Bayesian Statistics Made Simple

Allen B. Downey
Olin College

sites.google.com/site/simplebayes


Seabagel
Olin
College
KAZAK
2

Follow along at home

sites.google.com/site/simplebayes

The plan

From Bayes's Theorem to Bayesian inference.

A computational framework.

Work on example problems.

Goals

By the end, you should be ready to:

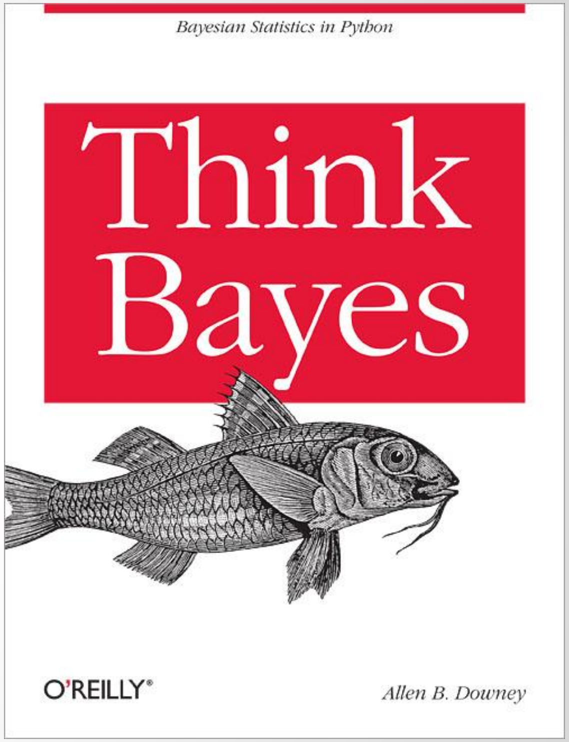
- Work on similar problems.
- Learn more on your own.

Think Bayes

This tutorial is based on my book,

Think Bayes

Bayesian Statistics in Python



Published by O'Reilly Media and available under a Creative Commons license from

`thinkbayes.com`

Probability

$p(A)$: the probability that A occurs.

$p(A|B)$: the probability that A occurs, given that B has occurred.

$$p(A \text{ and } B) = p(A)\, p(B|A)$$

Bayes's Theorem

By definition of conjoint probability:

$p(A \text{ and } B) = p(A)\ p(B|A) =$ (1)

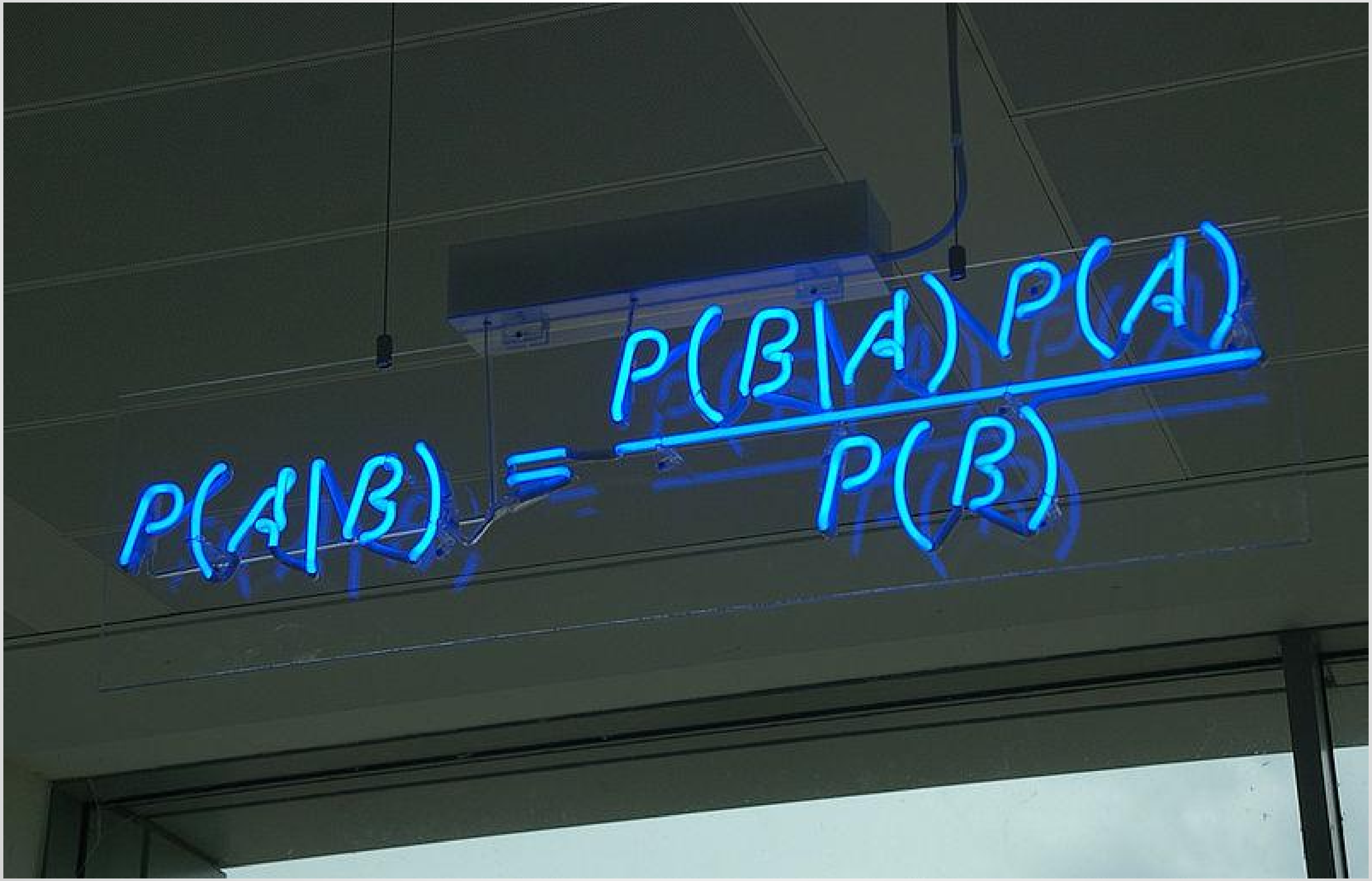
$p(B \text{ and } A) = p(B)\ p(A|B)$

Equate the right hand sides

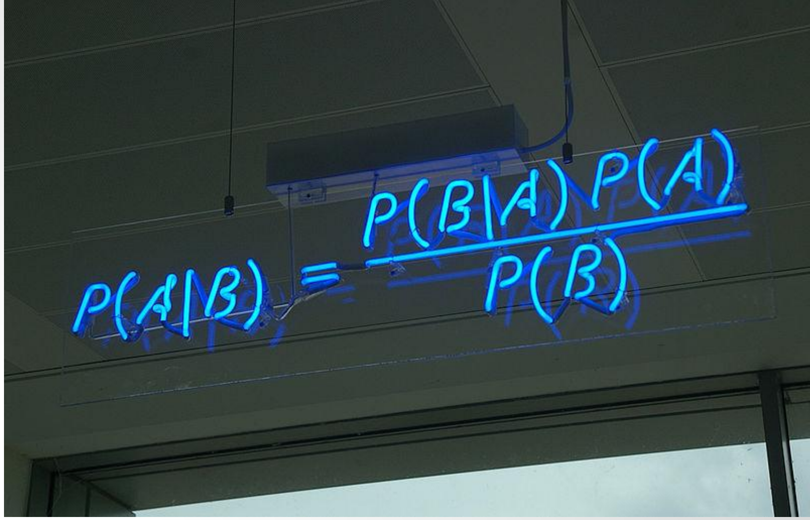
$p(B)\ p(A|B) = p(A)\ p(B|A)$ (2)

Divide by p(B) and ...

Bayes's Theorem



Bayes's Theorem



One way to think about it:

Bayes's Theorem is an algorithm to get from $p(B|A)$ to $p(A|B)$.

Useful if $p(B|A)$, $p(A)$ and $p(B)$ are easier than $p(A|B)$.

OR ...

Diachronic interpretation

H: Hypothesis

D: Data

Given $p(H)$, the probability of the hypothesis before you saw the data.

Find $p(H|D)$, the probability of the hypothesis after you saw the data.

A cookie problem

from Wikipedia

Suppose there are two bowls of cookies.

Bowl #1 has 10 chocolate and 30 vanilla.

Bowl #2 has 20 of each.

Fred picks a bowl at random, and then picks a cookie at random. The cookie turns out to be vanilla.

What is the probability that Fred picked from Bowl #1?

Cookie problem

H: Hypothesis that cookie came from Bowl 1.

D: Cookie is vanilla.

Given $p(H)$, the probability of the hypothesis before you saw the data.

Find $p(H|D)$, the probability of the hypothesis after you saw the data.

Diachronic interpretation

$p(H|D) = p(H)\, p(D|H) / p(D)$

$p(H)$: prior

$p(D|H)$: conditional likelihood of the data

$p(D)$: total likelihood of the data

Diachronic interpretation

$p(H|D) = p(H)\ p(D|H)\ /\ p(D)$

$p(H)$: prior = 1/2

$p(D|H)$: conditional likelihood of the data = 3/4

$p(D)$: total likelihood of the data = 5/8

Diachronic interpretation

$p(H|D) = (1/2)(3/4) / (5/8) = 3/5$

$p(H)$: prior = 1/2

$p(D|H)$: conditional likelihood of the data = 3/4

$p(D)$: total likelihood of the data = 5/8

A little intuition

$p(H)$: prior = 50%

$p(H|D)$: posterior = 60%

Vanilla cookie was more likely under H.

Slightly increases our degree of belief in H.

Computation

Pmf represents a Probability Mass Function

Maps from possible values to probabilities.

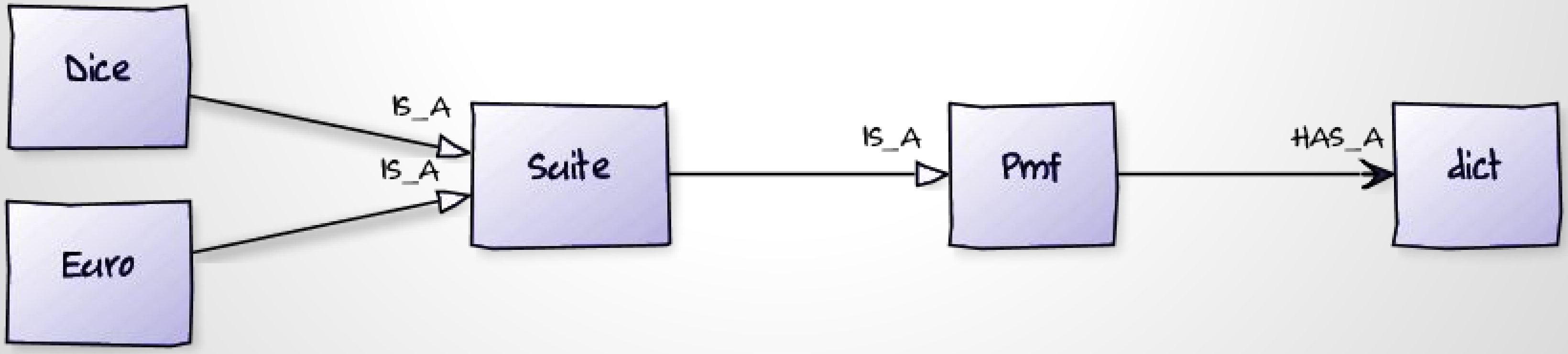


Diagram by yuml.me

Install test

How many of you got `install_test.py` running?

Don't try to fix it now!

Instead...

