The Indian Buffet Process

Eli Bingham[1] Matt Dickenson[2]

[1]University of North Carolina

[2]Duke University

February 10, 2014

Outline

Latent feature models

- Feature model: N items described by K features
- Dense feature model: every feature is present in every item, e.g. PCA
- Sparse feature model: only some features present in each item, and we can assume feature values and presence are independent:

$$
\begin{aligned}
\mathbf{F} &= \mathbf{A} \otimes \mathbf{Z} \\
P(\mathbf{F}) &= P(\mathbf{A})P(\mathbf{Z})
\end{aligned}
$$

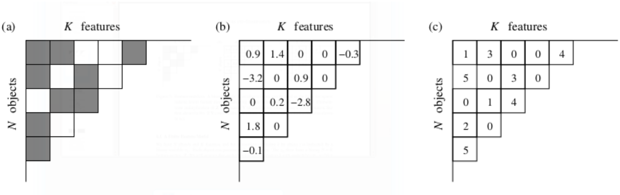


Figure : Griffiths and Ghahramani (2011) Figure 3

Example

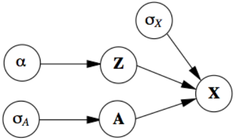


Figure : Griffiths and Ghahramani (2011) Figure 7

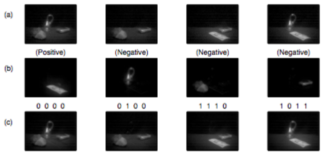


Figure : Griffiths and Ghahramani (2011) Figure 9

Motivation

- Problem with finite latent feature model: K is fixed
- Goal: construct nonparametric prior on $\mathbf{Z}$ so that K grows with the complexity of the dataset
- As with DPMMs, we can try to build one by taking $K \to \infty$ in a finite feature model

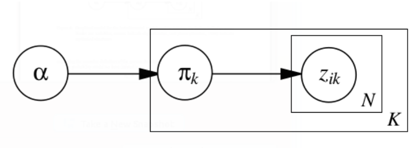


Figure : Griffiths and Ghahramani (2011) Figure 4

Background: Dirichlet Process

Finite version (Dirichlet distribution):

- Assignment of an object to a class is independent of all other assignments: $P(c|\theta) = \prod_{i=1}^{N} P(c_i|\theta) = \prod_{i=1}^{N} \theta_{c_i}$
- $\theta|\alpha \sim \text{Dirichlet}(\frac{\alpha}{K}, \ldots, \frac{\alpha}{K})$ (if symmetric)
- $c_i|\theta \sim \text{Discrete}(\theta)$, where Discrete : Bernoulli :: Multinomial : Binomial

Integrating out θ: $P(c) = \frac{\prod_{k=1}^{K} \Gamma(m_k + \frac{\alpha}{K})}{\Gamma(\frac{\alpha}{K})^K} \frac{\Gamma(\alpha)}{\Gamma(N+\alpha)}$

As $K \to \infty$, we get the CRP

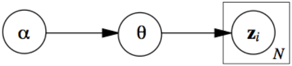


Figure : Griffiths and Ghahramani (2011) Figure 1

Background: Chinese Restaurant Process

1. N customers enter (in sequence) a restaurant with an infinite number of tables, each with infinite seating
2. First customer sits at first table with probability $\frac{\alpha}{\alpha} = 1$
3. i^{th} customer sits at the k^{th} table with probability $\frac{m_k}{i+\alpha-1}$, where m_k is the number of previous customers who sat at table k, or a new table with probability $\frac{\alpha}{i+\alpha-1}$

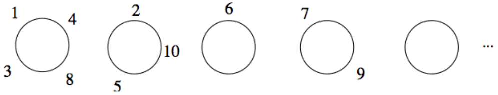


Figure : Griffiths and Ghahramani (2011) Figure 2

Limitation: each object (customer) can only belong to one class (table).

