SSC Working Notes

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1 Introduction

- The big goal is to use multiple observables to constrain cosmological parameters.
- The precision that observables can be measured is described by the covariance matrix.
- Super sample covariance (SSC) is the sample variance that arises from long wavelength modes (larger than the region the observable is measured in) that couple to modes within the observing region. It has been shown that the SSC is the dominate form of non-Gaussian covariance.
- Non-linear evolution of $\delta(\mathbf{x})$ accounts for non-gaussian contributions to the covariance matrix. In the non-linear regime different Fourier modes can couple together. This coupling accounts for off diagonal elements of the matter power spectrum covariance. This covariance can be described by the connected 4-point function of the matter field.
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2 Notations

We use the following index convention to identify objects:

- α, β are used for the long-wavelength modes of the density field, i.e. for the super mode δ_α;
- a, b are for the long-wavelength observables, for instance the tangential shear at the survey boundary. These objects are denoted as $\mathcal{O}_I$;
- I, J are for the short-wavelength observables, for example the power spectrum. These objects are denoted as $\mathcal{O}_a$;
- i, j are used for the cosmological parameters, $p_i = \{\Omega_m h^2, \Omega_b h^2, n_s, ...\}$

- i_z, j_z are used for redshift binning.

I also think it is a good idea that we use the same indexing in the code. For instance, when we create a for loop over redshift bins, we should use i_z as the index in the python code. In the code let us try and keep the nomenclature representative of the paper too. For instance we should call $\partial\bar{\delta}/\partial\delta_\alpha C_{\alpha\beta}\partial\bar{\delta}/\partial\delta_\beta$= **sigma2_SSC** in the code.

3 Basic Idea

We first define the linear response of a short-wavelength observable to the long-wavelength super modes

$$\mathcal{O}_I = \bar{\mathcal{O}}_I + \frac{\partial \mathcal{O}_I}{\partial \delta_\alpha}\delta_\alpha = \bar{\mathcal{O}}_I + T_{I\alpha}\delta_\alpha \ , \tag{1}$$

where the "bar" designates the observable evaluated with no super modes. We have also defined the transfer function $T_{I\alpha} = \partial\mathcal{O}_I/\partial\delta_\alpha$. The covariance matrix is defined by $C_{IJ} = \langle\mathcal{O}_I\mathcal{O}_J\rangle - \langle\mathcal{O}_I\rangle\langle\mathcal{O}_J\rangle$. Using Eq. 1 the covariance matrix is

$$\begin{aligned} C_{IJ} &= \bar{C}_{IJ} + T_{I\alpha}\langle\delta_\alpha\delta_\beta\rangle T_{\beta J} \\ &= \bar{C}_{IJ} + T_{I\alpha}C_{\alpha\beta}T_{\beta J} \ . \end{aligned} \tag{2}$$

Long-wavelength observables can be used to increase the Fisher information content. We first transform the Fisher information of the long-wavelength observables to the $\alpha - \beta$ basis by

$$F_{\alpha\beta} = \frac{\partial\mathcal{O}_a}{\partial\delta_\alpha}C^{-1}_{ab}\frac{\partial\mathcal{O}_b}{\partial\delta_\beta} \ . \tag{3}$$

The information content can be increased by adding Fisher information for each long-wavelength observable

$$F_{\alpha\beta} = F^{(0)}_{\alpha\beta} + F^{(1)}_{\alpha\beta} + F^{(2)}_{\alpha\beta} + F^{(3)}_{\alpha\beta} + ... \ , \tag{4}$$

where each matrix $F^{(i)}_{\alpha\beta}$, $i = 1, 2, 3, ...$ is constructed from a different long-wavelength observables in the surveys. The 0th Fisher matrix is reserved for $F^{(0)}_{\alpha\beta} = C^{-1}_{\alpha\beta} = \langle\delta_\alpha\delta_\beta\rangle^{-1}$. By increasing the information content from each long-wavelength observable, we decrease the covariance

$$C_{\alpha\beta} = \left[F^{(0)}_{\alpha\beta} + F^{(1)}_{\alpha\beta} + F^{(2)}_{\alpha\beta} + F^{(3)}_{\alpha\beta} + ...\right]^{-1} \ . \tag{5}$$

