(c) > 0$. On the other hand, Figure 3.17c suggests that if a point where $f'(c) = 0$ is not a relative extremum, then it must be an inflection point and $f''(c) = 0$ (if $f''(c)$ is defined). It follows that if $f'(c) = 0$ and $f''(c) < 0$, then $P(c, f(c))$ must be a relative maximum, while if $f'(c) = 0$ and $f''(c) > 0$, the corresponding critical point must be a relative minimum.

The use of the second derivative test is illustrated in the following example.

EXAMPLE 2.3

Use the second derivative test to find the relative maxima and minima of the function $f(x) = 2x^3 + 3x^2 - 12x - 7$.

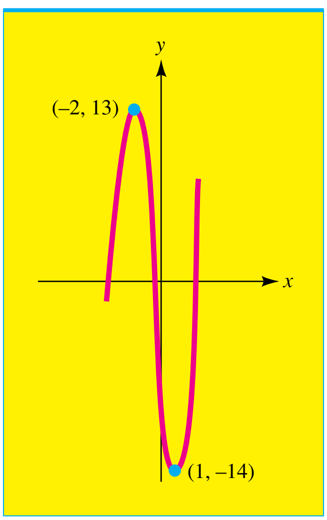


FIGURE 3.18 The graph of $f(x) = 2x^3 + 3x^2 - 12x - 7$.

Solution

Since the first derivative

$$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$$

is zero when $x = -2$ and $x = 1$, the corresponding points $(-2, 13)$ and $(1, -14)$ are the first-order critical points of f. To test these points, compute the second derivative

$$f''(x) = 12x + 6$$

and evaluate it at $x = -2$ and $x = 1$. Since

$$f''(-2) = -18 < 0$$

it follows that the critical point $(-2, 13)$ is a relative maximum, and since

$$f''(1) = 18 > 0$$

it follows that the critical point $(1, -14)$ is a relative minimum. For reference, the graph of f is sketched in Figure 3.18.

Note

Although it was easy to use the second derivative test to classify critical points in Example 2.3, the test does have some limitations. For some functions, the work involved in computing the second derivative is time-consuming, which may diminish the appeal of the test. Moreover, the test applies only to critical points at which the derivative is zero and not to those where the derivative is undefined. Finally, if both $f'(c)$ and $f''(c)$ are zero, the second derivative test tells you nothing about the nature of the critical point. This is illustrated in Figure 3.19, which shows the graphs of three functions whose first and second derivatives are both zero when $x = 0$. When it is inconvenient or impossible to apply the second derivative test, you can always use the first derivative test described in Section 1 to classify critical points.

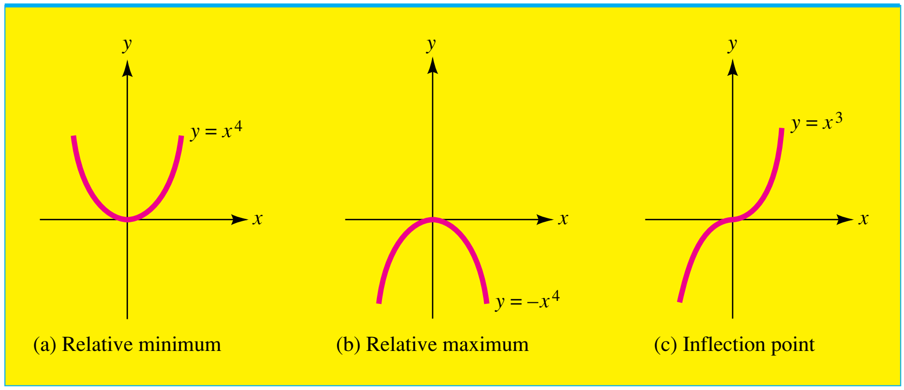


FIGURE 3.19 Three functions whose first and second derivatives are zero at $x = 0$.

In our next example, we return to the questions of worker efficiency and diminishing returns examined in the illustration at the beginning of this section. Our goal is to maximize a worker's *rate* of production; that is, the derivative of the worker's output. Hence, we will set to zero the *second* derivative of output and find an inflection point of the output function, which we interpret as the point of diminishing returns for the output.

