ES 101
Introduction
Number of operations

Matrix operations

$$c_{ij} = \sum_{s=1}^{k} a_{is} b_{sj}$$

Vectors and Matrices

$$a_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ . \\ . \\ a_n \end{bmatrix}$$

Linear Equations

$$\begin{cases} a_{11}\,x_1 + a_{12}x_2 + a_{13}x_3 + \ldots a_{1n}x_n = b_1 \\ a_{21}\,x_1 + a_{22}x_2 + a_{23}x_3 + \ldots a_{2n}x_n = b_2 \\ a_{31}\,x_1 + a_{32}x_2 + a_{33}x_3 + \ldots a_{3n}x_n = b_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ a_{n1}\,x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots a_{nn}x_n = b_n \end{cases}$$

Linear Algebra

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \ldots\ldots a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} \ldots\ldots a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34}, \ldots\ldots a_{3n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ a_{m1} a_{m2} & a_{m3} & a_{m4} & \ldots\ldots a_{mn} \end{bmatrix}$$

Approximate Representation: will use the first n terms of the infinite series

$$f(x) = \sum_{n=0}^{N} \frac{f^{n}(0)}{n!} x^{n} + R_{N+1}$$

$$R_{N+1} = \sum_{N+1}^{\infty} \frac{f^{n}(0)}{n!} x^{n} = \frac{f^{N+1}(\xi)}{(N+1)!} \xi^{n} \Big|_{max}$$ Truncation Error

And $0 < \xi < x$

Examples : $f(x) = x^{b}$ at $x = 1$

With $b = 2.5$, find R_4

MatLab

```
>> 3^2.5
```

15.5885

```
>> syms x b
>> T = taylor(x^b, x, 'ExpansionPoint', 1)
```

b*(x - 1) - (- b^2/2 + b/2)*(x - 1)^2 - (x - 1)^3*(b*(- b^2/6 + b/4) - b/3 + b^2/4) - (x - 1)^4*(b/4 - (b*(- b^2/6 + b/4))/2 + b*(b*(- b^2/24 + b/12) - b/6 + b^2/12) - b^2/6) - (x - 1)^5*((b*(- b^2/6 + b/4))/3 - b/5 - (b*(b*(- b^2/24 + b/12) - b/6 + b^2/12))/2 + b*(b/8 - (b*(- b^2/24 + b/12))/2 + b*(b*(- b^2/120 + b/48) - b/18 + b^2/48) - b^2/18) + b^2/8) + 1

```
>> T = taylor(x^(2.5), x, 'ExpansionPoint', 1)
```

T =

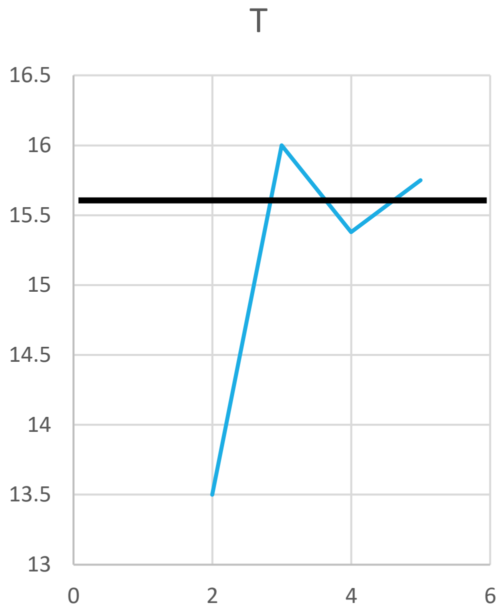
(5*x)/2 + (15*(x - 1)^2)/8 + (5*(x - 1)^3)/16 - (5*(x - 1)^4)/128 + (3*(x - 1)^5)/256 - 3/2

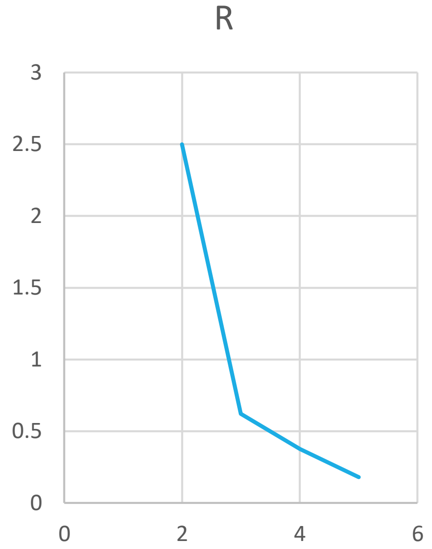
>>T2 = (5*x)/2 + (15*(x - 1)^2)/8 - 3/2 = 13.5
>> R2 = (5*(x - 1)^3)/16 = 2.5