Finite latent feature models

The basic finite distribution on $z_{i,k}$s:

$$\begin{aligned} \pi_k|\alpha &\sim \text{Beta}(\frac{\alpha}{K}, 1) \\ z_{i,k}|\pi_k &\sim \text{Bernoulli}(\pi_k) \end{aligned}$$

As with DPMMs, we can marginalize out latent feature presence probabilities π_k to obtain a distribution on matrices $\mathbf{Z} \in \{0, 1\}^{N \times K}$:

$$\begin{aligned} P(\mathbf{Z}) &= \prod_{k=1}^{K} \int \left(\prod_{i=1}^{N} P(z_{ik}|\pi_k) \right) P(\pi_k) d\pi_k \\ &= \prod_{k=1}^{K} \frac{\frac{\alpha}{K}\Gamma(m_k + \frac{\alpha}{K})\Gamma(N - m_k + 1)}{\Gamma(N + 1 + \frac{\alpha}{K})} \end{aligned}$$

The $K \to \infty$ limit

- $lof(\mathbf{Z})$ is the matrix obtained by ordering the columns of $\mathbf{Z}$ as N-digit binary numbers
- To define a probability over infinitely wide binary matrices using de Finetti's Theorem, we need exchangeable symmetry, so we define *lof* equivalence classes by modding out column order:

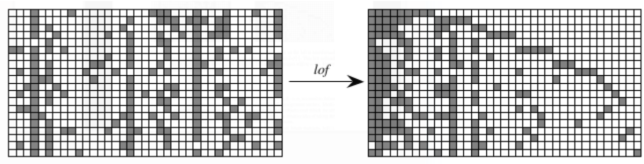


Figure : Griffiths and Ghahramani (2011) Figure 5

Indian Buffet Process

Indian Buffet Process:

1. N customers enter (in sequence) a buffet restaurant with an infinite number of dishes
2. First customer fills her plate with Poisson(α) number of dishes
3. i^{th} customer samples dishes in proportion to their popularity, with probability $\frac{m_k}{i}$, where m_k is the number of previous customers who sampled dish k
4. i^{th} customer then samples $K_1^{(i)} \sim \text{Poisson}(\frac{\alpha}{i})$ number of new dishes

Resulting probability distribution on matrices:

$$P(\mathbf{Z}) = \frac{\alpha^{K_+}}{\prod_{i=1}^{N} K_1^{(i)}!} \exp(\alpha H_N) \prod_{k=1}^{K_+} \frac{(N - m_k)!(m_k - 1)!}{N!}$$

Alternative derivation: Beta Process

The CRP is obtained by marginalizing over a Dirichlet process:

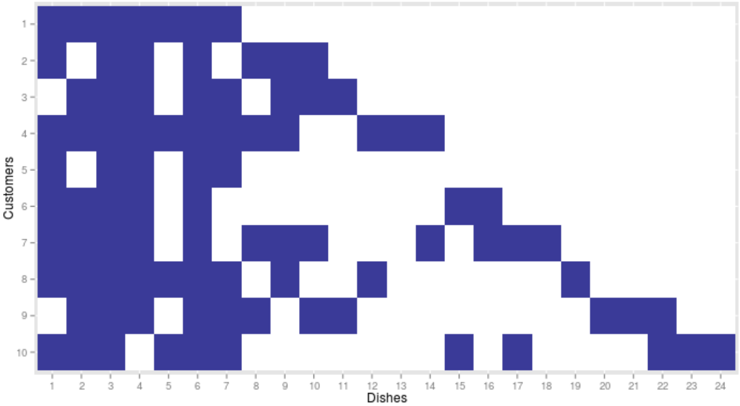
$$\begin{aligned}
\theta|\alpha, G_0 &\sim \mathrm{DP}(\alpha, G_0) \\
C|\theta &\sim \mathrm{Multinomial}(\theta) \\
CRP(\alpha) &\sim \int P(C|\theta)P(\theta|\alpha, G_0)d\theta
\end{aligned}$$