The mitigated covariance is now

$$C_{IJ} = \bar{C}_{IJ} + T_{I\alpha}\left[F^{(0)}_{\alpha\beta} + F^{(1)}_{\alpha\beta} + F^{(2)}_{\alpha\beta} + F^{(3)}_{\alpha\beta} + ...\right]^{-1} T_{\beta J} \ . \tag{6}$$

The last step in the process is to compute the Fisher information with respect to cosmological parameters

$$F_{ij} = \frac{\partial \mathcal{O}_I}{\partial p_i} C^{-1}_{IJ} \frac{\partial \mathcal{O}_J}{\partial p_j} \ . \tag{7}$$

4 Code Specific Material

4.1 cosmopie module

The **cosmopie** module provides cosmology related objects, like the growth factor, distances, and cosmological parameters. This module is usually passed to just about every routine in the code. It is initialized at the beginning and in this way, all objects are referring back to the same cosmology.

4.2 Halo mass function module

The **hmf** module calculates all objects related to the halo mass function. This includes the halo mass function, average number of collapsed objects, the halo bias. Currently this module can only perform the Sheth-Tormen mass function.

4.3 The basis module

The basis module takes in the following set of parameters:

- n the number of zeros;
- l_α the long wavelength angular mode;
- $R_{\max}$ the maximum radius of the field (should be larger than the survey depth).

To calculate $\langle \delta_\alpha \delta_\beta \rangle$ the basis module considers δ_α as a one-dimensional array ordered as $\delta_\alpha[l_\alpha, m_\alpha, n]$, so that the first entries corresponds to $[0,0,0], ..., [0,0,n], [1,-1,1], .., [1,-1,n], [1,0,0], ...$ (**it should be verified that this is the right ordering in the code**).This will be called α ordering. The covariance is ordered in the same way $[l_\alpha, m_\alpha, n] \times [l_\beta, m_\beta, n]$.

The basis module contains objects:

- **Get_C_alpha_beta** returns $C^{(0)}_{\alpha\beta}$, the covariance matrix composed of the long wavelength modes;

- **Get_F_alpha_beta** returns $F^{(0)}_{\alpha\beta}$, the Fisher matrix constructed from the long wavelength modes;

- **ddelta_bar_ddelta_alpha** a function that returns $\partial\bar{\delta}/\partial\delta_\alpha$ given a region of the universe.

4.4 The short wavelength array

The $\mathcal{O}_I$ array is ordered as $[x, z_{\mathrm{avg},i_z}]$, where x is any of the following $k, l, \{k, \mu\}$, and this is called I ordering. The covariance matrix built from $\mathcal{O}_I$ follows the same ordering.

4.5 The long wavelength array

The $\mathcal{O}_a$ array is ordered as $[x, z_{\mathrm{avg},i_z}]$, where x is any of the following $k, l, \{k, \mu\}$, and this is called a ordering. The covariance matrix built from $\mathcal{O}_a$ follows the same ordering. However, all functions that relate to $\mathcal{O}_a$ build the matrix $F_{\alpha\beta} = \frac{\partial\mathcal{O}_a}{\partial\delta_\alpha}C^{-1}_{ab}\frac{\partial\mathcal{O}_b}{\partial\delta_\beta}$, which is α ordered.

4.6 Increased Fisher Matrix

The function **Get_SSC_covar** builds the covariance matrix $C_{IJ} = \bar{C}_{IJ} + T_{I\alpha}C_{\alpha\beta}T_{J\beta}$. This matrix is I ordered. The first routine that **Get_SSC_covar** should build is the Fisher matrices for each long wavelength observable. Each Fisher matrix should then be added and this value should be stored as this should be stored as the object **F_alpha_beta**. The inverse of **F_alpha_beta** is then taken and this object is stored as **C_alpha_beta**. Then for each short wavelength observable $\frac{\partial\bar{\delta}}{\partial\delta_\alpha}C_{\alpha\beta}\frac{\partial\bar{\delta}}{\partial\delta_\beta}$ is calculated. This object is stored as **sigma2_SSC**. The final object calculated is $\frac{\partial\mathcal{O}_I}{\partial\bar{\delta}}\frac{\partial\bar{\delta}}{\partial\delta_\alpha}C_{\alpha\beta}\frac{\partial\bar{\delta}}{\partial\delta_\beta}\frac{\partial\mathcal{O}_I}{\partial\bar{\delta}}$. This object is stored as **C_SSC_IJ**.

