

t -Test

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One-sample t -test

One-sample t -tests look at just one parameter of interest μ (in addition to σ) as data-generation mechanism. (see shinyApp).

Statistical model

- $Y_i = \mu + \epsilon_i$; $\epsilon_i \stackrel{i.i.d.}{\sim} N(0, \sigma^2)$ (or with ϵ_i i.i.d. and “large” sample size), $i = 1, \dots, n$
- $Y_i \stackrel{i.i.d.}{\sim} N(\mu, \sigma^2)$ (equivalent)

This is the simplest example of $Y_i = \beta_1 + \beta_2 x_{i2} + \dots + \beta_p x_{ip} + \epsilon_i$, the **only intercept model** with $\beta_1 = \mu$ (see IntroLinearModels).

Intro in sample size calculation

Let us first ask the **how many do I need**-question. There are two kinds of *sample size calculations*, the power approach and the precision approach.

Consider the problem $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$.

Power approach

A priori sample size calculation, see pages 77-78 AMT. Let β be the Type II error (The probability of **not** rejectig H_0 **if** a specified alternative H_1 is true. $1 - \beta$ is the probability of the complement, of rejectig H_0 **if** a specified alternative H_1 is true).

Question: What n we need to assure

- to reject $H_0 : \mu = \mu_0$
- with probability $1 - \beta$
- with false-positive rate α
- **if** the specified alternative $H_1 : \mu = \mu_1$ is true?

Assume $\mu_0 = 0$, thus $\delta = \mu_1 - \mu_0 = \mu_1$. One can show hat

$$n \geq \frac{(z_{1-\alpha/2} + z_{1-\beta})^2}{(\delta/\sigma)^2}$$

The approximation $n \approx 8/(\delta/\sigma)^2$ holds for $\alpha = 0.05$ and $\beta = 0.2$.

Assume we estimate or “guess” σ from another study with $\hat{\sigma} = 22$ and we want that H_0 is rejected with probability 0.8 **if** $\mu_1 = 13$ is true.

```
(qnorm(.975)+qnorm(.8))^2/((mu1-mu0)/sigmahat)^2
```

```
[1] 22.478
```

In R, you can use the *exact* version, `power.t.test()`. Read the help file `help(power.t.test)`. You have to specify δ , σ , power and the type of test:

```
power.t.test(delta=mu1-mu0,sd=sigmahat,power=0.8,type="one.sample")
```

```
One-sample t test power calculation
```

```
      n = 24.469
delta = 13
  sd = 22
sig.level = 0.05
  power = 0.8
alternative = two.sided
```

Precision approach

In this approach, we do not need a Type II error. We have **not** to specify the alternative, which **most often makes more sense**, since there are **many (even infinite) options for the alternative** $\mu \neq \mu_0$.

We estimate the sample size by specifying the precision we want for the estimation, that is, we specify a priori the maximal width of the $100 \times (1 - \alpha)\%$ CI.

An approximative $100 \times (1 - \alpha)\%$ CI is given by $\bar{x} \pm \delta$, with $\delta = z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$. Solving for n gives

$$n \geq \frac{z_{1-\alpha/2}^2}{(\delta/\sigma)^2}$$

Using again $\hat{\sigma} = 22$:

```
delta<-c(2,3,4,5,8)
alpha<-0.05
data.frame(delta=delta,n=qnorm(1-alpha/2)^2*(sigmahat/delta)^2)
```

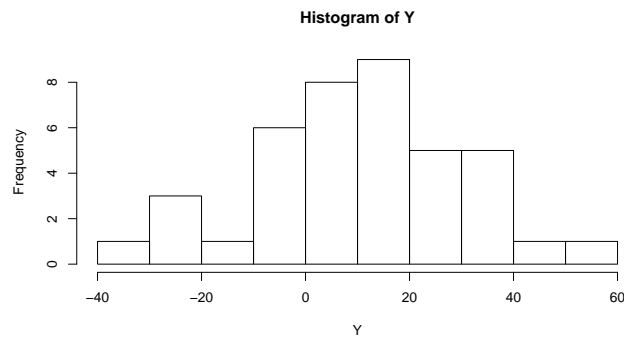
	delta	n
1	2	464.817
2	3	206.585
3	4	116.204
4	5	74.371
5	8	29.051