Partner up

- If you don't have a working environment, find a neighbor who does.
- Even if you do, try pair programming!
- Take a minute to introduce yourself.
- Questions? Ask your partner first (please).

Icebreaker

What was your first computer?

What was your first programming language?

What is the longest time you have spent finding a stupid bug?

Start your engines

1. You cloned BayesMadeSimple, right?
2. `cd` into that directory.
3. Start the Python interpreter.

```
$ python
>>> from thinkbayes import Pmf
```

Or IPython

1. `cd` into BayesMadeSimple.
2. Start IPython.
3. Create a new notebook.

```
$ ipython notebook --matplotlib inline
```

```
from thinkbayes import Pmf
```

Pmf

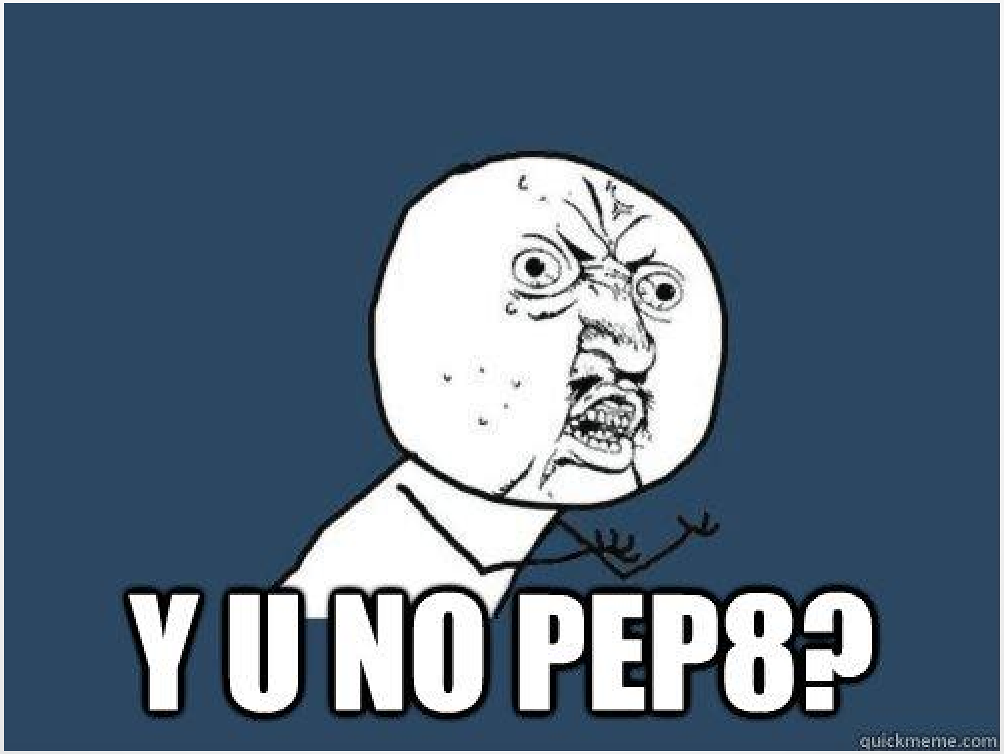
```
from thinkbayes import Pmf

# make an empty Pmf
d6 = Pmf()

# outcomes of a six-sided die
for x in [1,2,3,4,5,6]:
    d6.Set(x, 1)
```

Pmf

```
d6.Print()

d6.Normalize()

d6.Print()

d6.Random()
```

Style

I know what you're thinking.



The Bayesian framework

1) Build a Pmf that maps from each hypothesis to a prior probability, $p(H)$.

2) Multiply each prior probability by the likelihood of the data, $p(D|H)$.

3) Normalize, which divides through by the total likelihood, $p(D)$.

Prior

```
pmf = Pmf()
pmf.Set('Bowl 1', 0.5)
pmf.Set('Bowl 2', 0.5)
```

Update

```
p(Vanilla | Bowl 1) = 30/40
p(Vanilla | Bowl 2) = 20/40
```

```
pmf.Mult('Bowl 1', 0.75)
pmf.Mult('Bowl 2', 0.5)
```

Normalize

```
pmf.Normalize()
0.625     # return value is p(D)

print pmf.Prob('Bowl 1')
0.6
```

Exercise

What if we select another cookie, and it’s chocolate?

The posterior (after the first cookie) becomes the prior (before the second cookie).

Exercise

What if we select another cookie, and it’s chocolate?

```
pmf.Mult('Bowl 1', 0.25)
pmf.Mult('Bowl 2', 0.5)
pmf.Normalize()
pmf.Print()
Bowl 1 0.43
Bowl 2 0.573
```

Summary

Bayes's Theorem,
Cookie problem,
Pmf class.

The dice problem

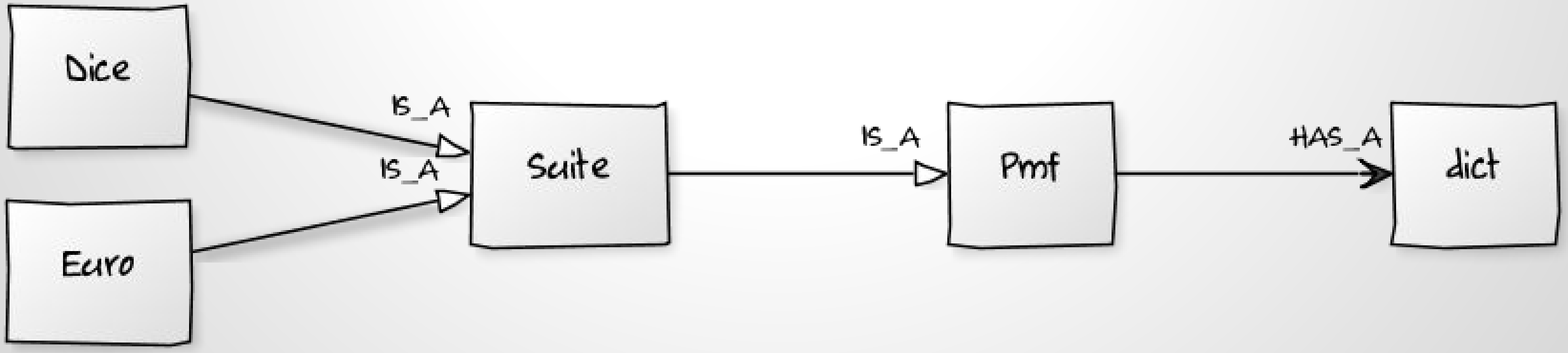
I have a box of dice that contains a 4-sided die, a 6-sided die, an 8-sided die, a 12-sided die and a 20-sided die.

Suppose I select a die from the box at random, roll it, and get a 6. What is the probability that I rolled each die?

Hypothesis suites

A suite is a mutually exclusive and collectively exhaustive set of hypotheses.

Represented by a `Suite` that maps
    hypothesis → probability.



Suite

```
class Suite(Pmf):
    "Represents a suite of hypotheses and
     their probabilities."

    def __init__(self, hypos):
        "Initializes the distribution."
        for hypo in hypos:
            self.Set(hypo, 1)
        self.Normalize()
```

Suite

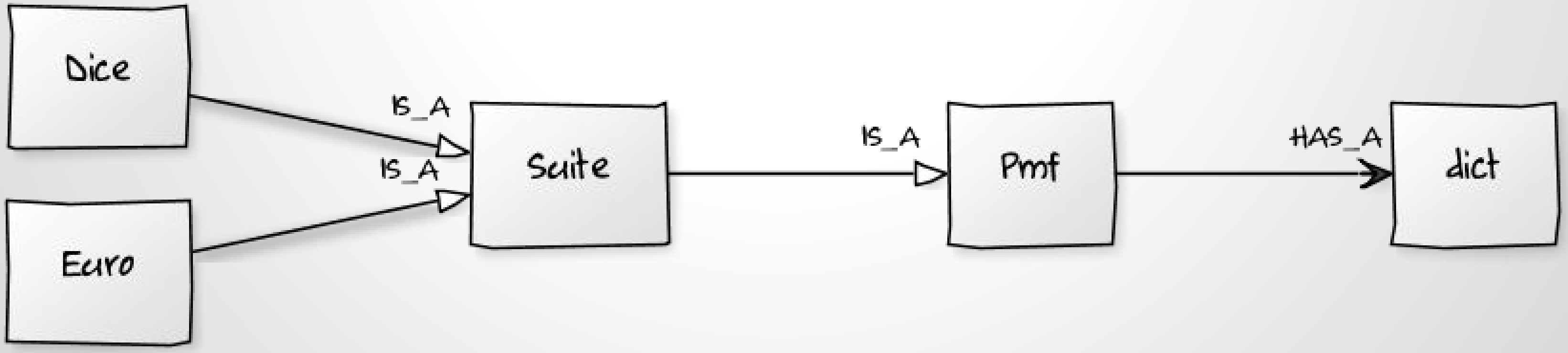
```
def Update(self, data):
    "Updates the suite based on data."

    for hypo in self.Values():
        like = self.Likelihood(data, hypo)
        self.Mult(hypo, like)

    self.Normalize()
```

self.Likelihood?

Suite

`Likelihood` is an abstract method.

Child classes inherit `Update`, provide `Likelihood`.



Likelihood

Outcome: 6

- What is the likelihood of this outcome on a six-sided die?
- On a ten-sided die?
- On a four-sided die?

What is the likelihood of getting *n* on an *m*-sided die?

Likelihood

```
# hypo is the number of sides on the die
# data is the outcome

class Dice(Suite):

    def Likelihood(self, data, hypo):
        # write this method!
```

Write your solution in `dice.py`

Likelihood

```
# hypo is the number of sides on the die
# data is the outcome

class Dice(Suite):

    def Likelihood(self, data, hypo):
        if hypo < data:
            return 0
        else:
            return 1.0/hypo
```

Dice

```
# start with equal priors
suite = Dice([4, 6, 8, 12, 20])

# update with the data
suite.Update(6)

suite.Print()
```

Dice

Posterior distribution:

```
4  0.0
6  0.39
8  0.30
12 0.19
20 0.12
```

More data? No problem...

Dice

```
for roll in [8, 7, 7, 5, 4]:
    suite.Update(roll)

suite.Print()
```

Dice

Posterior distribution:

```
4   0.0
6   0.0
8   0.92
12  0.080
20  0.0038
```

Summary

Dice problem,

Likelihood function,

Suite class.

Recess!

http://ksdcitizens.org/wp-content/uploads/2010/09/recess_time.jpg

Trains

The trainspotting problem:

- You believe that a freight carrier operates between 100 and 1000 locomotives with consecutive serial numbers.
- You spot locomotive #321.
- How many locomotives does the carrier operate?

Modify `train.py` to compute your answer.

Trains

- If there are m trains, what is the chance of spotting train #n?
- What does the posterior distribution look like?
- How would you summarize it?

Train

```
print suite.Mean()
print suite.MaximumLikelihood()
print suite.CredibleInterval(90)
```

Trains

- What if we spot more trains?
- Why did we do this example?

Trains

- Practice using the Bayesian framework, and figuring out Likelihood().

- Example that uses sparse data.

- It’s a non-trivial, real problem.

Tanks

The German tank problem.

http://en.wikipedia.org/wiki/German_tank_problem

Good time for questions

A Euro problem

"*When spun on edge 250 times, a Belgian one-euro coin came up heads 140 times and tails 110. 'It looks very suspicious to me,' said Barry Blight, a statistics lecturer at the London School of Economics. 'If the coin were unbiased, the chance of getting a result as extreme as that would be less than 7%.'* "

From "The Guardian" quoted by MacKay, *Information Theory, Inference, and Learning Algorithms*.

A Euro problem

MacKay asks, *"But do these data give evidence that the coin is biased rather than fair?"*

Assume that the coin has probability x of landing heads.