4.7 Response of power spectrum to background density

The following relations are used to calculate $dP(k)/d\bar{\delta}$ for $P(k)$ computed linearly, in standard perturbation theory, and using halo fit. These routines are calculated in the python model **power_response.py**. **We still need to include the form for the HOD power spectrum.**

- linear

$$\frac{d\log P(k)}{d\bar{\delta}} = \frac{68}{21} - \frac{1}{3}\frac{d\log k^3P(k)}{d\log k}\ , \tag{8}$$

- 1-loop perturbation theory

$$\frac{d\log P(k)}{d\bar{\delta}} = \frac{68}{21} - \frac{1}{3}\frac{d\log k^3 P(k)}{d\log k} + \frac{26}{21}\frac{P_{22}(k) + P_{13}(k)}{P(k)} \ , \tag{9}$$

- halo fit power spectrum

$$\frac{d\log P(k)}{d\bar{\delta}} = \frac{13}{21}\frac{d\log P(k)}{d\log \sigma_8} + 2 - \frac{1}{3}\frac{d\log k^3 P(k)}{d\log k} \ , \tag{10}$$

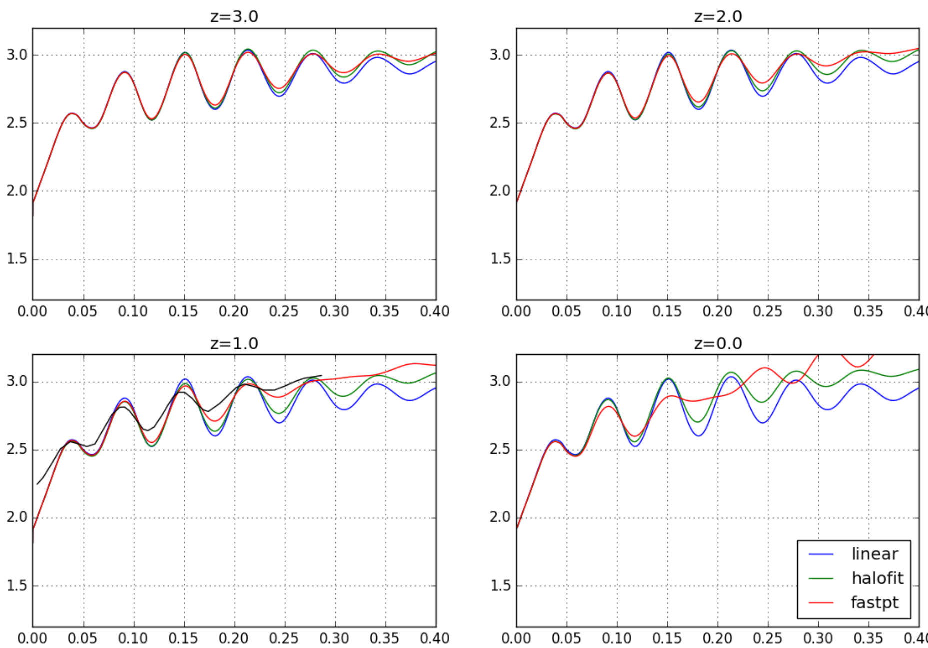

Figure 1: The power spectrum response $d\log P/d\bar{\delta}$ for linear, SPT, and halofit power spectra.

5 Importance of Super Sample Covariance

5.1 Super Sample Covariance in the Literature

Who been playing SSC?

[1] showed that the sample variance of the power spectrum is dominated by modes larger than the survey region. They also showed that halo sample variance and super sample covariance are related by the squeezed configurations of the trispectrum. Fig. 3 of their paper shows that the diagonal non-Gaussian covariance in the power spectrum is dominated by the SSC effect, for $k > 1$ h/Mpc the SSC is up to two orders of magnitude larger than the trispectrum contribution.

[2] compared the direct quantification of the SSC effect on the covariance of the power spectrum generated in subvolumes of large simulation to the SSC model of the power spectrum response to the background mode. Fig. 6 of their paper shows that the diagonal non Gaussian covariance in from the sub does matches the SSC model + small boxes, where small boxes are computed with periodic boundary conditions (i.e. they have no SSC-periodic boundary conditions can not capture SSC effects).