T3 = T2 + (5*(x - 1)^3)/16 = 16
R3 = (5*(x - 1)^4)/128 = 5/8 = 0.62

T4 = T3 - (5*(x - 1)^4)/128 = 15.38
R4 = (3*(x - 1)^5)/256 = 3/8 = 0.375

T5 = T4 + (3*(x - 1)^5)/256 = 15.75
R5 = 0.18





Matrices

A matrix A with size or dimension $m \times n$, with

m as number of rows

n number of columns

has the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \dots \dots . a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} \dots \dots . a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34}, \dots \dots . a_{3n} \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots . \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots . \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots . \\ a_{m1} a_{m2} & a_{m3} & a_{m4} & \dots \dots . a_{mn} \end{bmatrix} = [a_{i,j}] =$$

$a_{i,j} = a_{ij}$

i refers to row i

and

j refers to column j.

The number of elements for matrix (size or dimension) A is

$m \times n$.

Vectors

Vector x_i can be a row or a column:

$$\text{Column vector } x_i = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = (n \times 1)$$

$$\text{Row vector } x_i = [x_1 \quad x_2 \quad \dots \dots . x_n] = (1 \times n)$$

Matrix S is a square matrix when $m = n$, that is number of rows and columns are equal with size n^2

$$S = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \dots \dots . a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} \dots \dots . a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34}, \dots \dots . a_{3n} \\ \dots & \dots & \dots & \dots \dots \dots \dots . \\ \dots & \dots & \dots & \dots \dots \dots \dots . \\ \dots & \dots & \dots & \dots \dots \dots \dots . \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} \dots \quad \dots . a_{nn} \end{bmatrix}$$

And

$$\begin{array}{ccc} & & \text{J}^{\text{th}}\text{-column} \\ S_{ij}^C & = & \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \dots & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} \dots & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34}, \dots & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} \dots & \dots & a_{nn} \end{bmatrix} \\ \text{i}^{\text{th}}\text{-row} & & \end{array}$$

OUT

Determinant of square matrix S

$$\det [S] = |S|$$

$$|S| = \sum_i \sum_j a_{ij} \, M_{ij} (-1)^{i+j}$$

With M_{ij} = cofactor matrix

=det (S – i$^{\text{th}}$-row – J$^{\text{th}}$-column)

S – i$^{\text{th}}$-row – J$^{\text{th}}$-column = S_{ij}^C

And $M_{ij} = \left|S_{ij}^C\right|$

N_s =Number of operations to calculate det $[S]$. With n^2 cells,

$$N_s = n^2 N_{S_{ij}^C}$$

With $S_{ij}^C = [(\mathrm{n}-1) \times (n-1)]$,

then

$$N_s = n^2 (n-1)^2 N_{S_{ij}^{CC}}$$

$$N_s = n^2 (n-1)^2 (n-2)^2 N_{S_{ij}^{CCC}}$$

$$N_s = \dots\dots\dots\dots\dots\dots$$

$$N_s = (n!)^2 \text{ !!!!!!}$$

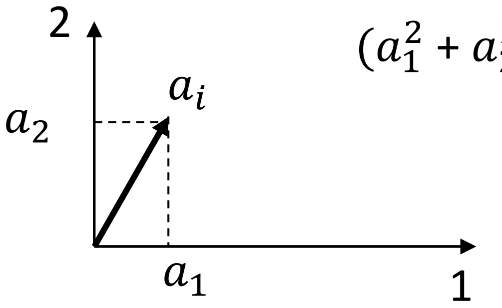
Ex – n = 3, 5, 100

Unit Vector

$$a_i = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Vector $a_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ . \\ . \\ a_n \end{bmatrix}$ has a unit length

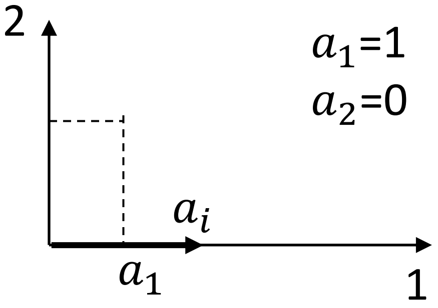
with $|a_i| = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} = 1$