(Forget that x is a probability; just think of it as a physical characteristic.)

A Euro problem

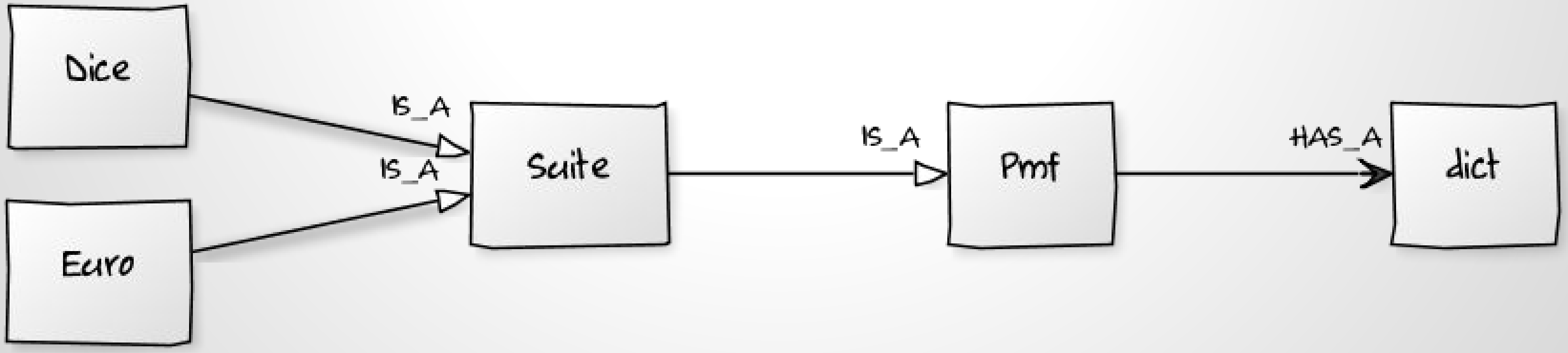
Estimation: Based on the data (140 heads, 110 tails), what is *x*?

Hypothesis testing: What is the probability that the coin is fair?

Euro

We can use the Suite template again.

We just have to figure out the likelihood function.



Likelihood

```
# hypo is the prob of heads (1-100)
# data is a string, either 'H' or 'T'

class Euro(Suite):

    def Likelihood(self, data, hypo):
        # one more, please!
```

Modify `euro.py` to compute your answer.

Likelihood

```
# hypo is the prob of heads (1-100)
# data is a string, either 'H' or 'T'

class Euro(Suite):

    def Likelihood(self, data, hypo):
        x = hypo / 100.0
        if data == 'H':
            return x
        else:
            return 1-x
```

Prior

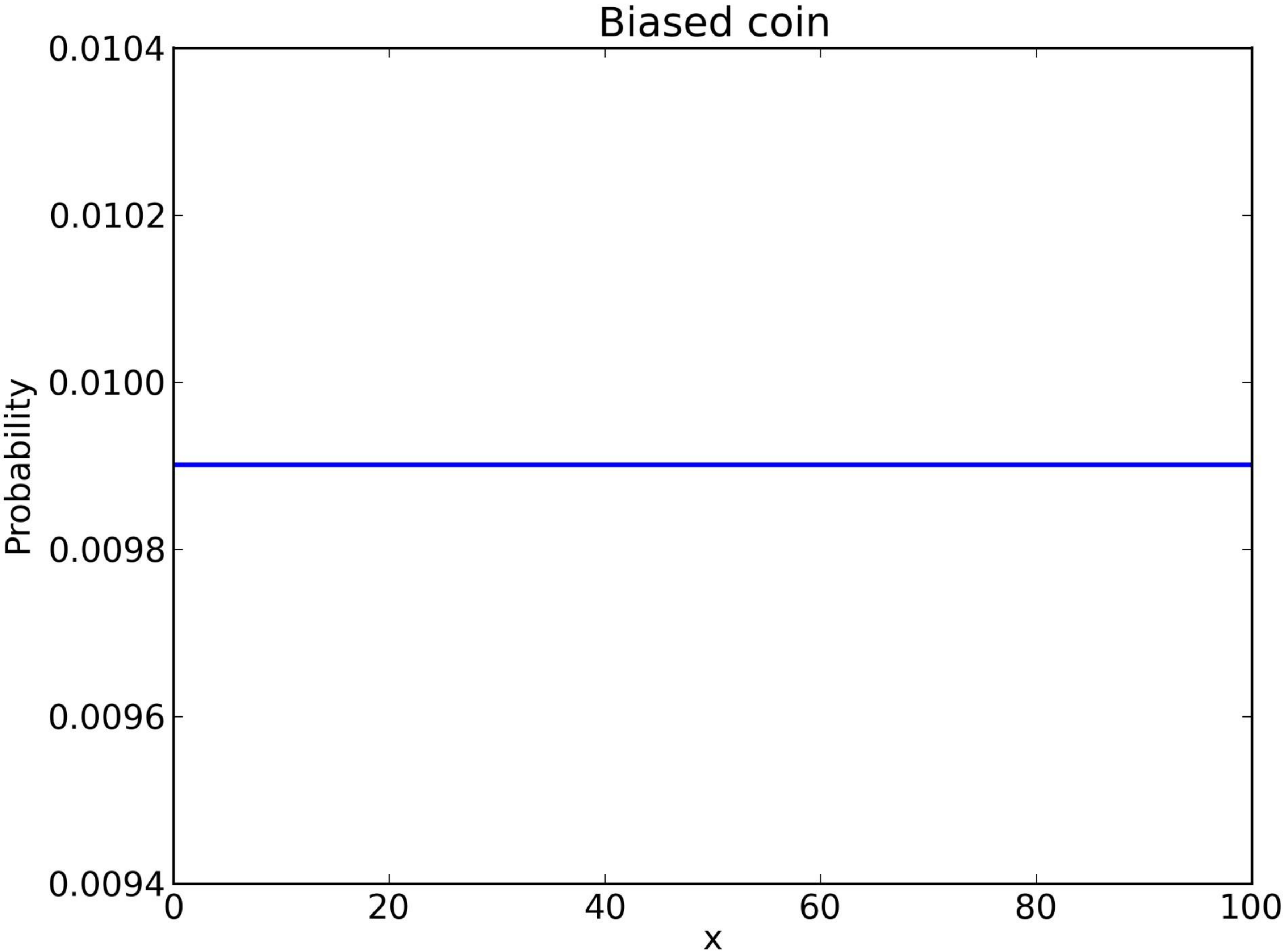
What do we believe about x before seeing the data?

Start with something simple; we'll come back and review.

Uniform prior: any value of x between 0% and 100% is equally likely.

Prior

```
suite = Euro(range(0, 101))
```



Biased coin
Probability
0.0104
0.0102
0.0100
0.0098
0.0096
0.0094
0
20
40
60
80
100
x

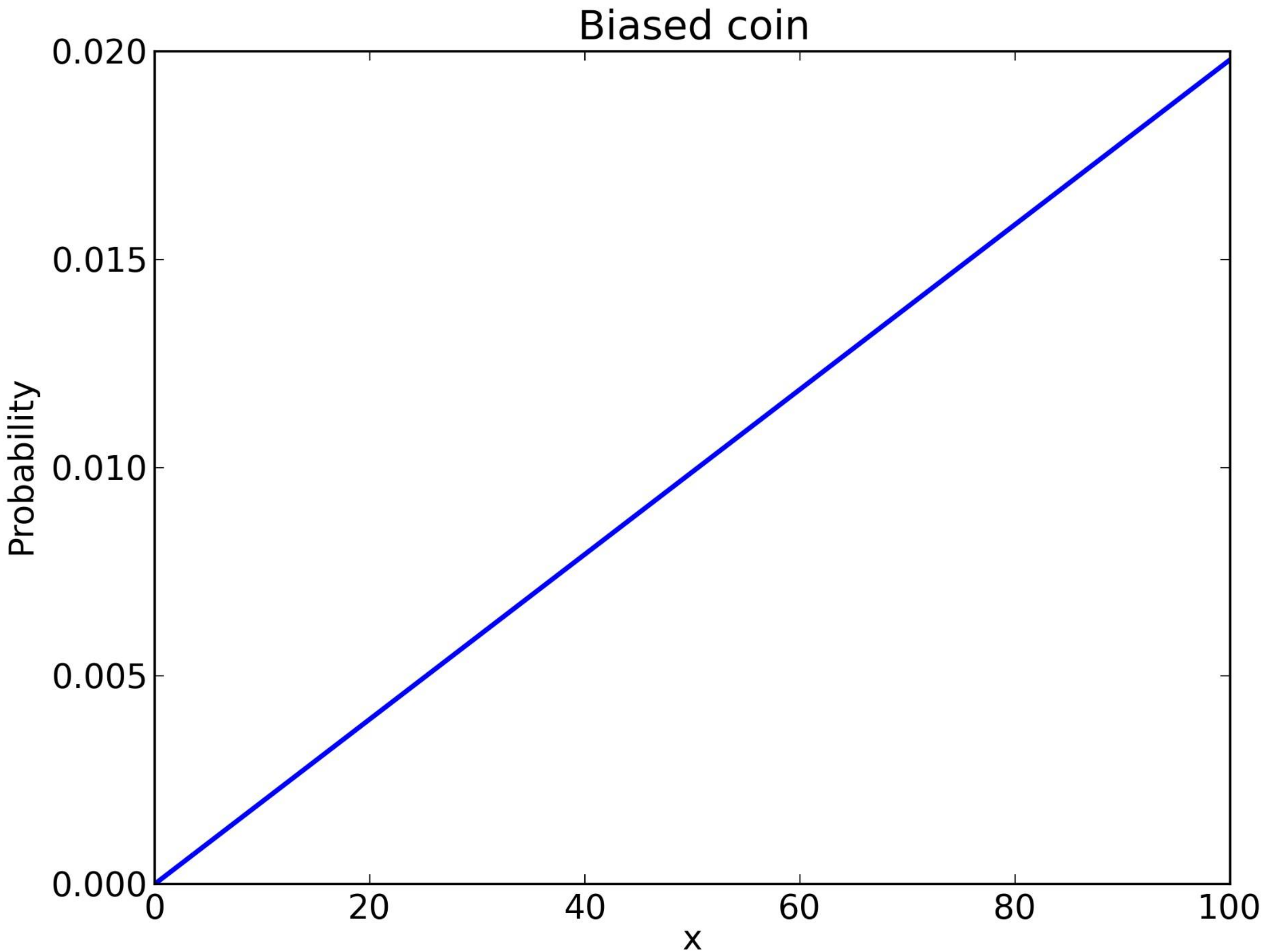
Update

Suppose we spin the coin once and get heads.

```
suite.Update('H')
```

What does the posterior distribution look like?

Hint: what is $p(x=0\% \mid D)$?