[3] compared the bias computed from the halo mass function in separate universe simulations with the bias computed from ratios of correlation functions. Fig. 4 and 5 of their paper show that the average bias and bias computed from the response of the halo mass function compared to the bias computed from correlations functions is in well agreement. They find 1-2 % for average bias in the 1 to 4 range and 4-5% agreement for the average bias at bias value of 8. There work provides a consistency between a one-point functions (the halo mass functions), and two-point functions (the halo-mass functions and mass-mass functions).

6 Theory

6.1 Trispectrum Consistency Relation

The trispectrum is defined as

$$\delta(\mathbf{k}_1)\delta(\mathbf{k}_2)\delta(\mathbf{k}_3)\delta(\mathbf{k}_4)\rangle = (2\pi)^3\delta_{\rm D}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\ . \tag{11}$$

$$T(\mathbf{k}_1,\mathbf{q}-\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}-\mathbf{k}_2) = \tag{12}$$

6.2 Spherical basis functions

For spherical geometry we use the following basis, composed of the spherical Bessel function and spherical harmonic:

$$\psi_\alpha(r,\theta,\phi) = j_{l_\alpha}(k_\alpha r) Y^{\rm R}_{l_\alpha m_\alpha}(\theta,\phi) \ , \tag{13}$$

where $Y^{\rm R}_{lm}(\theta,\phi)$ is the real spherical harmonic. The real spherical harmonics are defined as

$$Y^{\rm R}_{lm} = \begin{cases} \frac{i}{\sqrt{2}}\left(Y_{lm} - (-1)^m Y_{l-1m}\right) & \text{if } m<0 \\ Y_{l0} & \text{if } m=0 \\ \frac{1}{\sqrt{2}}\left(Y_{l-m} + (-1)^m Y_{lm}\right) & \text{if } m>0 \ . \end{cases} \tag{14}$$

The bias functions are defined within a ball, with a maximum radius $R_{\rm max}$ that corresponds to a redshift larger than any survey redshift. The k_α are determined by

$$j_{l_\alpha}(k_\alpha R_{\rm max}) = 0. \tag{15}$$

We write a super survey mode as

$$\delta_\alpha = \frac{1}{N_\alpha}\int \delta(\mathbf{r}) j_{l_\alpha}(k_\alpha r) Y^{\rm R}_{l_\alpha m_\alpha}(\hat{r}) d^3\mathbf{r} \ . \tag{16}$$

The normalization to the super mode is

$$N_\alpha = \int_0^{R_{\rm max}} dr r^2 j_{l_\alpha}(k_\alpha r) j_{l_\alpha}(k_\alpha r) \ . \tag{17}$$

6.2.1 Mean background density in a region

The mean background density in a region is composed of the super modes:

$$\begin{aligned} \bar{\delta} &= \frac{1}{\text{volume}} \int_{r_1,\theta_1,\phi_1}^{r_2,\theta_2,\phi_2} d^3\mathbf{x}\delta(\mathbf{x}) \\ &= \sum_\alpha \frac{3}{r_2^3 - r_1^3}\int_{r_1}^{r_2} dr\ r^2 j_{l_\alpha}(k_\alpha r)\delta_\alpha \frac{1}{2\sqrt{\pi}a_{00}} \underbrace{\int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2} d\theta d\phi\ \sin(\theta) Y^{\rm R}_{l_\alpha m_\alpha}(\hat{r})}_{a_{l_\alpha m_\alpha}} \ . \end{aligned} \tag{18}$$

Note that the integral is not over the ball that we define our basis functions in. It is over a survey region or slice of the survey.

