The Copeland Rule and Condorcet's Principle
Author(s): Christian Klamler
Source: *Economic Theory*, Vol. 25, No. 3 (Apr., 2005), pp. 745-749
Published by: Springer
Stable URL: http://www.jstor.org/stable/25055909
Accessed: 01/03/2014 13:48

Economic Theory **25**, 745–749 (2005)
DOI: 10.1007/s00199-004-0467-7

Economic Theory

Exposita Notes

The Copeland rule and Condorcet's principle*

Christian Klamler

Institute of Public Economics, University of Graz, Universitaetsstr. 15/E4, 8010 Graz, AUSTRIA
(e-mail: christian.klamler@uni-graz.at)

Received: July 30, 2003; revised version: December 31, 2003

Summary. The purpose of this note is to shed some light on the relationship between the Copeland rule and the Condorcet principle in those cases where there does not exist a Condorcet winner. It will be shown that the Copeland rule ranks alternatives according to their distances to being a Condorcet winner.

Keywords and Phrases: Copeland rule, Condorcet's principle, Distance functions.

JEL Classification Numbers: D70, D71.

1 Introduction

In two papers, Saari and Merlin [12,15] provide an exhaustive investigation into the properties and flaws of the widely used Copeland rule. It generates a complete and transitive binary relation by ranking the alternatives according to the difference between the number of alternatives they beat and the number of alternatives they loose against. One of the properties it satisfies is the well known Condorcet principle, which states that the Condorcet winner, i.e. the alternative that wins against all other alternatives in a pairwise contest, should be considered best in the set of alternatives. There is a strong intuitive appeal for this property in the sense that it respects the idea of democratic decision making.[1]

The purpose of this note is to shed light on the intimate relationship between the Copeland rule and the Condorcet principle even in those cases where there does not exist a Condorcet winner. It will be shown that the Copeland rule ranks alternatives according to their distances to being Condorcet winners. The use of

* I am very grateful to Daniel Eckert and Nick Baigent for their helpful comments.

[1] See Fishburn [7] for a more detailed discussion.

distance information in preference aggregation problems in a finite framework goes back to Dodgson [4] and distance-based aggregation rules are discussed in recent papers by Saari and Merlin [16], Ratliff [13,14] and Klamler [9,10].[2]

The next section sets out the formal framework. In Section 3 it will first be proved that there exist two equivalent definitions of the Kemeny metric on the set of complete binary relations. Then, it will be shown that the Copeland ranking is equivalent to the "closeness to Condorcet" ranking.[3] Section 4 concludes the paper.

2 Formal framework

Let X denote a finite set of n alternatives, and $R \subseteq X^2$ be a binary relation on X, where $\succ_R$ denotes the asymmetric part of R and $\sim_R$ the symmetric part of R. The restriction of R to any subset S of X is written $R|S$. Let $\mathcal{B}$ be the set of all complete binary relations on X. For all $x \in X$, the set of all complete binary relations having alternative x uniquely on top is given by $M_x = \{R \in \mathcal{B} : (\forall y \in X \setminus \{x\})\, x \succ_R y\}$. Distance between any two binary relations $R, R' \in \mathcal{B}$ will be measured by the Kemeny metric $\delta : \mathcal{B} \times \mathcal{B} \to \mathbb{R}_+$, where $\delta(R, R') = |(R - R') \cup (R' - R)|$, i.e. the distance between two complete binary relations is equal to the cardinality of their symmetric difference (Kemeny [8]).

A basic way in which two binary relations differ can be seen in the number of pairs of alternatives on which they have opposite strict preferences or differ by one having an indifference where the other has a strict preference. Hence, three functions, which count the number of pairs displaying the respective types of inversions, will be introduced. Superscripts denote the respective change on pairs of alternatives, e.g. for all $R, R' \in \mathcal{B}$, $\lambda^{IP}(R, R')$ counts the number of pairs $x, y \in X$ such that x and y are indifferent in R and x is strictly preferred to y in R'.[4]

$$\begin{aligned}
\lambda^{IP} : \mathcal{B} \times \mathcal{B} &\to \mathbb{R}_+ \quad \text{such that for all} \\
R, R' \in \mathcal{B},\ \lambda^{IP}(R, R') &= \left|\left\{(x,y) \in X^2 : x \sim_R y \& x \succ_{R'} y\right\}\right| \\
\lambda^{PI} : \mathcal{B} \times \mathcal{B} &\to \mathbb{R}_+ \quad \text{such that for all} \\
R, R' \in \mathcal{B}, \lambda^{PI}(R, R') &= \left|\left\{(x,y) \in X^2 : x \succ_R y \& x \sim_{R'} y\right\}\right| \\
\lambda^{PP} : \mathcal{B} \times \mathcal{B} &\to \mathbb{R}_+ \quad \text{such that for all} \\
R, R' \in \mathcal{B}, \lambda^{PP}(R, R') &= \left|\left\{(x,y) \in X^2 : x \succ_R y \& y \succ_{R'} x\right\}\right|
\end{aligned}$$

[2] The preservation of proximity, both in an ordinal and in the cardinal framework of distance functions and metrics, is also used as a property for social choice rules (see Baigent [1] and Eckert and Lane [5]). This use is partly justified by the analogy of proximity preservation to the continuity condition used in topological social choice theory (see Baigent [2] and Lauwers [11] for surveys).

