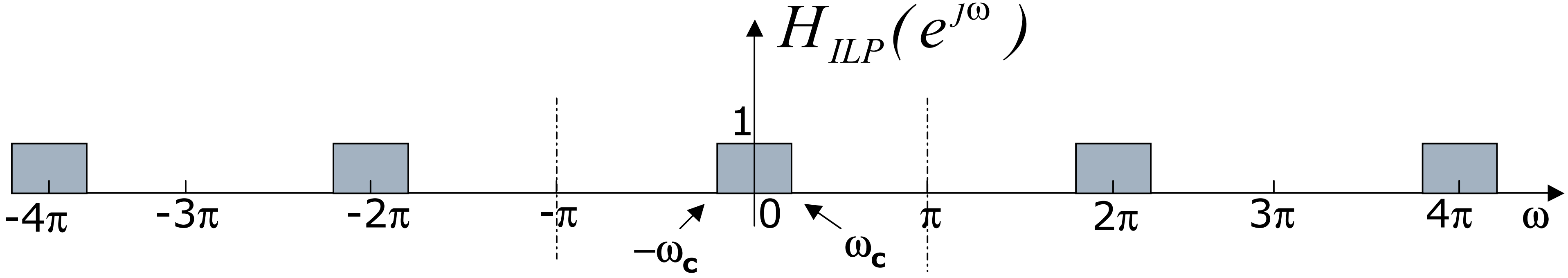
DIGITAL FILTERS

Vedat Tavşanoğlu

Frequency Response of Ideal Filters

Low-pass Filters

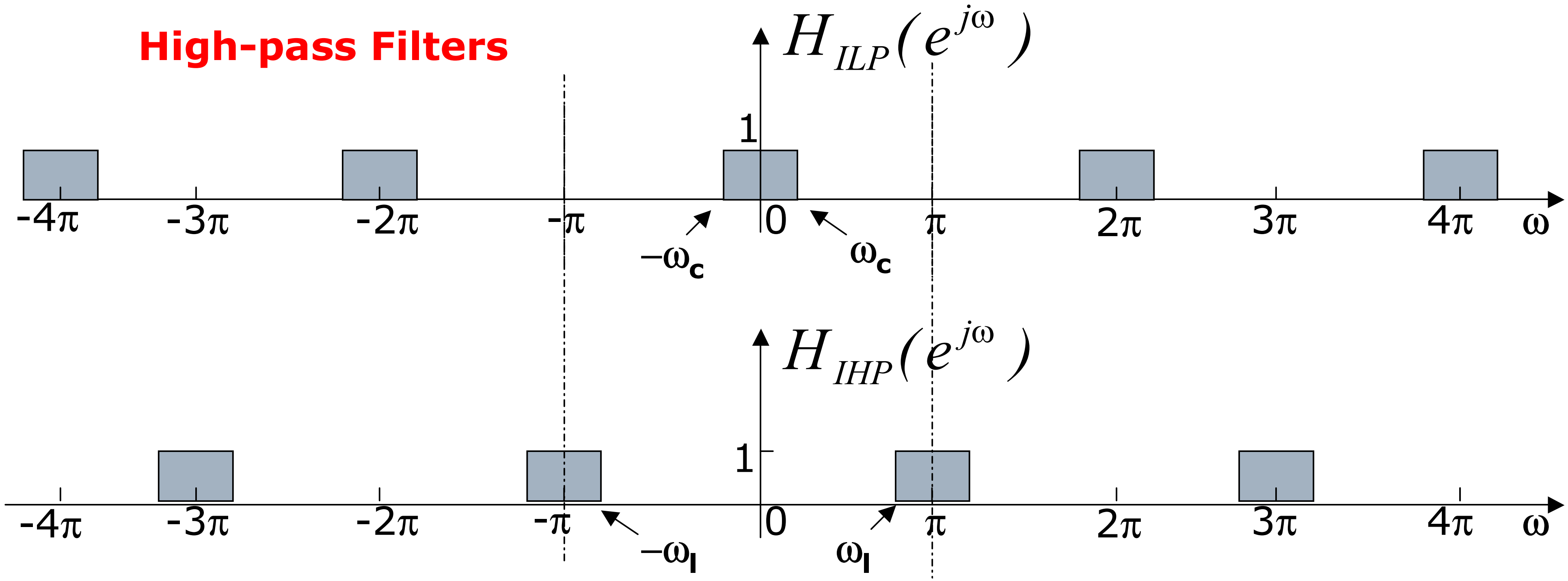


$$H_{ILP}(e^{j\omega}) = \begin{cases} 1 & for \quad -\omega_c \le \omega \le \omega_c \\ 0 & for \quad -\pi \le \omega \le -\omega_c \quad and \quad \omega_c \le \omega \le \pi \end{cases}$$

$$H_{ILP}(e^{j\omega}) = H_{ILP}(e^{j(\omega - 2k\pi)})\,;\, k = \pm 1, \pm 2, \ldots\ldots\ldots \pm \infty$$

Frequency Response of Ideal Filters

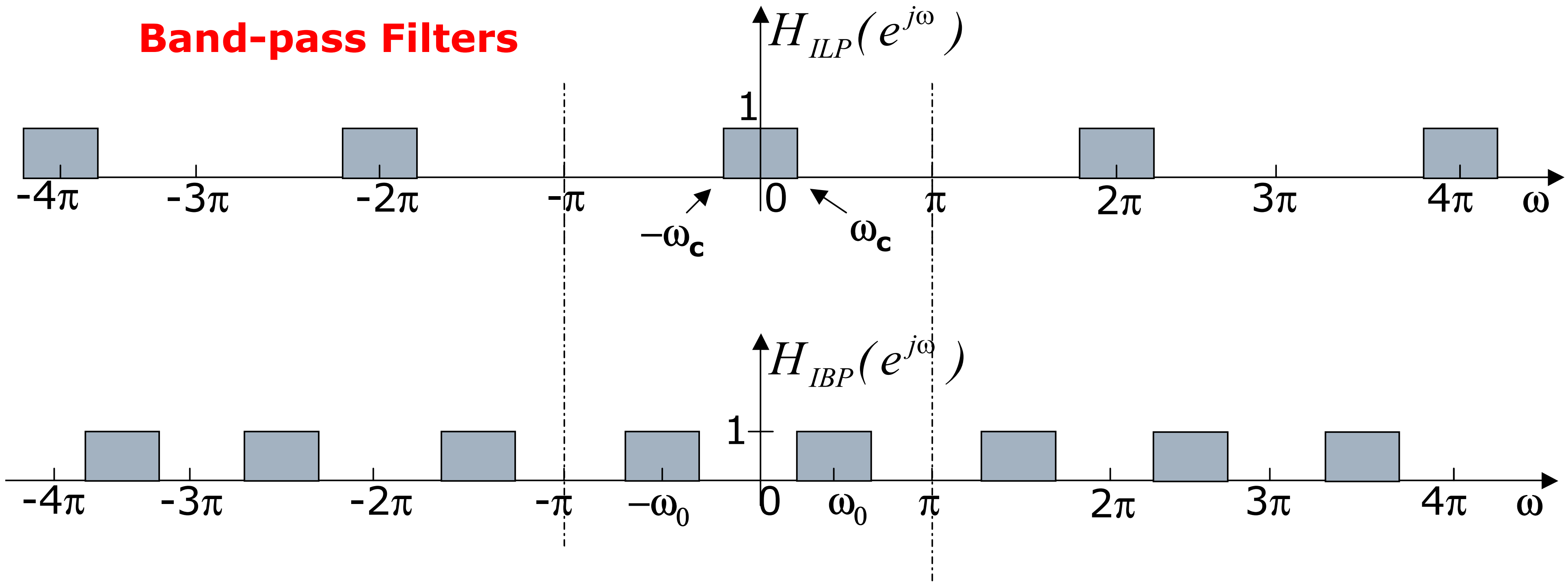
High-pass Filters



$$H_{IHP}(e^{j\omega}) = H_{ILP}(e^{j(\omega-\pi)})\,; \qquad \omega_c = \pi - \omega_l$$

Frequency Response of Ideal Filters

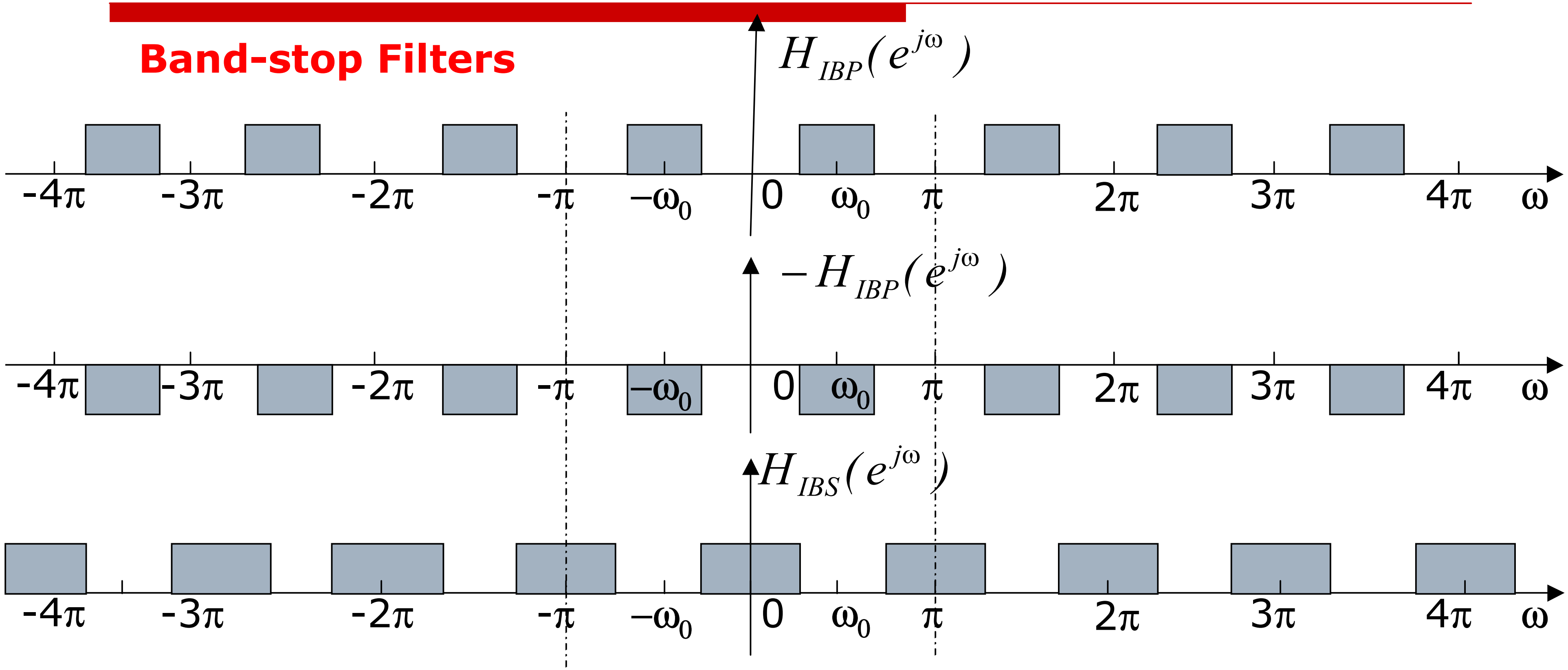
Band-pass Filters



$$H_{IBP}(e^{j\omega}) = H_{ILP}(e^{j(\omega-\omega_0)}) + H_{ILP}(e^{j(\omega+\omega_0)})$$

Frequency Response of Ideal Filters

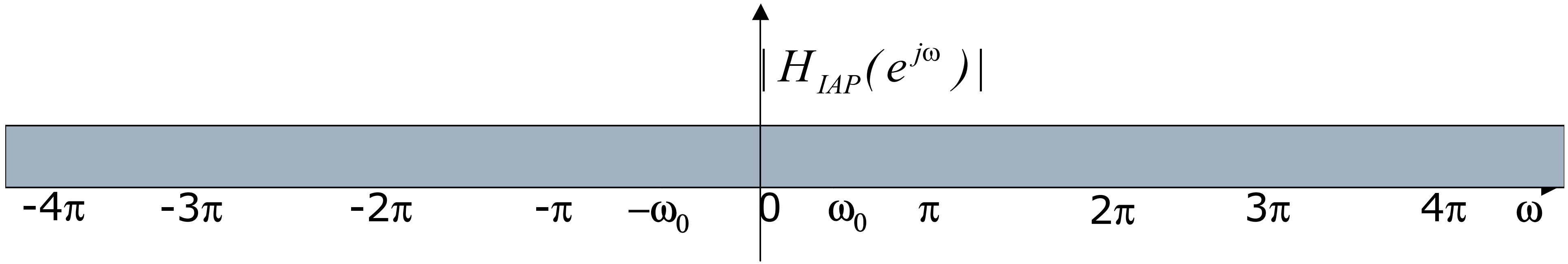
Band-stop Filters



$$H_{IBS}(e^{j\omega}) = 1 - H_{IBP}(e^{j\omega})$$

Frequency Response of Ideal Filters

All-pass Filters



Impulse Response of Ideal Low-pass Filters

$$H_{ILP}(e^{j\omega}) = \begin{cases} 1 & for \quad -\omega_c \le \omega \le \omega_c \\ 0 & for \quad -\pi \le \omega \le -\omega_c \quad and \quad \omega_c \le \omega \le \pi \end{cases}$$

$$H_{ILP}(e^{j\omega}) = H_{ILP}(e^{j(\omega - 2k\pi)})\,;\, k = \pm 1, \pm 2, \ldots\ldots\ldots \pm \infty$$

$$h_{ILP}(n) = F^{-1}\{H_{ILP}(e^{j\omega})\}$$

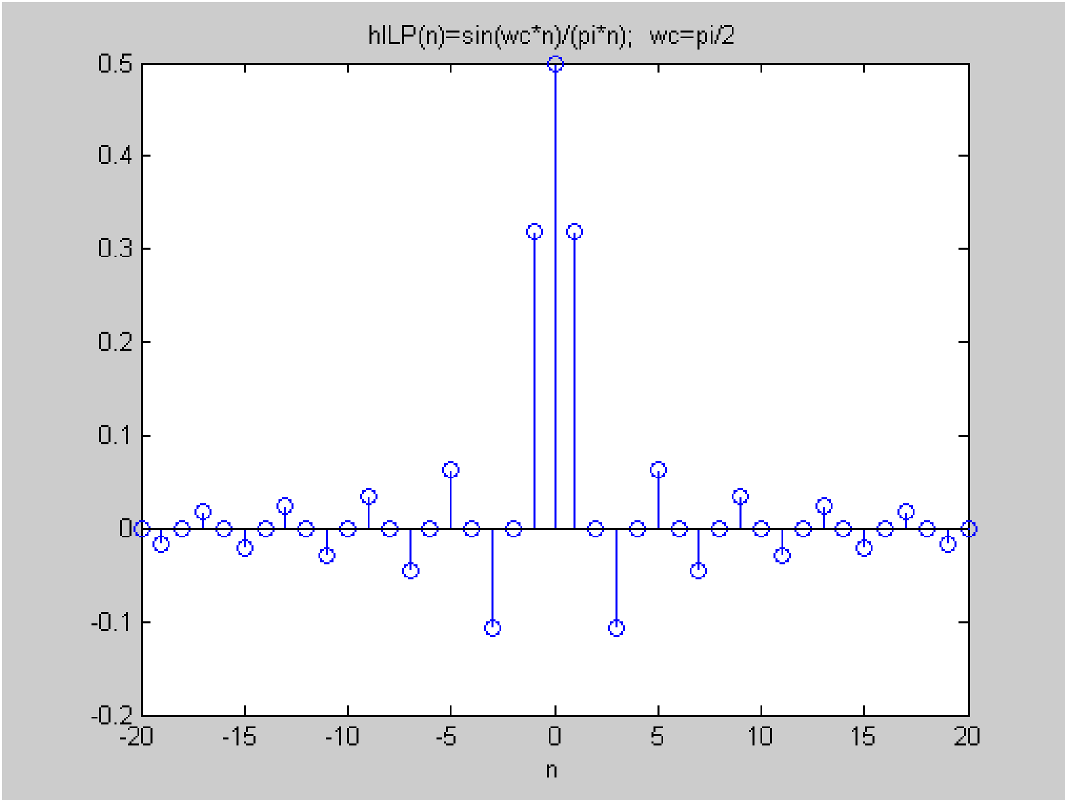
Impulse Response of Ideal Low-pass Filters

$$h_{ILP}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{ILP}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$$