In practice we are interested in derivatives of observables with respect to the δ_α. This is accomplished by chain rule

$$\frac{\partial f}{\partial \delta_\alpha} = \frac{\partial f}{\partial \bar{\delta}}\frac{\partial \bar{\delta}}{\partial \delta_\alpha} \ . \tag{19}$$

The derivative is computed from Eqn. 18

$$\frac{\partial\bar{\delta}}{\partial\delta_\alpha} = \frac{3}{r_2^3 - r_1^3}\int_{r_1}^{r_2} dr\ r^2 j_{l_\alpha}(k_\alpha r)\frac{1}{2\sqrt{\pi}a_{00}}\int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2} d\theta d\phi\ \sin(\theta) Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{r})\ . \tag{20}$$

6.2.2 Covariance of super modes

In our basis the super survey mode is defined as

$$\begin{aligned}\delta_\alpha &= \int \delta(\mathbf{r}) j_{l_\alpha}(k_\alpha r) Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{r}) d^3\mathbf{r} \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\delta(\mathbf{k})\int_0^{R_{\max}} dr^2 j_{l_\alpha}(k_\alpha r)\int d^2\hat{r} e^{ikr\hat{k}\cdot\hat{r}} Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{r}) \\ &= 4\pi i^{l_\alpha}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\delta(\mathbf{k}) Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{k})\int_0^{R_{\max}} dr^2 j_{l_\alpha}(k_\alpha r) j_{l_\alpha}(kr)\ ,\end{aligned} \tag{21}$$

where in the second equality $\delta(\mathbf{r}) = (2\pi)^{-3}\int d^3\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{r})\delta(\mathbf{k})$ was used and in the third equality the following identity was used

$$\int_{S^2} d^2\hat{r} Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{r}) e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi i^{l_\alpha} j_{l_\alpha}(kr) Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{k})\ . \tag{22}$$

Including a normalization

$$N_\alpha = \int d^3\mathbf{r} Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{r}) Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{r}) j_\alpha(k_\alpha r) j_\alpha(k_\alpha r) \tag{23}$$

$$= I_\alpha(k_\alpha, R_{\max})\int_\Omega Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{r}) Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{r})\hat{r} \tag{24}$$

$$= I_\alpha(k_\alpha, R_{\max}) \tag{25}$$

where $I_\alpha(k_\alpha, r_{\max})$ is as defined in (33) and $\int_\Omega Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{r}) Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{r})\hat{r} = 1$, we have

$$\delta_\alpha = \frac{4\pi i^{l_\alpha}}{N_\alpha}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\delta(\mathbf{k}) Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{k})\int_0^{R_{\max}} dr^2 j_{l_\alpha}(k_\alpha r) j_{l_\alpha}(kr)\ . \tag{26}$$

To get the covariance of the super survey field we take the ensemble average:

$$\begin{aligned}\langle\delta_\alpha\delta_\beta\rangle &= \frac{(4\pi)^2}{N_\alpha N_\beta}\int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\frac{d^3\mathbf{k}_2}{(2\pi)^3}\langle\delta(\mathbf{k}_1)\delta^*(\mathbf{k}_2)\rangle Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{k_1}) Y^{\mathrm{R}}_{l_\beta m_\beta}(\hat{k_2}) \\ &\quad\times\int_0^{R_{\max}} drr^2 j_{l_\alpha}(k_\alpha r) j_{l_\alpha}(k_1 r)\int_0^{R_{\max}} drr^2 j_{l_\beta}(k_\beta r) j_{l_\beta}(k_2 r) \\ &= \frac{(4\pi)^2}{N_\alpha N_\beta}\int \frac{d^3\mathbf{k}}{(2\pi)^3} P(k) I_\alpha(k, R_{\max})\times I_\beta(k, R_{\max}) Y^{\mathrm{R}}_{l_\alpha m_\alpha}(\hat{k_1}) Y^{R}_{l_\beta m_\beta}(-\hat{k_1}) \\ &= \frac{2}{\pi N_\alpha N_\beta}\int k^2 dk P(k)\delta_{l_\alpha, l_\beta}\delta_{m_\alpha, m_\beta} I_\alpha(k, R_{\max})\times I_\beta(k, R_{\max})\ ,\end{aligned} \tag{27}$$

