L3 Linear Equations

Number of operations

Linear Systems

Gabriel Cramer

Gauss Elimination

Carl Friedric Gauss

Gauss –Jordan Elimination

Camille Jordan

LU Factorization

Norms and Conditions

Matrix Algebra - $A = [A] = a_{ij} = a_i$

with a_i is column i

Addition and subtraction

$$A \pm B = C$$

$$a_{ij} \pm b_{ij} = c_{ij}$$

Commutation $A + B = B + A$

Association $[A + B] + C = [A] + [B + C]$

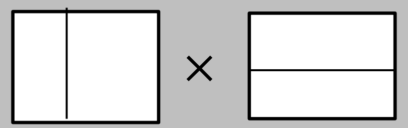
Multiplication

$$A * B = C$$

With $A = [n, k]$ and $B = [k, m]$, then $C = [n, m]$ and

$$c_{ij} = \sum_{s=1}^{k} a_{is} b_{sj}$$

$c_{ij} =$



Distribution: $A * (B + C) = A * B + A * C$

Division

For $A * B = I$,

then $B = A^{-1}$. Matix B is inverse of matrix A

For $A * B = C$,

to determine B, mulitipy both sides by inverse of A, A^{-1}

$$(A^{-1} * A) * B = A^{-1} * C$$

With $A^{-1} * A = I$, and $I * B = B$, then

$$B = A^{-1} * C$$

Linear equations- n linear equations for n unknown x_i. With $x_i = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_n \end{bmatrix}$

The linear equations will be in the form

$$\begin{cases} a_{11}\,x_1 + a_{12}x_2 + a_{13}x_3 + \ldots a_{1n}x_n = b_1 \\ a_{21}\,x_1 + a_{22}x_2 + a_{23}x_3 + \ldots a_{2n}x_n = b_2 \\ a_{31}\,x_1 + a_{32}x_2 + a_{33}x_3 + \ldots a_{3n}x_n = b_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ a_{n1}\,x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots a_{nn}x_n = b_n \end{cases}$$

Coefficient of matrix A have the form $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \ldots\ldots a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} \ldots\ldots a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34}, \ldots\ldots a_{3n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ a_{n1}a_{n2} & a_{n3} & a_{n4} \ldots\ldots a_{nn} \end{bmatrix}$

The inhomogeneous vector b has the form $b_i = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ . \\ b_n \end{bmatrix}$

General matrix form and Solution

$$Ax = b$$

$$x = A^{-1}\mathrm{b}$$

Problem(?)

Cramer's Rule

$$Ax = b$$

Then

$$x_j = \frac{|A_{sj}|}{|A|}$$

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \dots \dots . \, a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} \dots \dots . \, a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34}, \dots \dots . \, a_{3n} \\ \dots & \dots & \dots & \dots \dots \dots \dots \dots \dots . \\ \dots & \dots & \dots & \dots \dots \dots \dots \dots \dots . \\ \dots & \dots & \dots & \dots \dots \dots \dots \dots \dots . \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} \dots \dots . \, a_{nn} \end{bmatrix}$$

$|A|$ is determinant of A and A_{sj} is defined as associated j matrix of A

A_{sj} is matrix A with th jth column substituted by inhomogeneous vector b.

$$A_{sj} = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & a_{13} \dots & b_1 & \dots \dots . \, a_{1n} \\ a_{21} & a_{22} & a_{23} \dots & b_2 & \dots \dots . \, a_{2n} \\ a_{31} & a_{32} & a_{33} \dots & b_3, & \dots \dots . \, a_{3n} \\ \dots & \dots & \dots \dots & \dots & \dots \dots . \\ \dots & \dots & \dots \dots & \dots & \dots \dots . \\ \dots & \dots & \dots \dots & \dots & \dots \dots . \\ a_{n1} & a_{n2} & a_{n3} \dots . & b_n & \dots \dots . \, a_{nn} \end{bmatrix} \\ J\text{th column} \end{array}$$

Problem with Cramer's rule(?)

How many operations?

Properties of linear equations

$Ax = b$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

1- Scaling- any equation can be multiplied by an arbitrary constant α without changing values of unknown vector x.

$$\sum_{j=1}^{n} \alpha\, a_{ij} x_j = \alpha b_i$$

2- Switching – interchanging any two rows, and in solving the equations, it is referred to as pivoting.

$$\begin{cases} a_{11}\, x_1 + a_{12} x_2 + a_{13} x_3 + \ldots a_{1n} x_n = b_1 \\ a_{21}\, x_1 + a_{22} x_2 + a_{23} x_3 + \ldots a_{2n} x_n = b_2 \\ a_{31}\, x_1 + a_{32} x_2 + a_{33} x_3 + \ldots a_{3n} x_n = b_3 \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\ a_{n1}\, x_1 + a_{n2} x_2 + a_{n3} x_3 + \ldots a_{nn} x_n = b_n \end{cases}$$

