The δ_α basis and covariance $\langle \delta_\alpha \delta_\beta \rangle$

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1 Spherical basis functions

For spherical geometry we use the following basis, composed of the spherical Bessel function and spherical harmonic:

$$\psi_\alpha(r,\theta,\phi) = j_{l_\alpha}(k_\alpha r) Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\theta,\phi) \ , \tag{1}$$

where $Y^{\mathrm{R}}_{lm}(\theta,\phi)$ is the real spherical harmonic. The real spherical harmonics are defined as

$$Y^{\mathrm{R}}_{lm} = \begin{cases} \frac{i}{\sqrt{2}} \left(Y_{lm} - (-1)^m Y_{l-1m} \right) & \text{if } m < 0 \\ Y_{l0} & \text{if } m = 0 \\ \frac{1}{\sqrt{2}} \left(Y_{l-m} + (-1)^m Y_{lm} \right. & \text{if } m > 0 \ . \end{cases} \tag{2}$$

We write a super survey mode as

$$\delta_\alpha(k_\alpha) = \int \delta(\mathbf{r}) j_{l_\alpha}(k_\alpha r) Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{r}) d^3\mathbf{r} \ . \tag{3}$$

2 Mean background density in a region

The mean background density in a region is composed of the super modes:

$$\bar{\delta} = \sum_\alpha \frac{3}{r^3_{\mathrm{max}} - r^3_{\mathrm{min}}} \int_{r_{\mathrm{max}}}^{r_{\mathrm{min}}} dr \ r^2 j_{l_\alpha}(k_\alpha r) \delta_\alpha(k_\alpha) \frac{1}{4\pi\Omega} \underbrace{\int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} d\theta d\phi \ \sin(\theta) Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{r})}_{a_{l_\alpha m_\alpha}} \ , \tag{4}$$

where $\Omega = a_{00}$.

In practice we are interested in derivatives of observables with respect to the δ_α. This is accomplished by chain rule

$$\frac{\partial f}{\partial \delta_\alpha} = \frac{\partial f}{\partial \bar{\delta}} \frac{\partial \bar{\delta}}{\partial \delta_\alpha} \ . \tag{5}$$

The derivative is computed from Eqn. 4

$$\frac{\partial \bar{\delta}}{\partial \delta_\alpha} = \frac{3}{r_{\rm max}^3 - r_{\rm min}^3} \int_{r_{\rm max}}^{r_{\rm min}} dr \ r^2 j_{l_\alpha}(k_\alpha r) \frac{1}{4\pi\Omega} \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} d\theta d\phi \ \sin(\theta) Y^{\rm R}_{l_\alpha m_\alpha}(\hat{r}) \ . \tag{6}$$

3 Covariance of super modes

In our basis the super survey mode is defined as

$$\begin{aligned} \delta_\alpha(k_\alpha) &= \int \delta(\mathbf{r}) j_{l_\alpha}(k_\alpha r) Y^{\rm R}_{l_\alpha m_\alpha}(\hat{r}) d^3\mathbf{r} \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \delta(\mathbf{k}) \int_0^{r_{\rm max}} dr^2 j_{l_\alpha}(k_\alpha r) \int d^2\hat{r} e^{ikr\hat{k}\cdot\hat{r}} Y^{\rm R}_{l_\alpha m_\alpha}(\hat{r}) \\ &= 4\pi i^{l_\alpha} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \delta(\mathbf{k}) Y^{\rm R}_{l_\alpha m_\alpha}(\hat{k}) \int_0^{r_{\rm max}} dr^2 j_{l_\alpha}(k_\alpha r) j_{l_\alpha}(kr) \ , \end{aligned} \tag{7}$$

where in the second equality $\delta(\mathbf{r}) = (2\pi)^{-3} \int d^3\mathbf{k} \exp(i\mathbf{k}\cdot\mathbf{r})\delta(\mathbf{k})$ was used and in the third equality the following identity was used

$$\int_{S^2} d^2\hat{r} Y^{\rm R}_{l_\alpha m_\alpha}(\hat{r}) e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi i^{l_\alpha} j_{l_\alpha}(kr) Y^{\rm R}_{l_\alpha m_\alpha}(\hat{k}) \ . \tag{8}$$