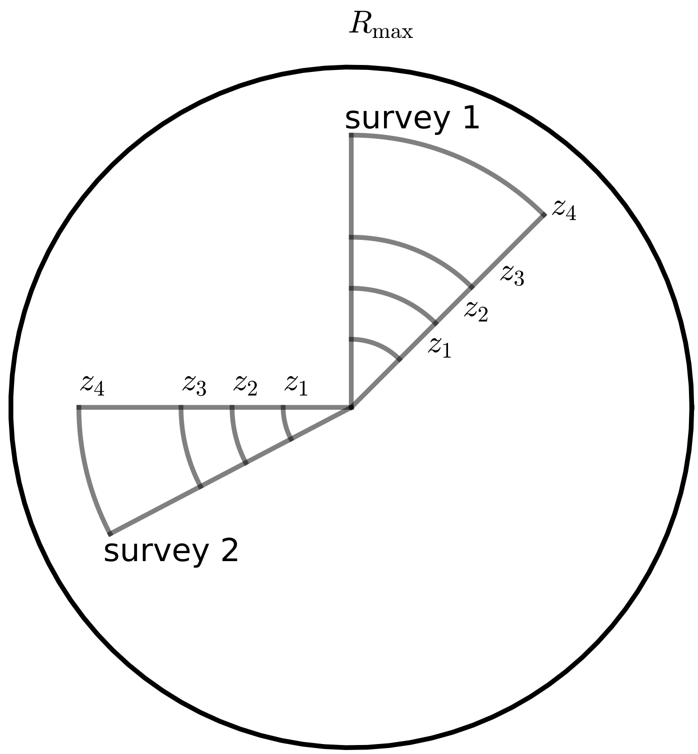

Figure 2: Cartoon depiction of our basis definition scenario. The basis ψ_α is defined on a ball with an $R_{\max}$ that exceeds the depth of any survey. Each $\bar{\delta}$ is calculated in a redshift slice of the a survey.

where $(2\pi)^3\delta^3_{\rm D}(\mathbf{k}+\mathbf{k}')P(k) = \langle \delta(\mathbf{k})\delta^*(\mathbf{k}')\rangle$ was used along with the definition

$$\begin{aligned}
I_\alpha(k, R_{\max}) &= \int_0^{R_{\max}} dr r^2 j_{l_\alpha}(k_\alpha r) j_{l_\alpha}(kr) \\
&= \frac{\pi}{2}\sqrt{\frac{1}{k_\alpha k}} \int_0^{R_{\max}} dr r J_{l_\alpha+1/2}(k_\alpha r) J_{l_\alpha+1/2}(kr) \\
&= \frac{\pi}{2}\frac{R_{\max}}{\sqrt{k_\alpha k}} \frac{\left[k_\alpha J_{l_\alpha+1/2}(kR_{\max}) J'_{l_\alpha+1/2}(k_\alpha R_{\max}) - k J_{l_\alpha+1/2}(k_\alpha R_{\max}) J'_{l_\alpha+1/2}(kR_{\max})\right]}{k^2 - k_\alpha^2} \\
&= \frac{\pi}{2}\frac{R_{\max}}{\sqrt{k_\alpha k}} \frac{\left[k_\alpha J_{l_\alpha+1/2}(kR_{\max}) J_{l_\alpha-1/2}(k_\alpha R_{\max}) - k J_{l_\alpha+1/2}(k_\alpha R_{\max}) J_{l_\alpha-1/2}(kR_{\max})\right]}{k^2 - k_\alpha^2} \\
&= \frac{\pi}{2}\frac{R_{\max}}{\sqrt{k_\alpha k}} \frac{k_\alpha J_{l_\alpha+1/2}(kR_{\max}) J_{l_\alpha-1/2}(k_\alpha R_{\max})}{k^2 - k_\alpha^2}\ ,
\end{aligned} \tag{28}$$

where in the last step we have use $J_{l_\alpha+1/2}(k_\alpha R_{\max})=0$, from the defining property of k_α. In the special case $k=k_\alpha$, we can simplify $N_\alpha = I_\alpha(k, R_{\max})$:

$$N_\alpha = I_\alpha(k_\alpha, R_{\max}) = \lim_{k\to k_\alpha} \frac{\pi}{2}\frac{R_{\max}}{\sqrt{k_\alpha k}} \frac{k_\alpha J_{l_\alpha+1/2}(kR_{\max}) J_{l_\alpha-1/2}(k_\alpha R_{\max})}{k^2-k_\alpha^2} \tag{29}$$

$$= \lim_{k\to k_\alpha} \frac{\pi R_{\max}}{2} \frac{J_{l_\alpha+1/2}(R_{\max}) J_{l_\alpha-1/2}(k_\alpha R_{\max})}{k^2-k_\alpha^2} \tag{30}$$

