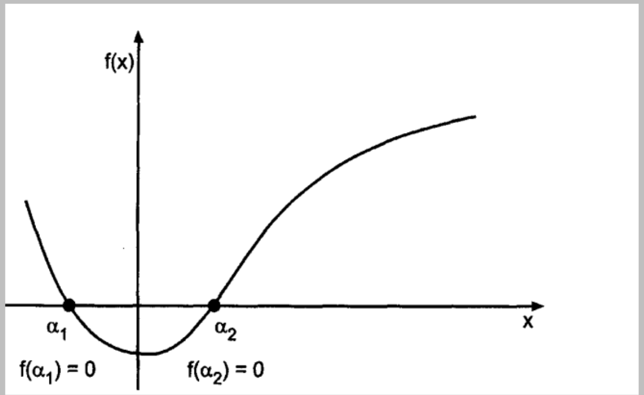
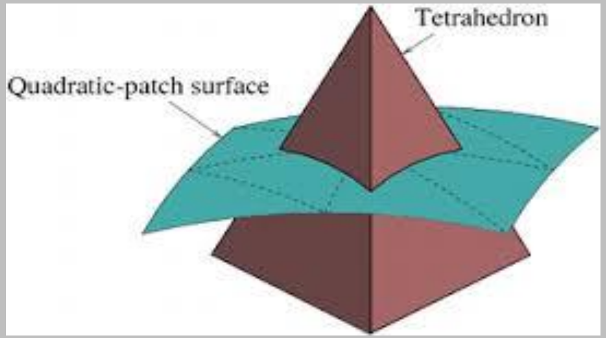
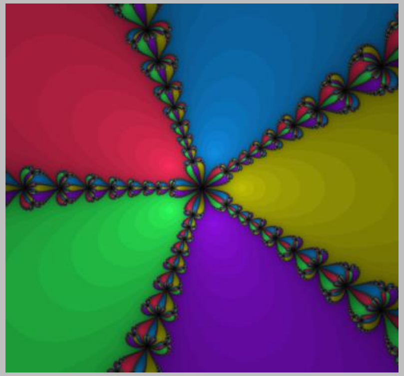
Roots of Nonlinear Functions



Surface Intersections



Fractals in use of Newton Method for complex functions

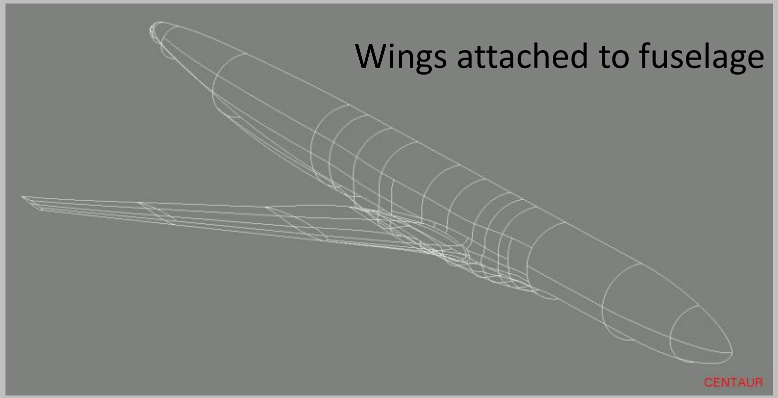


<u>MatLab</u>

root = fzero('$f(x)$', x_o)
root = fzero (@$f(x)$, $[x_2, x_1]$)

With c $\rightarrow$ coefficient vector for polynomial equation

$\alpha =$ roots (c)



Isaac Newton

Newton's Method

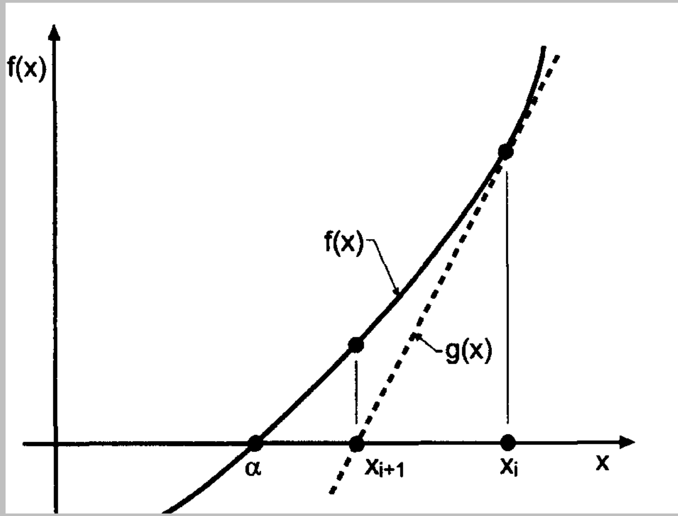
$$f'(x) = \frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}$$