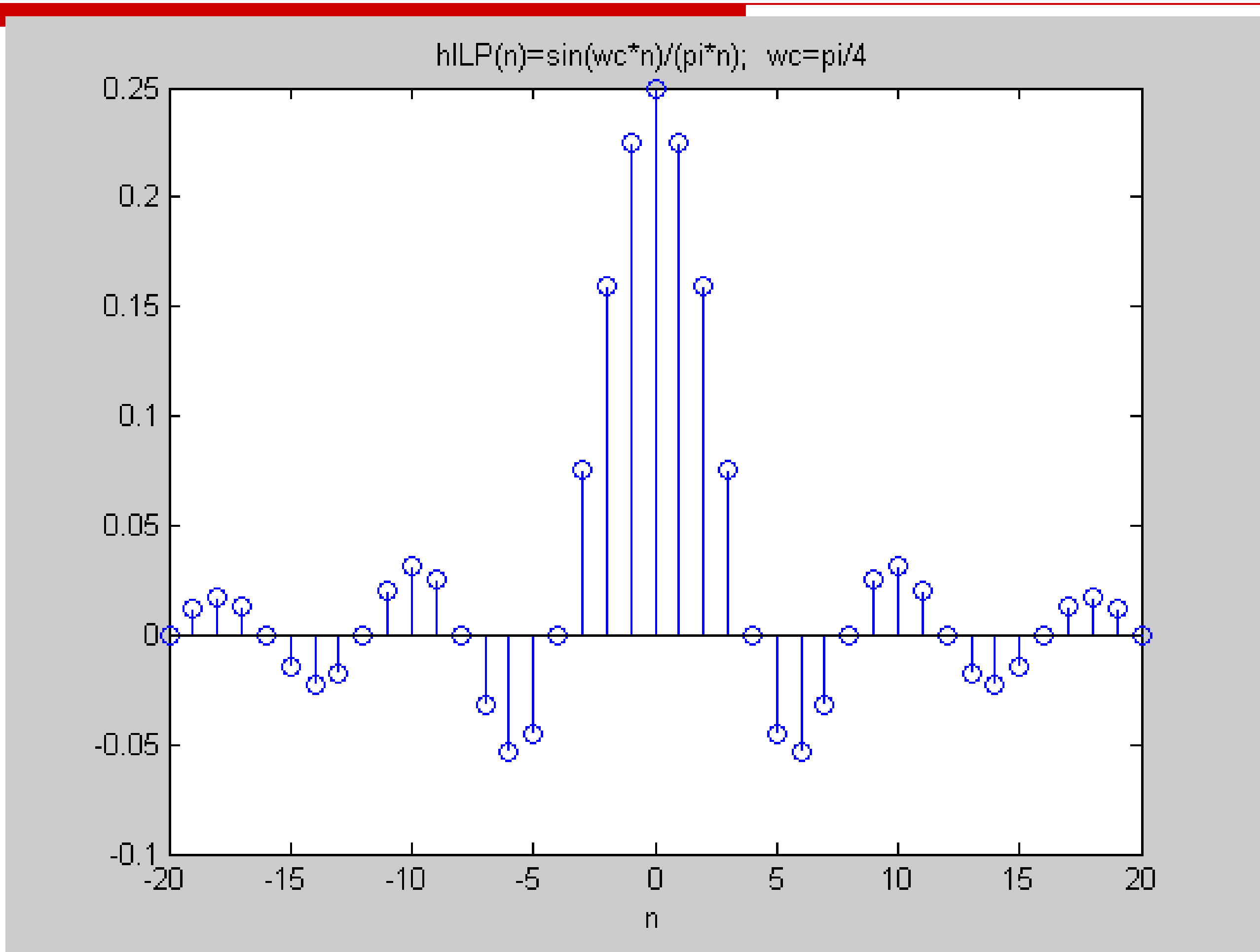
$$h_{ILP}(n) = \frac{1}{2\pi jn}(e^{j\omega_c n} - e^{-j\omega_c n}) = \frac{\sin\omega_c n}{\pi n}$$

Impulse Response of Ideal Low-pass Filters

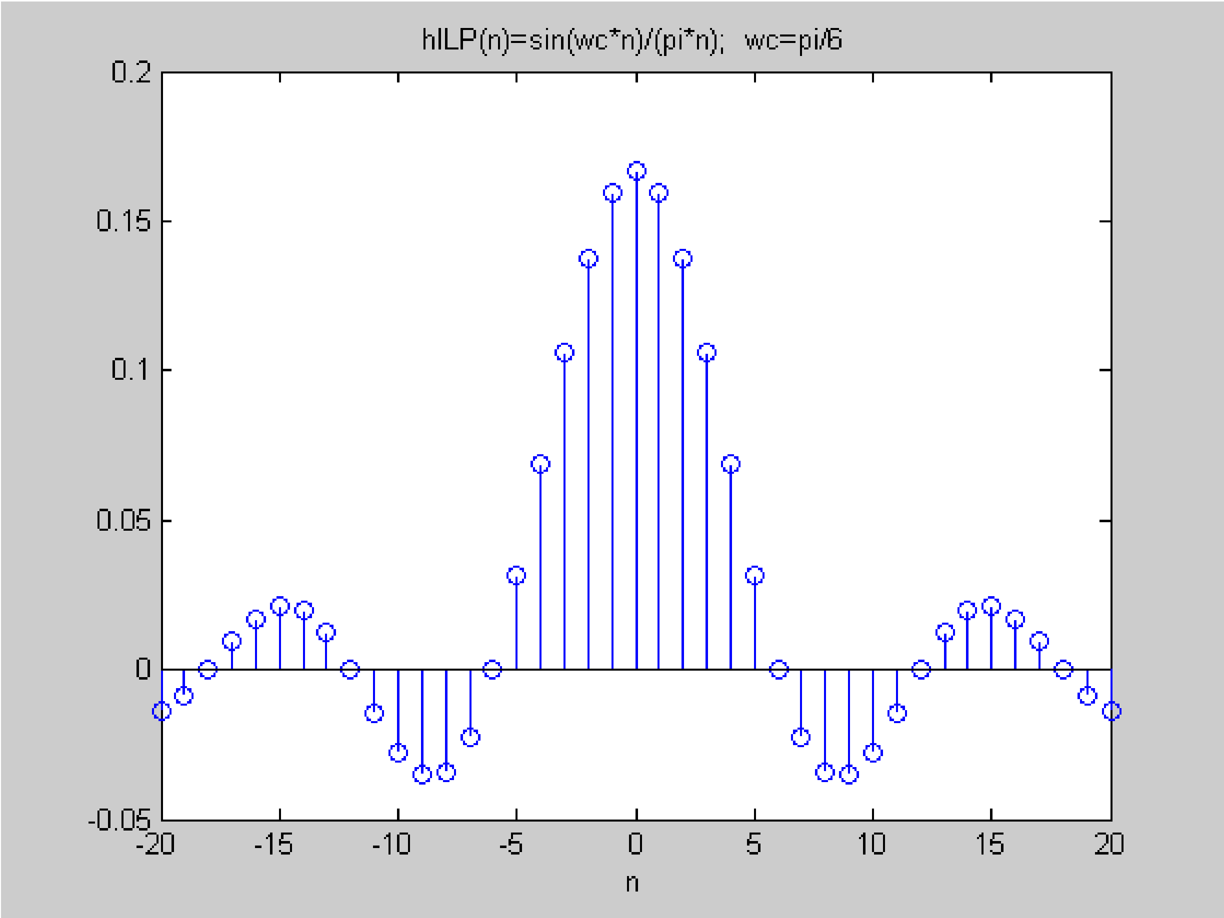
```
close all
n=-20:1:20;
wc=pi/2;
x=sin(wc*(n+.0001));
h=x./(pi*(n+.0001));
stem(n,h)
title('h(n)=sin(wc*n)/(pi*n);
wc=pi/2')
xlabel('n')
```



Impulse Response of Ideal Low-pass Filters



Impulse Response of Ideal Low-pass Filters



Impulse Response of Ideal Low-pass Filters

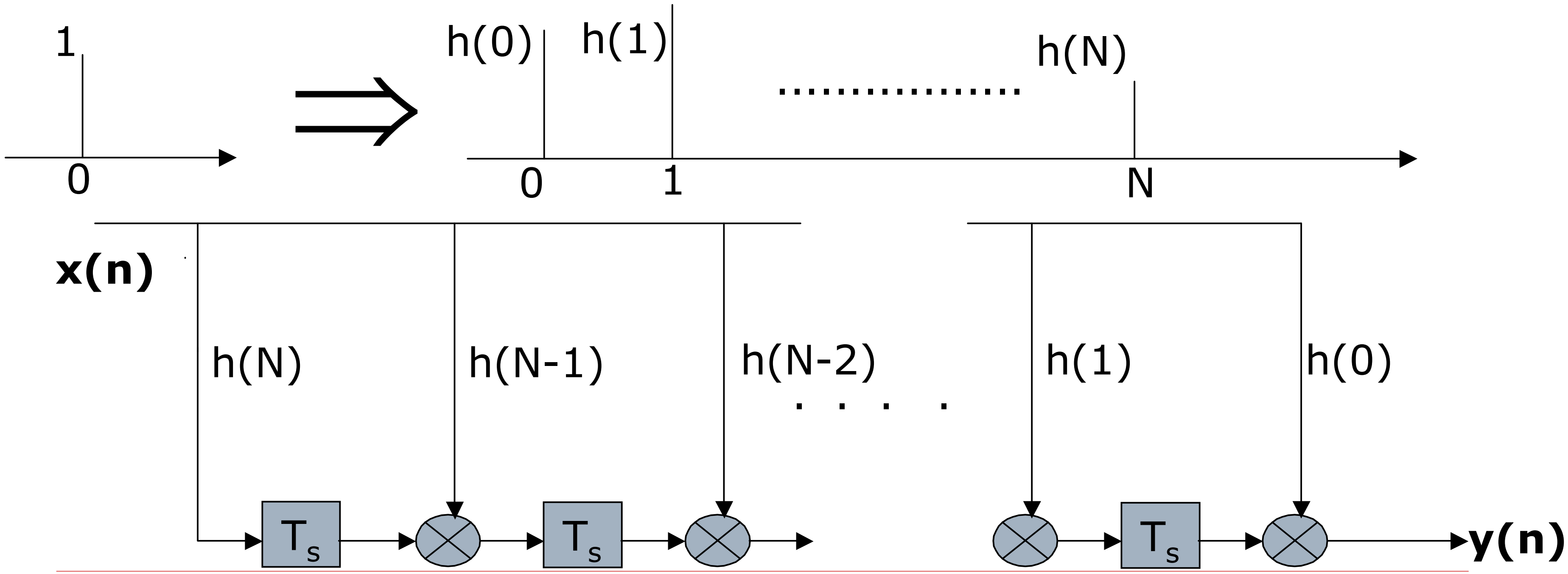
Amplitude values of $h_{ILP}(n) = \dfrac{\sin\omega_c n}{\pi n}$ **for** $\omega_c = \pi/2$**,** $\pi/4$**, and** $\pi/6$

ω_c \ n	0	±1	±2	±3	±4	±5	±6	±7	±8	±9	±10
$\pi/2$	0.5	0.32	0	-0.11	0	0.06	0	-0.05	0	0.04	0
$\pi/4$	0.25	0.23	0.16	0.08	0	-0.05	-0.05	-0.03	0	0.03	0.03
$\pi/6$	1/6	0.16	0.14	0.11	0.07	0.03	0	-0.02	-0.03	-0.04	-0.03

Design of Low-pass Filters

Let x(n)=δ(n),i.e., a unit sample, then the output of the system Shown below is given as:

$$h(n)=\{h(0),h(1),\ldots\ldots\ldots,h(N-1),h(N)\}$$



Design of Low-pass Filters

The two features of such a system are:

- **The response to the unit-sample is of finite duration.**
- **The response to the unit-sample starts at time zero,i.e., at n=0.**

Design of Low-pass Filters

However, the impulse response of the ideal low-pass filter,

$$h_{ILP}(n) = \frac{\sin \omega_c n}{\pi n}$$

is of infinite duration and extends symmetrically to infinity on both sides of n=0.

Design of Low-pass Filters

This implies that the impulse response of the ideal low-pass filter is not realisable with such a system.

The question is: How can we make such an impulse response realisable?

Design of Low-pass Filters

Obviously by making it:

- **to have finite duration.**
- **to start at time zero.**

This can be acomplished by:

- **first truncating it on both ends to make it finite duration.**
- **secondly shifting to the right to make it start at time zero .**

Design of Low-pass Filters

The Fourier transform of

$$h_{ILP}(n) = \frac{\sin\omega_c n}{\pi n}$$

is given as:

$$H_{ILP}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_{ILP}(n) e^{-j\omega n}$$

Design of Low-pass Filters

This can also be interpreted as the Fourier series expansion of $H_{ILP}(e^{j\omega})$, which is the periodic square waveform with duty cycle $2\omega_c$. Since $H_{ILP}(e^{j\omega})$ is symmetrical around n=0, we can write:

$$H_{ILP}(e^{j\omega}) = h_{ILP}(0) + 2\sum_{n=0}^{\infty} h_{ILP}(n)\cos\omega n$$

Now consider the truncation of $h_{ILP}(n)$ at both ends.

Design of Low-pass Filters

Truncated impulse response is given by:

$$h_t(n) = h_{ILP}(n)[u(n+N) - u(n-N)]$$

The Fourier transform of $h_t(n)$ is obtained as:

$$H_t(e^{j\omega}) = h_{ILP}(0) + 2\sum_{n=1}^{N} h_{ILP}(n)\cos\omega n$$

Design of Low-pass Filters

We have

$$H_t(e^{j\omega}) = \frac{\omega_c}{\pi} + \frac{2}{\pi}\sum_{n=1}^{N}\frac{\sin\omega_c n}{n}\cos\omega n$$

Assuming $\omega_c = \dfrac{\pi}{2}$

$$H_t(e^{j\omega}) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega - \frac{1}{3}\cos 3\omega + \ldots + \frac{1}{N}\cos N\omega\right)$$

Design of Low-pass Filters

In the following we will deal with

the Frequency Response of
Truncated Impulse Response

or equivalently we can say

Truncated Fourier Series Expansion
of Ideal Filter Frequency Response.

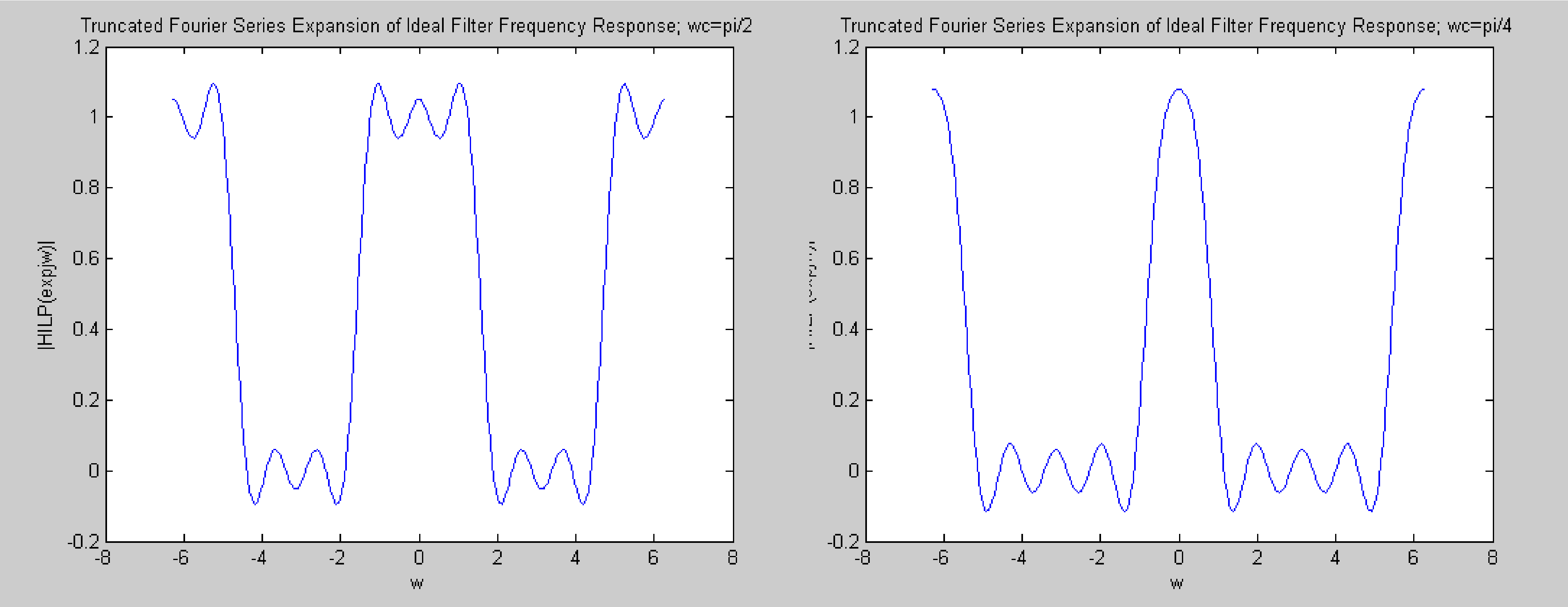
Design of Low-pass Filters

MATLAB Program to Compute the Frequency Response of Truncated Impulse Response from $n = \pm 6$

```
close all
w=-2*pi:pi/64:2*pi;
wc=pi/2;
h=wc/pi;
n=1;
while n<6
x=sin(wc*(n+.0001));
x1=x./(pi*(n+.0001));
x2=2*x1*cos(w*n);
h=h+x2;
n=n+1;
end
figure,plot(w,h);
title('Truncated Fourier Series
Expansion of Ideal Filter
Frequency Response');
xlabel('w');
ylabel('|HILP(expjw)|')
```