Simulation of example data

Let us simulate some data from population with **assumed known parameters** μ and σ^2 . We simulate to understand the emergence of data. Of course, in reality, we do not know the data-generating process.

```
set.seed(55)
mu<-10
sigma<-20
n<-40
Y<-rnorm(n,mu,sigma)
```

```
hist(Y)
```



```
summary(Y)
```

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
-31.505	-0.214	11.220	11.309	25.104	57.107

Analysis

As one-sample t -Test

```
t.test(Y)
```

One Sample t-test

```
data: Y
t = 3.66, df = 39, p-value = 0.00074
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 5.0604 17.5574
sample estimates:
mean of x
 11.309
```

As linear model

This is equivalent with a linear model with only an intercept as parameter.

```
summary(mod<-lm(Y~1))
```

Call:

```
lm(formula = Y ~ 1)
```

Residuals:

Min	1Q	Median	3Q	Max
-42.81	-11.52	-0.09	13.80	45.80

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	11.31	3.09	3.66	0.00074

Residual standard error: 19.5 on 39 degrees of freedom

Of course, the results are the same. The t -test is a simple linear model.

Estimation of σ “by hand” (this quantity is given in the output above):

```
sqrt(sum(mod$residuals^2)/(n-1))
```

```
[1] 19.538
```

Confidence interval for μ :

```
confint(mod)
```

	2.5 %	97.5 %
(Intercept)	5.0604	17.557

Small samples without normality

In the example above, the assumptions are met since we simulated data from a normal distribution. In the absence of normality, this assumption of the t -test is not met. We can then perform a non-parametric test.

- The `wilcox.test` is the non-parametric alternative to the t -test.

- With the actual data, results are similar since we have simulated from a normal distribution.

```
wilcox.test(Y,conf.int=TRUE)
```

Wilcoxon signed rank test

```
data: Y
V = 655, p-value = 0.00068
alternative hypothesis: true location is not equal to 0
95 percent confidence interval:
 5.141 17.842
sample estimates:
(pseudo)median
11.487
```

Two-sample t -Test

(see shinyApp).

Statistical model

- Means parameterization:

- $Y_{ij} = \mu_i + \epsilon_{ij}$, $i = 1, 2$; $j = 1, \dots, n_i$; $\epsilon_{ij} \stackrel{i.i.d.}{\sim} N(0, \sigma^2)$
- $Y_{ij} \stackrel{i.i.d.}{\sim} N(\mu_i, \sigma^2)$ (equivalent)

- Effects parameterization:

- $Y_{ij} = \mu + \alpha_i + \epsilon_{ij}$
- $Y_{ij} \stackrel{i.i.d.}{\sim} N(\mu + \alpha_i, \sigma^2)$ (equivalent)

- In R regression language:

- $Y_i = \beta_0 + \beta_1 I_{group_i=2} + \epsilon_i$, with
 - $i = 1, \dots, n$
 - $\beta_0 = \mu_1$, $\beta_1 = \mu_2 - \mu_1$
 - $I_{group_i=2} = 1$, if $group_i = 2$; $I_{group_i=2} = 0$, if $group_i = 1$ (=“Dummy” variable)
- (see IntroLinearModels).