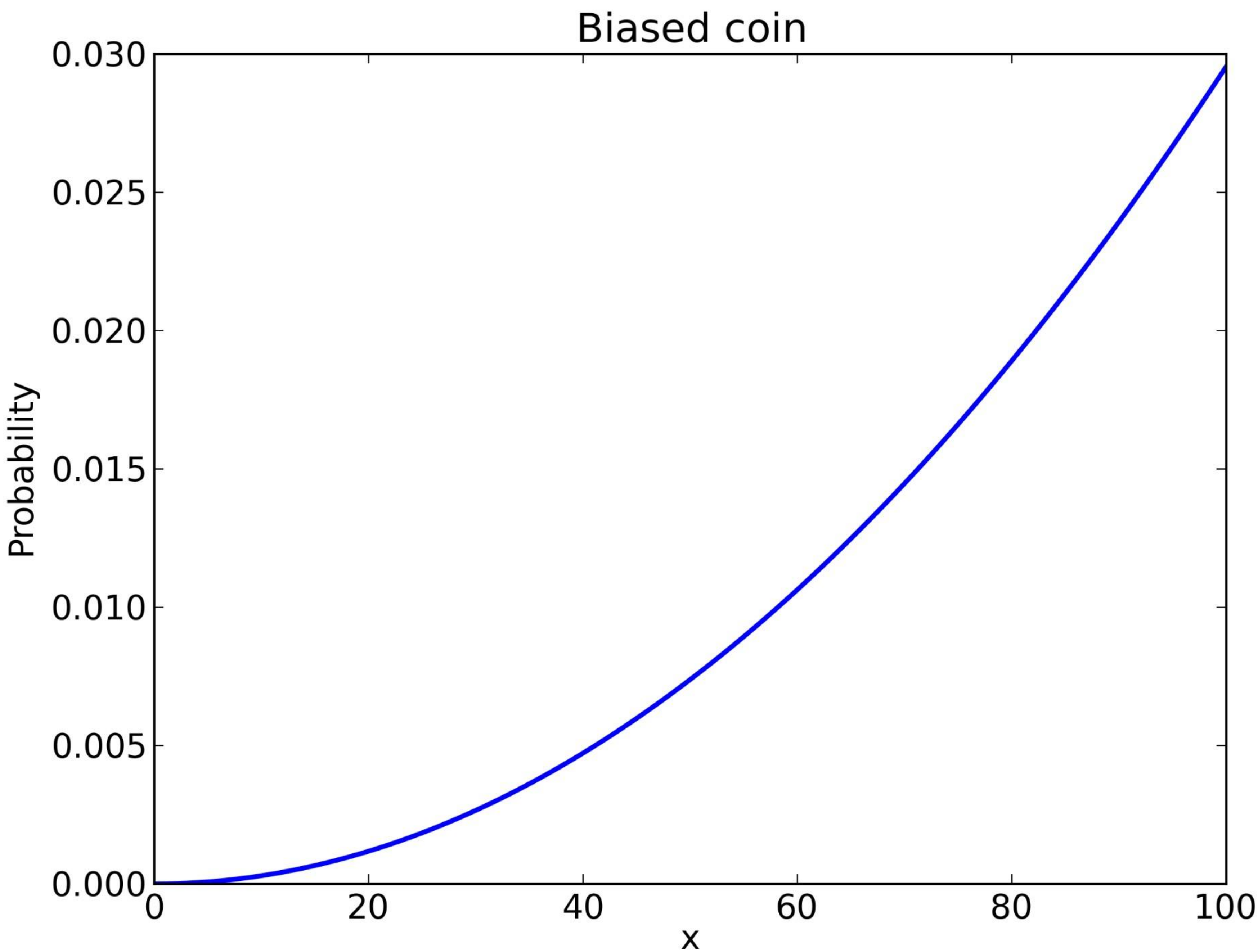

Biased coin
Probability
x
0.000
0.005
0.010
0.015
0.020
0
20
40
60
80
100

Update

Suppose we spin the coin again, and get heads again.

```
suite.Update('H')
```

What does the posterior distribution look like?



Biased coin
Probability
x
0.030
0.025
0.020
0.015
0.010
0.005
0.000
0
20
40
60
80
100

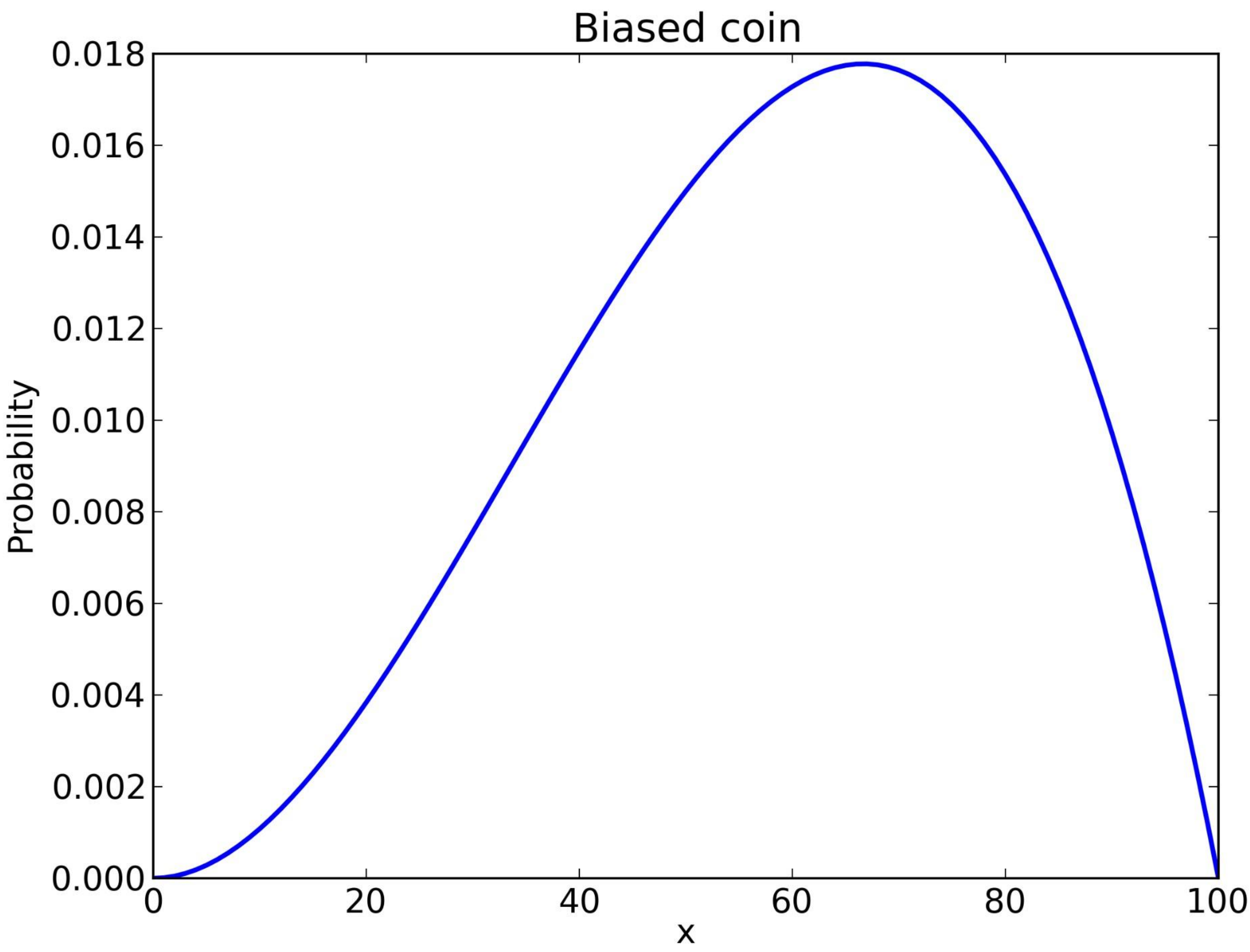
Update

Suppose we spin the coin again, and get tails.

```
suite.Update('T')
```

What does the posterior distribution look like?

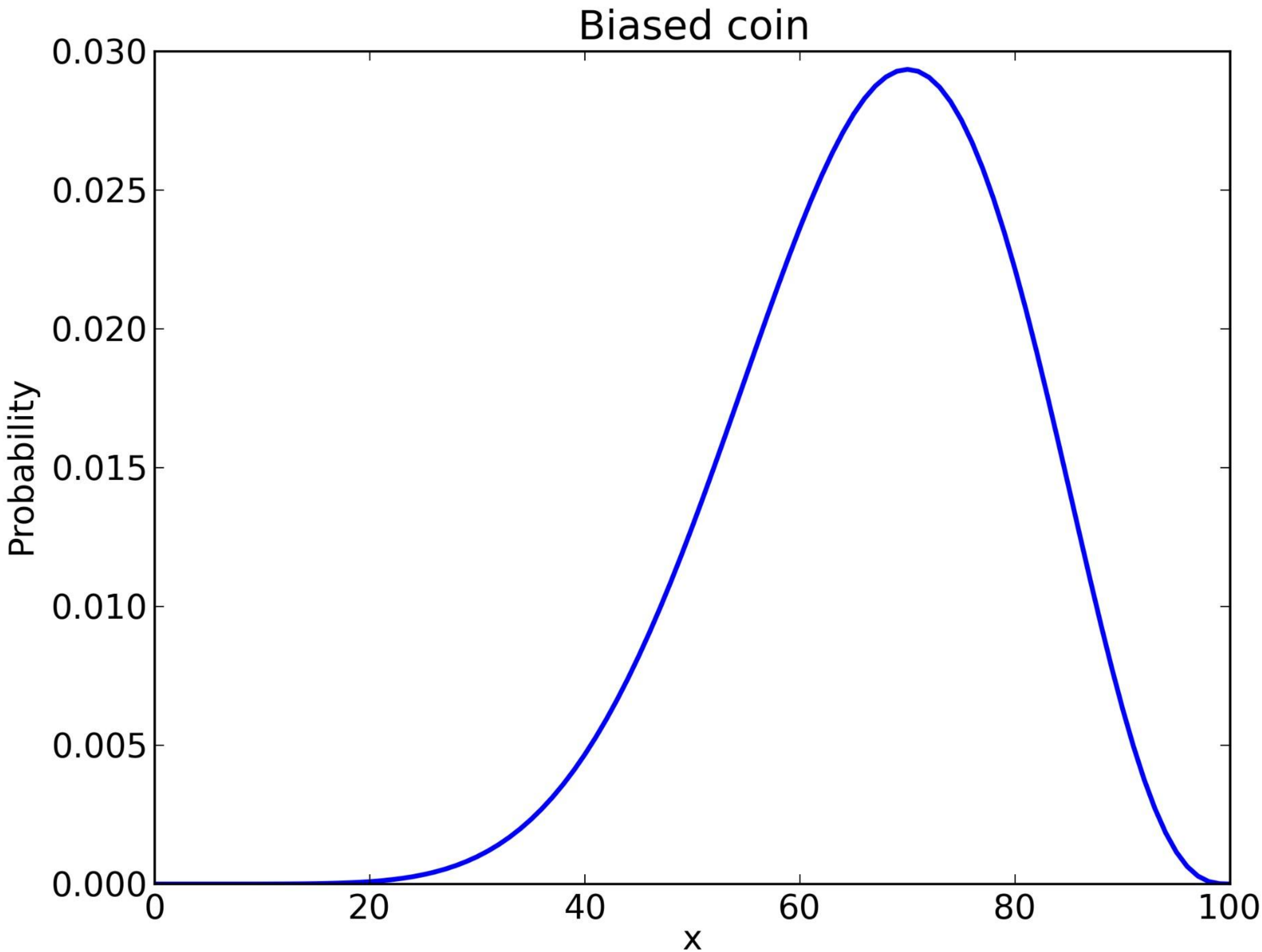
Hint: what's $p(x=100\% \mid D)$?