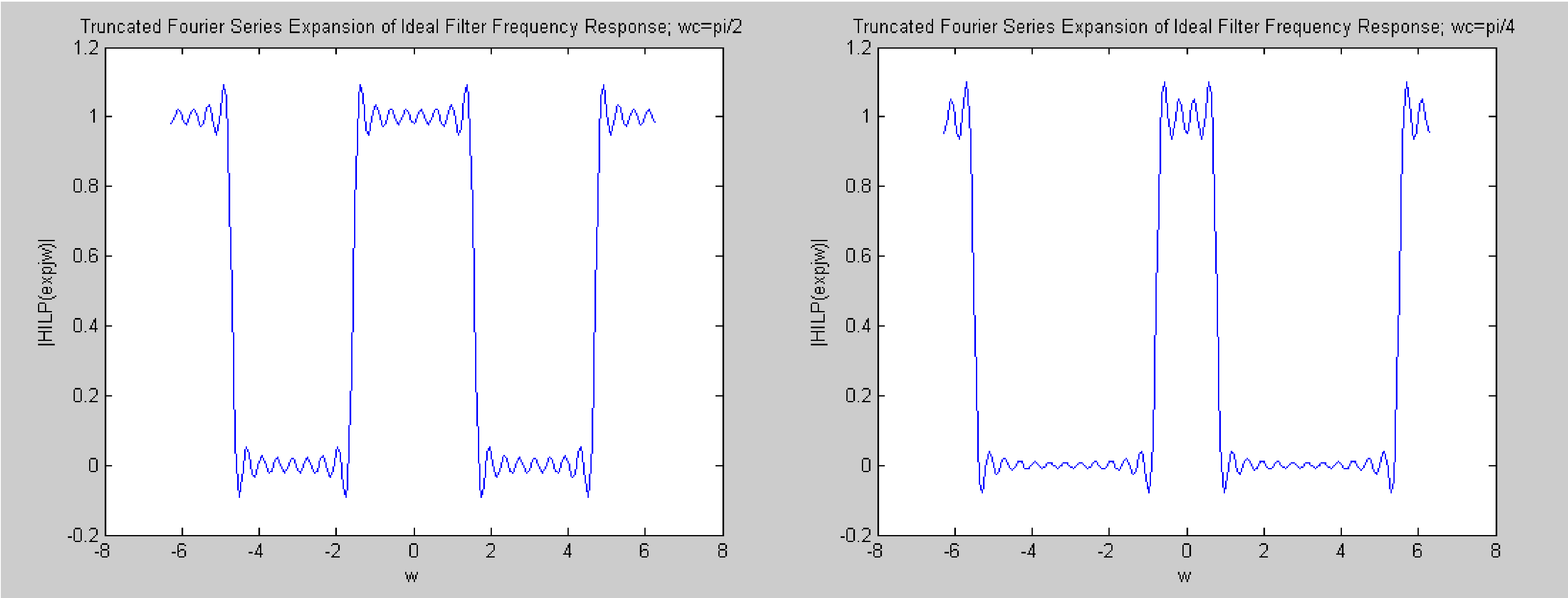
Design of Low-pass Filters

Frequency Response of Truncated Impulse Response from $n = \pm 6$ **plotted for** $-2\pi < \omega < 2\pi$



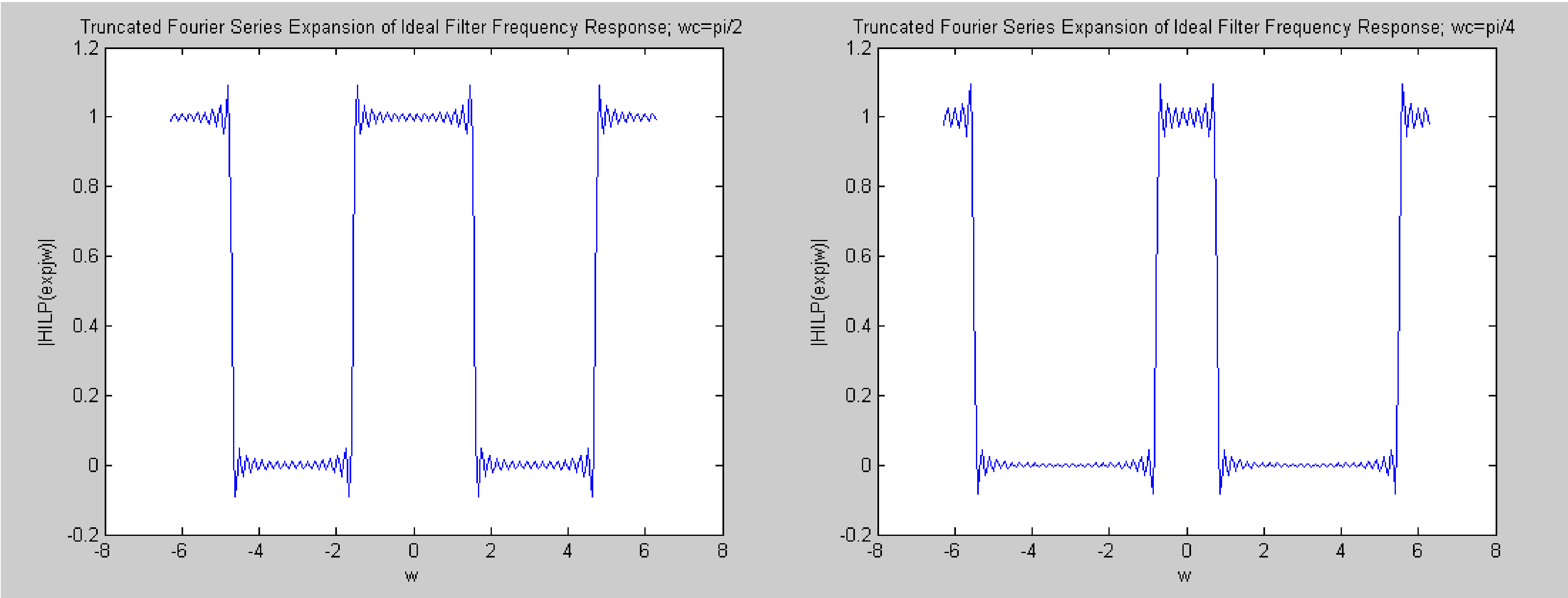
Design of Low-pass Filters

Frequency Response of Truncated Impulse Response from $n = \pm 16$ **plotted for** $-2\pi < \omega < 2\pi$



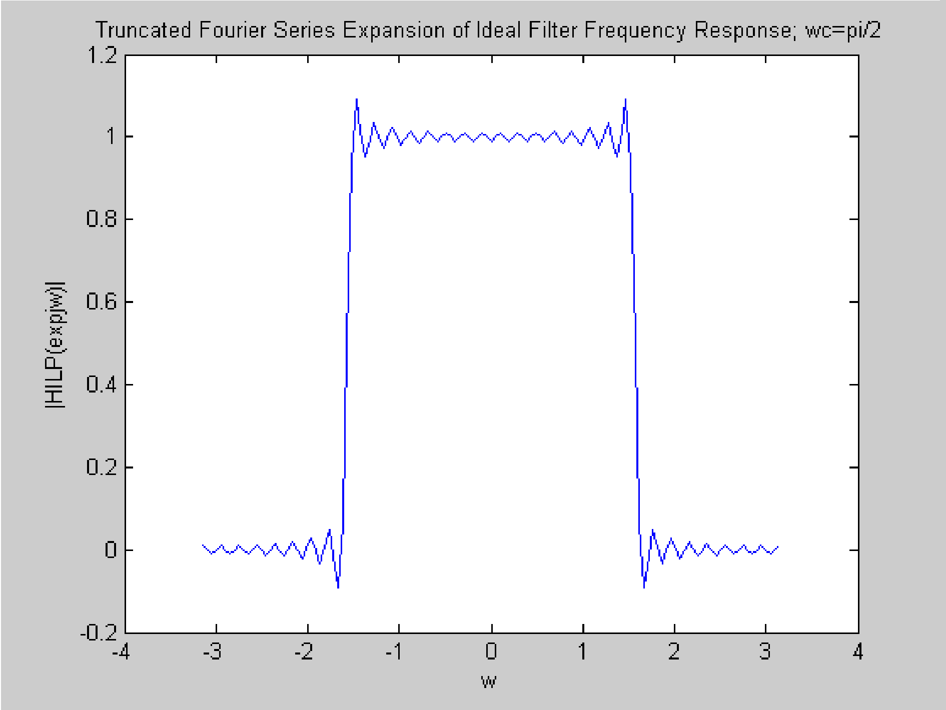
Design of Low-pass Filters

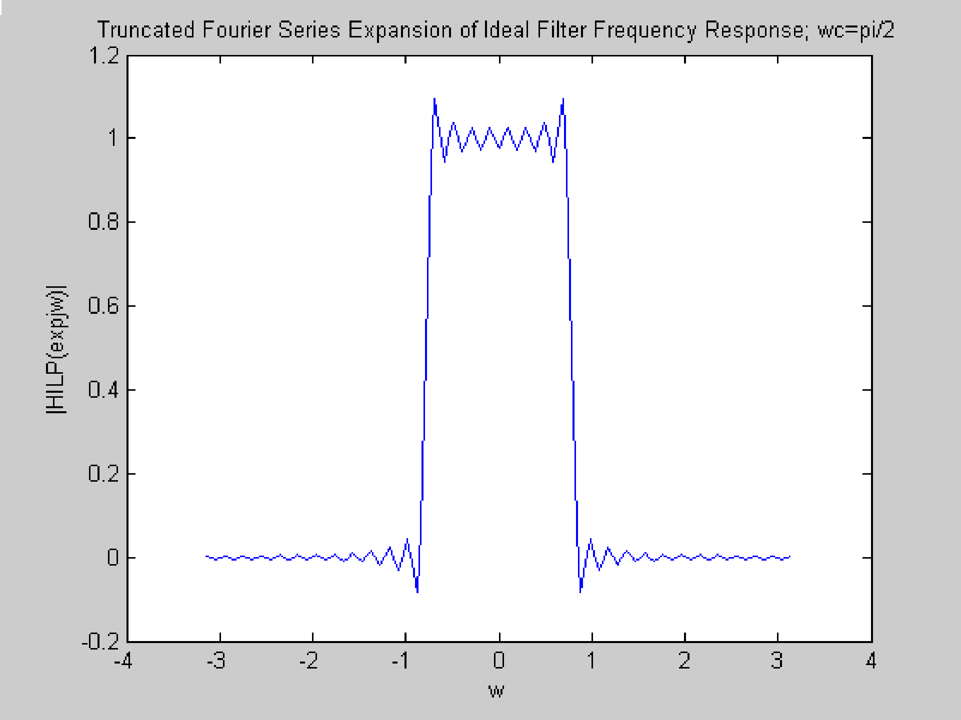
Frequency Response of Truncated Impulse Response from $n = \pm 32$ **plotted for** $-2\pi<\omega<2\pi$



Design of Low-pass Filters

Frequency Response of Truncated Impulse Response from $n = \pm 32$ **plotted for** $-\pi<\omega<\pi$





Design of Low-pass Filters

What remains to be done to obtain a realisable frequency response is to shift $h_t(n)$ to the right by N, i.e., the realisable impulse response is given as:

$$h(n) = h_t(n - N)$$

Since:

$$F\{h_t(n - N)\} = e^{-jN\omega} H_t(e^{j\omega})$$

we obtain

$$F\{h(n)\} = e^{-jN\omega} H_t(e^{j\omega})$$

Design of Low-pass Filters

$$H(e^{j\omega}) = (h_{ILP}(0) + 2\sum_{n=0}^{N} h_{ILP}(n)\cos\omega n)e^{-jN\omega}$$

$$H(e^{j\omega}) = (\frac{\omega_c}{\pi} + \frac{2}{\pi}\sum_{n=1}^{N} \frac{\sin\omega_c n}{n}\cos\omega n)e^{-jN\omega}$$

Given ω_c, the design of the truncated low-pass filter is straightforward.

Design of HP, BP and BS Filters

The low-pass design is also used in the design of HP,BP and BS filters. For such filters the frequency specifications are first translated to the frequency specifications of the corresponding low-pass filter.Then the impulse response of the required filter is obtained through relationships given above:

Design of HP, BP and BS Filters

High-pass Filters

The relation between the frequency responses of high- and low-pass filters was given as:

$$H_{HP}(e^{j\omega}) = H_{LP}(e^{j(\omega-\pi)})$$

We know that:

$$H_{LP}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_{LP}(n) e^{-j\omega n}$$

Using the above relationship we can write

Design of HP, BP and BS Filters

$$H_{HP}(e^{j\omega}) = H_{LP}(e^{j(\omega-\pi)}) = \sum_{n=-\infty}^{\infty} h_{LP}(n) e^{-j(\omega-\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} h_{LP}(n) e^{-j\omega n} e^{j\pi n} = \sum_{n=-\infty}^{\infty} (-1)^n h_{LP}(n) e^{-j\omega n}$$