Including a normalization[1] $N_\alpha = \int_\Omega Y^{\rm R}_{l_\alpha m_\alpha}(\hat{r}) Y^{\rm R}_{l_\alpha m_\alpha}(\hat{r}) d^3\mathbf{r}$ we have

$$\delta_\alpha = \frac{4\pi i^{l_\alpha}}{N_\alpha} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \delta(\mathbf{k}) Y^{\rm R}_{l_\alpha m_\alpha}(\hat{k}) \int_0^{r_{\rm max}} dr^2 j_{l_\alpha}(k_\alpha r) j_{l_\alpha}(kr) \ . \tag{9}$$

To get the covariance of the super survey field we take the ensemble average:

$$\begin{aligned} \langle \delta_\alpha \delta_\beta \rangle &= \frac{(4\pi)^2}{N_\alpha N_\beta} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{d^3\mathbf{k}_2}{(2\pi)^3} \langle \delta(\mathbf{k}_1) \delta^*(\mathbf{k}_2) \rangle Y^{\rm R}_{l_\alpha m_\alpha}(\hat{k_1}) Y^{\rm R}_{l_\alpha m_\alpha}(\hat{k_2}) \\ &\quad \times \int_0^{r_{\rm max}} drr^2 j_{l_\alpha}(k_\alpha r) j_{l_\alpha}(k_1 r) \int_0^{r_{\rm max}} drr^2 j_{l_\alpha}(k_\alpha r) j_{l_\alpha}(k_2 r) \\ &= \frac{(4\pi)^2}{N_\alpha N_\beta} \int \frac{d^3\mathbf{k}}{(2\pi)^3} P(k) I_\alpha(k, r_{\rm max}) \times I_\beta(k, r_{\rm max}) \ , \end{aligned} \tag{10}$$

[1] In our case N_α is 1.

where $(2\pi)^3\delta^3_{\rm D}(\mathbf{k}+\mathbf{k}')P(k) = \langle\delta(\mathbf{k})\delta^*(\mathbf{k}')\rangle$ was used along with the definition

$$\begin{aligned}
I_\alpha(k, r_{\rm max}) &= \int_0^{r_{\rm max}} dr r^2 j_{l_\alpha}(k_\alpha r) j_{l_\alpha}(kr) \\
&= \frac{\pi}{2}\sqrt{\frac{1}{k_\alpha k}} \int_0^{r_{\rm max}} dr r J_{l_\alpha+1/2}(k_\alpha r) J_{l_\alpha+1/2}(kr) \\
&= \frac{\pi}{2}\frac{r_{\rm max}}{\sqrt{k_\alpha k}} \frac{\left[J_{l_\alpha+1/2}(kr_{\rm max})J'_{l_\alpha+1/2}(k_\alpha r_{\rm max}) - J_{l_\alpha+1/2}(k_\alpha r_{\rm max})J'_{l_\alpha+1/2}(kr_{\rm max})\right]}{k^2-k_\alpha^2} \ .
\end{aligned} \tag{11}$$

4 The computation of basis objects

The radial basis function should be zero at the sector boundary. Thus, fiven the maximum radius of the sector $r_{\rm max}$ the super wave vector k_α is found by

$$j_{l_\alpha}(k_\alpha r_{\rm max}) = 0. \tag{12}$$

5 How to obtain derivatives by finite differences

We need to calculate objects like $\partial\mathcal{O}_I/\partial\delta_\alpha$. This is done by applying the chain rule.

$$\frac{\partial\mathcal{O}_I}{\partial\delta_\alpha} = \frac{\partial\mathcal{O}_I}{\partial\Theta^i}\frac{\partial\Theta^i}{\partial\bar{\delta}}\frac{\partial\bar{\delta}}{\delta_\alpha} \ . \tag{13}$$

When the observable is to complicated that an analytic method for the derivative is not possible, or is too cumbersome to attempt, we can use a finite difference approach. We can take $\bar{\delta} = \pm\epsilon$. As long as ϵ is small we should be in the linear regime (which is the region we are limiting our analysis too). Then we can take

$$\frac{\partial\mathcal{O}_I}{\partial\Theta^i} = \frac{\mathcal{O}_I(+\epsilon) - \mathcal{O}_I(-\epsilon)}{2\delta\Theta^I(\epsilon)} \ , \tag{14}$$

and in the above $\mathcal{O}_I(+\epsilon)$ means that the observable is calculated in the separate universe approch, where $\bar{\delta} = \pm\epsilon$ corresponds to a change in cosmological parameters. The figure below illustrates this approach. The point being, is that we can choose ϵ arbitrarily (as long as we are in the linear regime) and we will still get the correct derivative, because it is just a slope.

Figure 1: Finite difference approach.