Calculating the Radial Component of the Tidal Field, I think

Daniel Martens

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We want to calculate:

$$S_{rr} = \hat{r}^i \hat{r}^j \nabla_i \nabla_j \nabla^{-2} \psi_\alpha \tag{1}$$

In the basis where:

$$\psi_\alpha = j_l(k_\alpha r) Y_{l_\alpha m_\alpha}(\theta, \phi) \tag{2}$$

Because we are in k-space, the inverse Laplacian operator is simple:

$$\nabla^2 \psi_\alpha = -k_\alpha^2$$
$$\nabla^{-2} \psi_\alpha = \frac{-1}{k_\alpha^2} \tag{3}$$

Before acting on the remaining terms with the directional gradients, it helps to first dot with the directional unit vectors:

$$\hat{r}^i \hat{r}^j \nabla_i \nabla_j = \hat{r}^i (\nabla_i \hat{r}^j + \left[\hat{r}^j, \nabla_i\right]) \nabla_j$$
$$= \hat{r}^i \nabla_i \hat{r}^j \nabla_j - \hat{r}^i \nabla_i \hat{r}^j \nabla_j = \frac{d^2}{dr^2} - \hat{r}^i (\frac{1}{r}(\delta_i^j - \hat{r}^j \hat{r}_i)) \nabla_j$$
$$= \frac{d^2}{dr^2} \tag{4}$$

So we see that:

$$S_{rr} = \frac{-1}{k_\alpha^2} Y_{l_\alpha m_\alpha}(\theta, \phi) \frac{d^2}{dr^2} \left[j_l(k_\alpha r)\right] \tag{5}$$

Using the recursion relation of the spherical bessel functions of the first kind:

$$2 \frac{dj_l(x)}{dx} = j_{l-1}(x) - j_{l+1}(x) \tag{6}$$

And the chain rule:

$$\frac{d}{dr} j_l(k_\alpha r) = k_\alpha \frac{d}{d(k_\alpha r)} j_l(k_\alpha r) \tag{7}$$

We end with our result:

$$S_{rr} = \frac{Y_{l_\alpha m_\alpha}(\theta, \phi)}{4} \left[2 j_l(k_\alpha r) - j_{l+2}(k_\alpha r) - j_{l-2}(k_\alpha r)\right] \tag{8}$$