Set $f(x_{i+1}) = 0$ and solve for x_{i+1}

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Convergence

$$|\Delta x_i| = |x_{i+1} - x_i| < \varepsilon$$



- $f'(x)$ exist in the calculation domain
- $f'(x) \neq 0$, if $f'(x) \to 0$, convergence slow as root is double
- $f'(x)$ is changing sign. Oscillations in calculations and slow convergence.
- $f(x)$ is defined discretely and not analytically. Define incremental distance $h = 10^{-3}$ or smaller,
- and determine f'

$$f'(x_i) = \frac{f(x_i+h)-f(x)}{h}$$

- $f'(x)$ can be approximated from two points x_1 and x_2

$$f'(x) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

- Error estimation

$e_i = \alpha - x_i$ $\qquad e_{i+1} = O(e_i^2)$

Imaginary Roots

In Newton's method, starting initial point is complex

$$x^o = a + ib$$

Double and Multiple Roots

With m repeated roots

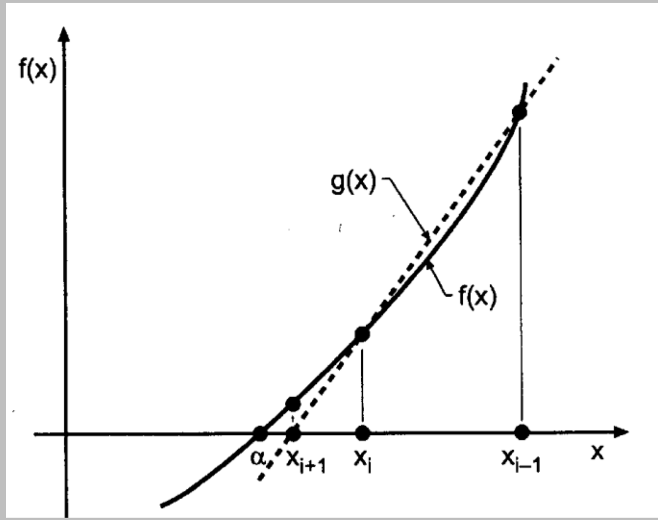
$$x_{i+1} = x_i - m\frac{f(x_i)}{f'(x_i)}$$

With this modification, Newton's method converges as e_i^2, as opposed to e_i

For roots of function $f(x) = x^3 - a$ And a = 155 $\alpha = 5.37$
Using Newton's method

Scant Method

Similar to Newton's method, with $f'(x_i)$ is estimated from straight line passing through two points x_i and x_{i-1}



With two points x_i and x_{i-1}, $f'(x_i)$ is approximated as

$$f'(x_i) \approx g'(x_i) = \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

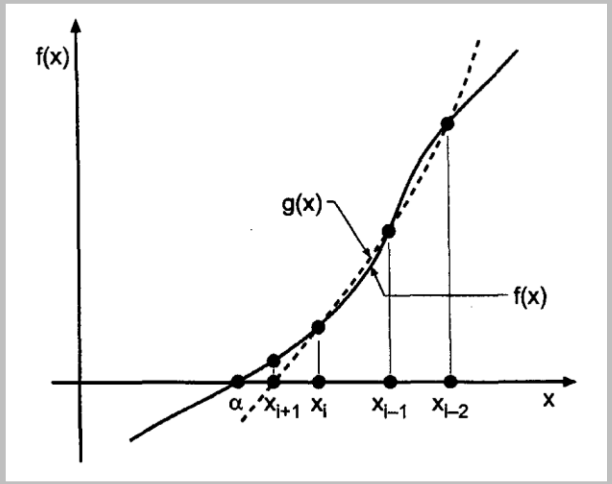
$$x_{i+1} = x_i - \frac{f(x_i)}{g'(x_i)}$$

Convergence : $e_{i+1} = O(e_i^{1.62})$

Muller Method

Similar to Scant method, a second order equation passing through three points x_i , x_{i-1} and x_{i-2} .