EXAMPLE 2.4

An efficiency study of the morning shift at a factory indicates that an average worker who starts at 8:00 A.M. will have produced $Q(t) = -t^3 + 9t^2 + 12t$ units t hours later. At what time during the morning is the worker performing most efficiently?

Solution

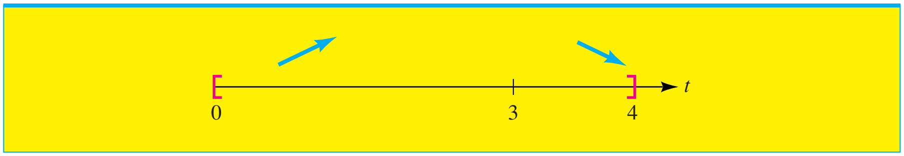
The worker's rate of production is the derivative

$$R(t) = Q'(t) = -3t^2 + 18t + 12$$

of the output function $Q(t)$. Assuming that the morning shift runs from 8:00 A.M. until noon, the goal is to find the largest rate $R(t)$ for $0 \leq t \leq 4$. The derivative of the rate function is

$$R'(t) = Q''(t) = -6t + 18$$

which is zero when $t = 3$, positive for $0 < t < 3$, and negative for $3 < t < 4$, as indicated in the arrow diagram shown.



Thus, the rate of production $R(t)$ increases for $0 < t < 3$, decreases for $3 < t < 4$, and has its maximum value when $t = 3$; that is, at 11:00 A.M. The graphs of the production function $Q(t)$ and the rate of production function $R(t)$ are shown in Figure 3.20.

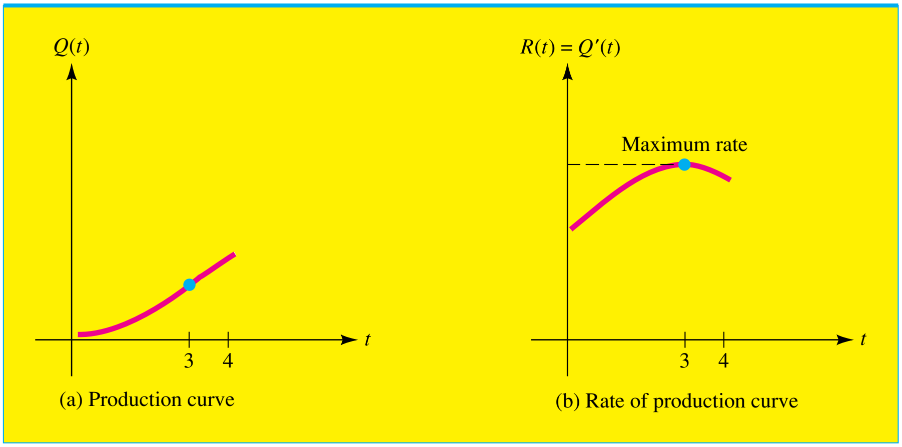
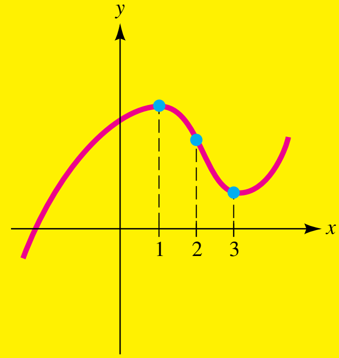


FIGURE 3.20 The production of an average worker.

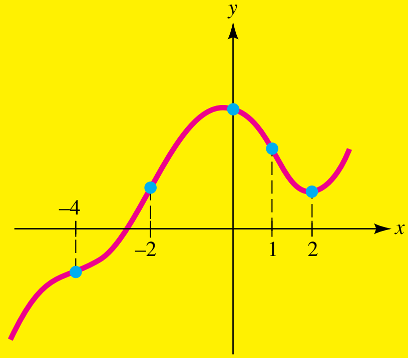
P . R . O . B . L . E . M . S 3.2

In Problems 1 and 2, determine where the second derivative of the function is positive and where it is negative.