$(a_1^2 + a_2^2)^{1/2}$ =1

If $a_k = 1$ and $a_i = 0$ for $i \neq k$, then vector a_i will be unit vector in k direction

$$a_i = \begin{array}{c l} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{array}{l} \\ \\ kth \\ \\ \\ \\ \end{array} \end{array}$$



a_1=1

a_2=0

Diagonal Matrix D_{ij}

$$D_{ij} = \begin{cases} a_{ij} \neq 0 & i = j \\ a_{ij} = 0 & i \neq j \end{cases}$$

$$D_{ij} = \begin{bmatrix} a_{11}\ 0\quad 0\quad 0 \ldots \quad \ldots .0 \\ 0\quad a_{22}\ \ 0\quad 0 \ldots \ldots .0 \\ 0\quad 0\ \ a_{33}\ \ 0, \ldots \ldots .0 \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\ 0\quad 0\quad 0\quad 0 \ldots \ldots \quad .a_{nn} \end{bmatrix}$$

D_{ij} is a square matric

$$|D_{ij}| = \prod_{k=1}^{n} a_{ii} = a_{11}a_{22}a_{33} \ldots .\ a_{nn}$$

Identy Matrix I

$$I = \begin{cases} a_{ij} = 1 & i = j \\ a_{ij} = 0 & i \neq j \end{cases}$$

$$I = \begin{bmatrix} 1\quad 0\quad 0\quad 0 \ldots \ldots .0 \\ 0\quad 1\quad 0\quad 0 \ldots \ldots .0 \\ 0\quad 0\quad 1\quad 0, \ldots \ldots .0 \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\ 0\quad 0\quad 0\quad 0 \ldots \ldots .1 \end{bmatrix}$$

If A is a square matrix, then

$$I \times A = A \times I = A$$

Lower Triangular Matrix L_{ij} - Non-zero elements below the diagonal elements

$$a_{ij} \neq 0 \quad i \leq j$$
$$a_{ij} = 0 \quad i > j$$

$$L_{ij} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \ldots & \ldots .0 \\ a_{21} & a_{22} & 0 & 0 \ldots \ldots & .0 \\ a_{31} & a_{32} & a_{33} & 0, \ldots \ldots & .0 \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\ a_{n1} & a_{n2} & a_{n3} & \ldots & \ldots . a_{nn} \end{bmatrix}$$

$$|L_{ij}| = \prod_{k=1}^{n} a_{ii} = a_{11}a_{22}a_{33} \ldots .\ a_{nn}$$

Upper Triangular Matrix U_{ij} - Non-zero elements above diagonal elements

$$a_{ij} = 0 \quad i < j$$
$$a_{ij} \neq 0 \quad i \leq j$$

$$U_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \ldots \ldots . a_{1n} \\ 0 & a_{22} & a_{23} & a_{24} \ldots \ldots . a_{2n} \\ 0 & 0 & a_{33} & a_{34} \ \ldots \ \ . a_{3n} \\ 0 & 0 & 0 & a_{44}\, a_{45} \ \ a_{4n} \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\ 0 & 0 & 0 & 0 \ldots \ \ \ldots . a_{nn} \end{bmatrix}$$

$$|U_{ij}| = \prod_{k=1}^{n} a_{ii} = a_{11}a_{22}a_{33} \ldots .\ a_{nn}$$

Empty Matrix - with $a_{ij} = 0$ for all i's and all j's

Transpose Matrix - A^T of Matrix A. Switching of rows and columns.

$$a_{ij}^T = a_{ji}$$

Symmetric Matrix $a_{ij} = a_{ji}$

If A symmetric, then $A^T = A$

Sparse Matrix - Vary large matrix with few non-zero elements

Diagonally Dominant Matrix – such that $|a_{ii}| \gg \sum_{i \neq j} a_{ik}$

Tridiagonal Matrix – three non-zero diagonal elements

$$\mathbf{T} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix}$$

Pantadiagonal Matrix – five non-zero diagonal elements

Banded Matrix B_{ij} - Nonzero elements on selected or handful of diagonal elements

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} & 0 \\ a_{21} & a_{22} & a_{23} & 0 & a_{25} \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ a_{41} & 0 & a_{43} & a_{44} & a_{45} \\ 0 & a_{52} & 0 & a_{54} & a_{55} \end{bmatrix}$$