where we used the identity

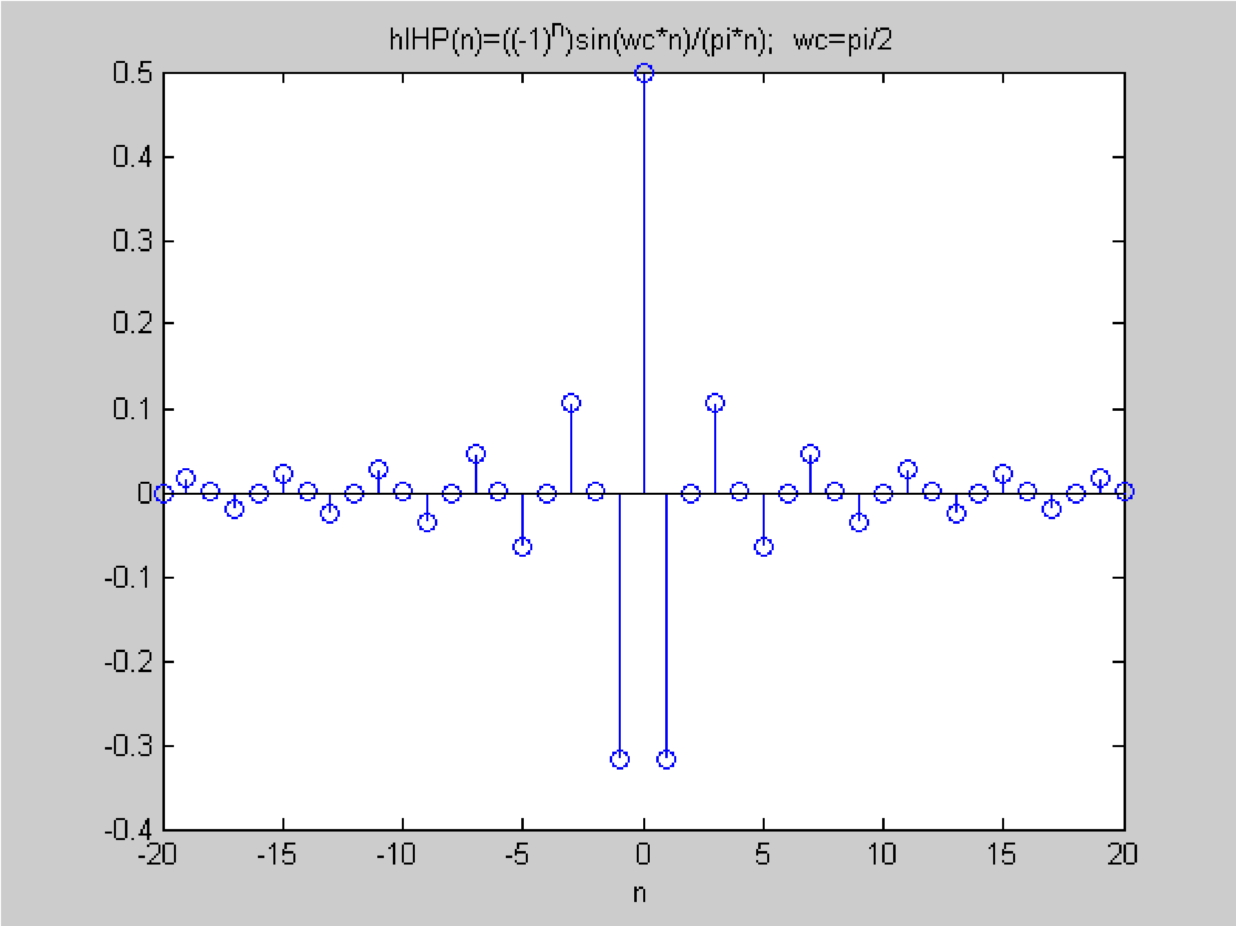
$$e^{j\pi n} = (-1)^n$$

Design of HP, BP and BS Filters

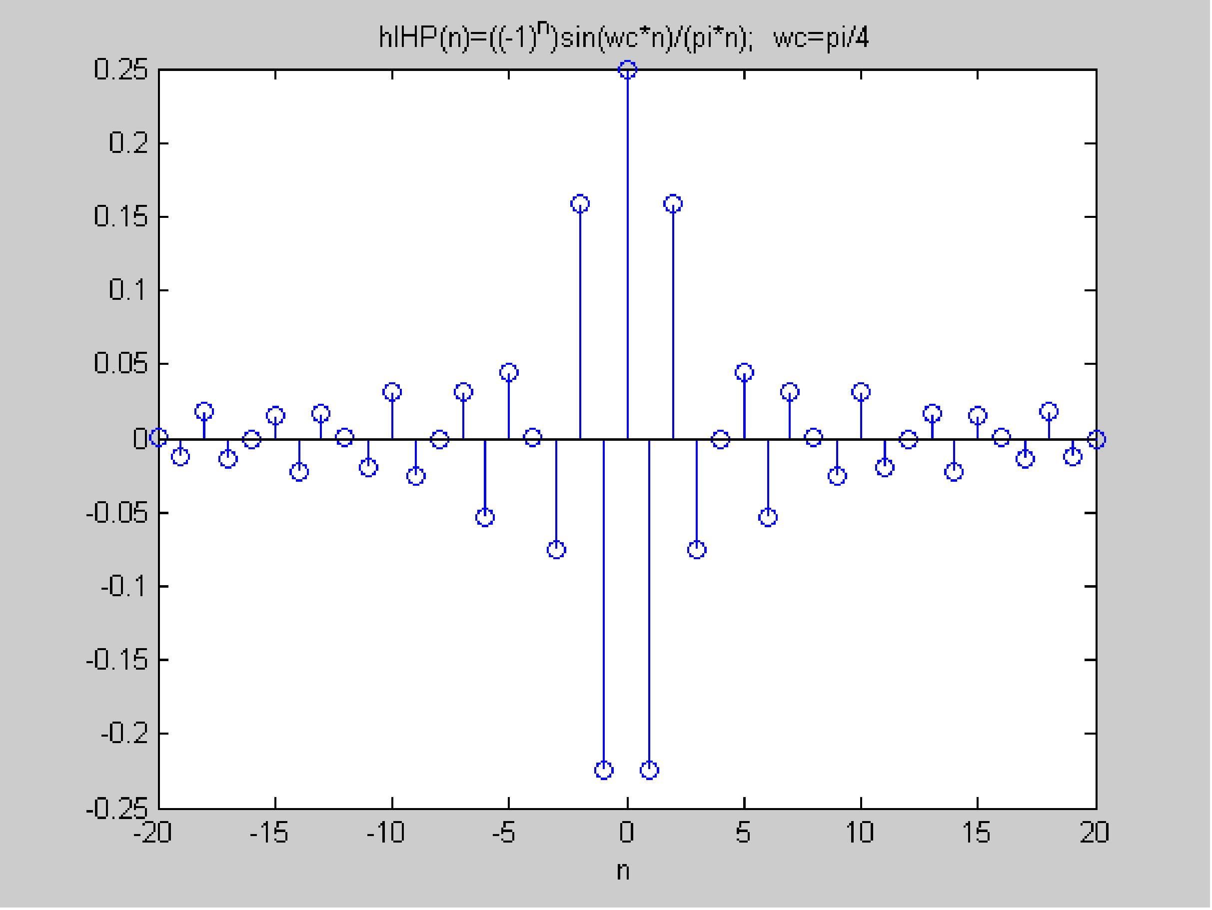
We can conclude that:

$$\boxed{h_{HP}(n) = (-1)^n h_{LP}(n)}$$

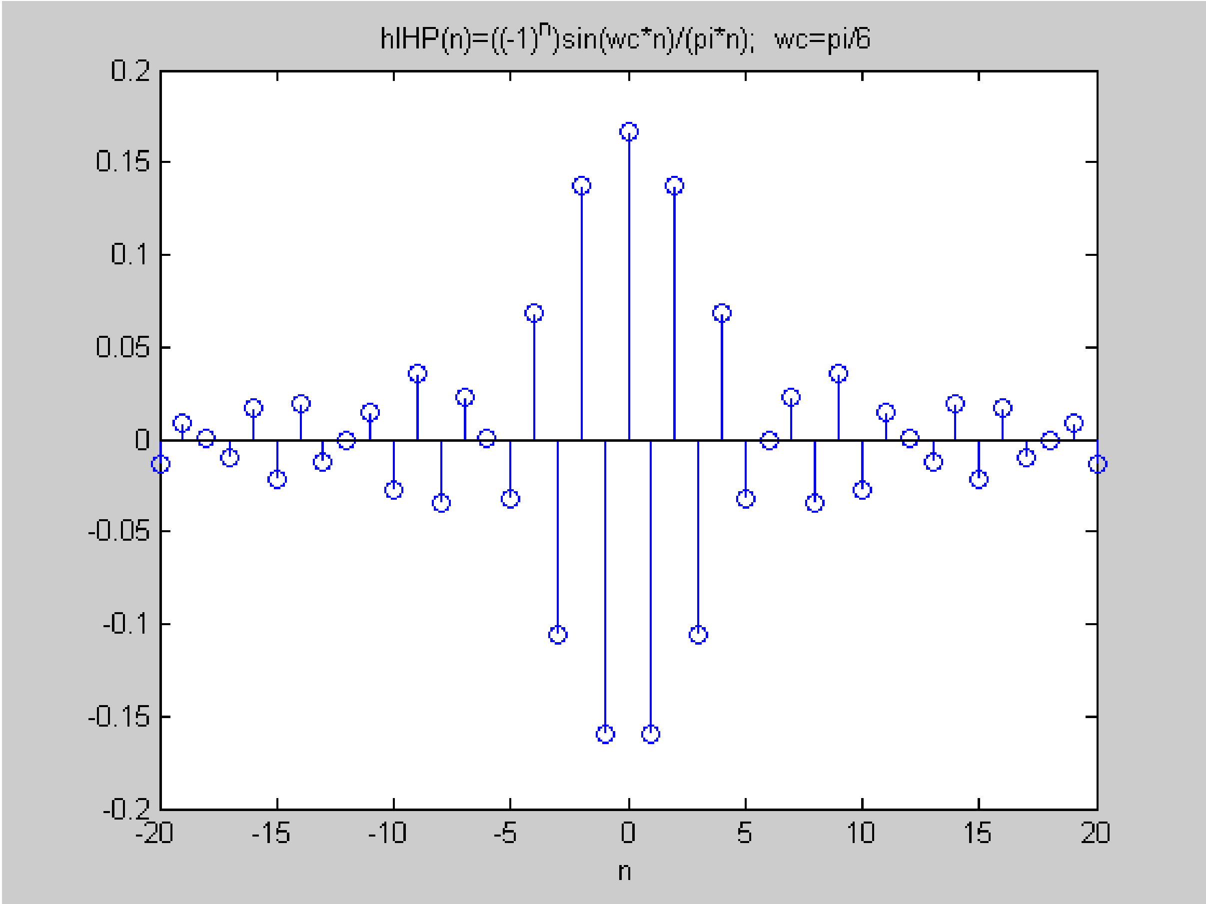
Design of HP, BP and BS Filters



Design of HP, BP and BS Filters



Design of HP, BP and BS Filters



Design of HP, BP and BS Filters

Amplitude values of

$$h_{IHP}(n) = (-1)^n \frac{\sin \omega_c n}{\pi n} \quad for\ \omega_c = \pi/2,\ \pi/4,\ \pi/6$$

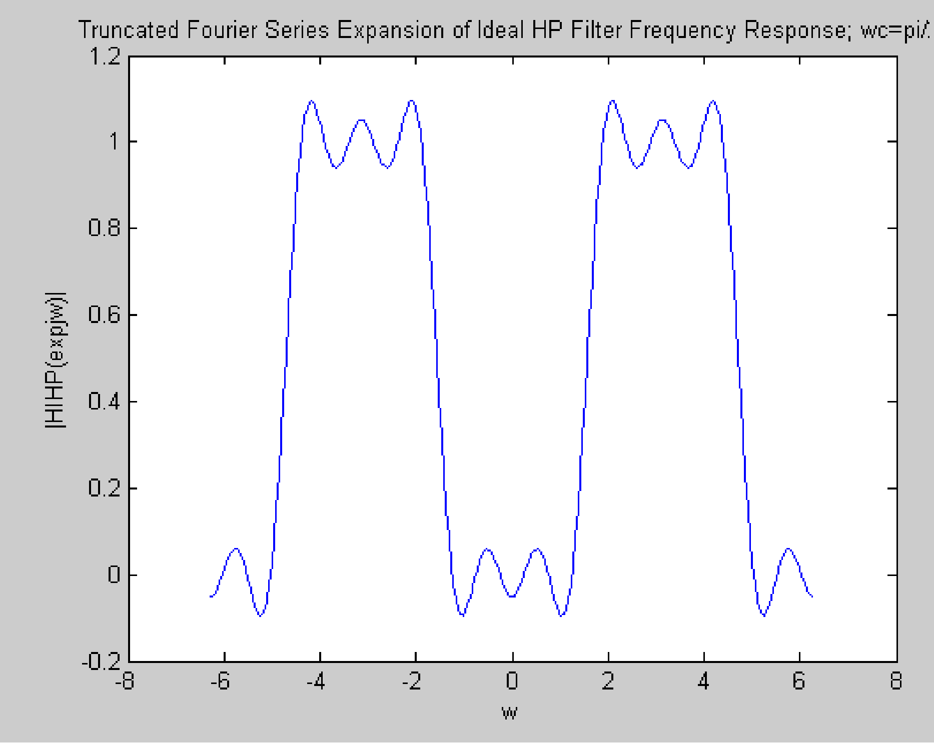
ω_c \ n	0	± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8	± 9	± 10
$\pi/2$	0.5	-0.32	0	-0.11	0	-0.06	0	0.05	0	-0.04	0
$\pi/4$	0.25	-0.23	0.16	-0.08	0	0.05	-0.05	0.03	0	-0.03	0.03
$\pi/6$	1/6	-0.16	0.14	-0.11	0.07	-0.03	0	0.02	-0.03	0.04	-0.03

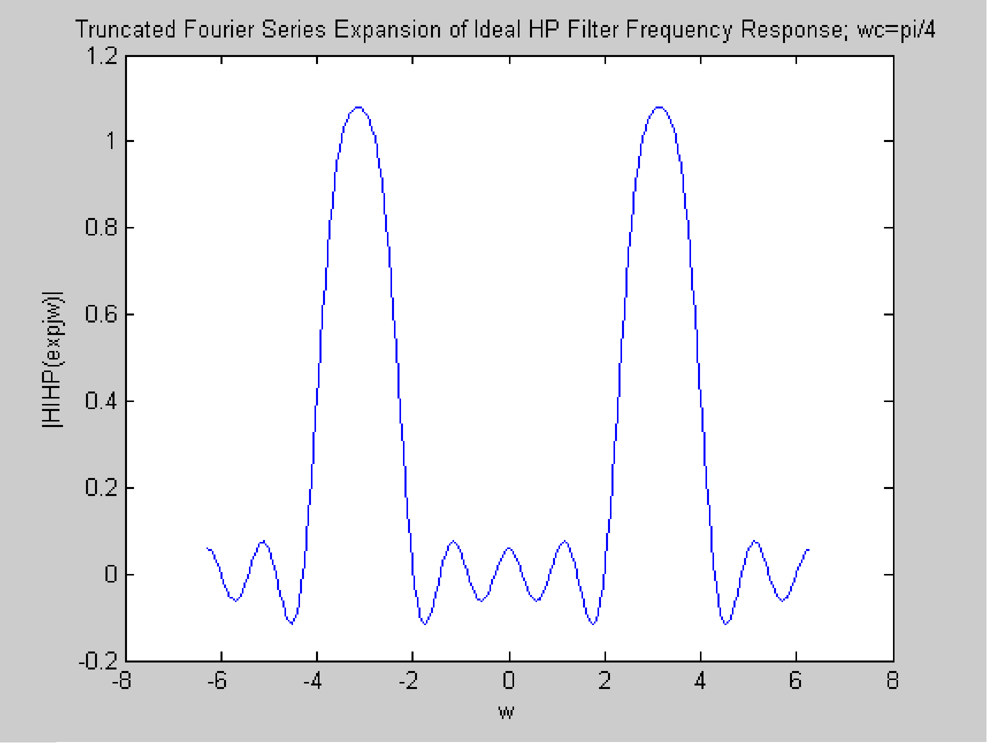
Design of HP, BP and BS Filters

```
close all
w=-2*pi:pi/64:2*pi;
wc=pi/2;
h=wc/pi;
n=1;
while n<6
x=((-1).^n)*sin(wc*(n+.0001));
x1=x./(pi*(n+.0001));
x2=2*x1*cos(w*n);
h=h+x2;
n=n+1;
end
figure,plot(w,h);
title('Truncated Fourier Series Expansion of Ideal HP Filter
     Frequency Response; wc=pi/2');
xlabel('w');
ylabel('|HIHP(expjw)|')
```

Design of HP, BP and BS Filters

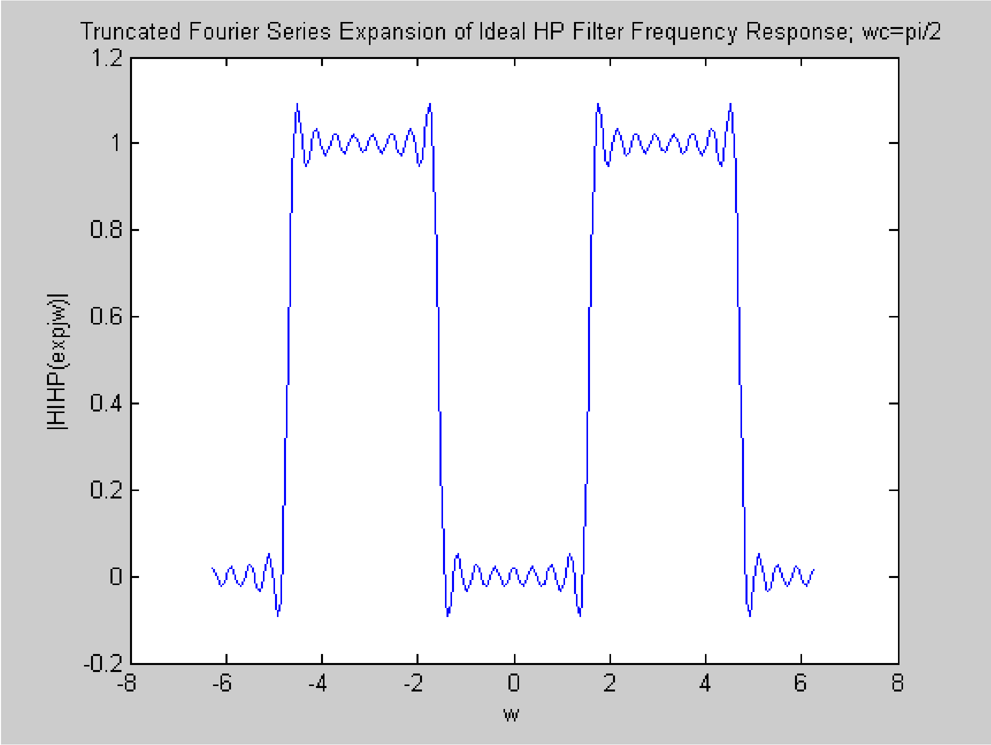
Frequency Response of Truncated Impulse Response from $n = \pm 6$ **plotted for** $-2\pi<\omega<2\pi$