1.



2.



In Problems 3 through 20, determine where the given function is increasing and decreasing, and where its graph is concave up and concave down. Find the relative extrema and inflection points, and sketch the graph of the function.

3. $f(x) = \frac{1}{3}x^3 - 9x + 2$

4. $f(x) = x^3 + 3x^2 + 1$

5. $f(x) = x^4 - 4x^3 + 10$

6. $f(x) = x^3 - 3x^2 + 3x + 1$

7. $f(x) = (x - 2)^3$

8. $f(x) = x^5 - 5x$

9. $f(x) = (x^2 - 5)^3$

10. $f(x) = (x - 2)^4$

11. $f(s) = 2s(s + 4)^3$

12. $f(x) = (x^2 - 3)^2$

13. $f(x) = (x + 1)^{1/3}$

14. $f(x) = \frac{x^2 - 3x}{x^2 + 1}$

15. $f(x) = (x + 1)^{4/3}$

16. $f(x) = (x + 1)^{2/3}$

17. $g(x) = \sqrt{x^2 + 1}$

18. $g(x) = (x + 1)^{5/3}$

19. $f(x) = \frac{x}{x^2 + x + 1}$

20. $f(x) = x^4 + 6x^3 - 24x^2 + 24$

In Problems 21 through 32 use the second derivative test to find the relative maxima and minima of the given function.

21. $f(x) = x^3 + 3x^2 + 1$

22. $f(x) = x^4 - 2x^2 + 3$

23. $f(x) = (x^2 - 9)^2$

24. $f(x) = x + \frac{1}{x}$

25. $f(x) = 2x + 1 + \frac{18}{x}$

26. $f(x) = \frac{x^2}{x - 2}$

27. $f(x) = x^2(x - 5)^2$

28. $f(x) = \left(\frac{x}{x + 1}\right)^2$

29. $h(t) = \dfrac{2}{1 + t^2}$

30. $f(s) = \dfrac{s + 1}{(s - 1)^2}$

31. $f(x) = \dfrac{(x - 2)^3}{x^2}$

32. $h(t) = \dfrac{(t + 3)^3}{(t - 1)^2}$

In Problems 33 through 36, the first derivative $f'(x)$ *of a certain function* $f(x)$ *is given. In each case,*

(a) Find intervals on which f *is increasing and decreasing.*
(b) Find intervals on which the graph of f *is concave up and concave down.*
(c) Find the x *coordinates of the relative extrema and inflection points of* f.
(d) Sketch a possible graph for $f(x)$.

33. $f'(x) = x^2 - 4x$

34. $f'(x) = x^2 - 2x - 8$

35. $f'(x) = 5 - x^2$

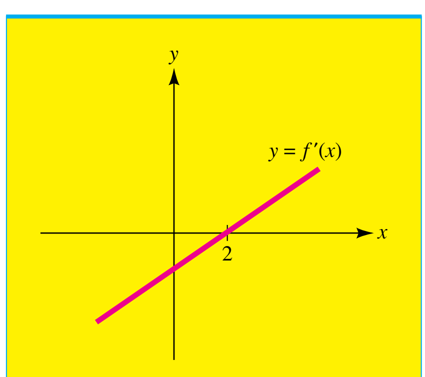
36. $f'(x) = x(1 - x)$

37. Sketch the graph of a function that has all of the following properties:
(a) $f'(x) > 0$ when $x < -1$ and when $x > 3$
(b) $f'(x) < 0$ when $-1 < x < 3$
(c) $f''(x) < 0$ when $x < 2$
(d) $f''(x) > 0$ when $x > 2$