This representation follows from de Finetti's Theorem, which gives the existence of conditionally independent representations of infinite exchangeable joint distributions. The IBP is obtained by marginalizing over a Beta process:

$$\begin{aligned}
\theta|\alpha, \beta, G_0 &\sim \mathrm{BP}(\alpha, \beta, G_0) \\
z_i|\theta &\sim \mathrm{BeP}(\theta) \\
\mathrm{IBP}(\alpha, \beta) &\sim \int \prod_i P(z_i|\theta)P(\theta|\alpha, \beta, G_0)d\theta
\end{aligned}$$

Demo

mcdickenson.shinyapps.io/ibp-demo



Alternative derivation: Stick-Breaking

1. Recursively break (an initially unit-length) stick, breaking off a Beta($\alpha, 1$) portion at each step
2. Let each portion of the "stick", π_k represent the probability of each feature (sorted from largest to smallest)

This helps to show the relation between the Dirichlet process and the IBP. The stick-breaking construction is also useful for defining inference algorithms.

Properties of the Resulting Distribution

- The "effective" dimension $K_+ \sim \text{Poisson}(\alpha H_N)$
- The number of dishes on each customer's plate is distributed Poisson(α) (by exchangeability)
- $\mathbf{Z}$ remains sparse as $K \to \infty$: effective dimensions of $\mathbf{Z}$ are $N \times K_+$, and the expected number of entries is $N\alpha$

Inference by Gibbs Sampling

Given some data $\mathbf{X}$, we want to sample from a marginal posterior

$$P(z_{i,k} = 1|\mathbf{Z}_{-(ik)}, \mathbf{X}) \propto P(\mathbf{X}|\mathbf{Z})P(z_{i,k} = 1|\mathbf{Z}_{-(ik)})$$

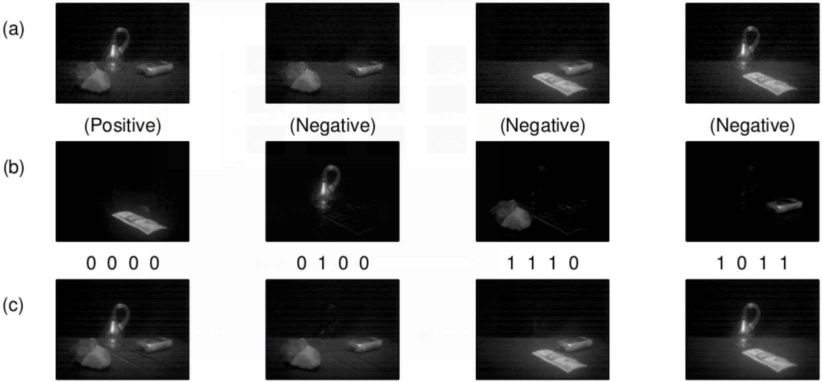
Iterate continuously over the rows $\mathbf{z}_i$ $(i = 1 \dots N)$ of $\mathbf{Z}$:

1. For each column k of $\mathbf{Z}$, if $m_{-i,k} = 0$ (i.e. the rest of the column is empty) delete the column; otherwise set $z_{i,k} = 1$ with probability

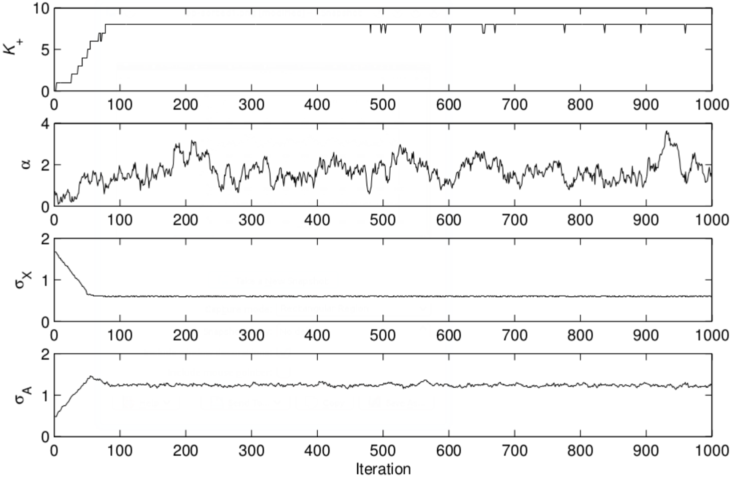
$$P(\mathbf{X}|\mathbf{Z})P(z_{i,k} = 1|\mathbf{z}_{-i,k}) = P(\mathbf{X}|\mathbf{Z})\frac{m_{-i,k}}{N}$$

