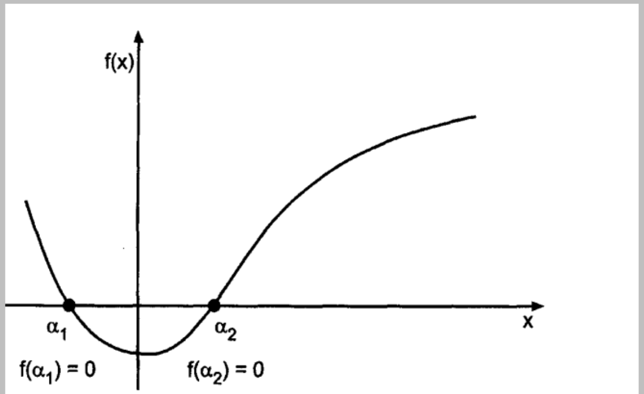
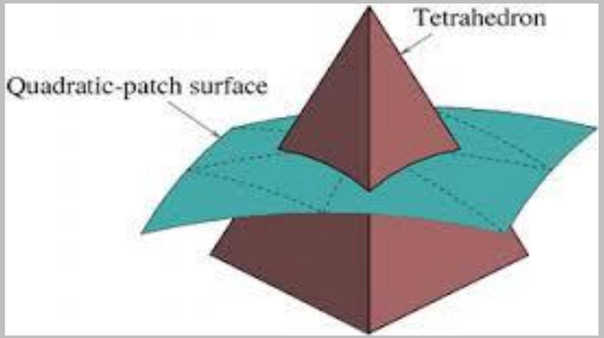
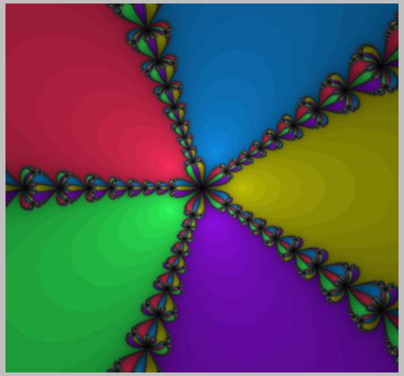
Roots of Nonlinear Functions



Surface Intersections



Fractals in use of Newton Method for complex functions

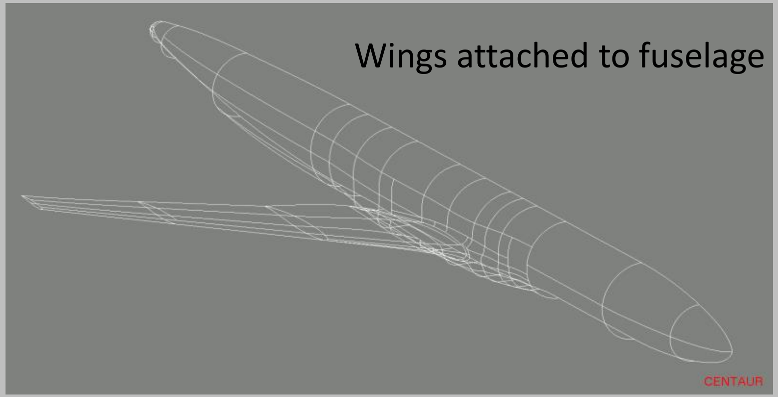


MatLab

root = fzero('$f(x)$', x_o)
root = fzero (@$f(x)$, $[x_2, x_1]$)