Simulation of example data

We simulate to understand the emergence of data. Of course, in reality, we do not know the data-generating process.

```
n<-30 ##n per group
muInt<-21 ##True mean Int
muCont<-20 ##True mean Cont
Delta<-muInt-muCont ##True mean difference
Delta
```

```
[1] 1
```

```
sigma<-2 ##True standard deviation
set.seed(10)
dataInt <- rnorm(n=n,mean=muInt,sd=sigma) ##Sample from Int
dataCont <- rnorm(n=n,mean=muCont,sd=sigma) ##Sample from Cont
group<-gl(n=2,k=n,labels=c("Cont","Int"))##grouping variable
```

```
response<-c(dataCont,dataInt)##the outcome
mydata <- data.frame(response=response,group=group)##the data.frame
str(mydata)
```

```
'data.frame': 60 obs. of 2 variables:
 $ response: num 16.3 19.8 21.9 20.4 17.2 ...
 $ group : Factor w/ 2 levels "Cont","Int": 1 1 1 1 1 1 1 1 1 1 ...
```

```
head(mydata)
```

```
  response group
1  16.293  Cont
2  19.844  Cont
3  21.937  Cont
4  20.370  Cont
5  17.240  Cont
6  17.129  Cont
```

```
summary(mydata)
```

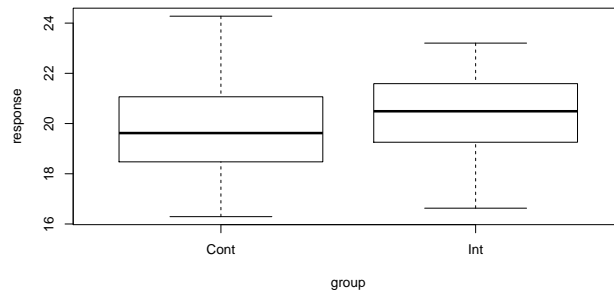
```
      response      group
Min.   :16.3   Cont:30
1st Qu.:18.7   Int :30
Median :20.3
Mean   :20.0
3rd Qu.:21.2
Max.   :24.3
```

```
by(mydata,mydata$group,summary)
```

```
mydata$group: Cont
      response      group
Min.   :16.3   Cont:30
1st Qu.:18.5   Int : 0
Median :19.6
Mean   :19.8
3rd Qu.:21.1
Max.   :24.3
```

```
-----
mydata$group: Int
      response      group
Min.   :16.6   Cont: 0
1st Qu.:19.3   Int :30
Median :20.5
Mean   :20.3
3rd Qu.:21.5
Max.   :23.2
```

```
boxplot(response~group,mydata)
```



Analysis

Hypothesis

$H_0: \mu_I - \mu_C = 0$ vs. $H_1: \mu_I - \mu_C \neq 0$

```
test <- t.test(x=dataInt,y=dataCont,var.equal=TRUE)
test
```

Two Sample t-test

```
data: dataInt and dataCont
t = 1.14, df = 58, p-value = 0.26
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -0.40249  1.47525
sample estimates:
mean of x mean of y
 20.311    19.774
```

```
test$estimate ##Estimate of muInt and muCont
```

Extract information from object test

```
mean of x mean of y
 20.311    19.774
```

```
test$statistic ##t-statistic
```

```
t
1.1436
```

```
test$stderr ##standard error
```

```
[1] 0.46903
```

```
((test$estimate[1]-test$estimate[2])-0)/test$stderr ##t-statistic "by hand"
```

```
mean of x
 1.1436
```

```
test$p.value
```

```
[1] 0.25749
```

Other Hypotheses

- $H_0: \mu_I - \mu_C \leq 0$ vs. $H_1: \mu_I - \mu_C > 0$

```
test2 <- t.test(x=dataInt,y=dataCont,alternative="greater",var.equal=TRUE)
test2
```

Two Sample t-test

```
data: dataInt and dataCont
t = 1.14, df = 58, p-value = 0.13
alternative hypothesis: true difference in means is greater than 0
95 percent confidence interval:
 -0.24763      Inf
sample estimates:
mean of x mean of y
 20.311    19.774
```

- $H_0: \mu_I - \mu_C \leq -1$ vs. $H_1: \mu_I - \mu_C > -1$

```
test3 <- t.test(x=dataInt,y=dataCont,mu=-1,alternative="greater",var.equal=TRUE)
test3
```