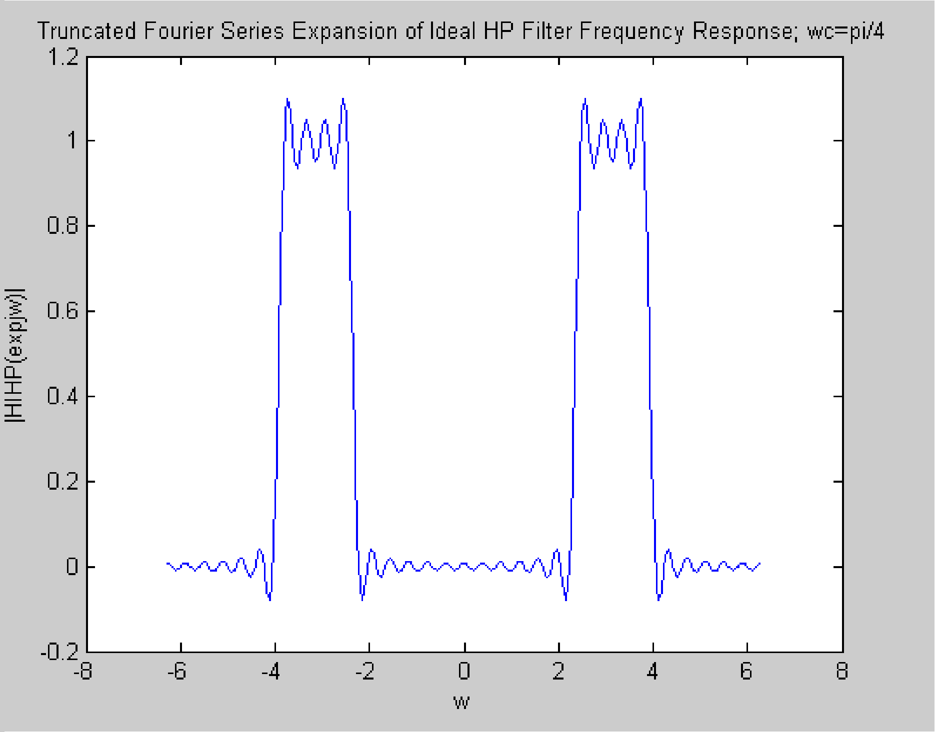




Design of HP, BP and BS Filters

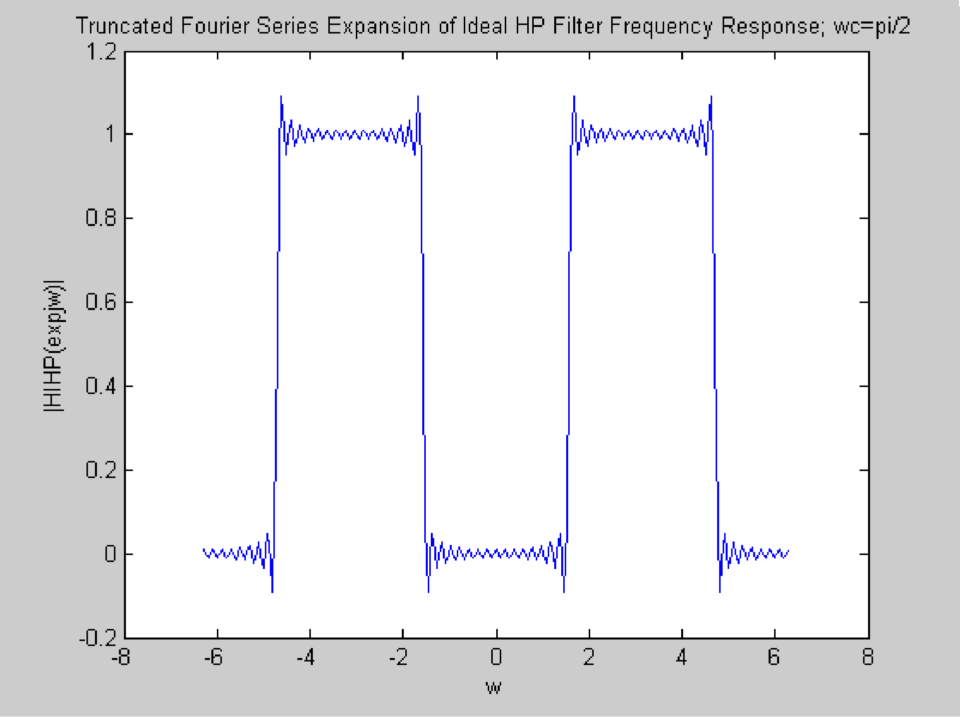
Frequency Response of Truncated Impulse Response from $n = \pm 16$ **plotted for** $-2\pi < \omega < 2\pi$

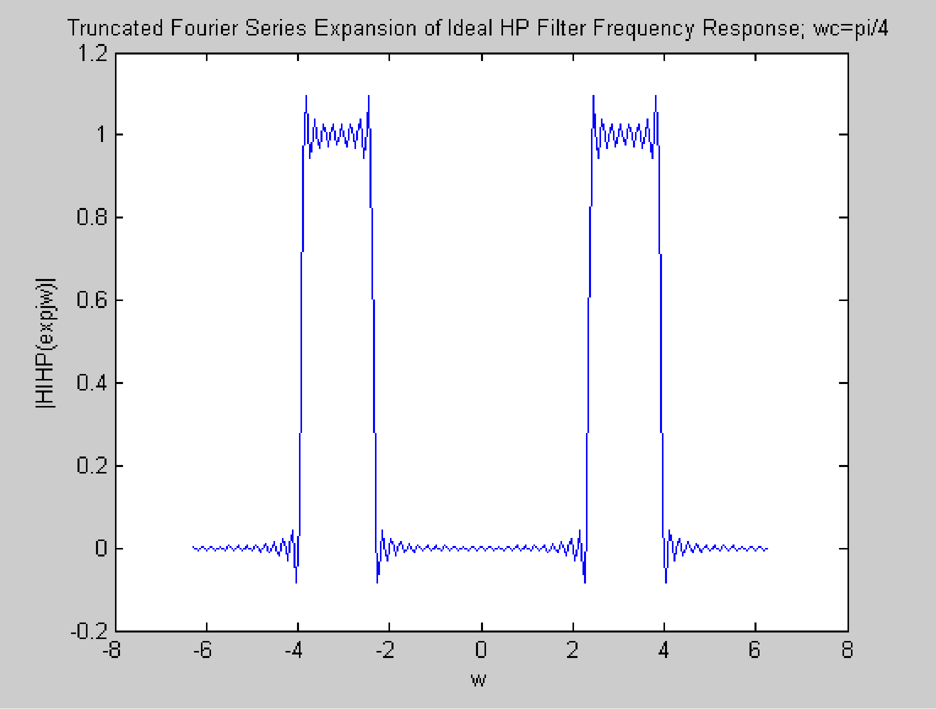




Design of HP, BP and BS Filters

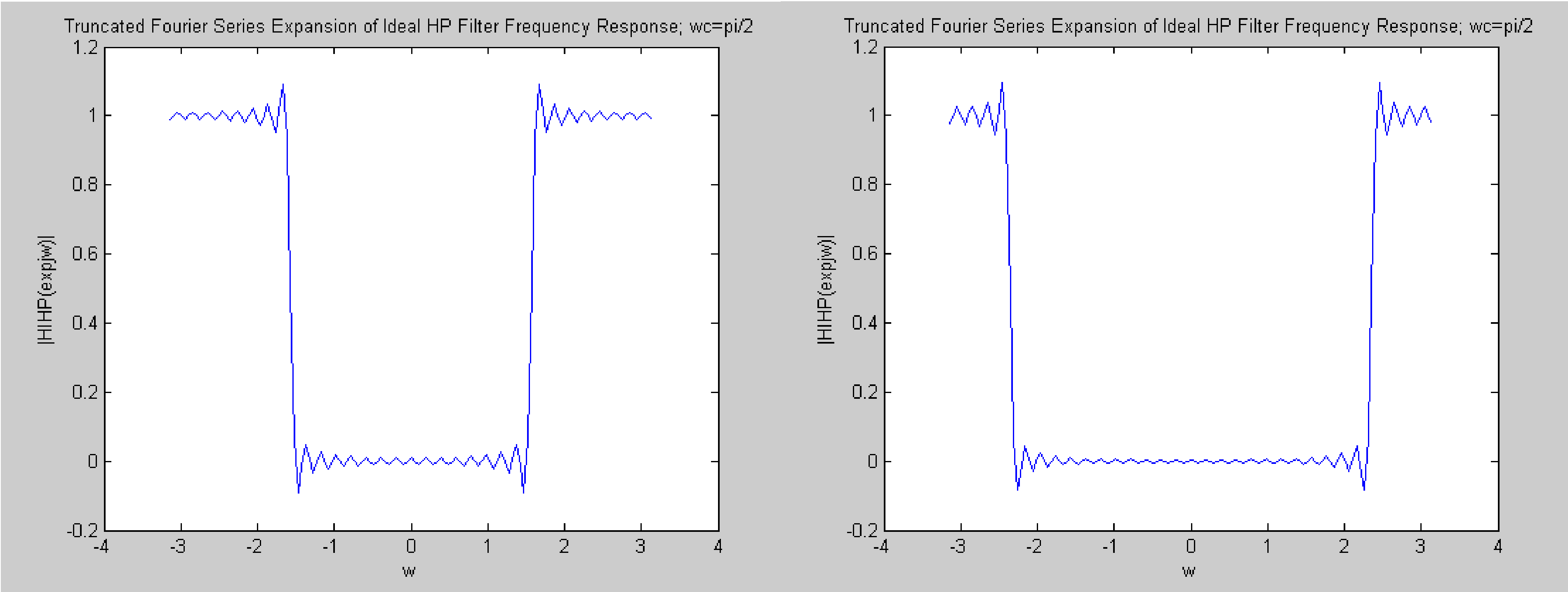
Frequency Response of Truncated Impulse Response from $n = \pm 32$ **plotted for** $-2\pi<\omega<2\pi$





Design of HP, BP and BS Filters

Frequency Response of Truncated Impulse Response from $n = \pm 32$ plotted for $-\pi<\omega<\pi$



Design of HP, BP and BS Filters

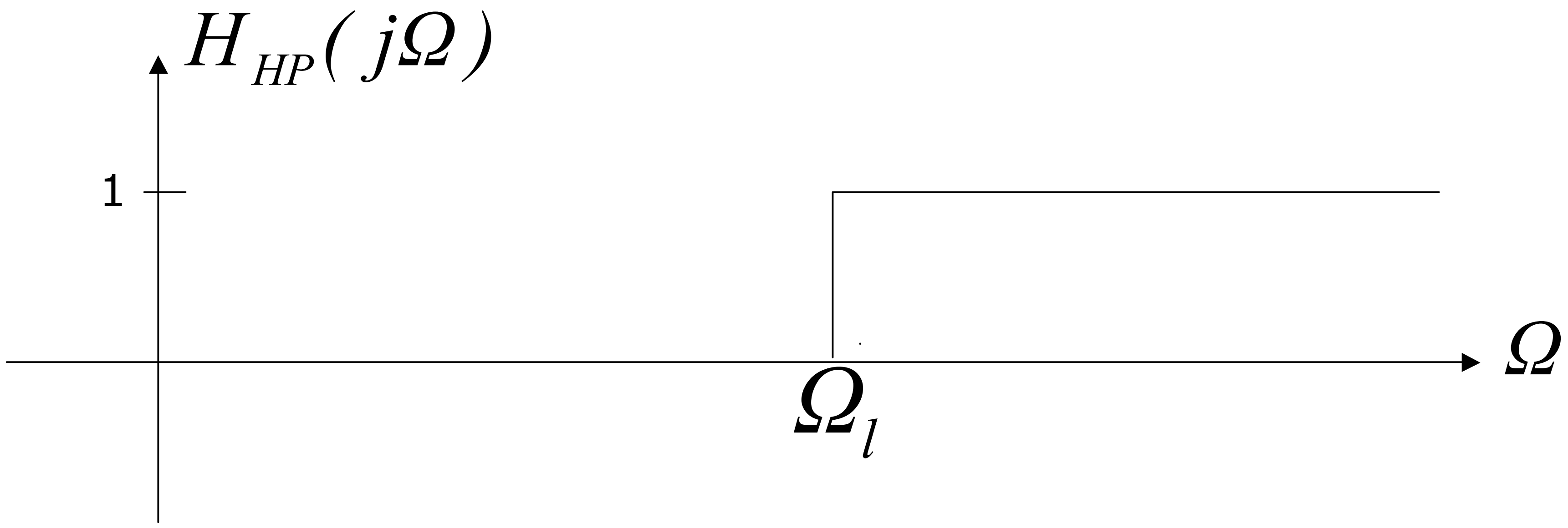
Q.1 An analogue high-pass filter approximating the frequency characteristic given in the figure below is to be designed using the system given on the next page which is composed of an A/D converter, a 5-coefficient causal Finite Impulse Response (FIR) digital filter and a D/A converter, where A/D and D/A converters are assumed to be perfect.

The lower cutoff frequency of the analogue high-pass filter and the maximum frequency in the analogue input signal are given as: F_l=60 kHz and F_{max}=80kHz.

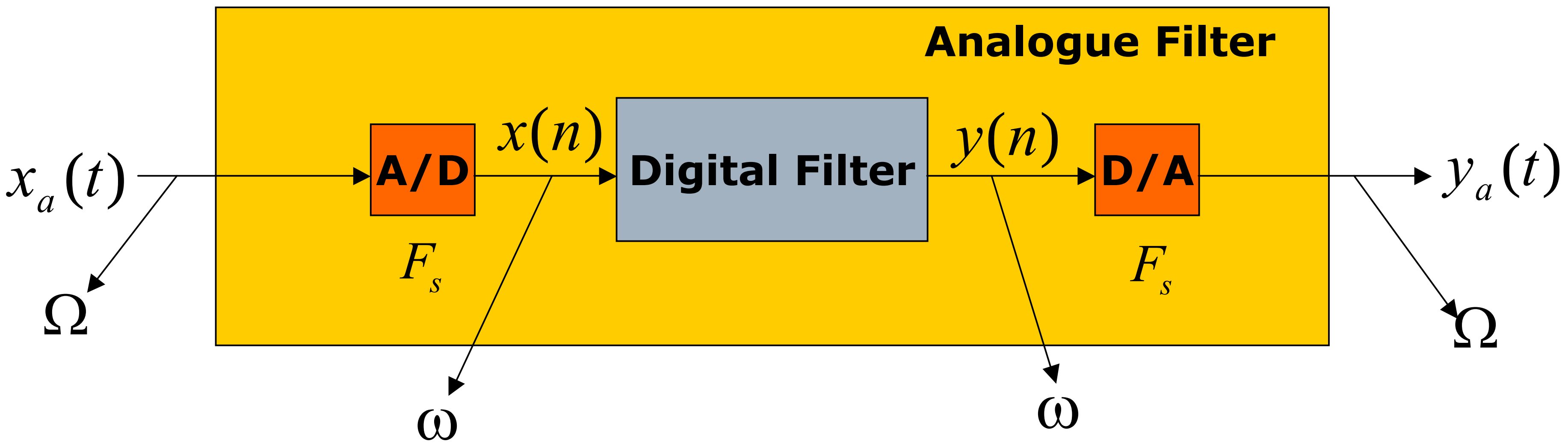
Design of HP, BP and BS Filters

(a)Find and plot the impulse response hILP(n) of the corresponding ideal low-pass filter.
(b)Find and plot the impulse response $h_{LP}(n)$ of the 5-coefficient causal filter.
(c)Give the expression for $H_{LP}(e^{j\omega})$ and $|H_{LP}(e^{j\omega})|$ and sketch $|H_{LP}(e^{j\omega})|$ roughly.
(d)Give $h_{hP}(n)$ and $H_{HP}(e^{j\omega})$. Plot $|H_{LP}(e^{j\omega})|$ roughly.