Biased coin
Probability
x
0.000
0.002
0.004
0.006
0.008
0.010
0.012
0.014
0.016
0.018
0
20
40
60
80
100

Update

After 10 spins, 7 heads and 3 tails:

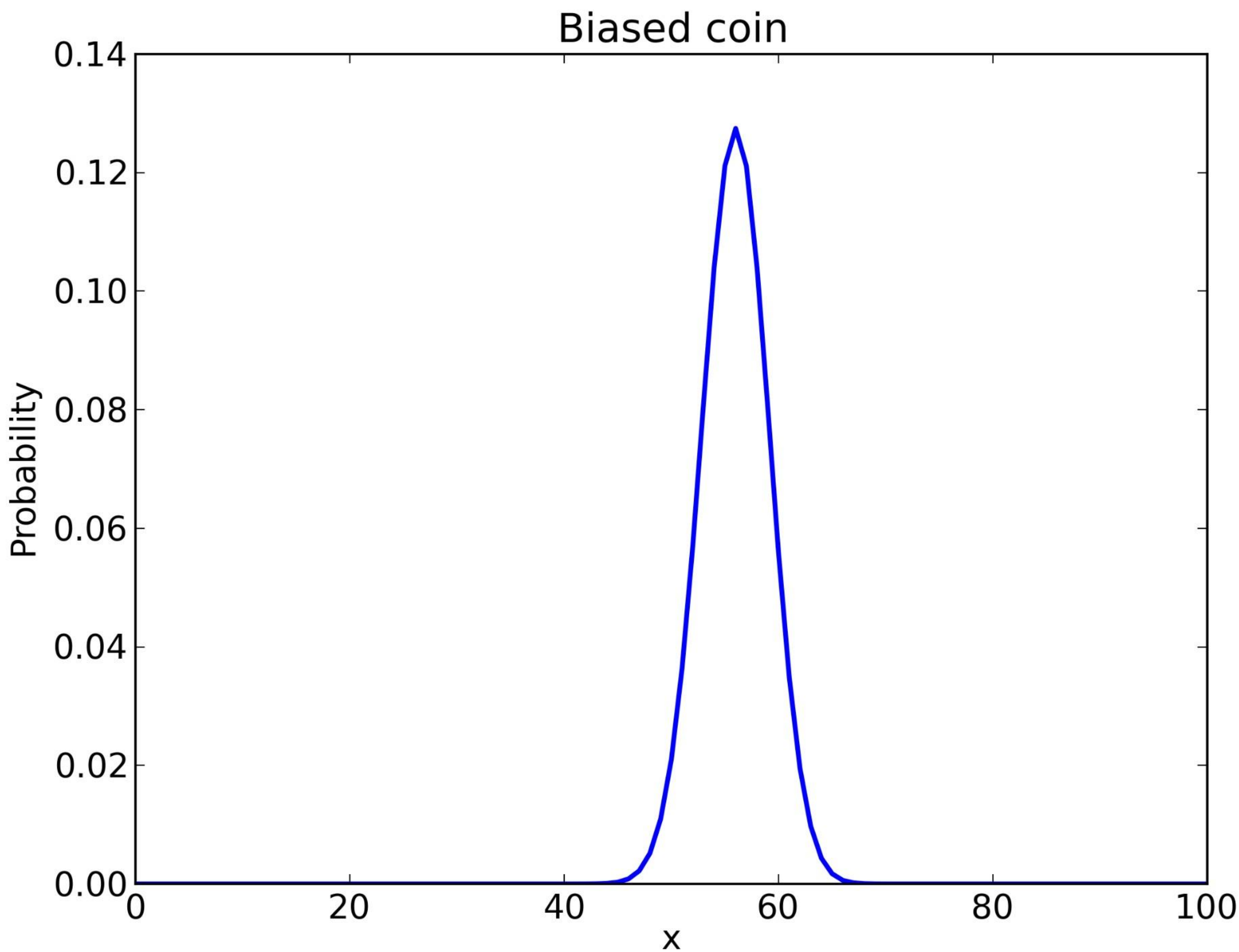
```
for outcome in 'HHHHHHHTTT':
    suite.Update(outcome)
```


Biased coin
Probability
x
0.030
0.025
0.020
0.015
0.010
0.005
0.000
0
20
40
60
80
100

Update

And finally, after 140 heads and 110 tails:

```
evidence = 'H' * 140 + 'T' * 110
for outcome in evidence:
    suite.Update(outcome)
```



Biased coin
Probability
x
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0
20
40
60
80
100

Posterior

- Now what?
- How do we summarize the information in the posterior Suite?

Posterior

Given the posterior distribution, what is the probability that x is 50%?

```
suite.Prob(50)
```

And the answer is... 0.021

Hmm. Maybe that's not the right question.

Posterior

How about the most likely value of x?

```
pmf.MaximumLikelihood()
```

And the answer is 56%.

Posterior

Or the expected value?

```
suite.Mean()
```

And the answer is 55.95%.

Posterior

Credible interval?

```
suite.CredibleInterval(90)
```

Posterior

The 5th percentile is 51.

The 95th percentile is 61.

These values form a 90% credible interval.

So can we say: "There's a 90% chance that *x* is between 51 and 61?"

Frequentist response

Thank you smbc-comics.com

Bayesian response

Yes, x is a random variable,
Yes, (51, 61) is a 90% credible interval,
Yes, x has a 90% chance of being in it.

Pro: Bayesian stats are amenable to decision analysis.

Con: The prior is subjective.

The prior is subjective

Remember the prior?

We chose it pretty arbitrarily, and reasonable people might disagree.

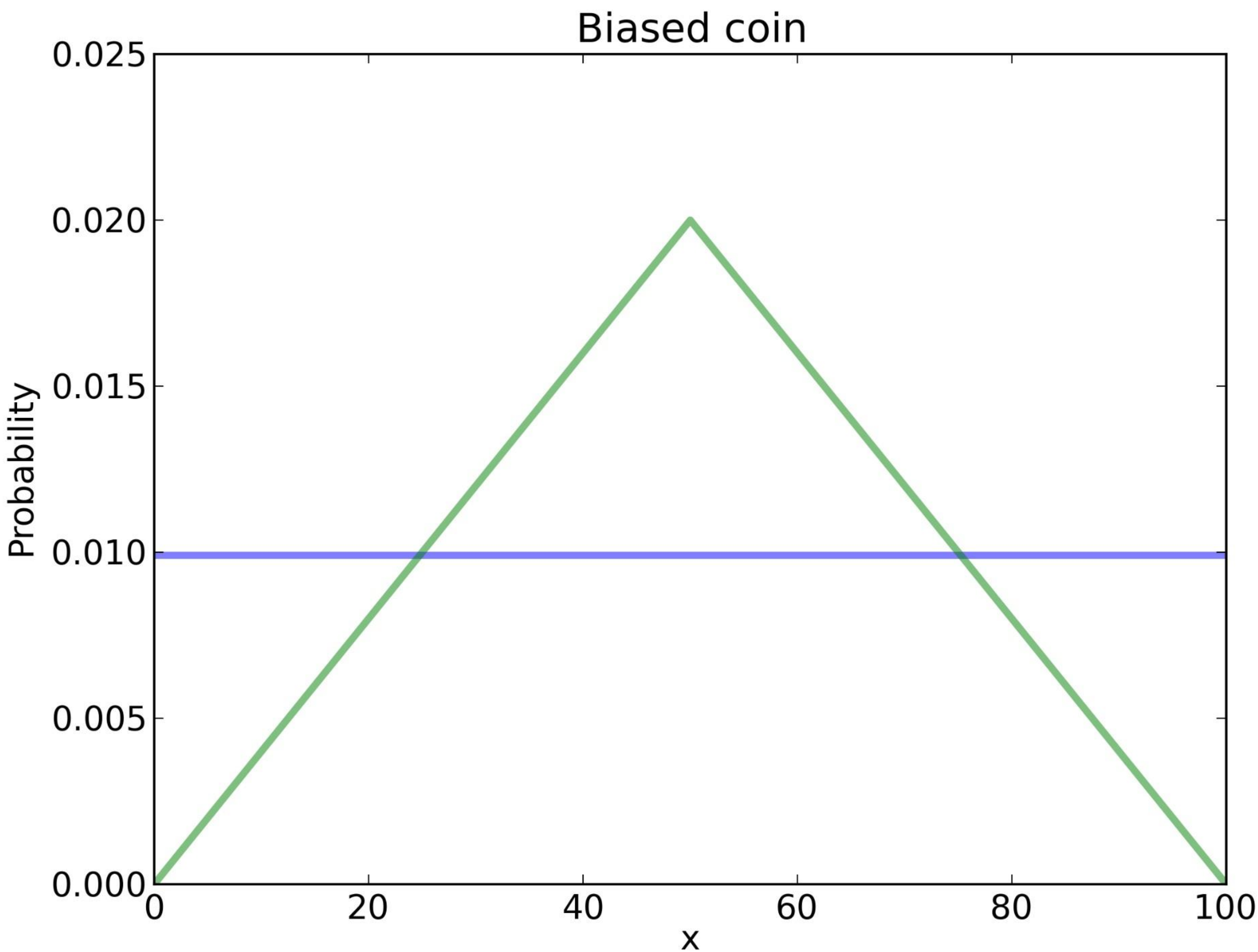
Is x as likely to be 1% as 50%?

Given what we know about coins, I doubt it.

Prior

How should we capture background knowledge about coins?

Try a triangle prior.

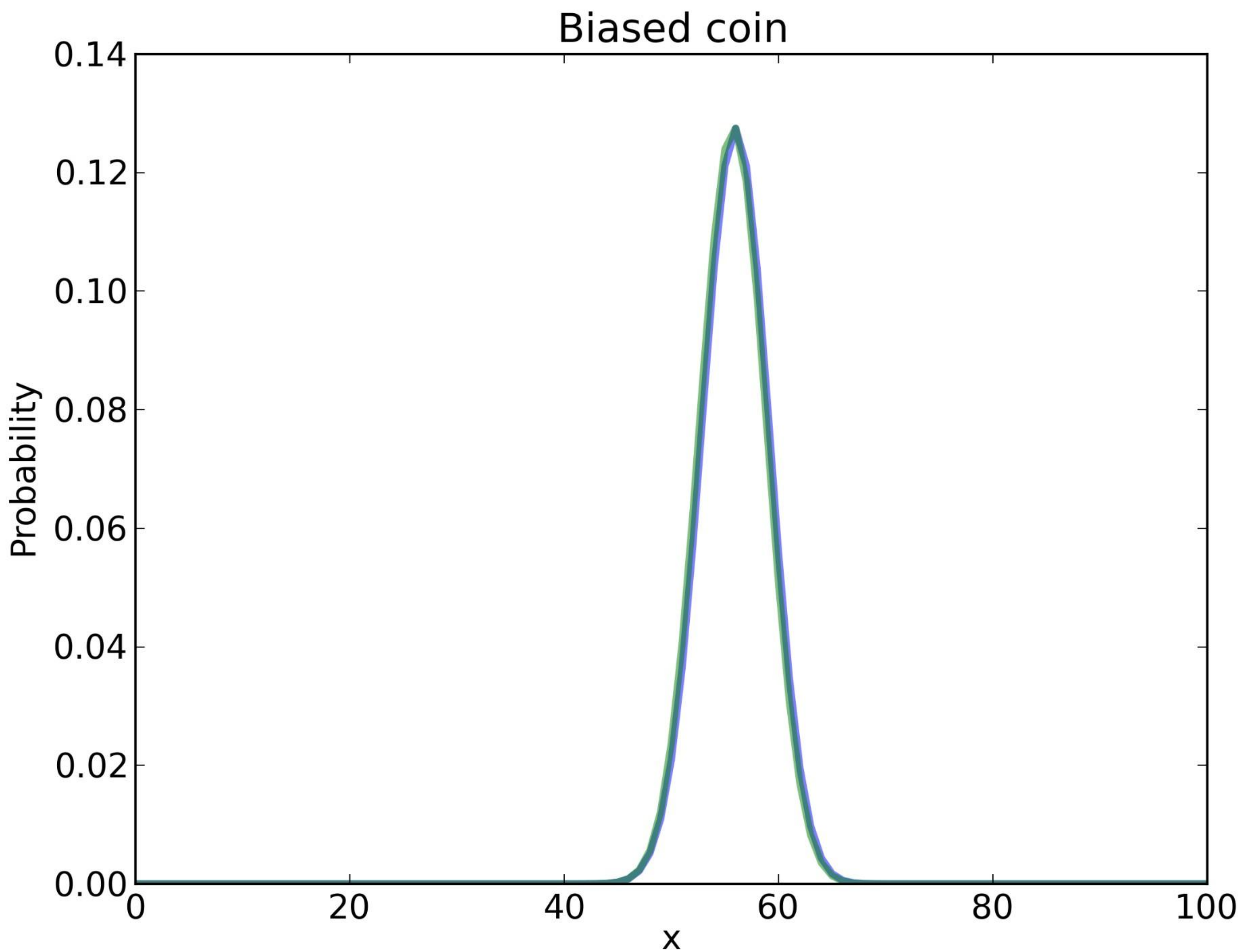


Biased coin
Probability
0.025
0.020
0.015
0.010
0.005
0.000
0
20
40
60
80
100
x

Posterior

What do you think the posterior distributions look like?