$$= \lim_{k\to k_\alpha} \frac{\pi R_{\max}}{2} \frac{r_{\max} J'_{l_\alpha+1/2}(kR_{\max}) J_{l_\alpha-1/2}(k_\alpha R_{\max})}{2k} \tag{31}$$

$$= \lim_{k\to k_\alpha} \frac{\pi R_{\max}^2}{2} \frac{\left[\frac{l_\alpha+1/2}{kR_{\max}} J_{l_\alpha+1/2}(kR_{\max}) - J_{l_\alpha+3/2}(k_\alpha R_{\max})\right] J_{l_\alpha-1/2}(k_\alpha R_{\max})}{2k} \tag{32}$$

$$= -\frac{\pi r_{\max}^2}{4k_\alpha} J_{l_\alpha+3/2}(k_\alpha R_{\max}) J_{l_\alpha-1/2}(k_\alpha R_{\max}) \ , \tag{33}$$

where between (30) and (31) we have used L'Hôpital's rule, and between (31) and (32) we have used a recurrence relation for the Bessel functions, $J'_n(x) = \frac{n}{x} J_n x - J_{n+1}(x)$.

7 Separate Universe Approach

7.1 Fermi Normal Coordinates

Here I would like to reserve the separate universe equations. It will be similar to Baldauf.

Fermi normal coordinate reference include Chp. 13 of [4], Chp. 1 of [5] the [6] Transport the tetrad ${e^a}_A$ along the geodesic γ, such that ${e^a}_0 = u^a(\tau)$, where $\mathbf{u} = d\gamma/d\tau$. And

$$g_{ab}{e^a}_A{e^b}_B = \eta_{AB} \ . \tag{34}$$

$$_{\mathbf{u}}e_A = u^a\nabla_a{e^b}_A = -u^a{\Omega_c}^b{e^c}_A \ , \tag{35}$$

where $\Omega^{ab} = a^a u^b - a^b u^a + u_c\omega_d\epsilon^{cdab}$. In the absence of acceleration and rotations the above equation gives

$$u^a\nabla_a{e^b}_A = 0 \ . \tag{36}$$

7.2 Power Spectrum Response

The response of correlation function to a $\bar{\delta}$ is computed by

$$\frac{d\xi(r,t)}{d\bar{\delta}} = \frac{\Phi[\xi_\mathrm{L}(r_\mathrm{L},t_\mathrm{L})] - \xi(r)}{\bar{\delta}} \ , \tag{37}$$

where Φ is the mapping from the L universe to the "standard" universe. The mapping is as follows

$$\begin{aligned}
\xi_\mathrm{L}(r_\mathrm{L},t_\mathrm{L}) &= \langle\delta_\mathrm{L}|\delta_\mathrm{L}\rangle_{r_\mathrm{L},t_\mathrm{L}} \\
&= \langle\delta(1+\bar{\delta})+1|\delta(1+\bar{\delta})+1\rangle_{r+\bar{\delta}/3,t_\mathrm{L}} \\
&= \langle\delta|\delta\rangle_{r+\bar{\delta}/3,t_\mathrm{L}} + 2\bar{\delta}\langle\delta|\delta\rangle_{r+\bar{\delta}/3,t_\mathrm{L}} + \mathcal{O}(\bar{\delta}^2) \\
&= \xi(r+\bar{\delta}/3,t_\mathrm{L}) + 2\bar{\delta}\xi(r+\bar{\delta}/3,t_\mathrm{L}) \\
&= \xi(r,t_\mathrm{L}) + \frac{\bar{\delta}}{3}r\partial_r\xi(r,t_\mathrm{L}) + 2\bar{\delta}\xi(r,t) + \mathcal{O}(\bar{\delta}^2) \ .
\end{aligned} \tag{38}$$

where the subscripts in the first few lines represent the position and time arguments. Plugging back into Eqn. 37 gives

$$\frac{d\xi(r,t)}{d\bar{\delta}} = \frac{\xi(r,t_\mathrm{L}) + \frac{\bar{\delta}}{3}r\partial_r\xi(r,t_\mathrm{L}) + 2\bar{\delta}\xi(r,t) - \xi(r,t)}{\bar{\delta}} \ . \tag{39}$$