Matrix Algebra - $A = [A] = a_{ij} = a_i$

with a_i is column i

Addition and subtraction

$$A \pm B = C$$
$$a_{ij} \pm b_{ij} = c_{ij}$$

Commutation $A + B = B + A$

Association $[A + B] + C = [A] + [B + C]$

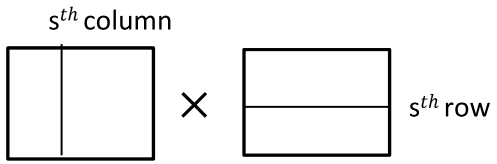
Multiplication

$$A * B = C$$

With $A = [n, k]$ and $B = [k, m]$, then $C = [n, m]$ and

$$c_{ij} = \sum_{s=1}^{k} a_{is} b_{sj}$$

$c_{ij} =$



Distribution: $A * (B + C) = A * B + A * C$

Division

For $A * B = I$,

then $B = A^{-1}$. Matix B is inverse of matrix A

For $A * B = C$,

to determine B, mulitipy both sides by inverse of A, A^{-1}

$$(A^{-1} * A) * B = A^{-1} * C$$

With $A^{-1} * A = I$, and $I * B = B$, then

$$B = A^{-1} * \mathrm{C}$$

Linear equations- n linear equations for n unknown x_i. With $x_i = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_n \end{bmatrix}$

The linear equations will be in the form

$$\begin{cases} a_{11}\, x_1 + a_{12}x_2 + a_{13}x_3 + \ldots a_{1n}x_n = b_1 \\ a_{21}\, x_1 + a_{22}x_2 + a_{23}x_3 + \ldots a_{2n}x_n = b_2 \\ a_{31}\, x_1 + a_{32}x_2 + a_{33}x_3 + \ldots a_{3n}x_n = b_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ a_{n1}\, x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots a_{nn}x_n = b_n \end{cases}$$

Coefficient of matrix A have the form $\mathrm{A} = \begin{bmatrix} a_{11} \ a_{12} \ a_{13} \ a_{14} \ldots \ldots . a_{1n} \\ a_{21} \ a_{22} \ a_{23} \ a_{24} \ldots \ldots . a_{2n} \\ a_{31} \ a_{32} \ a_{33} \ a_{34}, \ldots \ldots . a_{3n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ a_{n1} a_{n2} \ a_{n3} \ a_{n4} \ldots \ldots . a_{nn} \end{bmatrix}$

The inhomogeneous vector b has the form $b_i = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ . \\ b_n \end{bmatrix}$

General matrix form and Solution

$$Ax = b$$

$$x = A^{-1}\mathrm{b}$$

Problem(?)

Kramer's Rule

$$Ax = b$$

Then

$$x_j = \frac{|A_{sj}|}{|A|}$$

$|A|$ is determinant of A and A_{sj} is defined as associated j matrix of A

A_{sj} is matrix A with th jth column substituted by inhomogeneous vector b.

$$A_{sj} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \dots & b_1 \dots & \dots . & a_{1n} \\ a_{21} & a_{22} & a_{23} \dots & b_2 \dots & \dots . & a_{2n} \\ a_{31} & a_{32} & a_{33} \dots & b_3 , \dots & \dots . & a_{3n} \\ & \dots \dots \dots \dots & \dots \dots \dots \dots & \dots & \dots . & \\ & \dots \dots \dots \dots & \dots \dots \dots \dots & \dots & \dots . & \\ & \dots \dots \dots \dots & \dots \dots \dots \dots & \dots & \dots . & \\ a_{n1} & a_{n2} & a_{n3} \dots . & b_n \dots & \dots . & a_{nn} \end{bmatrix}$$

Jth column

Problem with Kramer's rule(?)

How many operations?