3- Linear Combinations

$$\sum_{j=1}^{n} \alpha_1 a_{1j} x_j = \alpha_1 b_1$$
$$\sum_{j=1}^{n} \alpha_2 a_{2j} x_j = \alpha_2 b_2$$

Summing up:

$$\sum_{j=1}^{n} (\alpha_1 a_{1j} + \alpha_2 a_{2j}) x_j = \alpha_1 b_1 + \alpha_2 b_2$$

This combination is called elimination procedure

Elimination Method

Using rows and their linear combinations to eliminate unknown variable x, as shown in example below:

$$
\begin{aligned}
a_{11}\,x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \qquad R_1\\
a_{21}\,x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \qquad R_2\\
a_{31}\,x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \qquad R_3
\end{aligned}
$$

1- eliminate x_1 from rows 2 by forming $R_2 - \frac{a_{21}}{a_{11}} R_1$. The ratio $\frac{a_{21}}{a_{11}}$ or its more general form $\frac{a_{ij}}{a_{ii}}$ is em (elimination multiplier - m_{ij}).

$$(a_{22} - \frac{a_{21}}{a_{11}})x_2 + (a_{23} - \frac{a_{21}}{a_{11}})x_3 = (b_2 - \frac{a_{21}}{a_{11}} b_1) \qquad R'_2$$

2- Repeat step 2 for row 3 to eliminate x_2 using the above equation by forming $R_3 - \frac{a_{32}}{(a_{22} - \frac{a_{21}}{a_{11}})} R'_2$

$$(a_{33} - \frac{a_{32}}{(a_{22} - \frac{a_{21}}{a_{11}})} x_3) = b_3 - \frac{a_{32}}{(a_{22} - \frac{a_{21}}{a_{11}})} (b_2 - \frac{a_{21}}{a_{11}} b_1)$$

3- x_3 is directly obtained from the above equation and successive back substitutions will produce x_2 and x_1.

Number of operations (?)

Notes

1- A is augmented by vector b in the form $A|b$

$$A|b = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{33} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

2- Modified equations can be written with modified coefficients in shorter forms

R'_2 $\quad a'_{22}\, x_2 + a'_{23} x_3 = b'_2$
or $\quad [\, a'_{22} \;\; a'_{23} \mid b'_2 \,]$

3- Multiple b vectors –

Augment matrix A with all b vectors

$$A|b = \left[\begin{array}{ccc|c|c|c} a_{11} & a_{12} & a_{13} & b_1 & c_1 & d_1 \;\; \ldots. \\ a_{21} & a_{22} & a_{33} & b_2 & c_2 & d_2 \;\ldots. \\ a_{31} & a_{32} & a_{33} & b_3 & c_3 & d_3 \;\ldots. \end{array}\right]$$

And proceed with elimination method.

4- Pivoting and switching when a_{ii} is zero or very small.
Pivoting and switching should be automatic, not pre-selected, in computer routines. Gauss Elimination

Gauss Elimination

To automate elimination method with pivoting and scaling with coefficient matrix A and inhomogeneous vector b, define an order vector o $(n \times 1)$ for rows which is initially $(1,2,3,\ldots,n)$.

1- In column 1, the largest coefficient be row k. Pivot row k to be the first row, with order vector will be $o = (k, 2,3,..,1,..n)$

2- Eliminate x_k from $(n-1)$ equations

3- In modified $(n-1)$ equations, pivot and eliminate $x_{k'}$, and $o = (k, k', 3,..,1,..n)$.

4- Upon completion, matrix A is transformed to an upper triangular matrix that could be solved for x direct backward substitution.

5- Number of operations(?)

Gauss –Jordan Elimination

This elimination procedure will find A^{-1} and x

1- Form an augmented matrix in the form

$$[A \mid b \mid I]$$

2- By elimination, transform A to I so that we have

$$[A \mid b \mid I] \rightarrow [I \mid b' \mid A']$$

3- $A' = A^{-1}$ and $b' = x$

4- No. of operations (?).

LU Factorization

$$L = \begin{bmatrix} a_{11} & 0 & 0 & 0 \ldots & \ldots .0 \\ a_{21} & a_{22} & 0 & 0 \ldots \ldots & .0 \\ a_{31} & a_{32} & a_{33} & 0, \ldots \ldots .0 & \\ \ldots & \ldots & \ldots & \ldots & \ldots . \\ \ldots & \ldots & \ldots & \ldots & \ldots . \\ \ldots & \ldots & \ldots & \ldots & \ldots . \\ a_{n1} & a_{n2} & a_{n3} & \ldots & \ldots . a_{nn} \end{bmatrix} \qquad U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \ldots \ldots . a_{1n} \\ 0 & a_{22} & a_{23} & a_{24} & \ldots \ldots . a_{2n} \\ 0 & 0 & a_{33} & a_{34} & \ldots \quad . a_{3n} \\ 0 & 0 & 0 & a_{44} \, a_{45} & a_{4n} \\ \ldots & \ldots & \ldots & \ldots & \ldots . \\ \ldots & \ldots & \ldots & \ldots & \ldots . \\ 0 & 0 & 0 & 0 \ldots & \ldots . a_{nn} \end{bmatrix}$$

For linear equation $Ax = b$, decompose matrix A as product of L and U

$$A = LU$$

$$LUx = b$$

Multiply both sides with L^{-1}

$$L^{-1}LUx = L^{-1}b \rightarrow IUx = L^{-1}b$$

With $IU = U$ and set $b' = L^{-1}b$

$$Ux = b'$$

This equation solved for x with direct backward substitution.