Two Sample t-test

```
data: dataInt and dataCont
t = 3.28, df = 58, p-value = 0.00089
alternative hypothesis: true difference in means is greater than -1
95 percent confidence interval:
 -0.24763      Inf
sample estimates:
mean of x mean of y
 20.311    19.774
```

Formula version

Back to the problem $H_0: \mu_I - \mu_C = 0$ vs. $H_1: \mu_I - \mu_C \neq 0$

```
test<- t.test(response~group,data=mydata,var.equal=TRUE)
test
```

Two Sample t-test

```
data: response by group
t = -1.14, df = 58, p-value = 0.26
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -1.47525  0.40249
sample estimates:
mean in group Cont mean in group Int
 19.774          20.311
```

As linear model


```
mod<-lm(response~group,data=mydata)
summary(mod)
```

Call:

```
lm(formula = response ~ group, data = mydata)
```

Residuals:

```
      Min       1Q   Median       3Q      Max
-3.681 -1.152  0.119  1.282  4.501
```

Coefficients:

```
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   19.774      0.332   59.62  <2e-16
groupInt       0.536      0.469    1.14    0.26
```

Residual standard error: 1.82 on 58 degrees of freedom

Multiple R-squared: 0.0221, Adjusted R-squared: 0.00519

F-statistic: 1.31 on 1 and 58 DF, p-value: 0.257

```
confint(mod)
```

```
              2.5 %  97.5 %
(Intercept) 19.11038 20.4381
groupInt    -0.40249  1.4753
```

What is the F -statistic

In the output of the model summary, there is an F -value. We will introduce the F -statistic in the context of ANOVA.

The aim is to test the null that we can omit **group** from the model. In this simple case, that is the same as the t -test for **group** ($\mu_I = \mu_C$). Very generally, we use F -tests as tests in the context of **model comparison**.

```
mod2<-lm(response~1,data=mydata) ## Intercept-only model
anova(mod,mod2) ## tests the null: "Effect of group is absent"
```

Analysis of Variance Table

Model 1: response ~ group

Model 2: response ~ 1

```
  Res.Df  RSS Df Sum of Sq    F Pr(>F)
1      58  191
2      59  196 -1    -4.32 1.31  0.26
```

Introduction in residual analysis

The **assumptions of linear models** are (see IntroLinearModels)

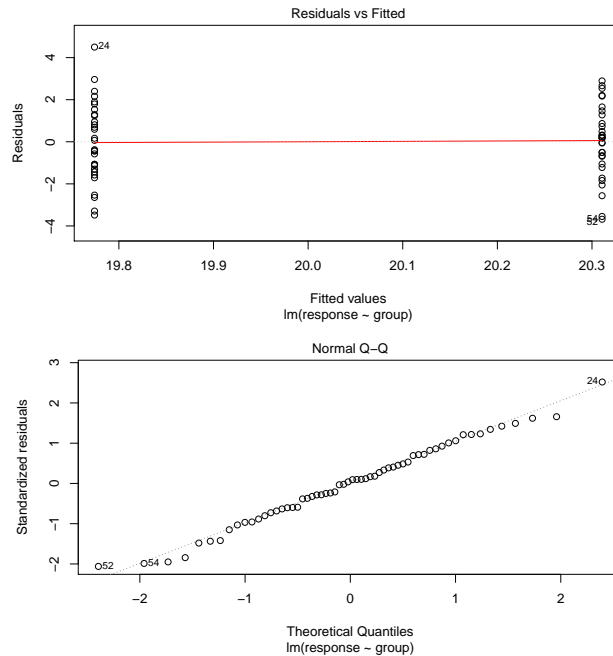
- ϵ_i are independent and identically distributed, i.i.d.
- $E(\epsilon_i) = 0$ for all i .
- $Var(\epsilon_i) = \sigma^2$ constant for all i .