Design of HP, BP and BS Filters



Design of HP, BP and BS Filters



Design of HP, BP and BS Filters

Solution:

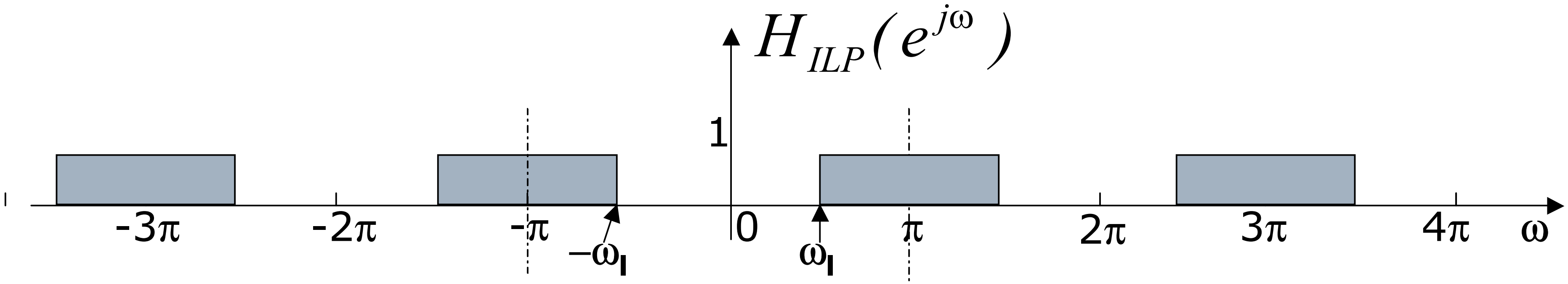
(a)Minimum sampling frequency is found as:

$$F_s = 2F_{max} = 80kHz$$

Using this frequency we find:

$$f_l = \frac{F_l}{F_s} = \frac{20}{80} = \frac{1}{4} \qquad \omega_l = \frac{\pi}{2}$$

Design of HP, BP and BS Filters



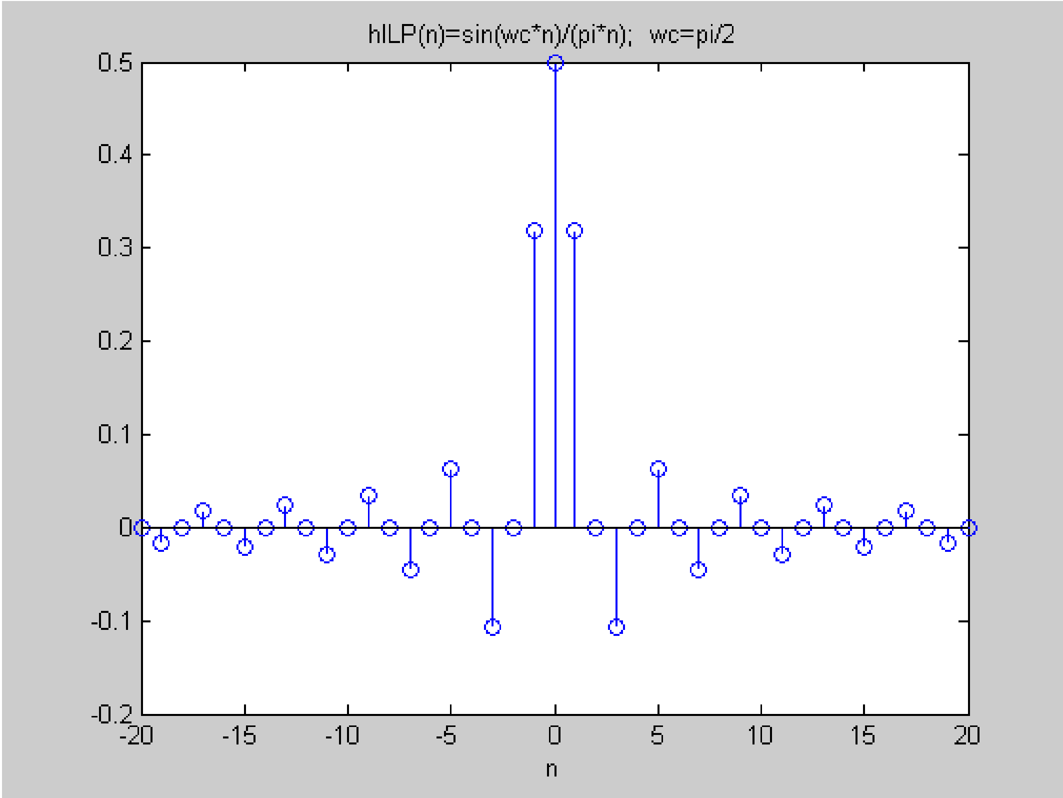
The cutoff frequency of the corresponding prototype lowpass filter can be found as:

$$\omega_c = \pi - \omega_l = \frac{\pi}{2}$$

The impulse response of the prototype lowpass filter:

Design of HP, BP and BS Filters

$$h_{ILP}(n) = \frac{\sin \omega_c n}{\pi n} = \frac{\sin \frac{\pi}{2} n}{\pi n}$$

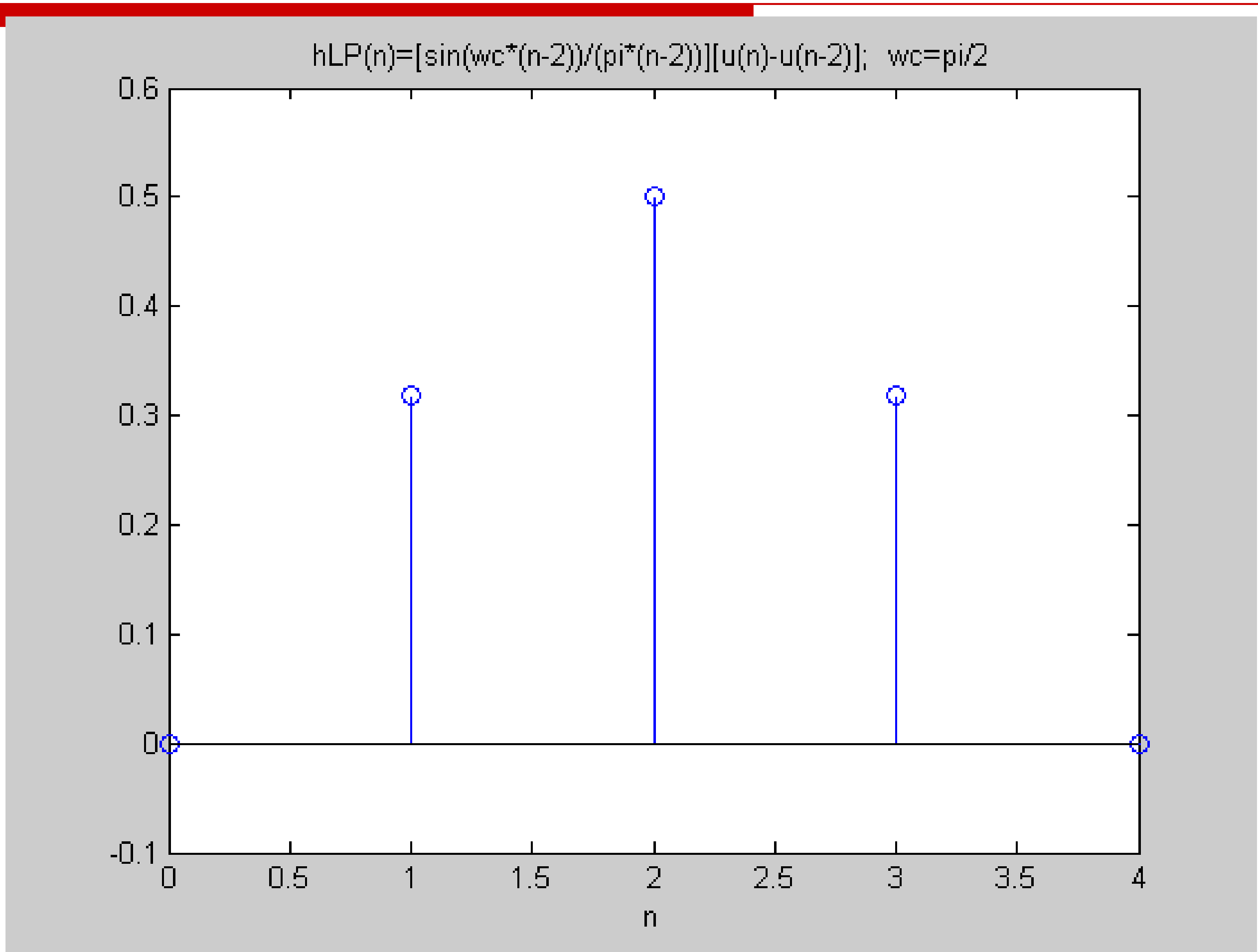


Design of HP, BP and BS Filters

(b)The impulse response of the 5-coefficient (truncated) causal filter is given as:

$$h_{LP}(n) = \frac{\sin\dfrac{\pi}{2}(n-2)}{\pi(n-2)}[u(n)-u(n-2)]$$

Design of HP, BP and BS Filters



Design of HP, BP and BS Filters

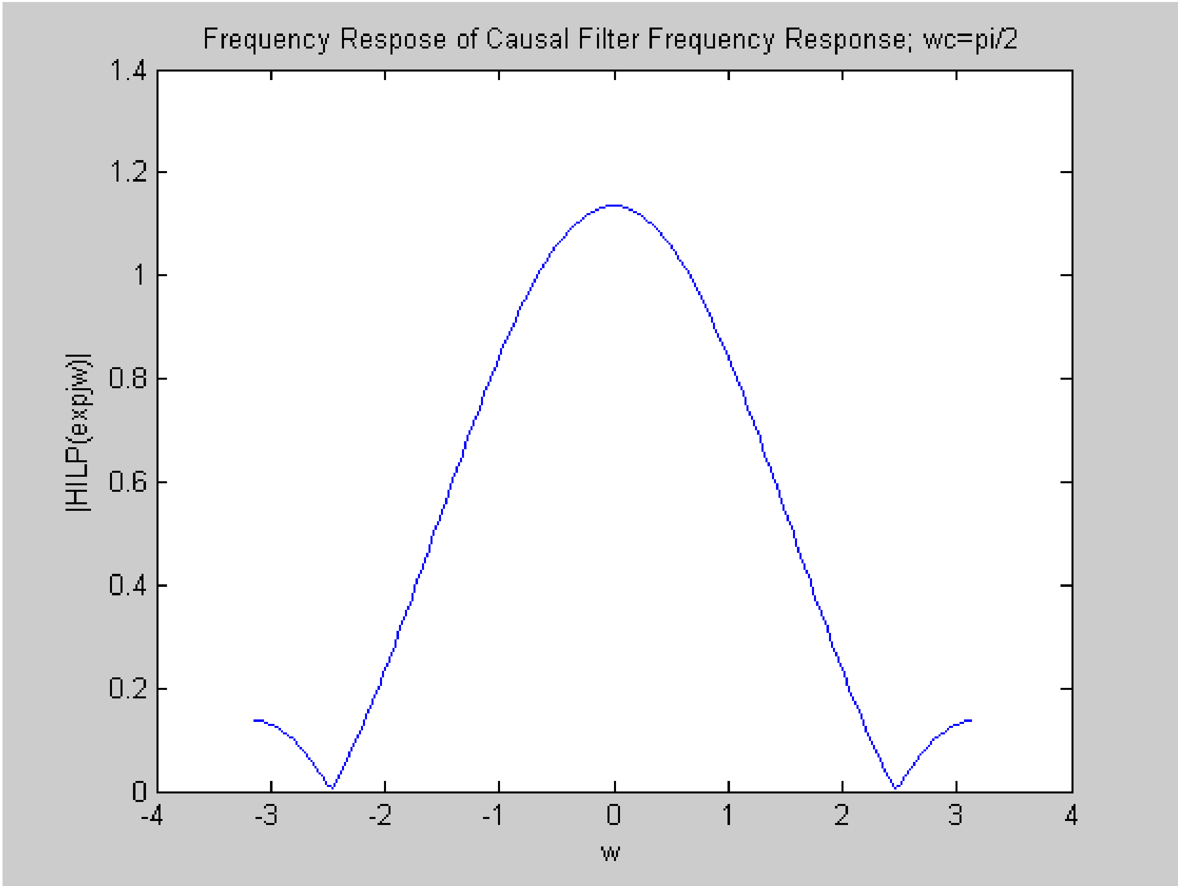
(c)

$$H_{LP}(e^{j\omega}) = [h_{LP}(0) + 2h_{LP}(1)\cos\omega]e^{-j2\omega}$$

$$= [0.5 + 0.64\cos\omega]e^{-j2\omega}$$

$$|H_{LP}(e^{j\omega})| = |0.5 + 0.64\cos\omega|$$

Design of HP, BP and BS Filters



Design of HP, BP and BS Filters

(d)

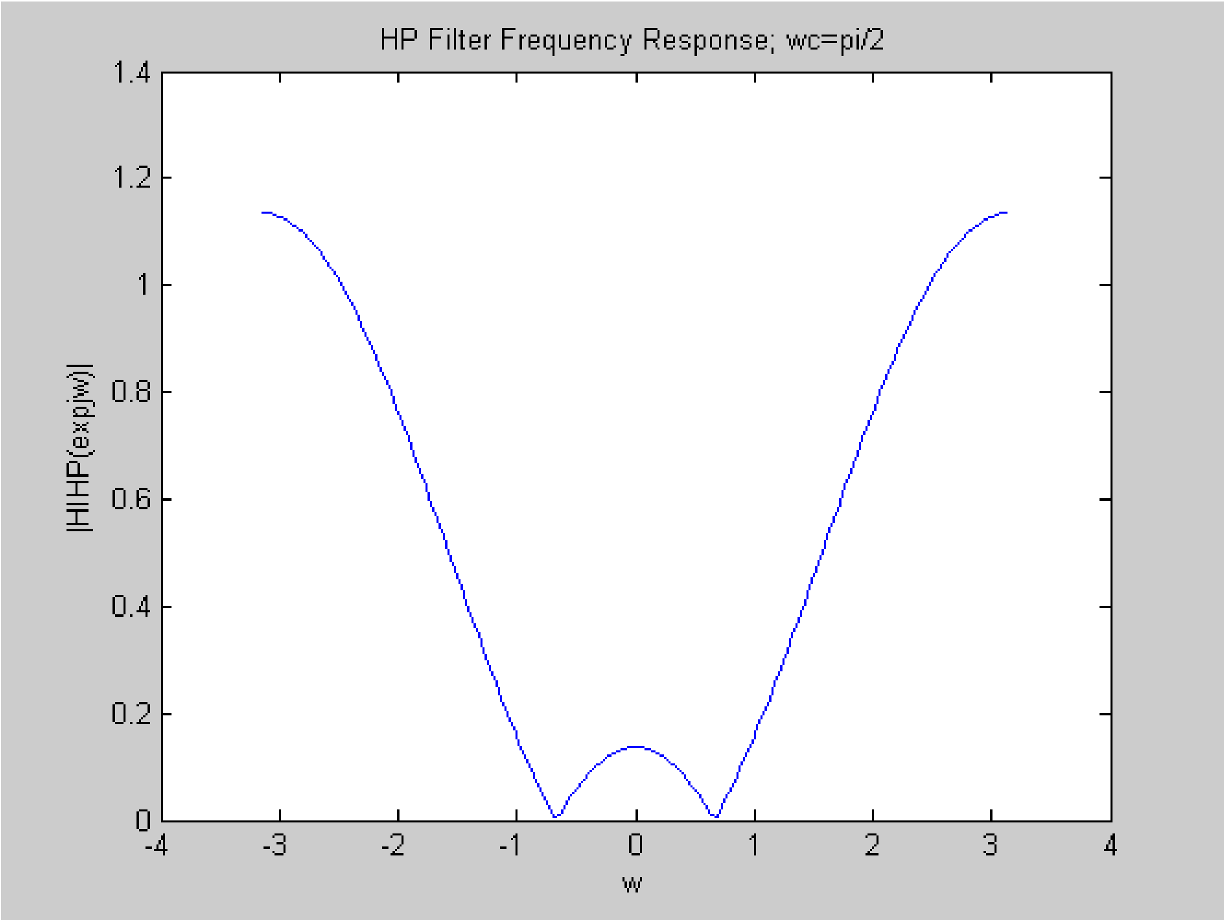
$$h_{HP}(n)=(-1)^{(n-2)}\frac{\sin\frac{\pi}{2}(n-2)}{\pi(n-2)}[u(n)-u(n-2)]$$

$$H_{HP}(e^{j\omega})=[h_{LP}(0)-2h_{LP}(1)\cos\omega]e^{-j2\omega}$$

$$=[0.5-0.64\cos\omega]e^{-j2\omega}$$

$$|H_{HP}(e^{j\omega})|=|0.5-0.64\cos\omega|$$

Design of HP, BP and BS Filters



Design of HP, BP and BS Filters

Band-pass Filters

The relation between the frequency responses of ideal band- and low-pass filters was given as:

$$H_{IBP}(e^{j\omega}) = H_{ILP}(e^{j(\omega-\omega_0)}) + H_{ILP}(e^{j(\omega+\omega_0)})$$

Using the above relationship we can write

$$H_{IBP}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_{ILP}(n)e^{-j(\omega-\omega_0)n} + \sum_{n=-\infty}^{\infty} h_{ILP}(n)e^{-j(\omega+\omega_0)n}$$