I was going to put an image here, but then I Googled "posterior". Never mind.



Biased coin
Probability
x
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0
20
40
60
80
100

Swamp the prior

With enough data,
reasonable people converge.

But if any $p(H_i) = 0$, no data
will change that.

Swamp the prior

Priors can be arbitrarily low, but avoid 0.

See wikipedia.org/wiki/Cromwell's_rule

"I beseech you, in the bowels of Christ, think it possible that you may be mistaken."

Summary of estimation

1. Form a suite of hypotheses, H_i.
2. Choose prior distribution, $p(H_i)$.
3. Compute likelihoods, $p(D|H_i)$.
4. Turn off brain.
5. Compute posteriors, $p(H_i|D)$.

Recess!

http://images2.wikia.nocookie.
net/__cb20120116013507/recess/images/4/4c/Recess_Pic_for_the_Int
ernet.png

Hypothesis testing

Remember the original question:

"But do these data give evidence that the coin is biased rather than fair?"

What does it mean to say that data give evidence for (or against) a hypothesis?

Hypothesis testing

D is evidence in favor of H if

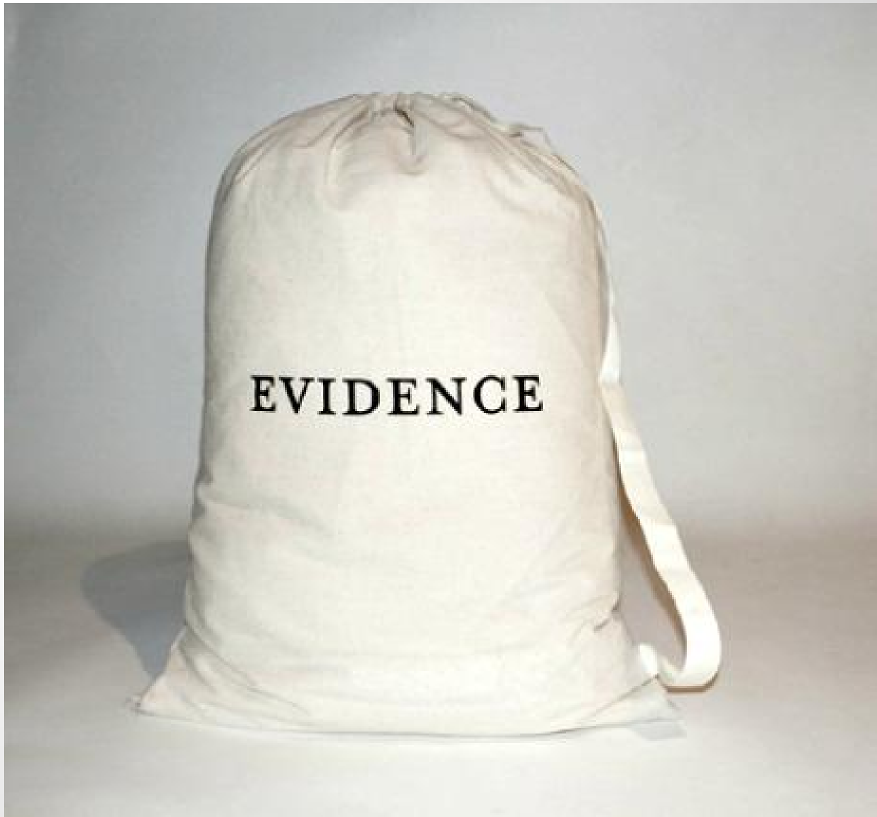
p(H|D) > p(H)

which is true if

p(D|H) > p(D|~H)

or equivalently if

p(D|H) / p(D|~H) > 1



Hypothesis testing

This term

$p(D|H) / p(D|\sim H)$

is called the likelihood ratio, or Bayes factor.

It measures the strength of the evidence.

Hypothesis testing

F: hypothesis that the coin is fair

B: hypothesis that the coin is biased

$p(D|F)$ is easy.

$p(D|B)$ is hard because B is underspecified.

Bogosity

Tempting: we got 140 heads out of 250 spins, so B is the hypothesis that $x = 140/250$.

But,

1. Doesn't seem right to use the data twice.
2. By this process, almost any data would be evidence in favor of B.

We need some rules

1. You have to choose your hypothesis before you see the data.
2. You can choose a suite of hypotheses, but in that case we average over the suite.


29
7
28
12
35
3
26
0
32
15
1-18
EVEN

Likelihood

```
def AverageLikelihood(suite, data):
    total = 0

    for hypo, prob in suite.Items():
        like = suite.Likelihood(data, hypo)
        total += prob * like

    return total
```

Hypothesis testing

F: hypothesis that $x = 50\%$.

B: hypothesis that x is not 50%, but might be any other value with equal probability.

Prior

```
fair = Euro()
fair.Set(50, 1)
```

Prior

```
bias = Euro()
for x in range(0, 101):
    if x != 50:
        bias.Set(x, 1)
bias.Normalize()
```

Bayes factor

```
data = 140, 110

like_fair = AverageLikelihood(fair, data)
like_bias = AverageLikelihood(bias, data)

ratio = like_bias / like_fair
```

Hypothesis testing

Read `euro2.py`.

Notice the new representation of the data, and corresponding Likelihood function.

Run it and interpret the results.

Hypothesis testing

And the answer is:

$p(D|B) = 2.6 \cdot 10^{-76}$
$p(D|F) = 5.5 \cdot 10^{-76}$

Likelihood ratio is about 0.47.

So this dataset is evidence against B.

Fair comparison?

- Modify the code that builds `bias`; try out a different definition of B and run again.

```
bias = Euro()
for x in range(0, 49):
    bias.Set(x, x)
for x in range(51, 101):
    bias.Set(x, 100-x)
bias.Normalize()
```

Conclusion

- The Bayes factor depends on the definition of B.
- Depending on what “biased” means, the data might be evidence for or against B.
- The evidence is weak either way (between 0.5 and 2).

Summary

Euro problem,

Bayesian estimation,

Bayesian hypothesis testing.

Recess!

http://ksdcitizens.org/wp-content/uploads/2010/09/recess_time.jpg

Word problem for geeks

ALICE: What did you get on the math SAT?

BOB: 760

ALICE: Oh, well I got a 780. I guess that means I'm smarter than you.

NARRATOR: Really? What is the probability that Alice is smarter than Bob?

Assume, define, quantify

Assume: each person has some probability, x, of answering a random SAT question correctly.

Define: "Alice is smarter than Bob" means $x_a > x_b$.

How can we quantify Prob { $x_a > x_b$ } ?

Be Bayesian

Treat *x* as a random quantity.

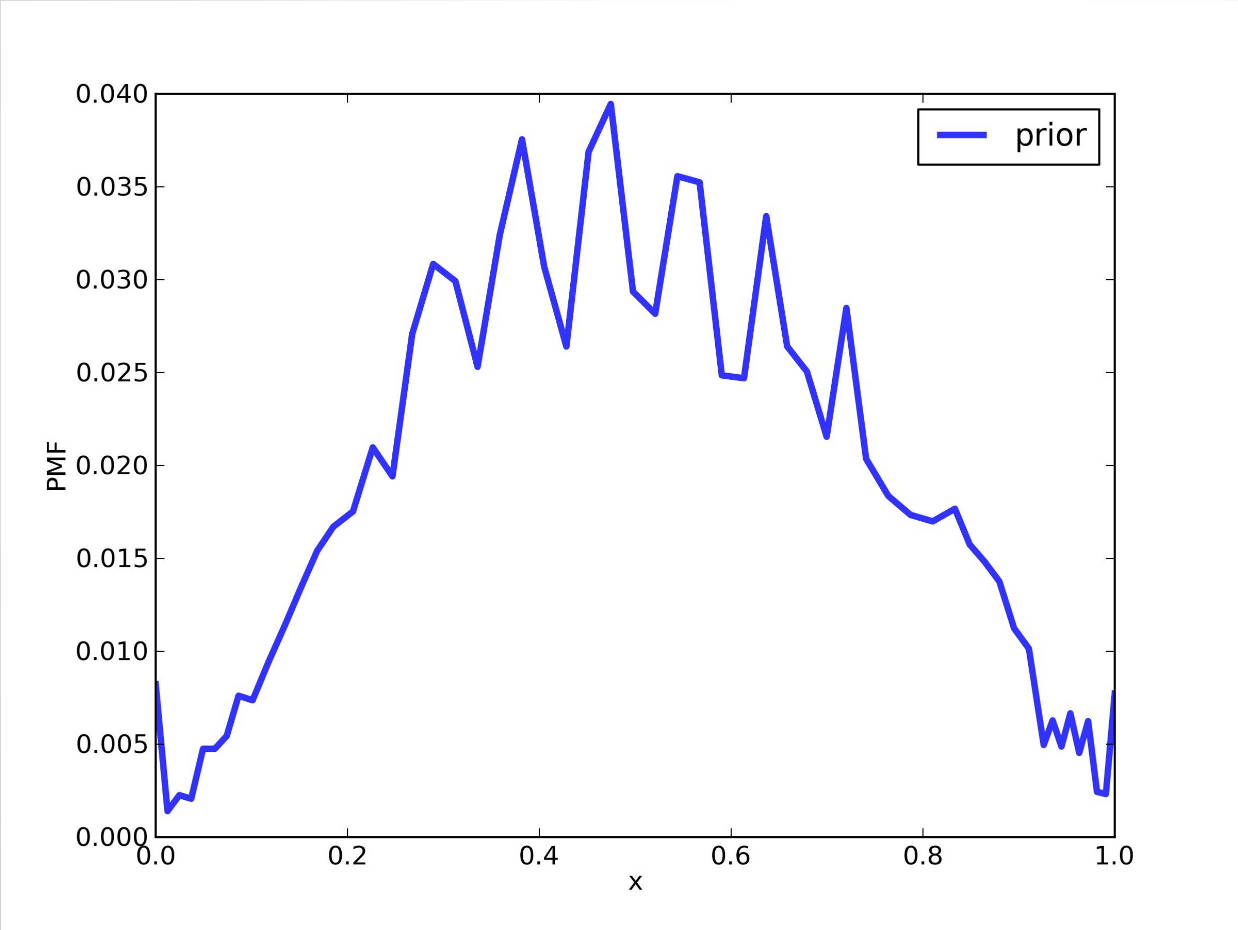
Start with a prior distribution.

Update it.

Compare posterior distributions.

Prior?

