Matt McDermott

Exam 1: In Class

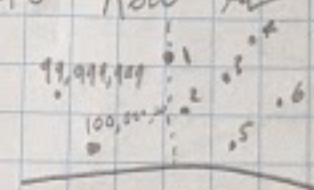
→ There are 10^8 stars visible on a clear sky

How many questions necessary to get on a designated star?

100,000,000

☆ number stars according to how far East of North

ex:



Binary Search

↳ Q_1: is the star # 1 - 50,000,000

yes ↙ ↘ no

Q_2: # 1 - 25,000,000

yes ↙ ↘ no

⋮

Q_2: # 50,000,000 - 75,000,000

yes ↙ ↘ no

⋮

$\log_2(10^8) = 26.5754$

↳ ∴ would need to ask at least 27 questions

↳ $\boxed{B}$

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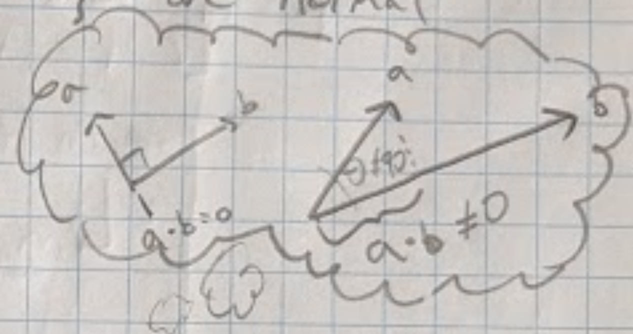
10/14/2020

ES-101: Exam 1 Take Home

$$A = [a_1; a_2; a_3; a_4] = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \\ 2 & 1 & 1 & 4 \end{bmatrix}$$

I) Determine if vectors a_2 and a_3 are normal to one another



$$a_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \quad a_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

※ if 2 vectors are normal, dot product = 0

$$a_2 \cdot a_3 = \begin{bmatrix} 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1\cdot1 + 1\cdot1 + 1\cdot2 + 1\cdot1 = 5$$

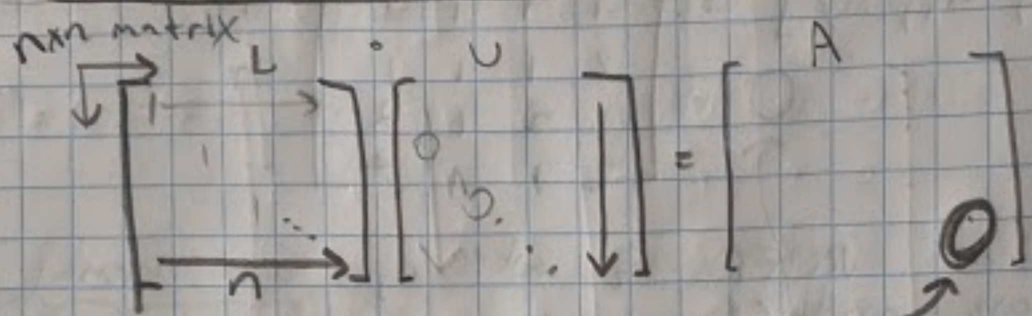
$$5 \neq 0$$

$$\downarrow$$

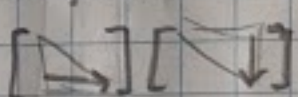
$\boxed{\therefore a_2 \text{ \& } a_3 \text{ } \underline{\text{not}} \text{ normal to eachother}}$

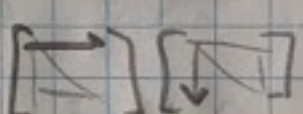
II.) Estimate the # of operations to perform LU decomposition of A

↳ Assume using Doolittle Factorization



each equation for element of array A takes:

↳ max: (n) multiplications $+$ $(n-1)$ additions $\overline{}$ $2n-1$ 

min: 1 multiplication $+$ 0 additions $\overline{}$ 1 

on average $= \dfrac{(2n-1)+1}{2} = n$ operations per element in A

$\left(n \text{ operations per element}\right) \times \left(n \times n \text{ elements in A}\right) \approx n^3$ operations

$\boxed{\therefore \text{ LU decomposition of A should take} \approx 64 \text{ operations using LU Doolittle}}$

III) Decompose Matrix A to LU

$LU = A$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \\ 2 & 1 & 1 & 4 \end{bmatrix}}_{A}$$

$U_{11} = 1$ $\quad U_{12} = 1$ $\quad U_{13} = 1$ $\quad U_{14} = 2$

$(L_{21})(U_{14})$

$L_{21}(1) = 2$ $\quad (2)(1) + U_{22} = 1$ $\quad (2)(1) + U_{23} = 1$ $\quad (2)(2) + U_{24} = 1$

$L_{21} = 2$ $\quad U_{22} = -1$ $\quad U_{23} = -1$ $\quad 4 + U_{24} = 1$

$U_{24} = -3$

$(L_{31})(1) = 3$ $\quad (3)(1) + L_{32}(-1) = 2$ $\quad (3)(1) + (1)(-1) + U_{33} = 1$

$L_{31} = 3$ $\quad L_{32} = 1$ $\quad U_{33} = -1$

$(3)(2) + (-3)(1) + U_{34} = 2$ $\quad (2)(2) + (1)(-3) + \cancel{(0)(-1)} + 1(U_{44}) = 4$

$\cancel{4} - 3 + U_{44} = \cancel{4}$

$U_{34} = -1$ $\quad U_{44} = 3$

$L_{41}(1) = 2$ $\quad (L_{41})(U_{12}) + (L_{42})(U_{22})$ $\quad (1)(2) + (1)(-1) + L_{43}(-1) = 1$

$L_{41} = 2$ $\quad 2 + (-1)L_{42} = 1$ $\quad L_{43} = 0$

$L_{42} = 1$

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}}$$

IV) Find smallest eigenvalue of A (with 0.01 accuracy) $\longrightarrow$ Inverse Power Method

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \\ 2 & 1 & 1 & 4 \end{bmatrix}$$

$$A^{-1} = ? \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ \\ \\ R_4 - R_2 \end{array} \left[\begin{array}{cccc|cccc} -1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} -R_1 - \frac{R_4}{3} \\ R_2 + R_1 \\ \\ \frac{R_4}{3} \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & \frac{2}{3} & 0 & \frac{1}{3} \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} \end{array}\right]$$

$$\begin{array}{c} \\ R_2 - R_1 \\ R_3 - 2R_4 \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 1 & 0 & 2 & -\frac{2}{3} & 0 & -\frac{1}{3} \\ 3 & 2 & 1 & 0 & 0 & \frac{2}{3} & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} \end{array}\right]$$

$$\begin{array}{c} \\ \\ R_3 - 3R_1 \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 1 & 0 & & & & \\ 0 & 2 & 1 & 0 & 3 & -\frac{4}{3} & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 1 & & & & \end{array}\right]$$

$$\begin{array}{c} \\ -R_2 + R_3 \\ \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & 1 & -\frac{4}{3} & 1 & -\frac{2}{3} \\ 0 & 2 & 1 & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \end{array}\right] \longrightarrow \begin{array}{c} \\ \\ R_3 - 2R_2 \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 1 & -\frac{4}{3} & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & 1 & \frac{4}{3} & -1 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} \end{array}\right]$$