With c $\rightarrow$ coefficient vector for polynomial equation

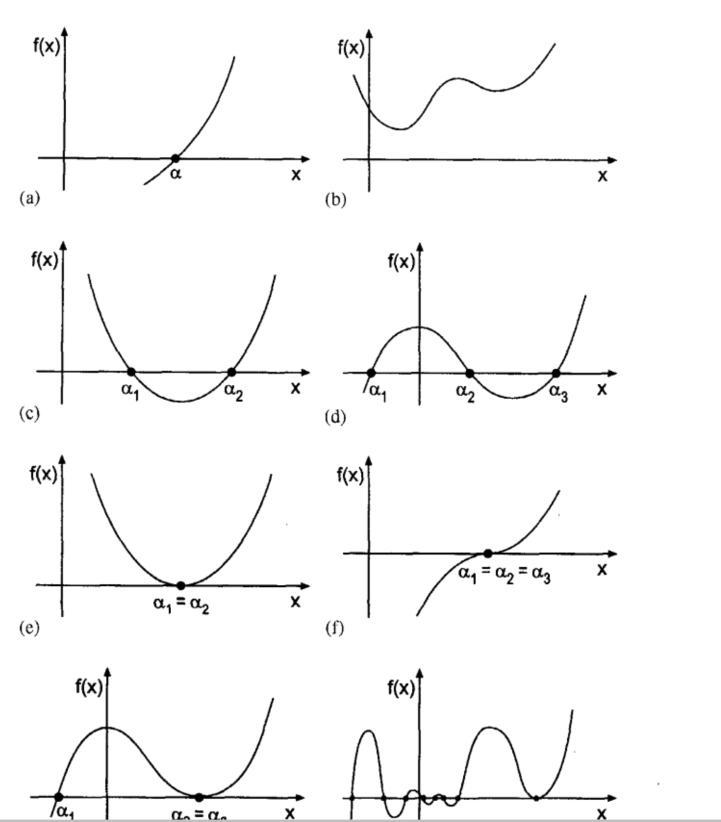
α = roots (c)



Isaac Newton

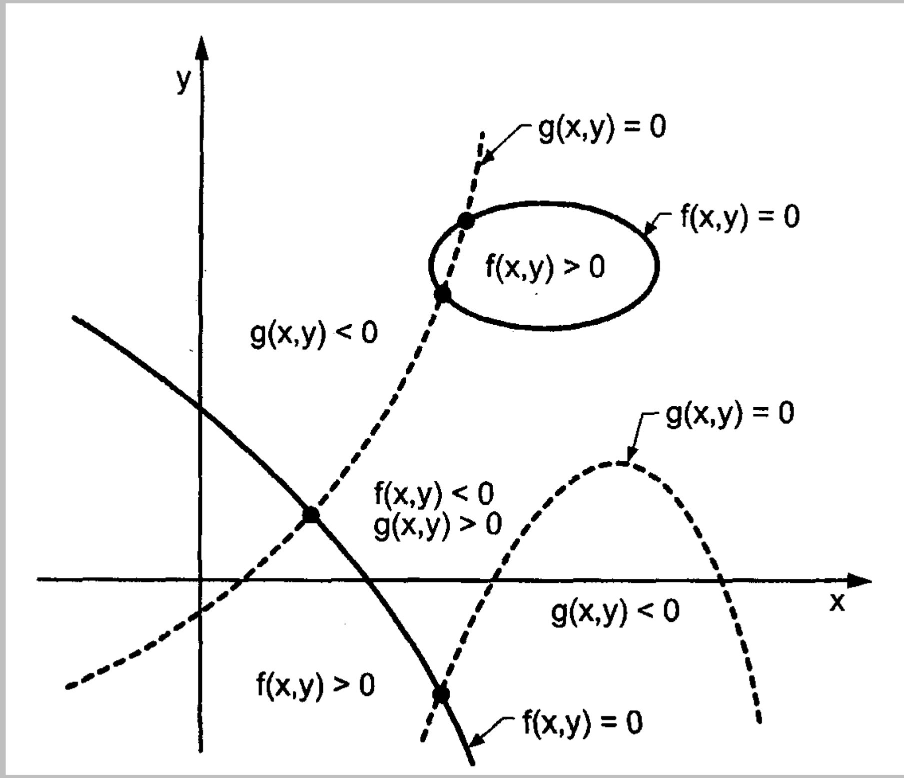
Roots of Nonlinear Functions

For function $f(x)$, find α such that $f(\alpha) = 0$ and with this condition, α is the root of f.



Remarks

- $f(x)$ is "smooth", finding α's will be straight forward, and convergence will be fast.
- If α is repeated or complex, convergence is slow or nonexistent.
- If α is sparse, finding all roots will be a difficult task.