Properties of linear equations

$$Ax = b \text{ or } \sum_{j=1}^{n} a_{ij} x_j = b_i$$

1- Scaling- any equation can be multipied by an arbitrary constant α without changing values of unknown vector x.

$$\sum_{j=1}^{n} \alpha\, a_{ij} x_j = \alpha b_i$$

2- Switching – interchanging any two rows, and in solving the equations, it is referred to as pivoting.

$$\begin{cases} a_{11}\, x_1 + a_{12}x_2 + a_{13}x_3 + \dots a_{1n}x_n = b_1 \\ a_{21}\, x_1 + a_{22}x_2 + a_{23}x_3 + \dots a_{2n}x_n = b_2 \\ a_{31}\, x_1 + a_{32}x_2 + a_{33}x_3 + \dots a_{3n}x_n = b_3 \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . \\ a_{n1}\, x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots a_{nn}x_n = b_n \end{cases}$$

3- Linear Combinations

$$\sum_{j=1}^{n} \alpha_1 a_{1j} x_j = \alpha_1 b_1$$

$$\sum_{j=1}^{n} \alpha_2 a_{2j} x_j = \alpha_2 b_2$$

Summing up:

$$\sum_{j=1}^{n} (\alpha_1 a_{1j} + \alpha_2 a_{2j}) x_j = \alpha_1 b_1 + \alpha_2 b_2$$

This combination is called elimination procedure

Elimination Method

Using rows and their linear combinations to eliminate unknown variable x, as shown in example below:

$$
\begin{aligned}
a_{11}\,x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \qquad R_1 \\
a_{21}\,x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \qquad R_2 \\
a_{31}\,x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \qquad R_3
\end{aligned}
$$

1- eliminate x_1 from rows 2 by forming $R_2 - \frac{a_{21}}{a_{11}} R_1$. The ratio $\frac{a_{21}}{a_{11}}$ or its more general form $\frac{a_{ij}}{a_{ii}}$ is em (elimination multiplier).

$$(a_{22} - \frac{a_{21}}{a_{11}})x_2 + (a_{23} - \frac{a_{21}}{a_{11}})x_3 = (b_2 - \frac{a_{21}}{a_{11}} b_1) \qquad R'_2$$

2- Repeat step 2 for row 3 to eliminate x_2 using the above equation by forming $R_3 - \frac{a_{32}}{(a_{22} - \frac{a_{21}}{a_{11}})} R'_2$

$$(a_{33} - \frac{a_{32}}{(a_{22} - \frac{a_{21}}{a_{11}})} x_3) = b_3 - \frac{a_{32}}{(a_{22} - \frac{a_{21}}{a_{11}})} (b_2 - \frac{a_{21}}{a_{11}} b_1)$$

3- x_3 is directly obtained from the above equation and successive back substitutions will produce x_2 and x_1.

Number of operations (?)

Notes

1- A is augmented by vector b in the form $A|b$

$$A|b = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{33} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

2- Modified equations can be written with modified coefficients in shorter forms

$R'_2 \qquad a'_{22}\,x_2 + a'_{23}x_3 = b'_2$

or $\qquad [\, a'_{22} \;\; a'_{23} \mid b'_2 \,]$

3- Multiple b vectors –

Augment matrix A with all b vectors

$$A|b = \left[\begin{array}{ccc|c|c|c} a_{11} & a_{12} & a_{13} & b_1 & c_1 & d_1 \;\; \ldots. \\ a_{21} & a_{22} & a_{33} & b_2 & c_2 & d_2 \;\ldots. \\ a_{31} & a_{32} & a_{33} & b_3 & c_3 & d_3 \;\ldots. \end{array}\right]$$

And proceed with elimination method.

4- Pivoting and switching when a_{ii} is zero or very small.
Pivoting and switching should be automatic, not pre-selected, in computer routines. Gauss Elimination

Gauss Elimination

To automate elimination method with pivoting and scaling with coefficient matrix A and inhomogeneous vector b, define an order vector o $(n \times 1)$ for rows which is initially $(1,2,3, \dots, n)$.

1- In column 1, the largest coefficient be row k. Pivot row k to be the first row, with order vector will be $o = (k, 2,3, .., 1, .. n)$

2- Eliminate x_k from $(n-1)$ equations

3- In modified $(n-1)$ equations, pivot and eliminate $x_{k'}$, and $o = (k, k', 3, .., 1, .. n)$.

4- Upon completion, matrix A is transformed to an upper triangular matrix that could be solved for x direct backward substitution.

5- Number of operations(?)

Gauss –Jordan Elimination

This elimination procedure will find A^{-1} and x

1- Form an augmented matrix in the form

$$[A \mid b \mid I]$$

2- By elimination, transform A to I so that we have

$$[A \mid b \mid I] \rightarrow [I \mid b' \mid A']$$

3- $A' = A^{-1}$ and $b' = x$

4- No. of operations (?).