[3] A comparable type of exercise can be found in Farkas & Nitzan [6]. They show that the alternative which is closest to being the top alternative in all individual rankings relative to the Kemeny metric, is precisely the top alternative in the ranking derived from the Borda rule.

[4] An application of this framework can also be found in Baigent and Klamler [3] who provide a characterization of the transitive closure rule.

3 Results

The following Lemma 3.1 shows that the Kemeny distance between two complete binary relations can be measured equivalently with the help of the above functions. The simple proof is left to the reader.

Lemma 3.1. For all $R, R' \in \mathcal{B}$,

$$\delta(R, R') = |(R - R') \cup (R' - R)| = 2\lambda^{PP}(R, R') + \lambda^{IP}(R, R') + \lambda^{PI}(R, R')$$

Example 3.2. Let $X = \{x_1, x_2, x_3, x_4\}$ and let $R \in \mathcal{B}$ be such that $x_1 \succ_R x_2$, $x_2 \succ_R x_3$, $x_3 \succ_R x_4$, $x_4 \succ_R x_1$, $x_1 \succ_R x_3$, $x_4 \succ_R x_2$. Let $R' \in \mathcal{B}$ be such that $x_1 \succ_{R'} x_2$, $x_2 \succ_{R'} x_3$, $x_3 \succ_{R'} x_4$, $x_1 \succ_{R'} x_4$, $x_1 \succ_{R'} x_3$, $x_4 \succ_{R'} x_2$. Then $\lambda^{PP}(R, R') = 1$ and $\lambda^{IP}(R, R') = \lambda^{PI}(R, R') = 0$ and hence $\delta(R, R') = 2$.

The following Lemma 3.3 establishes a distance-based connection between any binary relation R and any set M_x of binary relations with x as the unique top alternative. We will use the fact that for any $x \in X$ there are $n - 1$ pairs formed by x and some $y \in X \setminus \{x\}$. Hence, for every $R \in \mathcal{B}$ and $x \in X$, there exist some non-negative integers such that $d + e + f = n - 1$ where $|\{y \in X \setminus \{x\} : x \succ_R y\}| = d$, $|\{y \in X \setminus \{x\} : x \sim_R y\}| = e$ and $|\{y \in X \setminus \{x\} : y \succ_R x\}| = f$.

Lemma 3.3. Let $R \in \mathcal{B}$, let $x \in X$, and assume that $|\{y \in X \setminus \{x\} : x \sim_R y\}| = e$ and $|\{y \in X \setminus \{x\} : y \succ_R x\}| = f$, where $e, f \in \mathbb{Z}_+$. Then $\min_{R' \in M_x} \delta(R, R') = 2f + e$.

Proof. $R' \in M_x$ implies $|\{y \in X \setminus \{x\} : x \sim_{R'} y\}| = |\{y \in X \setminus \{x\} : y \succ_{R'} x\}| = 0$. From the definition of λ^{IP} and λ^{PP} we get that $\lambda^{IP}(R, R') \geq e$ and $\lambda^{PP}(R, R') \geq f$. This implies, together with Lemma 3.1, that $\delta(R, R') \geq 2f + e$. Consider now $R' \in M_x$ to be such that $R'|S = R|S$ for $S = X \setminus \{x\}$. This implies that $\delta(R, R') = 2f + e$. Hence, the lemma is true. □

The Copeland rule ranks the alternatives according to the difference between the number of alternatives they beat and the number of alternatives they loose against. Thus, we define, for all $R \in \mathcal{B}$ and all $x \in X$, the Copeland value $c_R(x)$ as $c_R(x) = |\{y \in X : x \succ_R y\}| - |\{y \in X : y \succ_R x\}|$.

For every $R \in \mathcal{B}$, the Copeland ranking C_R is now defined as follows:

For all $x, y \in X$, $xC_R y \Leftrightarrow c_R(x) \geq c_R(y)$.

We now turn to the idea of measuring each alternative's distance to being a Condorcet winner relative to the Kemeny metric. For all $R \in \mathcal{B}$ and all $x \in X$, the distance of x to being a Condorcet winner is measured by the smallest distance of $R \in \mathcal{B}$ to some $R' \in M_x$. It is given by $\tilde{c}_R(x)$, such that $\tilde{c}_R(x) = \min_{R' \in M_x} \delta(R, R')$.

Therefore, for every $R \in \mathcal{B}$, we define the "closeness to Condorcet" ranking[5] $\tilde{C}_R$ as follows:

[5] At first sight this ranking might seem similar to the Slater [17] ranking. However, the essential difference is that the Slater ranking is of shortest Kemeny distance to the original binary relation whereas the "closeness to Condorcet" ranking ranks the alternatives according to their distances from being Condorcet winners.