Intersection of two functions

With two arbitrary functions $f(x)$ and $g(x)$, their intersection(s) can be found according to

$$h(x) = f(x) - g(x) = 0$$

Roots of $h(x)$ are be intersections of $f(x)$ and $g(x)$.

Optimization or an Infection Point at x

To maximize or minimize $f(x)$

$$f'(x) = 0$$

At infection point

$$f''(x) = 0$$

- With root finding, these points can be determined
- Discrete differentiation will be covered later

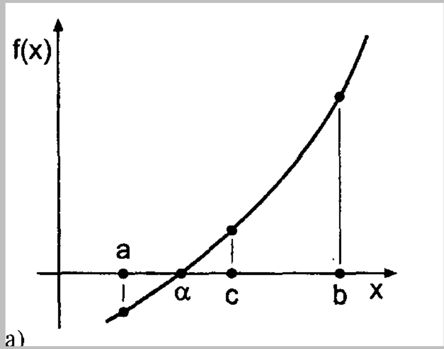
Root finding Methods- Iterations

- Open domain or closed domain
- Bounded or unbounded
- Initial guess
- Real or Complex roots
- One or two-point method
- Convergence time and convergence criteria

Two-point Closed Method

- Bisection
- False Position Method

Bisection method



- For $f(x) = 0$, points a and b are $f(b) > 0$ and $f(a) < 0$, such that

  $f(a)f(b) < 0.$

- Distance between a and b is $(b - a)$. Compute point c with $c = \frac{b-a}{2}$
- Determine $f(c)$. If

  $f(a)f(c) < 0.$ take $b = c$

  $f(a)f(c) > 0$ take a = c

- Iterate with new a and b and find new c until desired convergence.
- Number of iterations (?)

False Position Method

For two points at a and b, set a straight line passing through these two points. Find intersection of line with x axis.

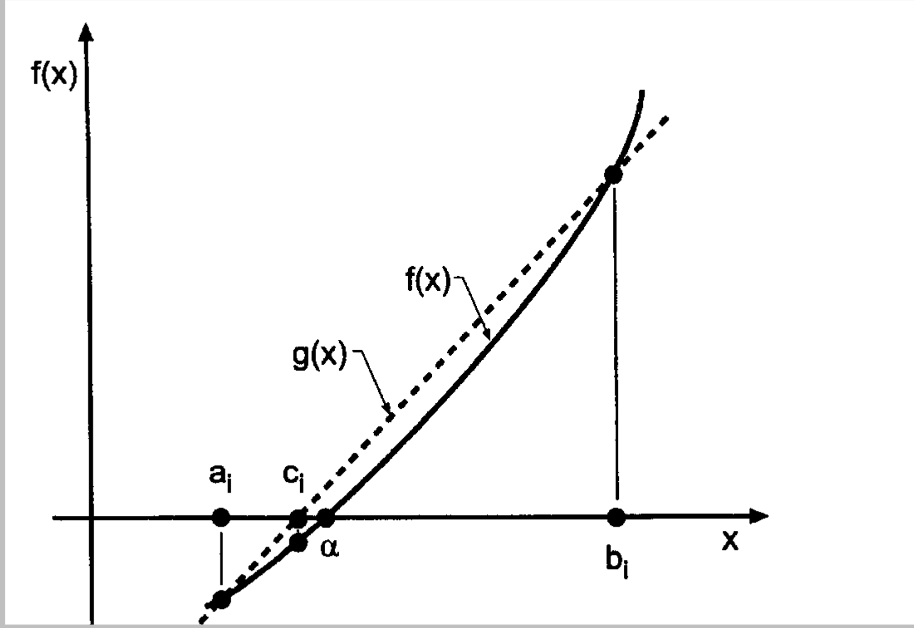
The straight line can be defined as

$$y = f(a) + \frac{y_a - y_b}{a - b}(a - x)$$

At line intersection with axis x, set $y = 0$ and $x = c$:

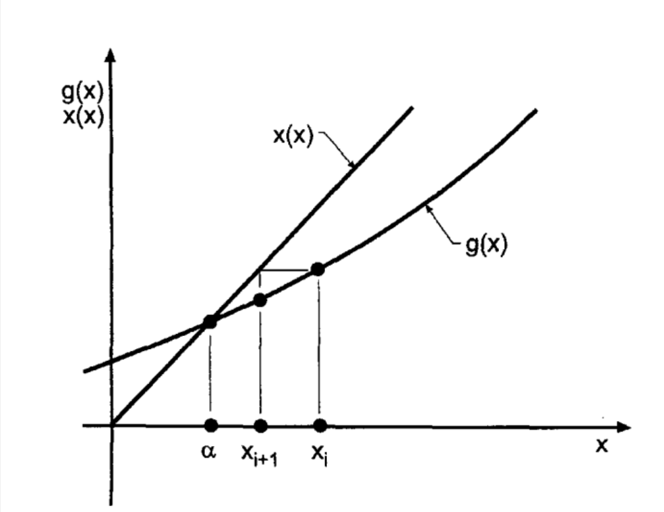
$$c = a - \frac{y_a(a - b)}{y_b - y_a}$$

From $f(a)f(c) < 0$, you can iterate like Bisection method.



Open Domain Iteration

- Fixed point method
- Newton's method
- Secant method
- Muller's method



Convergence Criterion

Differentiate $x = g(x)$ with x

$1 = g'(x)$

Convergence when $|g'(x)| < 1$

Fixed point method

For $f(x) = 0$, define a function $g(x)$ such that $f(x) = x - g(x)$

At root, then

$x - g(x) = 0$ or $x = g(x)$

For iterations, this equality is modified to

$x_{k+1} = g(x_k)$

Start with x_o, and $x_1 = g(x_o)$, and continue until convergence.

For roots of function $f(x) = x^2 - 3x + e^x - 2$

Determine $g(x)$ and $g'(x)$

Newton's Method

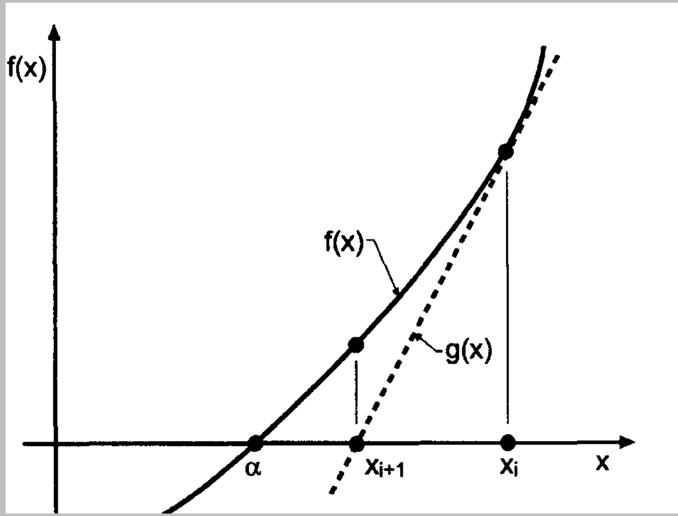
$$f'(x) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

Set $f(x_{i+1}) = 0$ and solve for x_{i+1}

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Convergence

$$|\Delta x_i| = |x_{i+1} - x_i| < \varepsilon$$



- $f'(x)$ exist in the calculation domain
- $f'(x) \neq 0$, if $f'(x) \to 0$, convergence slow as root is double
- $f'(x)$ is changing sign. Oscillations in calculations and slow convergence.
- $f(x)$ is defined discretely and not analytically. Define incremental distance $h = 10^{-3}$ or smaller,
- and determine f'

$$f'(x_i) = \frac{f(x_i + h) - f(x)}{h}$$

- $f'(x)$ can be approximated from two points x_1 and x_2

$$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- Error estimation

$e_i = \alpha - x_i$ $\qquad e_{i+1} = O(e_i^2)$

For roots of function $f(x) = x^2 - 3x + e^x - 2$

Using Newton's method

Imaginary Roots

In Newton's method, starting initial point is complex

$$x^o = a + ib$$

Double and Multiple Roots

With m repeated roots

$$x_{i+1} = x_i - m\frac{f(x_i)}{f'(x_i)}$$

With this modification, Newton's method converges as e_i^2, as opposed to e_i