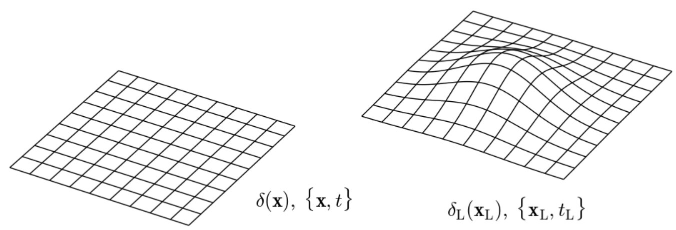

Figure 3: Illustration of the difference in correlations functions. The difference in any correlation (in real or Fourier space) is defined as difference in correlates on two different manifolds. The computation requires the appropriate mapping between manifolds.

I have not transformed the time coordinate. I will do this later. For the linear and SPT power spectrum, the time coordinate transformation will correspond to change in the growth factor. The derivative term in Fourier space is computed as follows:

$$
\begin{aligned}
r\partial_r\xi(r) = r\hat{r}\cdot\nabla\xi(r) &= \hat{r}\cdot\nabla\int\frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{r}}P(k) \\
&= r\hat{r}\cdot\int\frac{d^3\mathbf{k}}{(2\pi)^3}\; i\mathbf{k}\; e^{i\mathbf{k}\cdot\mathbf{r}}P(k) \\
&= \int\frac{d^3\mathbf{k}}{(2\pi)^3}\; i\mathbf{r}\; e^{i\mathbf{k}\cdot\mathbf{r}}\cdot\mathbf{k}P(k) \\
&= \int\frac{d^3\mathbf{k}}{(2\pi)^3}\;\nabla_{\mathbf{k}}\; e^{i\mathbf{k}\cdot\mathbf{r}}\cdot\mathbf{k}P(k) \\
&= -\int\frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{r}}\;\nabla_{\mathbf{k}}\;\cdot[\mathbf{k}P(k)] \\
&= -\int\frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{r}}[3P(k)+k\frac{dP(k)}{dk}]\ .
\end{aligned}
\tag{40}
$$

Therefore the power spectrum response is

$$
\frac{dP(k)}{d\bar{\delta}} = \frac{1}{\bar{\delta}}\left[P(k,t_{\rm L}) - \frac{\bar{\delta}}{3}[3P(k,t_{\rm L}) + k\frac{dP(k)}{dk}] + 2\bar{\delta}P(k,t_{\rm L}) - P(k,t)\right]\ . \tag{41}
$$

7.3 Linear power spectrum response

Linear growth factors in the L universe compared to the “standard” universe are

$$
G_{\rm L}(a[1-\frac{1}{3}\bar{\delta}]) = G(a)\left[1+\frac{13}{21}\bar{\delta}\right]\ . \tag{42}
$$

To relate the linear power spectrum at time $t_{\rm L}$ to time t requires the ratio of growth factors squared and we have

$$
P_{\rm lin}(k,t_{\rm L}) = \left[1+\frac{26}{21}\bar{\delta}\right]P(k,t) \tag{43}
$$

Plugging the above back into the master equation for power spectrum response gives

$$
\frac{dP(k)}{d\bar{\delta}} = \frac{68}{21}P(k) - \frac{1}{3}[3P(k)+k\frac{dP(k)}{dk}]\ . \tag{44}
$$

7.4 1 -loop perturbation theory

The 1-loop growth factor relation is

$$
P_{1-loop}(k,z) = G^2P_{11}(k) + G^4[P_{22}(k)+P_{13}(k)]\ , \tag{45}
$$

So the transformation is

$$P_{1-loop}(k,t_{\rm L}) = \left[1+\frac{26}{21}\bar{\delta}\right] P_{11}(k,t) + \left[1+\frac{52}{21}\bar{\delta}\right] \left[P_{22}(k,t)+P_{13}(k,t)\right] . \tag{46}$$

Plugging into the master equation gives

$$\frac{d\log P(k)}{d\bar{\delta}} = \frac{68}{21} - \frac{1}{3}\frac{d\log k^3 P(k)}{d\log k} + \frac{26}{21}\frac{P_{22}(k)+P_{13}(k)}{P(k)} \ , \tag{47}$$

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