For all $x, y \in X$, $x\tilde{C}_R y \Leftrightarrow \tilde{c}_R(x) \leq \tilde{c}_R(y)$

Theorem 3.4. For all $R \in \mathcal{B}$, the Copeland ranking and the "closeness to Condorcet" ranking are equivalent, i.e. $C_R = \tilde{C}_R$.

Proof. The theorem will be proved by showing that for all $R \in \mathcal{B}$ it is the case that for all $x, y \in X$, $c_R(x) \geq c_R(y) \Leftrightarrow \tilde{c}_R(x) \leq \tilde{c}_R(y)$. Assume $d, e, f, r, s, t \in \mathbb{Z}_+$ and $|X| = n = d + e + f + 1 = r + s + t + 1$. For any $R \in \mathcal{B}$, let, for some distinct $x, y \in X$, $|\{z \in X\backslash\{x\} : x \succ_R z\}| = d$, $|\{z \in X\backslash\{x\} : x \sim_R z\}| = e$, $|\{z \in X\backslash\{x\} : z \succ_R x\}| = f$, $|\{z \in X\backslash\{y\} : y \succ_R z\}| = r$, $|\{z \in X\backslash\{y\} : y \sim_R z\}| = s$ and $|\{z \in X\backslash\{y\} : z \succ_R y\}| = t$. This implies that $c_R(x) = d - f$ and $c_R(y) = r - t$. Furthermore, by Lemma 3.3, $\tilde{c}_R(x) = 2f + e$ and $\tilde{c}_R(y) = 2t + s$. As $e = n - 1 - d - f$ and $s = n - 1 - r - t$ this implies $\tilde{c}_R(x) = f - d + n - 1$ and $\tilde{c}_R(y) = t - r + n - 1$. Hence $\tilde{c}_R(x) \leq \tilde{c}_R(y)$ is equivalent to $f - d + n - 1 \leq t - r + n - 1$, which can be simplified to $f - d \leq t - r$ or $d - f \geq r - t$. As this is equivalent to $c_R(x) \geq c_R(y)$, this proves the theorem. $\square$

4 Conclusion

In this paper it has been shown in what sense closeness to being a Condorcet winner is implicit in the Copeland rule. Alternatives are ranked higher in the Copeland ranking whenever their distance to being a Condorcet winner relative to the Kemeny distance is smaller. This clarifies the relationship between the Copeland rule and Condorcet's principle. In addition, one could provide this as an argument for the Copeland rule being a good procedure to overcome problems resulting from voting cycles. It might also be taken as a starting point for comparing the Copeland rule to other aggregation procedures explicitly based on distance information such as the Slater rule, Kemeny's rule or Dodgson's rule.

References

1. Baigent, N.: Preference proximity and anonymous social choice. Quarterly Journal of Economics **102**, 161–169 (1987)
2. Baigent, N.: Topological theories of social choice. In: Arrow, K.J., Sen, A.K., Suzumura, K. (eds.) Handbook of social choice and welfare, vol. 2. Amsterdam: North-Holland 2003
3. Baigent, N., C. Klamler: Transitive closure, proximity and intransitivities. Economic Theory **23**, 175–181 (2004)
4. Dodgson, C.: A method of taking votes on more than two issues (1876). In: Black, D. (ed.) The theory of committees and elections. London: Cambridge University Press 1958
5. Eckert, D., Lane, B.: Anonymity, ordinal preference proximity and imposed social choices. Social Choice and Welfare **19**, 681–684 (2002)
6. Farkas, D., Nitzan, S.: The Borda rule and Pareto stability: a comment. Econometrica **47**, 1305–1306 (1979)
7. Fishburn, P.C.: Condorcet social choice functions. SIAM Journal of Applied Mathematics **33**, 469–489 (1977)
8. Kemeny, J.: Mathematics without numbers. Daedalus **88**, 577–591 (1959)
9. Klamler, C.: A comparison of the Dodgson method and the Copeland rule. Economics Bulletin **4**(8), 1–7 (2003)

10. Klamler, C.: The Dodgson ranking and its relation to Kemeny's method and the Slater rule. Social Choice and Welfare (forthcoming) (2004)
11. Lauwers, L.: Topological social choice. Mathematical Social Sciences **40**, 1–39 (2000)
12. Merlin, V.R., Saari, D.G.: Copeland method II. Manipulation, monotonicity, and paradoxes. Journal of Economic Theory **72**, 148–172 (1997)
13. Ratliff, T.C.: A comparison of Dodgson's method and Kemeny's rule. Social Choice and Welfare **18**, 79–90 (2001)
14. Ratliff, T.C.: A comparison of Dodgson's method and the Borda count. Economic Theory **20**, 357–372 (2002)
15. Saari, D.G., Merlin, V.R.: Copeland method I. Relationships and the dictionary. Economic Theory **8**, 51–76 (1996)
16. Saari, D.G., Merlin, V.R.: A geometric examination of Kemeny's rule. Social Choice and Welfare **17**, 403–438 (2000)
17. Slater, P.: Inconsistencies in a schedule of paired comparisons. Biometrica **48**, 303–312 (1961)