**Problem 2):** Let  $\mathcal{G}$  be a finite group and  $g \in \mathcal{G}$ . Now define  $\langle g \rangle \equiv g^0, g^1, g^2, \dots, g^k, \dots$ , where  $k \in \mathbb{N}$ . Beginning with the multiplicative case, let  $m, n \in \mathbb{N}$  so that we have

$$g^m g^n = g^{m+n}$$

Since  $m, n \in \mathbb{N}$  and  $\mathbb{N}$  is closed under addition,  $(m+n) \in \mathbb{N}$ , it is clear that  $g^{m+n} \in \langle g \rangle$ . Therefore,  $\langle g \rangle$  is closed under its operation. From our definition of  $\langle g \rangle$ , we know that  $g^0 \in \langle g \rangle$ . Additionally,  $g^0 \equiv e = 1$ ; therefore  $\langle g \rangle$  contains the identity element. Now, let  $m \in \mathbb{Z}^+$  and write  $g^{-m} g^m$ . Using  $g^{-m} \equiv (g^{-1})^m$ , this yields

$$g^{-m} g^m = (g^{-1})^m g^m = (g^{-1} g)^m = (e)^m = e = 1$$

which implies the existence of an inverse for each element in  $\langle g \rangle$ . Finally, let  $m, n, k \in \mathbb{N}$ , then we have

$$g^{m}(g^{n}g^{k}) = g^{m}(g^{n+k}) = g^{m+(n+k)}$$
 (2.1)

Since  $\mathbb N$  is associative under addition, the expression in 2.1 may be rewritten as

$$g^{m+(n+k)} = g^{(m+n)+k} = (g^{m+n}) g^k = (g^m g^n) g^k$$

thereby demonstrating the associativity of operations in  $\langle g \rangle$ . Since  $\mathcal G$  is finite, it has order  $m = |\mathcal G|$ . Therefore, the elements of  $\langle g \rangle$  will be repeats of elements in  $\mathcal G$  starting with  $g^{m+1}$ . Moreover, this means that  $\langle g \rangle \subseteq \mathcal G$ , thus satisfying the last condition for  $\langle g \rangle$  to be a sub-group of  $\mathcal G$ .

Continuing with the additive case, let  $m, n \in \mathbb{N}$  so that we have

$$m \times g \, n \times g = (m+n) \times g$$

Since  $m, n \in \mathbb{N}$  and  $\mathbb{N}$  is closed under addition,  $(m+n) \in \mathbb{N}$ , it is clear that  $(m+n) \times g \in \langle g \rangle$ . Therefore,  $\langle g \rangle$  is closed under its operation. From our definition of  $\langle g \rangle$ , we know that  $0 \times g \in \langle g \rangle$ . Additionally,  $0 \times g \equiv e = 0$ ; therefore  $\langle g \rangle$  contains the identity element. Now, let  $m \in \mathbb{Z}^+$  and write  $(-m) \times g \times g$ . Using  $(-m) \times g \equiv m \times (-g)^m$ , this yields

$$(-m) \times g \, m \times g = m \times (-g) \, m \times g = m \times (-g \, g) = m \times (e) = e = 0$$

which implies the existence of an inverse for each element in  $\langle g \rangle$ . Finally, let  $m, n, k \in \mathbb{N}$ , then we have

$$m \times g \ (n \times g \ k \times g) = m \times g \ ((n+k) \times g) = (m+(n+k)) \times g \tag{2.2}$$

Since  $\mathbb N$  is associative under addition, the expression in 2.2 may be rewritten as

$$(m+(n+k))\times g=((m+n)+k)\times g=(m+n)\times g\,k\times g=(m\times g\,n\times g)\,k\times g$$

thereby demonstrating the associativity of operations in  $\langle g \rangle$ . Since  $\mathcal G$  is finite, it has order  $m=|\mathcal G|$ . Therefore, the elements of  $\langle g \rangle$  will be repeats of elements in  $\mathcal G$  starting with  $(m+1)\times g$ . Moreover, this means that  $\langle g \rangle \subseteq \mathcal G$ , thus satisfying the last condition for  $\langle g \rangle$  to be a sub-group of  $\mathcal G$ .