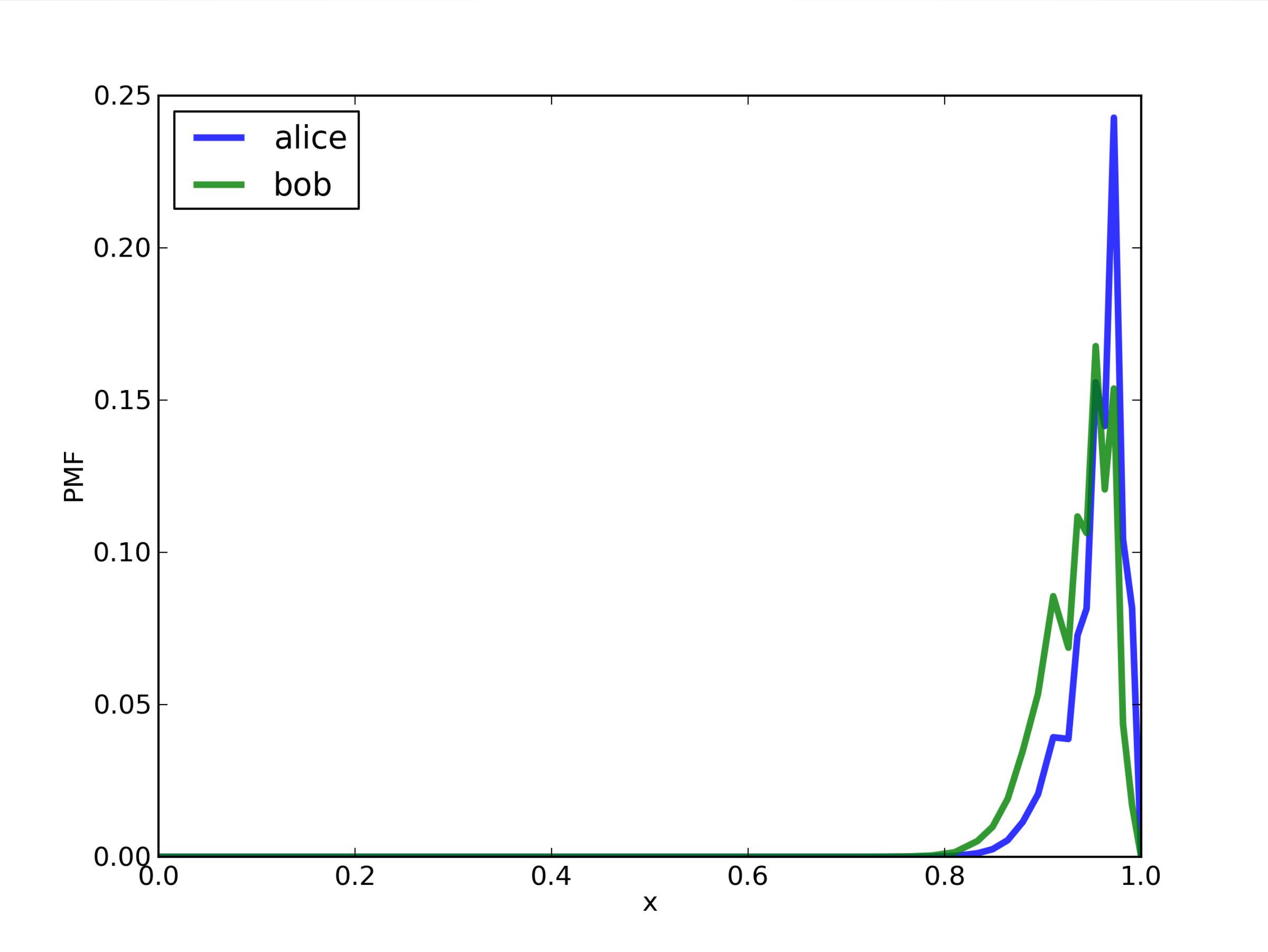
Distribution of raw scores.



Likelihood

```
def Likelihood(self, data, hypo):
    x = hypo
    score = data
    raw = self.exam.Reverse(score)

    yes, no = raw, self.exam.max_score - raw
    like = x**yes * (1-x)**no
    return like
```

Posterior



PmfProbGreater

```
def PmfProbGreater(pmf1, pmf2):
    """Returns the prob that a value from pmf1
     is greater than a value from pmf2."""
```

PmfProbGreater

```
def PmfProbGreater(pmf1, pmf2):
    """Returns the prob that a value from pmf1
     is greater than a value from pmf2."""

    Iterate through all pairs of values.
    Check whether the value from pmf1 is greater.
    Add up total probability of successful pairs.
```

PmfProbGreater

```
def PmfProbGreater(pmf1, pmf2):

    for x1, p1 in pmf1.Items():
        for x2, p2 in pmf2.Items():
            # FILL THIS IN!
```

PmfProbGreater

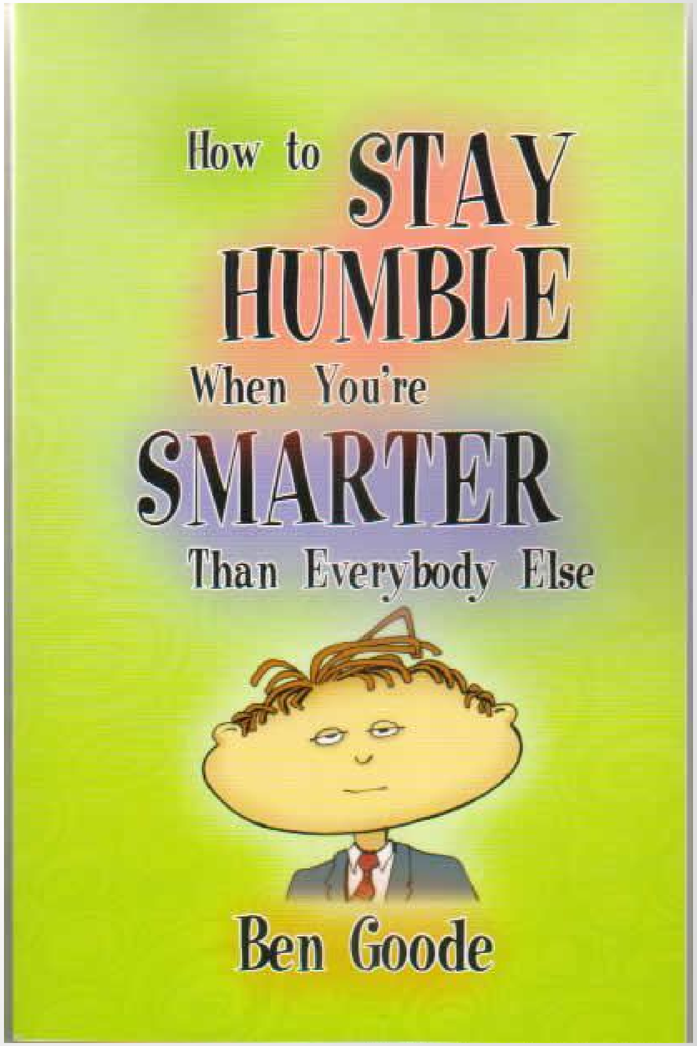
```
def PmfProbGreater(pmf1, pmf2):

    total = 0.0
    for x1, p1 in pmf1.Items():
        for x2, p2 in pmf2.Items():
            if x1 > x2:
                total += p1 * p2

    return total
```

And the answer is...

Alice: 780

Bob: 760

Probability that Alice is "smarter": 61%



Why this example?

Posterior distribution is often the input to the next step in an analysis.

Real world problems start (and end!) with modeling.

Modeling

- This result is based on the simplification that all SAT questions are equally difficult.
- An alternative (in the book) is based on item response theory.

Modeling

- For most real world problems, there are several reasonable models.
- The best choice depends on your goals.
- Modeling errors often dominate.

Modeling

Therefore:

- Don't mistake the map for the territory.
- Don't sweat approximations smaller than modeling errors.
- Iterate.

Recess!

http://images2.wikia.nocookie.
net/__cb20120116013507/recess/images/4/4c/Recess_Pic_for_the_Int
ernet.png

Case study

Problem: students sign up to participate in a community service project. Some fraction, q, of the students who sign up actually participate, and of those some fraction, r, report back.

Given a sample of students who sign up and the number who report back, we can estimate the product q*r, but don't learn about q and r separately.

Case study

If we can get a smaller sample of students where we know who participated and who reported, we can use that to improve the estimates of q and r.

And we can use that to compute the posterior distribution of the number of students who participated.

volunteer.py

```
probs = numpy.linspace(0, 1, 101)

hypos = []
for q in probs:
    for r in probs:
        hypos.append((q, r))

suite = Volunteer(hypos)
```

volunteer.py

```
# students who signed up and reported
data = 140, 50
suite.Update(data)

# students who signed up, participated,
# and reported
data = 5, 3, 1
suite.Update(data)
```

volunteer.py

```
class Volunteer(thinkbayes.Suite):

    def Likelihood(self, data, hypo):
        if len(data) == 2:
            return self.Likelihood1(data, hypo)
        elif len(data) == 3:
            return self.Likelihood2(data, hypo)
        else:
            raise ValueError()
```

volunteer.py

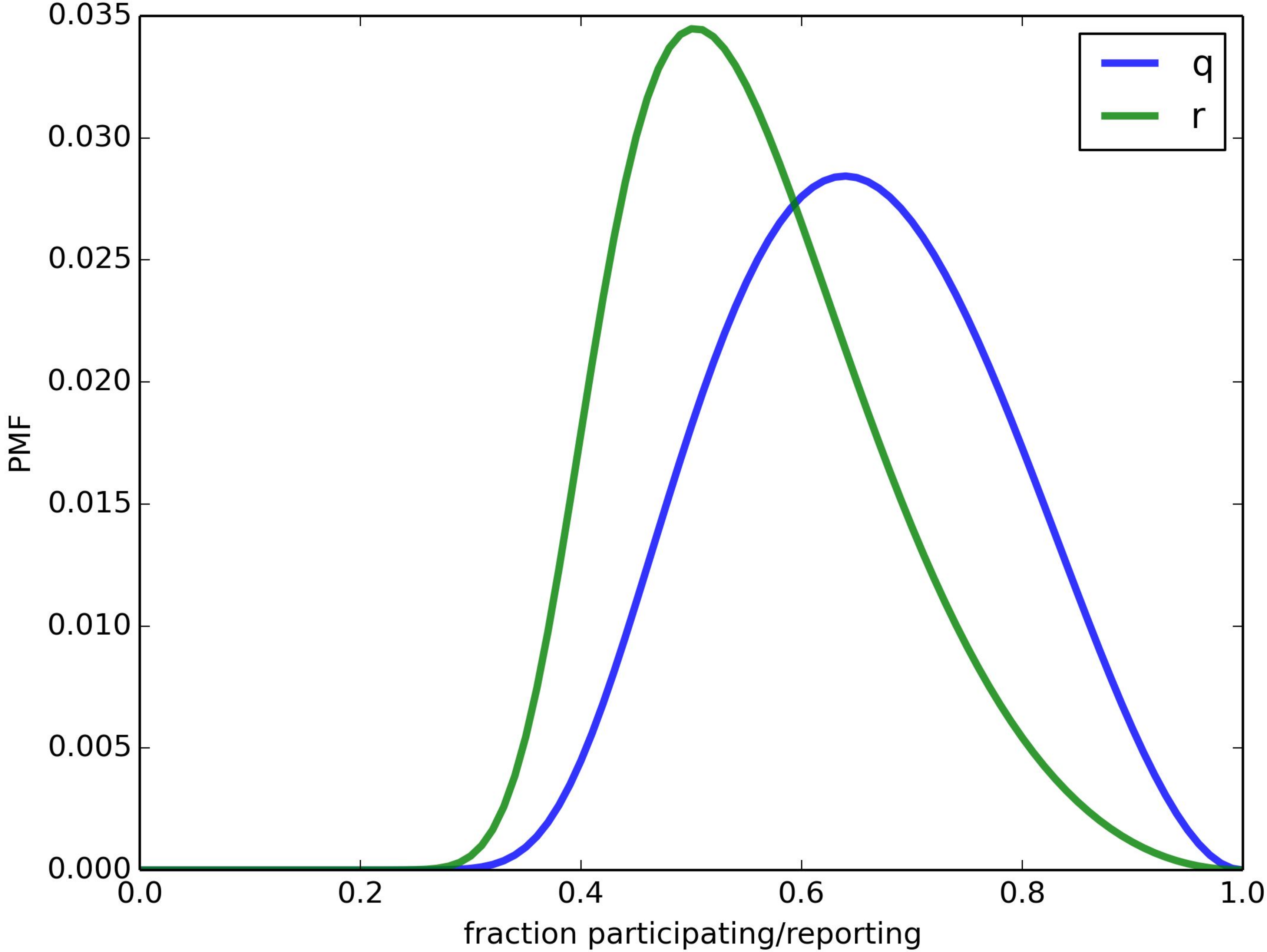
```
def Likelihood1(self, data, hypo):
     q, r = hypo
     p = q * r
     signed_up, reported = data
     yes = reported
     no = signed_up - reported

     like = p**yes * (1-p)**no
     return like
```