For the t -test, we have to assume **in addition**

- ϵ_i i.i.d. $\sim N(0, \sigma^2)$

In R, we can just call the `plot` function for the model object to check the assumptions.

```
plot(mod,which=c(1,2))
```



Small samples without normality

In the example above, the assumptions are met since we simulated data from a normal distribution. Again, in the absence of normality, this assumption of the t -test is not met. We can then perform a non-parametric test.

- The Mann-Whitney Test (`wilcox.test` in R) is the non-parametric alternative to the two-sample t -test.
- With the actual data, results are similar since we have simulated from a normal distribution.

```
wilcox.test(dataInt,dataCont,conf.int=TRUE)
```

Wilcoxon rank sum test

data: dataInt and dataCont

W = 530, p-value = 0.24

alternative hypothesis: true location shift is not equal to 0

95 percent confidence interval:

-0.38743 1.56444

sample estimates:

difference in location

0.57931

Equivalence testing

Philosophical background

Very often, it makes not much sense to test nulls such as $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$. Assume a theory predicts a range for μ , i.e. that μ lies in a region $[-\epsilon, +\epsilon]$.

You then have the following test situation:

$H_0 : \mu \leq -\epsilon$ OR $\mu \geq +\epsilon$ versus $H_1 : -\epsilon < \mu < \epsilon$, meaning that

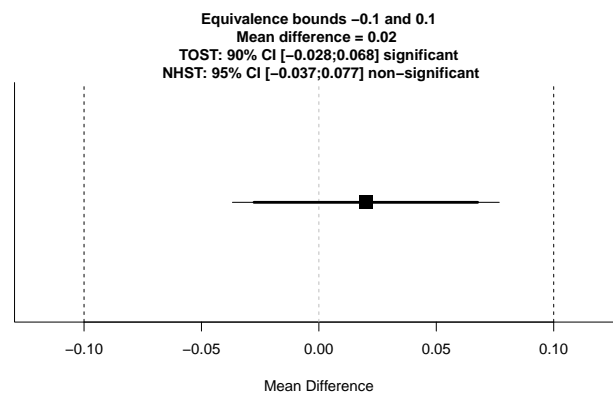
- Null: The true parameter is outside a tolerance region, “irrelevance”.
- Alternative: The true parameter is inside a tolerance region, “relevance”.

Rejecting H_0 now really means rejecting irrelevance (defined by the margins ϵ).

Implementation

This analysis can be performed with TOST (Two One-sided t -Tests), this is implemented in the package TOSTER. For the one-sample situation, we have the function `TOSTone.raw()`.

```
library(TOSTER)
## Test observed mean of 0.52 and standard deviation of 0.5 in sample of 300 participants
## against 0.5 given equivalence bounds in raw units of -0.1 and 0.1, with an alpha = 0.05.
TOSTone.raw(m=0.52,mu=0.5,sd=0.5,n=300,low_eqbound=-0.1, high_eqbound=0.1, alpha=0.05)
```



TOST results:

t-value lower bound: 4.16 p-value lower bound: 0.00002

t-value upper bound: -2.77 p-value upper bound: 0.003

degrees of freedom : 299

Equivalence bounds (raw scores):

low eqbound: -0.1

high eqbound: 0.1

TOST confidence interval:

lower bound 90% CI: -0.028

upper bound 90% CI: 0.068

NHST confidence interval:

lower bound 95% CI: -0.037

upper bound 95% CI: 0.077

Equivalence Test Result:

The equivalence test was significant, $t(299) = -2.771$, $p = 0.00297$, given equivalence bounds of -0.100 and 0.100.

Null Hypothesis Test Result:

The null hypothesis test was non-significant, $t(299) = 0.693$, $p = 0.489$, given an alpha of 0.05.

Based on the equivalence test and the null-hypothesis test combined, we can conclude that the observed mean difference is significantly different from zero.

Package TOSTER functions

If you need such analysis in the future, look at `help(package="TOSTER")` for other functions in the package, such as functions for sample size estimation.