Characteristic Problems

Number of operations

Power Method

Iterative Solutions

Inverse Method

Shifting Method

- Eigen Problems
- Eigen Vectors
- Eigenvalues

QR Method

- Iterative Solution
- General method
- Solving for all eigenvalues

Power Method

For $Ax = \lambda x$

And matrix A diagonizable
n distinct eigenvalues
orthonormal

Then

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \cdots |\lambda_n|$$

Power method determines the largest eigenvalue λ_1

1. Set $x^o = \begin{bmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}$
2. Calculate $Ax^{(o)} = y^{(1)}$
3. Divide $y^{(1)}$components by its largest element λ^1
   $y^{(1)} = \lambda^1 x^1$
4. λ^1 and x^1 are first estimate of largest λ and x^1 the corresponding to eigen vector
5. Continue the iteration: $Ax^{(k)} = y^{(k+1)}$ and determine $\lambda^{(k+1)}$ and $x^{(k+1)}$

$\lambda^{(k+1)}$ converges to λ_1

Convergence issues

Note that iterations go with the λ_1^k

1. Power method convergence according to $\frac{|\lambda_1|}{|\lambda_2|}$. Convergence slow when $|\lambda_1|$ and $|\lambda_2|$ are close and the ratio is close to unity.
2. If $|\lambda_1| < 1$ when $k \to \infty$ $\lambda_1^k \to 0$
3. If $|\lambda_1| > 1$ when $k \to \infty$ $\lambda_1^k \to \infty$
4. If λ_1 is complex, no convergence

Inverse Power method

For $Ax = \lambda x$ multiply both sides by A^{-1}

$$A^{-1}x = \frac{1}{\lambda}\, x$$

Use power method to solve for $\frac{1}{\lambda}$

$\frac{1}{\lambda}$ should be maximum eigenvalue, so λ is the minimum eignevalue

Shifted Power method

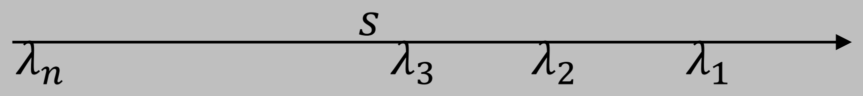
With s as a constant, deduct sI form both sides of the $Ax = \lambda x$

$$(A - sI)x = (\lambda - s)x$$

or

$$A_s x = \lambda_s x$$

With inverse method, the closest eigenvalue to s



With shifting s, iteration converges to λ_3

Use power inverse method to determine eigenvalues for matrix

$$A = [2,1; 1,2]$$

Use power shifted method with $s = 1.5$ to determine eigenvalues for matrix

$$A = [2,1; 1,2]$$

QR Method

Iteratively solves for all eigenvalues for $Ax = \lambda x$

A is $n \times n$ and when symmetric, all eigenvalues are real

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \dots \dots . a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} \dots \dots . a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34}, \dots \dots . a_{3n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} \dots \dots . a_{nn} \end{bmatrix} \text{ and } a_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \end{bmatrix}, a_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \cdot \\ \cdot \\ \cdot \\ a_{n2} \end{bmatrix} \text{ ..and } a_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \cdot \\ \cdot \\ \cdot \\ a_{nn} \end{bmatrix}$$

A= $(a_1, a_2, a_3, \dots.., a_n)$

Introduce Matrix Q

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \dots \dots . q_{1n} \\ q_{21} & q_{22} & q_{23} & q_{24} \dots \dots . q_{2n} \\ q_{31} & q_{32} & q_{33} & q_{34}, \dots \dots . q_{3n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ q_{n1} & q_{n2} & q_{n3} & q_{n4} \dots \dots . q_{nn} \end{bmatrix} \text{ and } q_1 = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{31} \\ \cdot \\ \cdot \\ \cdot \\ q_{n1} \end{bmatrix}, q_2 = \begin{bmatrix} q_{12} \\ q_{22} \\ q_{32} \\ \cdot \\ \cdot \\ \cdot \\ q_{n2} \end{bmatrix} \text{ ..and } q_n = \begin{bmatrix} q_{1n} \\ q_{2n} \\ q_{3n} \\ \cdot \\ \cdot \\ \cdot \\ q_{nn} \end{bmatrix}$$

Q= $(q_1, q_2, q_3, \dots.., q_n)$

Vector q's are orthonormal: $|q_i| = 1$ and $q_i \cdot q_j = 0$

Normal vectors have no projection on other vectors.

Matrix R

It is an upper triangular matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \dots \dots . r_{1n} \\ 0 & r_{22} & r_{23} & r_{24} \dots \dots . r_{2n} \\ 0 & 0 & r_{33} & r_{34} \dots \quad . r_{3n} \\ 0 & 0 & 0 & r_{44} \, r_{45} \quad r_{4n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \dots \quad \dots . r_{nn} \end{bmatrix}$$

If Q is orthonormal, from matrix A, can be transformed to matrix A' such that

$$A = QA'Q^T$$

A and A' are similar or equivalent with identical eigenvalues.

QR Procedure

1. Determine the matrix Q

$$q_1 = \frac{a_1}{|a_1|}$$

$$a'_2 = a_2 - q_1^T a_2 q_1 \text{ (projection)}$$

$$q_2 = \frac{a'_2}{|a'_2|}$$

$$a'_3 = a_3 - q_1^T a_3 q_1 - q_2^T a_3 q_2$$

Recurrence

$$a'_k = a_k - \sum_{i=1}^{k-1} q_i^T a_k q_i$$

$$q_k = \frac{a'_k}{|a'_k|}$$

2. Construct upper triangular matrix R

$r_{ii} = |a'_i|$ and $r_{ij} = q_i^T a_j$

Reconstruct A: $A^{(1)} = QR$

With new $A^{(1)}$, determine new $Q^{(1)} and\ R^{(1)}$

With convergence

Eigenvalues $\lambda_i = r_{ii}$

Use QR method to determine eigenvalues for matrix

$$A = [2,1; 1,2]$$