Design of HP, BP and BS Filters

$$H_{IBP}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_{ILP}(n) e^{-j(\omega-\omega_0)n} + \sum_{n=-\infty}^{\infty} h_{ILP}(n) e^{-j(\omega+\omega_0)n}$$

$$= \sum_{n=-\infty}^{\infty} h_{ILP}(n) e^{-j\omega n} e^{j\omega_0 n} + \sum_{n=-\infty}^{\infty} h_{ILP}(n) e^{-j\omega n} e^{-j\omega_0}$$

$$= \sum_{n=-\infty}^{\infty} h_{ILP}(n) e^{-j\omega n} (e^{j\omega_0 n} + e^{-j\omega_0})$$

$$= \sum_{n=-\infty}^{\infty} (2h_{ILP}(n) cos\,\omega_0 n)\, e^{-j\omega n}$$

where we used the identity

Design of HP, BP and BS Filters

We can make the following identification:

$$\boxed{h_{IBP} = 2h_{ILP}(n)\cos\omega_0 n}$$

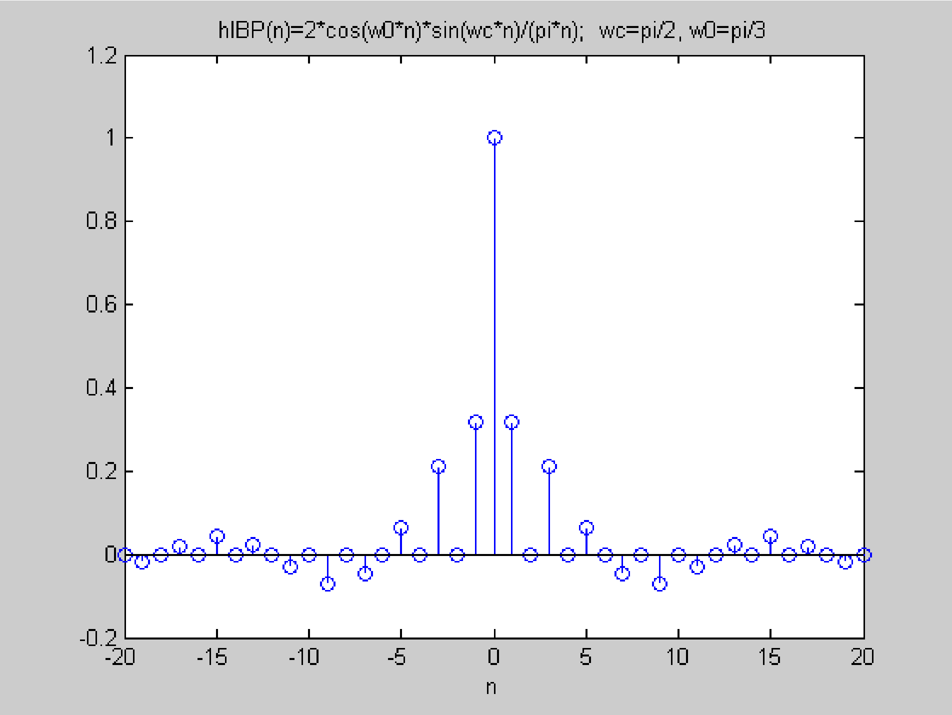
Design of HP, BP and BS Filters

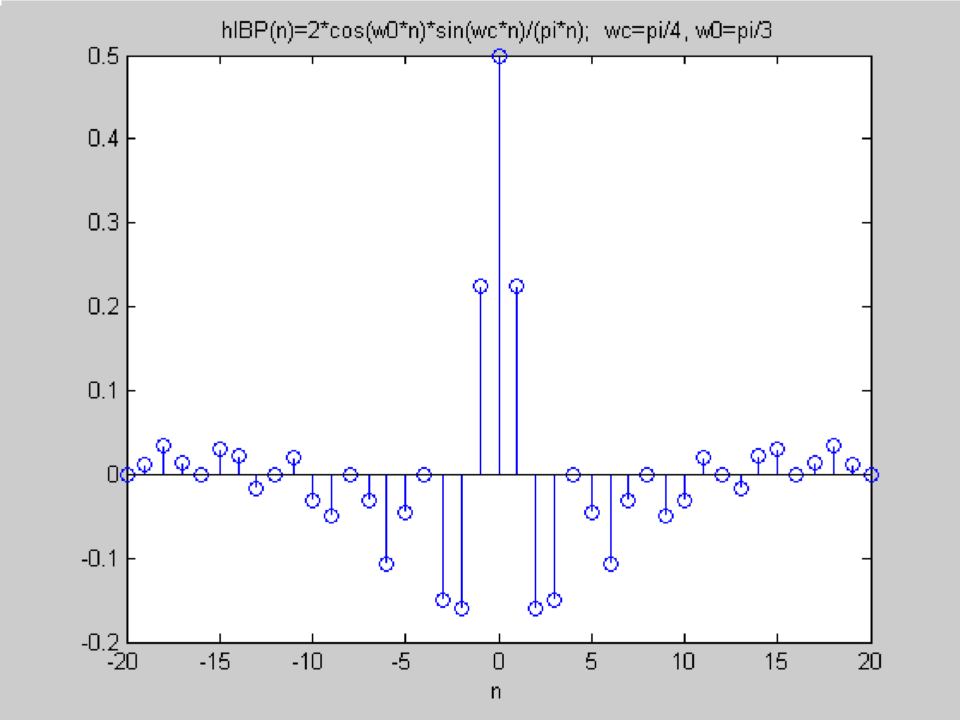
MATLAB Program Computing the Impulse Response of BP Filter with w_0=pi/3

```
close all
n=-20:1:20
   wc=pi/4
   w0=pi/3
x=sin(wc*(n+.0001));
h=x./(pi*(n+.0001));
h=2*h.*cos(w0*n)
stem(n,h)
title('hIBP(n)=2*cos(w0*n)*sin(wc*n)/(pi*n);
    wc=pi/4, w0=pi/3')
xlabel('n')
```

Design of HP, BP and BS Filters

Impulse Responses of BP Filters with w0=pi/3



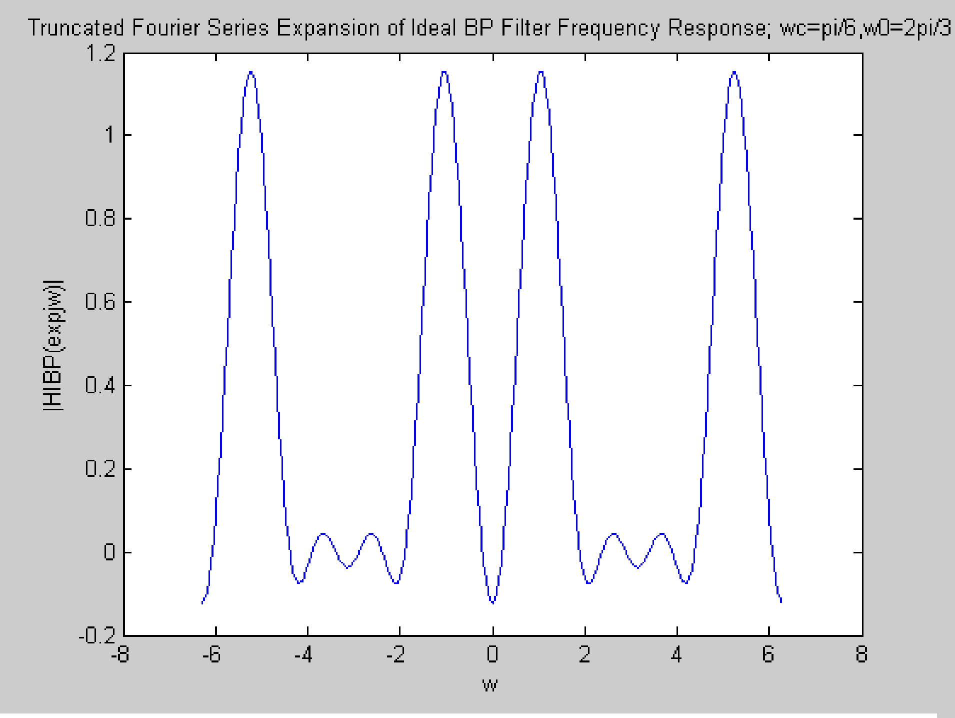


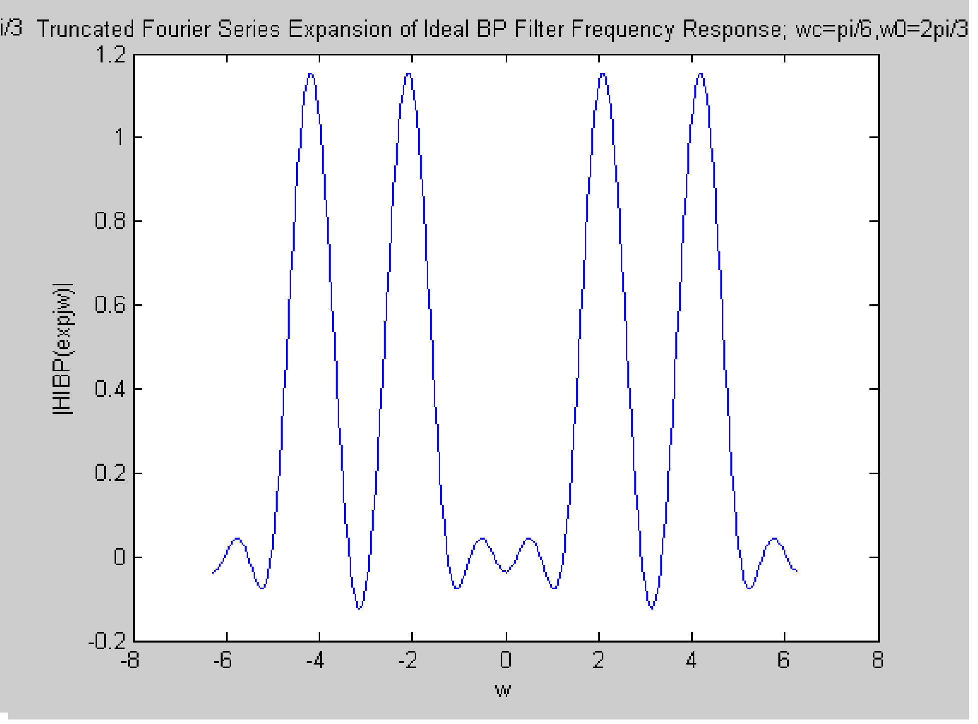
Design of HP, BP and BS Filters

```
close all
w=-2*pi:pi/64:2*pi;
wc=pi/6;
w0=2*pi/3;
h=2*wc/pi;
n=1;
while n<33
x=sin(wc*(n+.0001));
x1=x./(pi*(n+.0001));
x1=2*x1.*cos(w0*n)
x2=2*x1.*cos(w*n);
h=h+x2;
n=n+1;
end
figure,plot(w,h);
title('Truncated Fourier Series Expansion of Ideal BP Filter Frequency
      Response; wc=pi/6,w0=2pi/3');
xlabel('w');
ylabel('|HIBP(expjw)|')
```

Design of HP, BP and BS Filters

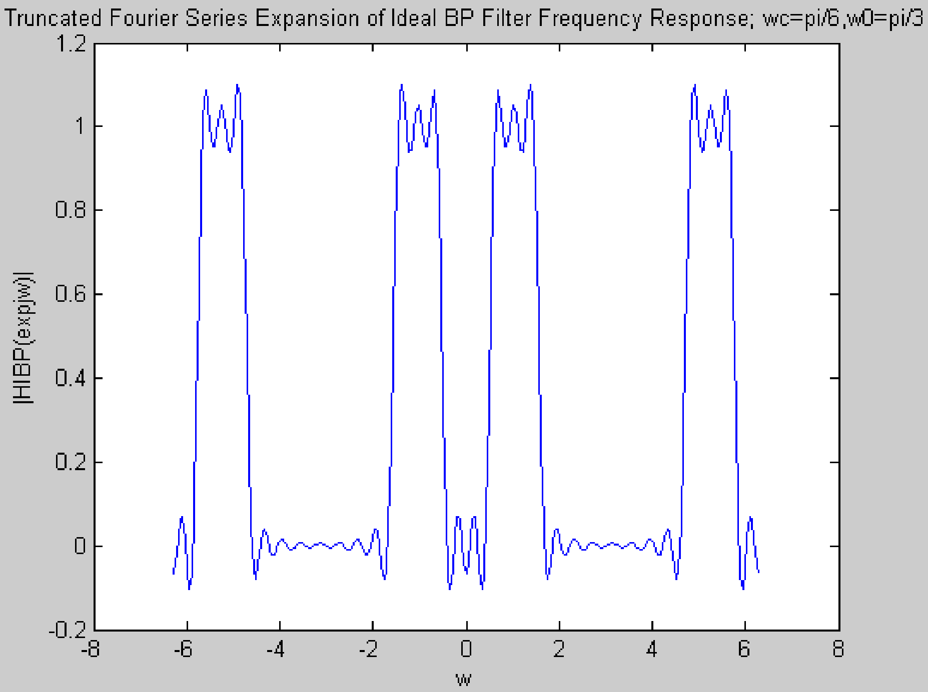
Frequency Response of Truncated Impulse Response from $n = \pm 6$ **plotted for** $-2\pi<\omega<2\pi$

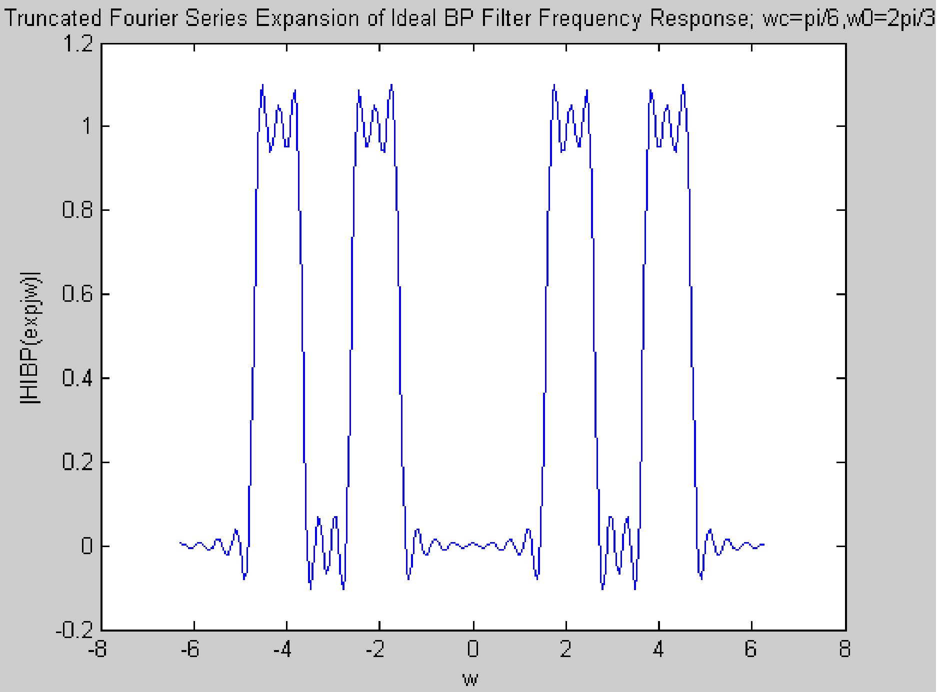




Design of HP, BP and BS Filters

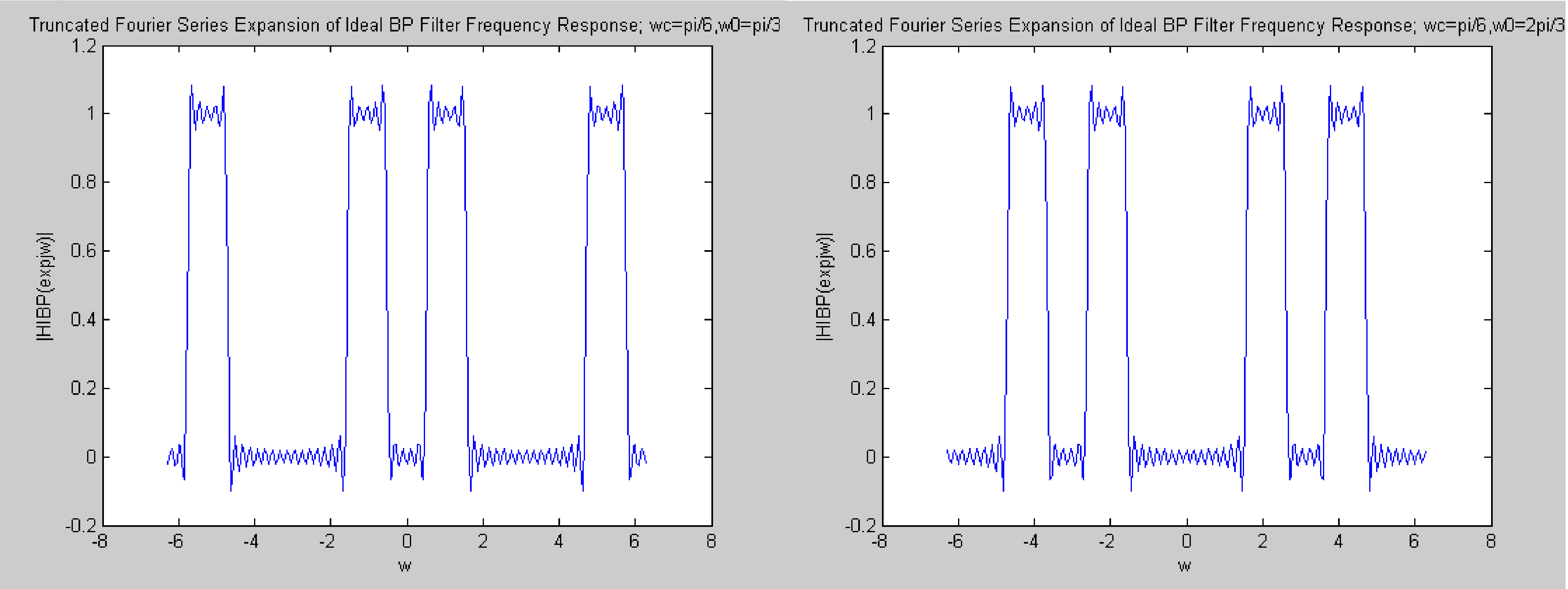
Frequency Response of Truncated Impulse Response from $n = \pm 16$ **plotted for** $-2\pi < \omega < 2\pi$