2. At the end of the row, add $K_1^{(i)} \sim P(\mathbf{X}|\mathbf{Z})\text{Poisson}(\frac{\alpha}{N})$ new columns with ones in row i

Inference by Gibbs sampling



Inference by Gibbs sampling



Other inference strategies

- Variational inference in latent feature models with a mean-field approximation
- Adding Metropolis-Hastings proposals to the Gibbs sampler to explore the space with larger jumps
- Maintaining posterior of weight matrix $\mathbf{A}$ instead of marginalizing out $\mathbf{A}$ allows for sampling with faster mixing and iterations taking $O(N)$ time
- This also allows parallelization via computing features of different objects independently

"A Choice Model with Infinitely Many Latent Features"
(Görür, Jäkel, and Rasmussen, ICML 2006)

- Customers compare items (e.g. cell phones) based on the (binary) features of each; more features are better
- Number of features is potentially infinite and ordering is not important, so IBP is used
- Celebrity example: "With whom would you prefer to spend an hour of conversation?"

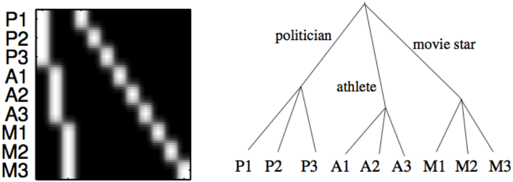


Figure : Görür, Jäkel, and Rasmussen (2006) Figure 3

Application 2: Topic Modeling

"The IBP Compound Dirichlet Process
and its Application to Focused Topic Modeling"
Williamson, Wang, Heller, and Blei (2010)

Stick-breaking construction:

$$
\begin{aligned}
\mu_k &\sim \text{Beta}(\alpha, 1) \\
\pi_k &= \prod_{j=1}^{k} \mu_j \\
b_{m,k} &\sim \text{Bernoulli}(\pi_k)
\end{aligned}
$$

Application 2: Topic Modeling

Focused topic model:

1. for $k = 1, 2, \ldots$
   - Sample stick length π_k
   - Sample relative mass $\phi_k \sim \text{Gamma}(\gamma, 1)$
   - Draw topic distribution over words: $\beta_k \sim \text{Dirichlet}(\eta)$
2. for $m = 1, \ldots, M$
   - Sample binary vector b_m
   - Draw total number of words $n^{(m)} \sim NB(\sum_k b_{m,k}\phi_k, 1/2)$
   - Sample distribution over topics $\theta_m \sim \text{Dirichlet}(b_m \cdot \phi)$
   - For each word $w_{m,i}, i = 1, \ldots, n^{(m)}$
     1. Draw topic index $z_{m,i} \sim \text{Discrete}(\theta_m)$
     2. Draw word $w_{m,i} \sim \text{Discrete}(\beta_{z_{m_i}})$

Application 2: Topic Modeling

An advantage of the focused topic model is that it separates the global topic proportions from the distribution over topics within a document. A rare topic within the corpus can be dominant within a document (e.g. baseball), and a frequent topic can be a small proportion of many documents.