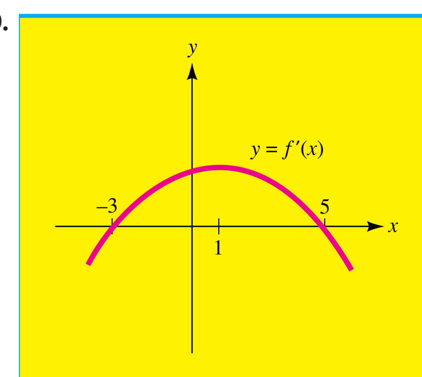
38. Sketch the graph of a function f that has all of the following properties:
(a) The graph has discontinuities at $x = -1$ and at $x = 3$
(b) $f'(x) > 0$ for $x < 1$, $x \neq -1$
(c) $f'(x) < 0$ for $x > 1$, $x \neq 3$
(d) $f''(x) > 0$ for $x < -1$ and $x > 3$ and $f''(x) < 0$ for $-1 < x < 3$
(e) $f(0) = 0 = f(2)$, $f(1) = 3$

In Problems 39 through 42 the graph of a derivative function $y = f'(x)$ *is given. Describe the function* $y = f(x)$ *and sketch a possible graph of* $f(x)$.

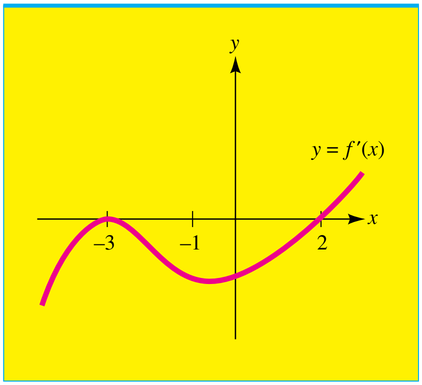
39.



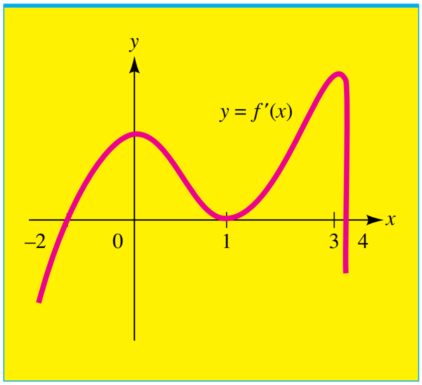
40.



41.



42.



MARGINAL ANALYSIS **43.** The cost of producing x units of a commodity per week is

$$C(x) = 0.3x^3 - 5x^2 + 28x + 200$$

(a) Find the marginal cost $MC(x) = C'(x)$. Sketch the graphs of $C(x)$ and $MC(x)$ on the same coordinate axes.
(b) Find where $C''(x) = 0$. How are these values of x related to the graph of $MC(x)$?

SALES **44.** A company estimates that when x thousand dollars are spent on the marketing of a certain product, $Q(x)$ units of the product will be sold, where

$$Q(x) = -4x^3 + 252x^2 - 3{,}200x + 17{,}000 \quad 10 \le x \le 40$$

Sketch the graph of $Q(x)$ for $10 \le x \le 40$. Where does the graph have an inflection point? What is the significance of the marketing expenditure that corresponds to this point?

WORKER EFFICIENCY **45.** An efficiency study of the morning shift (from 8:00 A.M. to 12:00 noon) at a factory indicates that an average worker who arrives on the job at 8:00 A.M. will have produced $Q(t) = -t^3 + \frac{9}{2}t^2 + 15t$ units t hours later.

(a) At what time during the morning is the worker performing most efficiently?
(b) At what time during the morning is the worker performing least efficiently?

WORKER EFFICIENCY **46.** An efficiency study of the morning shift (from 8:00 A.M. to 12:00 noon at a certain factory indicates that an average worker who arrives on the job at 8:00 A.M. will have assembled $Q(t) = -t^3 + 6t^2 + 15t$ transistor radios t hours later.
(a) At what time during the morning is the worker performing most efficiently?
(b) At what time during the morning is the worker performing least efficiently?

POPULATION GROWTH **47.** A 5-year projection of population trends suggests that t years from now, the population of a certain community will be $P(t) = -t^3 + 9t^2 + 48t + 50$ thousand.