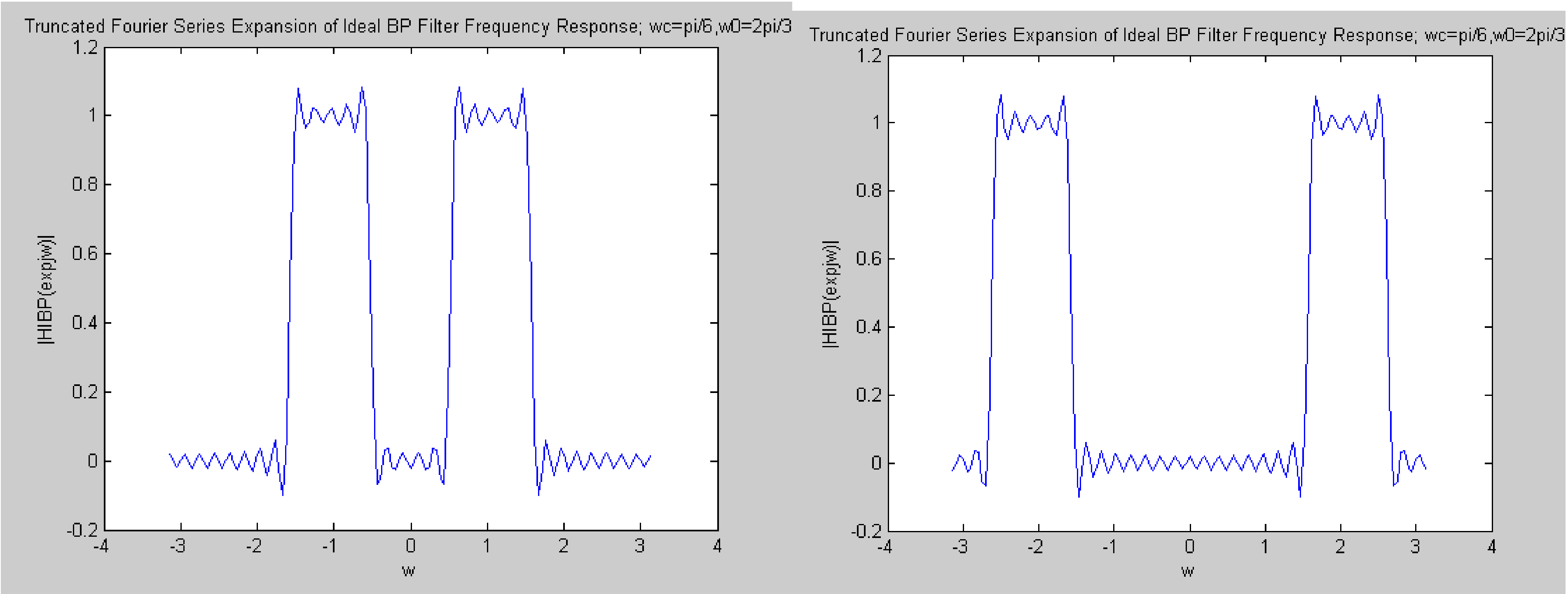
Design of HP, BP and BS Filters

Frequency Response of Truncated Impulse Response from $n = \pm 32$ **plotted for** $-2\pi < \omega < 2\pi$



Design of HP, BP and BS Filters

Frequency Response of Truncated Impulse Response from $n = \pm 32$ **plotted for** $-\pi<\omega<\pi$



Design of HP, BP and BS Filters

Band-stop (reject) Filters

The relation between the frequency responses of ideal band-pass and band-stop filters was given as:

$$H_{IBS}(e^{j\omega}) = 1 - H_{IBP}(e^{j\omega})$$

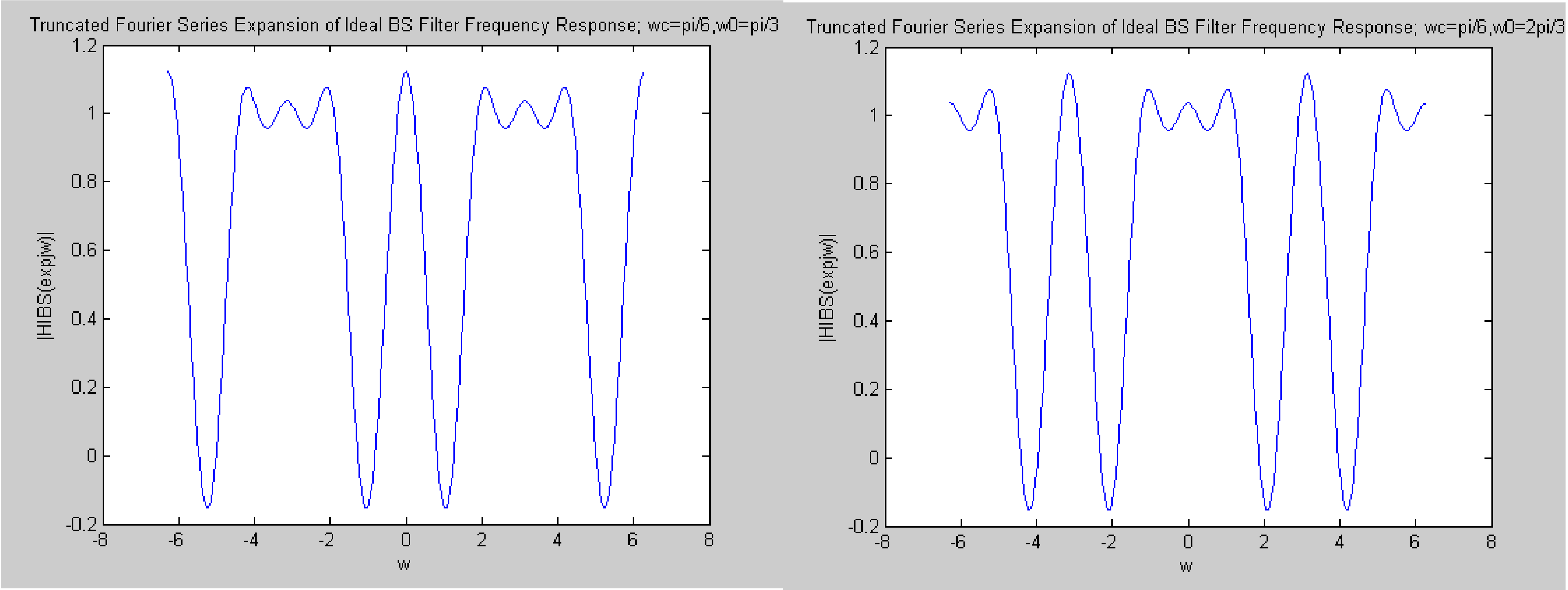
Taking the inverse Fourier transform of both sides,we obtain,

$$h_{IBS}(n) = \delta(n) - h_{IBP}(n)$$

$$\boxed{h_{IBS} = \delta(n) - 2h_{ILP}(n)\cos\omega_0 n}$$

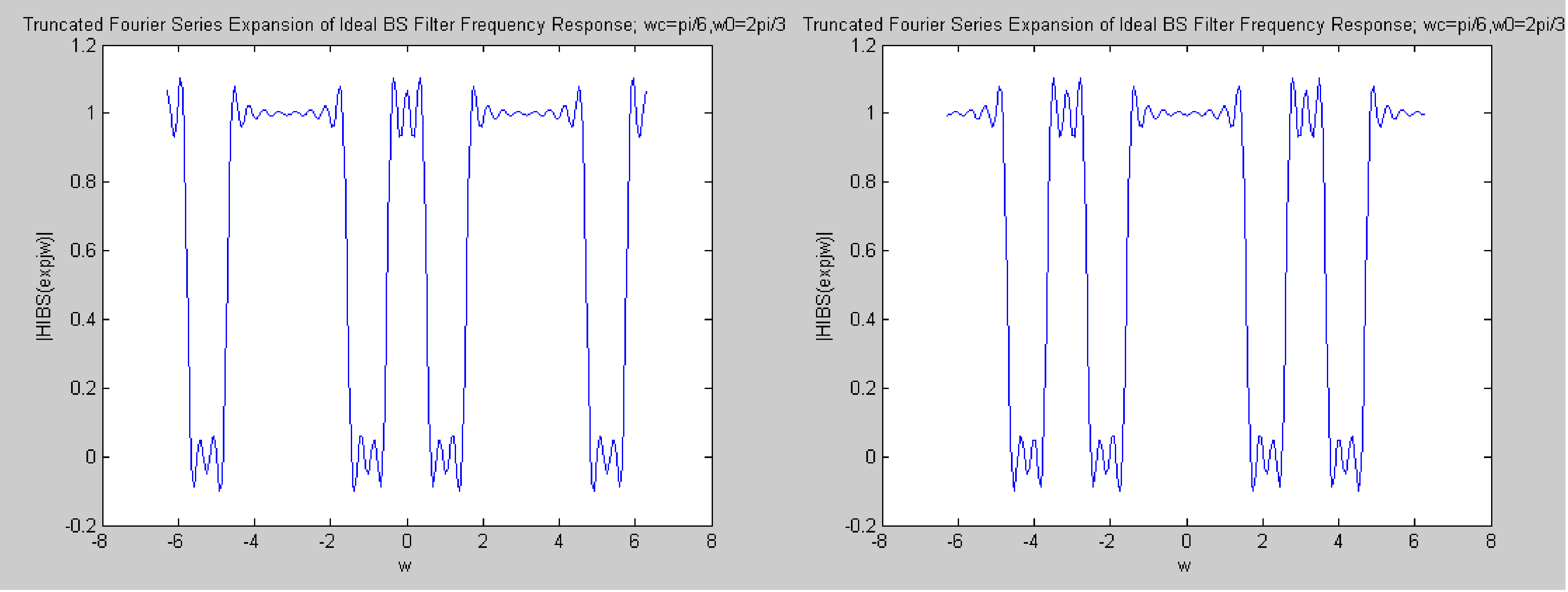
Design of HP, BP and BS Filters

Frequency Response of Truncated Impulse Response from $n = \pm 6$
plotted for $-2\pi<\omega<2\pi$



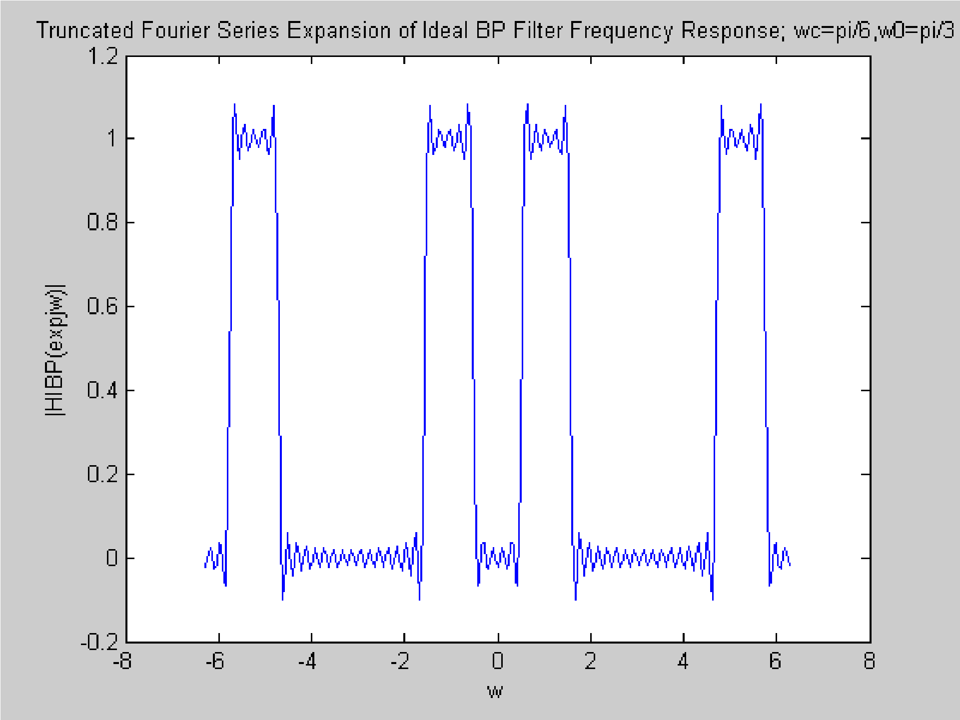
Design of HP, BP and BS Filters

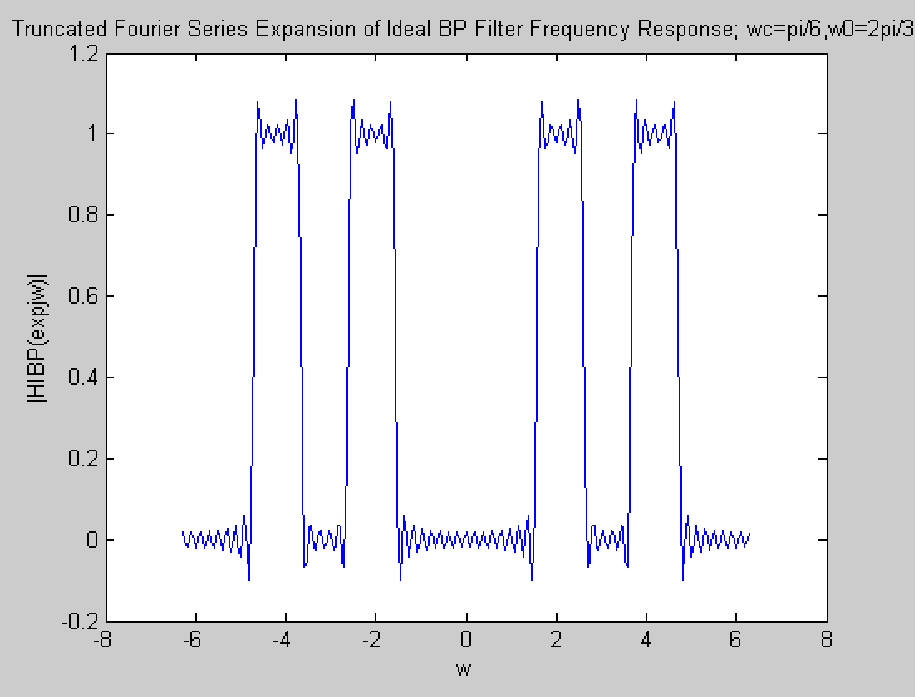
Frequency Response of Truncated Impulse Response from $n = \pm 16$ **plotted for** $-2\pi < \omega < 2\pi$



Design of HP, BP and BS Filters

Frequency Response of Truncated Impulse Response from $n = \pm 33$ **plotted for** $-2\pi<\omega<2\pi$





Design of HP, BP and BS Filters

Frequency Response of Truncated Impulse Response from $n = \pm 32$ **plotted for** $-\pi<\omega<\pi$