L4

Characteristic Problems

Number of operations

- Eigen Problems
- Eigen Vectors
- Eigenvalues

Power Method

Iterative Solutions

Inverse Method
Shifting Method

QR Method

- Iterative Solution
- General method
- Solving for all eigenvalues

Linear System

$Ax = b$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

With

A = system matrix
b = forcing function or vector
x = eigen vector

For system with no forcing function such that $b = 0$
The linear system will become $Ax = 0$

$|A| \neq 0$ then $x = 0$ Trivial solution

$|A| = 0$ then x can have infinite no. of solutions

Eigenvalue Problem

$$Ax = \lambda x$$

λ= eigenvalue

$$\boldsymbol{\lambda} = \lambda I$$
$$(A - \lambda I)\, x = 0$$

For non-trivial solution

$$|A - \lambda I| = 0$$

This equation generates a polynomial solution of degree n for n eigenvalues.

MatLab

$\text{eig}(A) = \lambda_i$
Or
c = poly(A)
λ = roots(c)

c = coeff's of polynomial equation for λ of order n

Rules of eigenvalue problems

- If matrix A has n distinctive eigenvalues, A is diagnosable
- If A symmetric — A has n real and independent eigenvalues
- If A is L or a U — $\lambda_i = a_{ii}$
  Determinant of A — $|A| = \prod_{i+1}^{n} \lambda_i = \lambda_1 \lambda_2 \lambda_3 \dots \lambda_n$
- Trace (A) = $\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$

Determine eigenvalues and eigenvectors for matrix

$$A = [2,1; 1,2]$$

This matrix has real eigenvalues (?)

Determine λ of this matrix

Determine the corresponding eigenvectors

Power Method

For $Ax = \lambda x$

And matrix A diagonizable
n distinct eigenvalues
orthonormal

Then

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \cdots |\lambda_n|$$

Power method determines the largest eigenvalue λ_1

1. Set $x^o = \begin{bmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}$
2. Calculate $Ax^{(o)} = y^{(1)}$
3. Divide $y^{(1)}$ components by its largest element λ^1
   $y^{(1)} = \lambda^1 x^1$
4. λ^1 and x^1 are first estimate of largest λ and x^1 the corresponding to eigen vector
5. Continue the iteration: $Ax^{(k)} = y^{(k+1)}$ and determine $\lambda^{(k+1)}$ and $x^{(k+1)}$

$\lambda^{(k+1)}$ converges to λ_1

Convergence issues

Note that iterations go with the λ_1^k

1. Power method convergence according to $\frac{|\lambda_1|}{|\lambda_2|}$. Convergence slow when $|\lambda_1|$ and $|\lambda_2|$ are close and the ratio is close to unity.
2. If $|\lambda_1| < 1$ when $k \to \infty$ $\lambda_1^k \to 0$
3. If $|\lambda_1| > 1$ when $k \to \infty$ $\lambda_1^k \to \infty$
4. If λ_1 is complex, no convergence

Use power method to determine eigenvalues for matrix

$$A = [2,1;1,2]$$

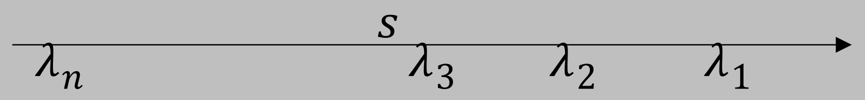
Inverse Power method

For $Ax = \lambda x$ multiply both sides by A^{-1}

$$A^{-1}x = \frac{1}{\lambda}x$$

Use power method to solve for $\frac{1}{\lambda}$

$\frac{1}{\lambda}$ should be maximum eigenvalue, so λ is the minimum eignevalue



Shifted Power method

With shifting s, iteration converges to λ_3

With s as a constant, deduct sI form both sides of the $Ax = \lambda x$

$$(A - sI)x = (\lambda - s)x$$

or

$$A_s x = \lambda_s x$$

With inverse method, the closest eigenvalue to s