(a) At what time during the 5-year period will the population be growing most rapidly?
(b) At what time during the 5-year period will the population be growing least rapidly?
(c) At what time is the rate of population growth changing most rapidly?

ELIMINATION OF HAZARDOUS WASTE

48. Certain hazardous waste products have the property that as the concentration of substrate (the substance undergoing change by enzymatic action) increases, there is a toxic inhibition effect. A mathematical model for this behavior is the **Haldane equation***

$$R(S) = \frac{cS}{a + S + bS^2}$$

where R is the specific growth rate of the substance (the rate at which cells divide); S is the substrate concentration; and a, b, and c are positive constants.
(a) Sketch the graph of $R(S)$. Does the graph appear to have a highest point? A lowest point? A point of inflection? What happens to the growth rate as S grows larger and larger? Interpret your observations.
(b) Read an article on hazardous waste management, and write a paragraph on how mathematical models are used to develop methods for eliminating waste. A good place to start is the reference cited here.

POPULATION GROWTH

49. Studies show that when environmental factors impose an upper bound on the possible size of a population $P(t)$, the population often tends to grow in such a way that the percentage rate of change of $P(t)$ satisfies

$$\frac{100\,P'(t)}{P(t)} = A - BP(t)$$

where A and B are positive constants. Where does the graph of $P(t)$ have an inflection point? What is the significance of this point? (Your answer will be in terms of A and B.)

THE SPREAD OF AN EPIDEMIC

50. Let $Q(t)$ denote the number of people in a city of population N_0 who have been infected with a certain disease t days after the beginning of an epidemic. Studies indicate that the rate $R(Q)$ at which an epidemic spreads is jointly proportional to the number of people who have contracted the disease and the number who have not, so $R(Q) = kQ(N_0 - Q)$. Sketch the graph of the rate function, and interpret your graph. In particular, what is the significance of the highest point on the graph of $R(Q)$?

51. Use calculus to show that the graph of the quadratic function $y = ax^2 + bx + c$ is concave upward if a is positive and concave downward if a is negative.

* Michael D. La Grega, Philip L. Buckingham, and Jeffrey C. Evans, *Hazardous Waste Management,* McGraw-Hill, New York, 1994, page 578.

52. Given the function $f(x) = 2x^3 + 3x^2 - 12x - 7$, complete the following steps:
(a) Graph using $[-10, 10]1$ by $[-10, 10]1$ and $[-10, 10]1$ by $[-20, 20]2$.
(b) Fill in the following table:

x	-4	-2	-1	0	1	2
$f(x)$						
$f'(x)$						
$f''(x)$						

(c) Find the x intercepts and the y intercepts.
(d) Approximate the relative maximum and relative minimum points to two decimal places.
(e) Find the intervals over which $f(x)$ is increasing.
(f) Find the intervals over which $f(x)$ is decreasing.
(g) Find any inflection points.
(h) Find the intervals over which the graph of $f(x)$ is concave upward.
(i) Find the intervals over which the graph of $f(x)$ is concave downward.
(j) Show that the concavity changes from upward to downward, or vice versa, when x moves from a little less than the point of inflection to a little greater than the point of inflection.
(k) Find the largest and smallest values for this function for $-4 \le x \le 2$.

53. Repeat Problem 52 for the function

$$f(x) = 3.7x^4 - 5.03x^3 + 2x^2 - 0.7$$

3 Limits Involving Infinity: Asymptotes

So far in this chapter, you have seen how to use the first derivative of a function $f(x)$ to find where the graph of $f(x)$ is rising and falling, and to use the second derivative to determine the graph's concavity. In this section, you will see how to sketch portions of the graph of $f(x)$ where x or $f(x)$ increase or decrease without bound. We then collect our various curve-sketching methods into a general procedure and use it to analyze several examples. First, however, we need to discuss what is meant by taking a limit of $f(x)$ as x increases or decreases without bound.

Limits Involving Infinity

The symbol ∞ is used in mathematics to represent a quantity that becomes large (or small) beyond any finite bound. It is important to remember that ∞ *never* represents a number. However, there are times when using the symbol ∞ in a limit expresses useful information.