A^{-1}

$$A^{-1}x^{(0)} = \begin{bmatrix} -1 & \frac{2}{3} & 0 & \frac{1}{3} \\ 1 & -\frac{4}{3} & 1 & -\frac{2}{3} \\ 1 & -\frac{4}{3} & -1 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$A^{-1}x^{(1)} = \begin{bmatrix} 1 & \frac{2}{3} & 0 & \frac{1}{3} \\ 1 & -\frac{4}{3} & 1 & -\frac{2}{3} \\ 1 & -\frac{4}{3} & -1 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} \\ \frac{4}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} \qquad \lambda_{inv} = -\frac{5}{3}$$

$$A^{-1}x^{(2)} = \begin{bmatrix} 1 & \frac{2}{3} & 0 & \frac{1}{3} \\ 1 & -\frac{4}{3} & 1 & -\frac{2}{3} \\ 1 & -\frac{4}{3} & -1 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -0.8 \\ -0.4 \\ 0.2 \end{bmatrix} = \begin{bmatrix} -1.6 \\ 1.9 \\ 0.4 \\ 0.2 \end{bmatrix} \qquad \lambda_{inv} = -1.6$$

$$A^{-1}x^{(3)} = A^{-1} \begin{bmatrix} 1 \\ -1.25 \\ -0.25 \\ -0.125 \end{bmatrix} = \begin{bmatrix} -1.791\overline{6} \\ 2.\overline{3} \\ -0.2083 \\ 0.\overline{3} \end{bmatrix} \qquad \lambda_{inv} = -1.79$$

$$A^{-1}x^{(4)} = A^{-1} \begin{bmatrix} 1 \\ -1.3023 \\ 0.11627 \\ -0.1760 \end{bmatrix} = \begin{bmatrix} -1.9302 \\ 2.9767 \\ -0.7907 \\ 0.37209\ldots \end{bmatrix} \qquad \lambda_{inv} = -1.93$$

$$A^{-1}x^{(5)} = A^{-1} \begin{bmatrix} 1 \\ -1.54 \\ 0.40 \\ -0.19 \end{bmatrix} = \begin{bmatrix} -2.092 \\ 3.6 \\ -1.9 \\ 0.45 \end{bmatrix} \qquad \lambda_{inv} = -2.09$$

$$A^{-1}x^{(6)} = A^{-1} \begin{bmatrix} 1 \\ -1.71 \\ 0.67 \\ -0.215 \end{bmatrix} = \begin{bmatrix} -2.21 \\ 4.1 \\ -1.85 \\ 0.5 \end{bmatrix} \qquad \lambda_{inv} = -2.21$$

$$A^{-1}X^{(7)} = A^{-1}\begin{bmatrix}1\\-1.8\\0.85\\0.225\end{bmatrix} = \begin{bmatrix}-2.31\\4.47\\-2.247\\0.542\end{bmatrix}$$

$\lambda_{inv} = -2.31$

$$A^{-1}X^{(8)} = A^{-1}\begin{bmatrix}1\\-1.9\\0.97\\-0.23\end{bmatrix} = \begin{bmatrix}-2.368\\4.7\\-2.47\\0.567\end{bmatrix}$$

$\lambda_{inv} = -2.368$

$$A^{-1}X^{(9)} = A^{-1}\begin{bmatrix}1\\-1.9\\1.05\\-0.24\end{bmatrix} = \begin{bmatrix}-2.405\\4.86\\-2.61\\0.58\end{bmatrix}$$

$\lambda_{inv} = -2.405$

$$A^{-1}X^{(10)} = A^{-1}\begin{bmatrix}1\\-2.01\\1.08\\-0.24\end{bmatrix} = \begin{bmatrix}-2.43\\4.95\\-2.70\\0.592\end{bmatrix}$$

$\lambda_{inv} = -2.43$

$$A^{-1}X^{(11)} = A^{-1}\begin{bmatrix}1\\-2.03\\1.11\\-0.24\end{bmatrix} = \underbrace{\begin{bmatrix}-2.44\\4.98\\-2.75\\0.59\end{bmatrix}}_{X^{(12)}}$$

$\lambda_{inv} = -2.44$

$$\lambda = \frac{1}{\lambda_{inv}} = -0.4098$$

$\boxed{\therefore \text{ eigenval with smallest absolute value} = -0.4098}$