Factorization Procedure

For $Ax = b$, use Gauss elimination to transform matrix A into U.

In this transformation, keep the elimination multipliers m_{ij} and

$l_{ij} = m_{ij}$.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \ldots & \ldots .0 \\ m_{21} & 1 & 0 & 0 \ldots \ldots & .0 \\ m_{31} & m_{32} & 1 & 0, \ldots \ldots .0 & \\ \ldots & \ldots & \ldots & \ldots & \ldots . \\ \ldots & \ldots & \ldots & \ldots & \ldots . \\ \ldots & \ldots & \ldots & \ldots & \ldots . \\ m_{n1} & m_{n2} & m_{n3} & \ldots & \ldots .1 \end{bmatrix}$$

This procedure is Doolittle factorization and with steps on the left, x can be determined.

MatLab

For A is defined $\quad [l, u] = lu(A)$

For A and P defined $[l, u, P] = lu(A)$
with $PA = LU$

More on determinants

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

For which del $[A]$ = $|A| = a_{11}\ a_{12} - \ a_{12}\, a_{21}$

1- Pivot matrix A to A^p

$A^p = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$.

$|A^p|$ = $a_{12}a_{21} \ - a_{11}\ a_{12} = -|A|$

Pivoting by i rows, $|A^p| = (1)^i |A|$

2- Multiply A by a constant α

$\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow |\alpha A| = \ \alpha |A|$

3- Set $R_1' = R_1 + \lambda R_2$

$R_1' = \begin{bmatrix} a_{11} + \lambda a_{21} & a_{12} + \lambda a_{21} \\ a_{21} & a_{22} \end{bmatrix}$

$|R_1'| = |A|$

Solving Tridiagonal Matrix

$$T = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \ \ldots & \ldots .0 \\ a_{21} & a_{22} & a_{23} & 0 \ \ldots\ldots & .0 \\ 0 & a_{32} & a_{33} & a_{34}, \ldots & \ldots .0 \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . & & & \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . & & & \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . & & & \\ 0 \quad 0 & 0 & a_{(n-1)n} & a_{nn} \end{bmatrix}$$

1- Gauss elimination to make it zero the lowest diagonal elements to make T a U. Back substitution to find x. In this transformation, the vector b should also be modified.

2- Use Gauss elimination to calculate em's and decompose T to L and U.

3- No. of operations (?)

4- T can be stored as a $3 \times n$ matrix to save memory space.

$$T = \begin{bmatrix} - & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{32} & a_{33} & a_{34} \\ & \ldots\ldots\ldots\ldots & \\ & \ldots\ldots\ldots\ldots & \\ & \ldots\ldots\ldots\ldots & \\ a_{(n-1)n} & a_{nn} & - \end{bmatrix}$$

Norms and conditions

For A a matrix and b a vector

Norm of $A = \|A\|$
Norm of $b = \|b\|$

For b

$\|b\|_1 = \sum_{i=1}^{n} |x_i|$
$\|b\|_2 = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2}$
$\|b\|_\infty = \max\limits_{i=1,n} x_i$

For A

$\|A\|_1 = \max\limits_{j=1,n} \sum_{i=1}^{n} |a_{i,j}|$ columns
$\|A\|_2 = \max\limits_{i=1,n} \sum_{i=1}^{n} |a_{i,j}|$ rows
$\|A\|_3 = \min\limits_{i=1,n} \lambda_i$ Special norm
$\|A\|_e = \left[\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j}^2\right]^{1/2}$ Euclidean norm

$\lambda_i = n$ eigenvalues of A

Norm Properties

$\|A\| \geq 0$ zero when $A = 0$
$\|kA\| = k\|A\|$
$\|A + B\| \leq \|A\| + \|B\|$
$\|AB\| \leq \|A\| \times \|B\|$

Conditioning number $C(A)$

For linear problem $Ax = b$, conditioning number defined as

$C(A) = \|A\| \, \|A^{-1}\|$

A variation of x as δx in terms of variation of matrix A as δA

$$\frac{\|\delta x\|}{\|x\|} \leq C(A) \frac{\|\delta A\|}{\|A\|}$$

$C(A)$ scales δx with δA

$C(A) << 1$ problem is well-conditioned
$C(A) \gg 1$ problem is ill-conditioned