volunteer.py

```
def Likelihood2(self, data, hypo):
    q, r = hypo
    signed_up, participated, reported = data
    yes = participated
    no = signed_up - participated
    like1 = q**yes * (1-q)**no

    yes = reported
    no = participated - reported
    like2 = r**yes * (1-r)**no

    return like1 * like2
```

volunteer.py

```
def MarginalDistribution(suite, index):

    pmf = thinkbayes.Pmf()
    for t, prob in suite.Items():
        pmf.Incr(t[index], prob)
    return pmf
```



0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
PMF
0.0
0.2
0.4
0.6
0.8
1.0
fraction participating/reporting
q
r

Summary

- The Bayesian approach is a divide and conquer strategy.
- You write Likelihood().
- Bayes does the rest.

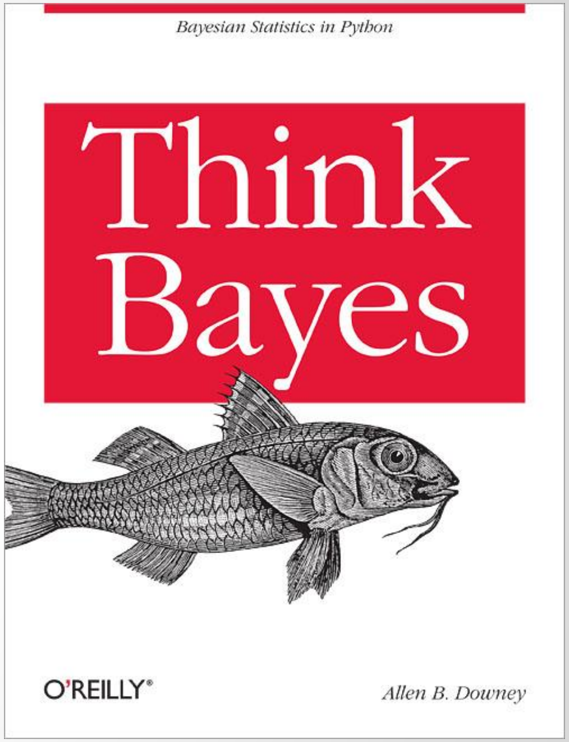
Think Bayes

This tutorial is based on my book,

Think Bayes

Bayesian Statistics Made Simple

Published by O'Reilly Media and available under a Creative Commons license from `thinkbayes.com`



Case studies

- Euro
- SAT
- Red line
- Price is Right
- Boston Bruins
- Paintball
- Variability hypothesis
- Kidney tumor growth
- Geiger counter
- Unseen species

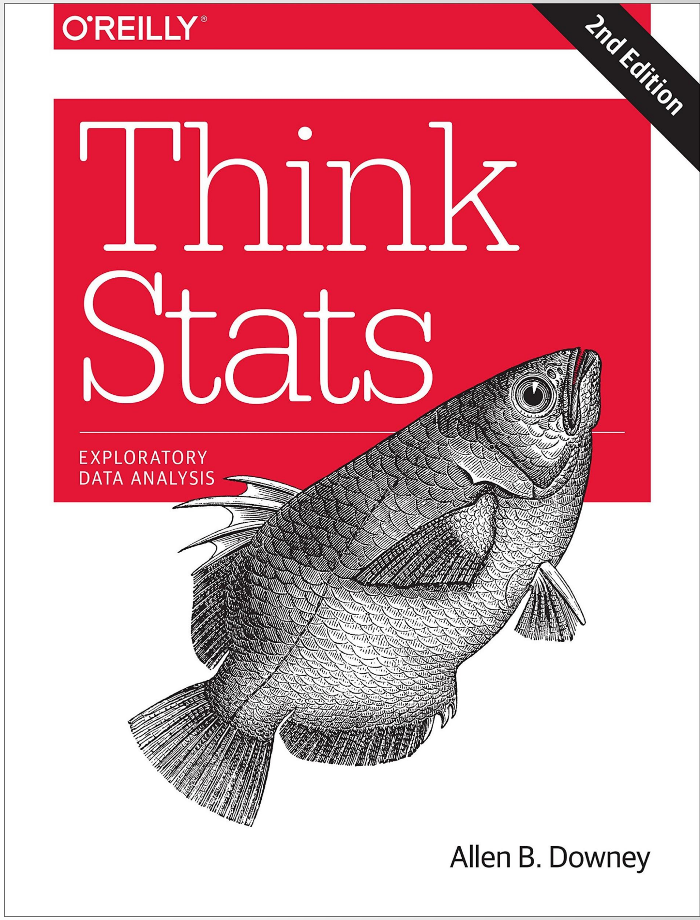
Think Stats

You might also like

Think Stats*, 2nd edition*

Exploratory Data Analysis

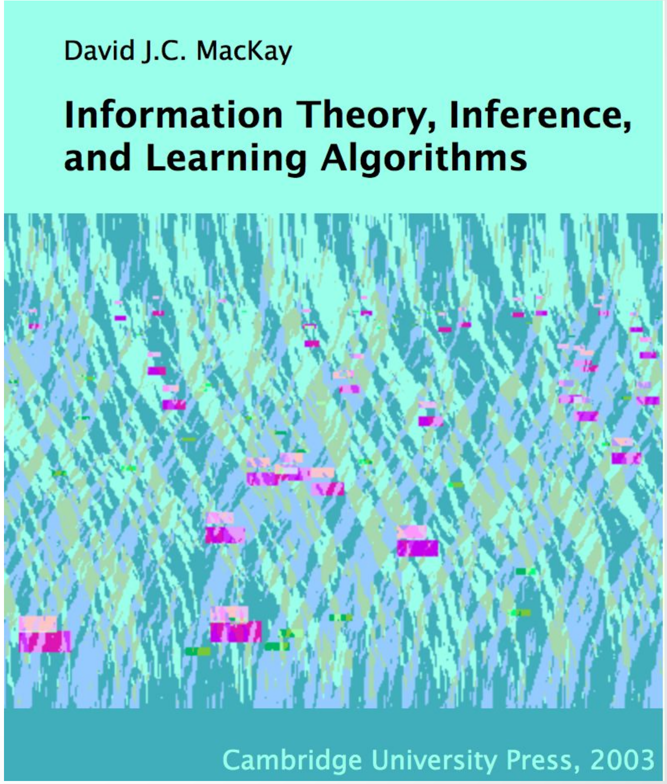
Published by O'Reilly Media
and available under a
Creative Commons license from
`thinkstats2.com`



More reading

MacKay, *Information Theory, Inference, and Learning Algorithms*

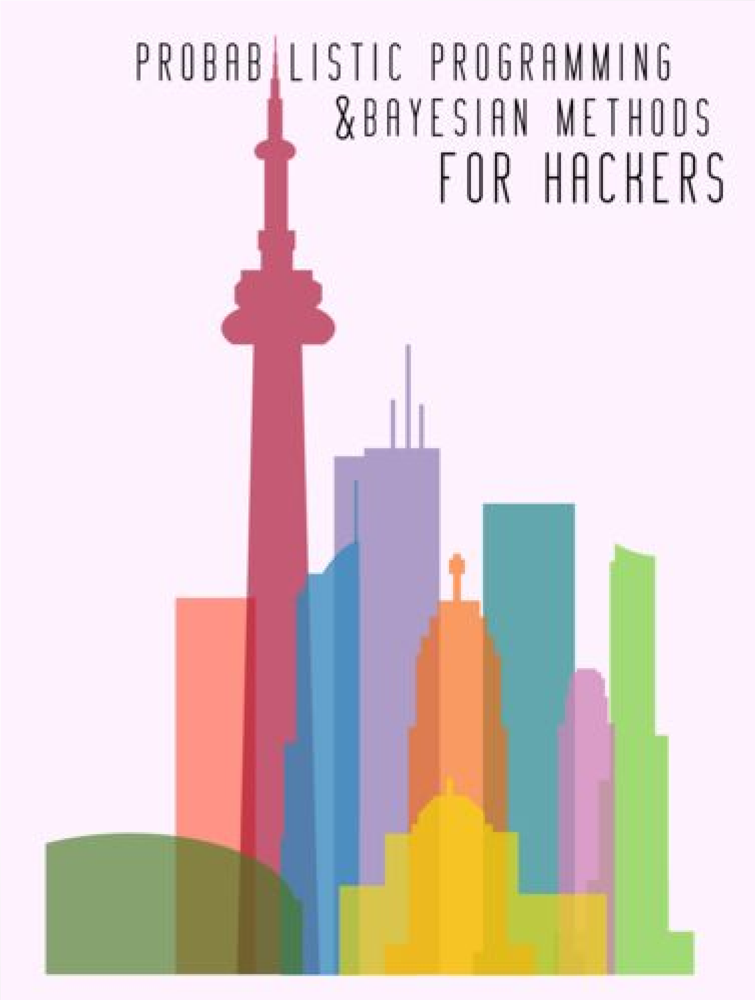
Free PDF.



More reading

Davidson-Pilon, *Bayesian Methods for Hackers*

On Github.



More reading (not free)

Howson and Urbach, *Scientific Reasoning: The Bayesian Approach*

Sivia, *Data Analysis: A Bayesian Tutorial*

Gelman et al, *Bayesian Data Analysis*

Where does this fit?

Usual approach:

- Analytic distributions.
- Math.
- Multidimensional integrals.
- Numerical methods (MCMC).

Where does this fit?

Problem:

- Hard to get started.
- Hard to develop solutions incrementally.
- Hard to develop understanding.

My theory

- Start with non-analytic distributions.
- Use background information to choose meaningful priors.
- Start with brute-force solutions.
- If the results are good enough and fast enough, stop.
- Otherwise, optimize (where analysis is one kind of optimization).
- Use your reference implementation for regression testing.

Need help?

I am always looking for interesting projects.

- Sabbatical June 2015 to August 2016.

Data Science at Olin

- January to May 2015.
- 30 students, 15 projects.
- External collaborators with data and questions.
- Exploratory data analysis and visualization.
- Focus on health, medicine, and fitness.

sites.google.com/site/datascience15/

Director of Programs

New York City

DataKind is looking for a **Director of Programs!**

We're a fast growing non-profit that tackles humanity's biggest problems through data science. We bring volunteer data scientists together with organizations that are dedicated to solving the world's most pressing challenges. Every day, mission driven organizations apply to us with their data science needs. We then work to understand their pain points, how their organization works and, most importantly, how their data and other data sources can be used to transform the way they work.

datakind.org


The QUESTION