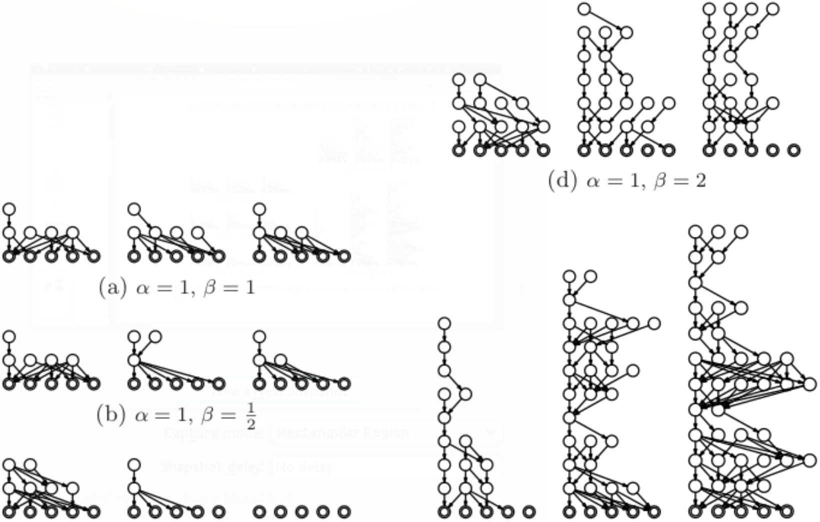
Application 3: CIBPs for Belief Network Structure Learning

- Adams et al construct a prior on infinitely wide, infinitely deep sparse belief network structures
- The Cascading Indian Buffet Process is an infinite sequence of binary matrices

$$\mathbf{Z}^{(m)} \sim \mathrm{IBP}(\alpha, \beta)$$

  where $Z^{(m+1)}$ has the same number of rows as columns in $Z^{(m)}$
- Culinary analogy: Each "dish" in the restaurant also corresponds to a "customer" in the next restaurant
- Theorem: the CIBP generative process stops growing at finite "depth"
- They also present MCMC algorithms for inference and test on images

CIBP prior samples



(a) $\alpha = 1,\ \beta = 1$

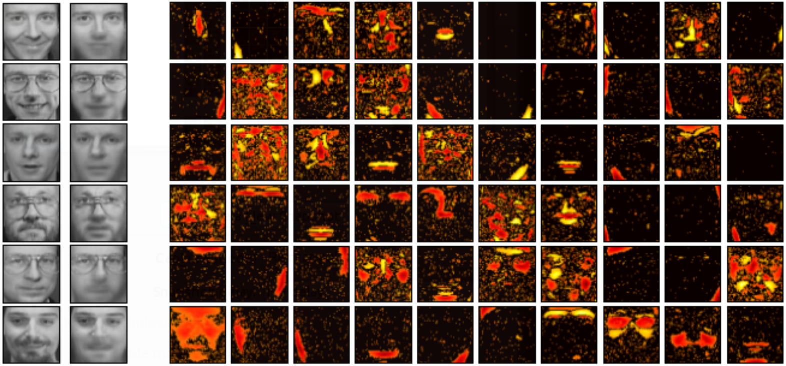
(b) $\alpha = 1,\ \beta = \frac{1}{2}$

(c) $\alpha = \frac{1}{2},\ \beta = 1$

(d) $\alpha = 1,\ \beta = 2$

(e) $\alpha = \frac{3}{2},\ \beta = 1$

CIBP Facial reconstruction



(a) (b)

Discussion

Limitations of IBP:

1. Coupling of average number of features α and total number of features $N\alpha$ (can be overcome with a two-parameter generalization)
2. Computationally complex, can be time-consuming

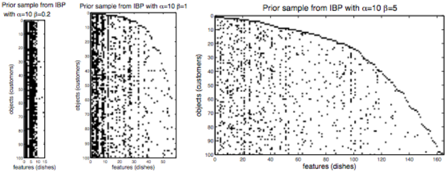


Figure : Griffiths and Ghahramani (2011) Figure 10