More calculations and better stability.



$$g(x_i) = a(x - x_i)^2 + b(x - x_i) + c$$

With $f_i = f(x_i)$, $f_{i-1} = f(x_{i-1})$, $f_{i-2} = f(x_{i-2})$

By setting $g_i = f_i$, $g_{i-1} = f_{i-1}$, $g_{i-2} = f_{i-2}$

With $c = f_i$, a and b determined from a linear set of equations

$$\Delta x_{i-1} = x_{i-1} - x_i$$
$$\Delta x_{i-2} = x_{i-2} - x_i$$

$$\begin{cases} (\Delta x_{i-2})^2 a + \Delta x_{i-2}\, b = f_i - f_{i-2} \\ (\Delta x_{i-1})^2 a + \Delta x_{i-1}\, b = f_i - f_{i-1} \end{cases}$$

Upon determining a and b, set $g(x_{i+1}) = 0$, x_{i+1} is determined by solving resulting quadradic equation

$$x_{i+1} = x_i + \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}$$

Multipe Roots – Modified Newton's method

With $f(x)$ having m roots, define deflated function $h(x)$

$$f(x) = (x-\alpha)^m h(x)$$

α not a root for $h(x)$, with $h(\alpha) \neq 0$

Define $u(x)$ as $$u(x) = \frac{f(x)}{f'(x)}$$

$$u(x) = \frac{(x-\alpha)h(x)}{mh(x)+(x-\alpha)h\prime(x)}$$

The function $u(x)$ has only one simple root and the same as $f(x)$
Use Newton's method to find the root for $u(x)$

Recurrence Equation

$$x_{i+1} = x_i - \frac{f_i f'_i}{(f'_i)^2 - f_i f''_i}$$

Roots of Polynomials

Consider a polynomial $P_n(x)$ of degree n in the form

$$P_n(x) = a_o + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

Descartes Rule

With the coefficient vector: $(a_o, a_1, a_2, \ldots \ldots a_n)$,
$N =$ number of sign switches

The number of positive real roots = $N - 2k$, with k is any positive natural number

Newton's method can be used to find the roots

Polynomial Deflation- reducing the order of the polynomial

$P_n(x)$ has a simple root at $x = \alpha$, and define the reduced polynomial as $P'_n(x)$

$$P_n(x) = (x - \alpha)\, P'_n(x)$$

$$P'_n(x) = b_1 + b_2 x + b_3 x^2 + b_4 x^3 + \cdots + b_n x^{n-1}$$

Compare $P'_n(x)$ and $P_n(x)$, and the following relationship between the coefficients

$$b_n = a_n$$
$$b_{n-1} = a_{n-1} - \alpha b_n$$
$$b_{n-2} = a_{n-2} - \alpha b_{n-1}$$

$$b_1 = a_1 - \alpha b_2$$

With a's and α known, above equations can be used successively to determine b's , starting with b_n

Example: $f(x) = 4x^3 - 5x^2 + 3x - 2 = 0$

- How many positive real roots?
- With $x = 1$ as one of the roots, determine the rest of the roots

Bairstow method

With $P_n(x) = a_o + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$
Define $Q_{n-2}(x)$ as

$$P_n(x) = (x^2 - rx - s)Q_{n-2}(x) + remainder$$

$$Q_{n-2}(x) = b_n x^{n-2} + b_{n-1} x^{n-3} + \cdots + b_3 x + b_2$$

And remainder = $b_1(x - r) + b_o$
Both b_0 and b_1 *should be zero for* $Q_{n-2}(x)$ *to be exact solution*

The relationship between coefficients are

$$b_n = a_n$$
$$b_{n-1} = a_{n-1} + rb_n$$
$$b_{n-2} = a_{n-2} + rb_{n-1} + sb_n$$

$$b_1 = a_1 + rb_2 + sb_3$$
$$b_0 = a_o + rb_1 + sb_2$$

With b_0 and b_1 functions of (r, s)

Example: $f(x) = 4x^3 - 5x^2 + 3x - 2 = 0$
with $r_o = 2; s_o = -2$

Procedure

Start with initial guesses r and s.

Define coefficients c in the following way

$$c_n = b_n$$
$$c_{n-1} = b_{n-1} + rc_n$$
$$c_{n-2} = b_{n-2} + rc_{n-1} + sc_n$$

$$c_2 = b_2 + rc_3 + sc_4$$
$$c_1 = b_1 + rc_2 + sc_3$$

Variations of coefficients r and s will be Δr and Δs and they satisfy the equations

$$c_2 \Delta r + c_3 \Delta s = -b_1$$
$$c_1 \Delta r + c_2 \Delta s = -b_o$$

With the above equations we can determine Δr and Δs
Start with two guess's r_o and s_o, then

$$r_1 = r_o + \Delta r$$
$$s_1 = s_o + \Delta s$$

Recurrence equations:

$$r_{i+1} = r_i + \Delta r_i$$
$$s_{i+1} = s_i + \Delta s_i$$

Continue until convergence