

Problem 1): Let $x, y, e, x^{-1} \in \mathcal{G}$ where $e \in \mathcal{G}$ is the identity element of \mathcal{G} and x^{-1} is such that both $x x^{-1} = e = x^{-1} x$ and $y x^{-1} = e = x^{-1} y$ hold. Therefore we have

$$x x^{-1} = y x^{-1} \quad (1.1)$$

$$x x^{-1} = x^{-1} y \quad (1.2)$$

$$x^{-1} x = y x^{-1} \quad (1.3)$$

$$x^{-1} x = x^{-1} y \quad (1.4)$$

By applying the cancelation rule ($ab = ac \Rightarrow b = c$ for $a, b, c \in \mathbb{G}$ for any group \mathbb{G}) to the expression in 1.1 and 1.4, it is clear that we have

$$x = y \quad (1.5)$$

Since \mathcal{G} is abelian, we may rewrite the expression in 1.2 as

$$x x^{-1} = x^{-1} x = x^{-1} y$$

or

$$x x^{-1} = y x^{-1} = x^{-1} y$$

From either expression, the application of the cancelation rule yields the same result as in expression 1.5. Similarly, we use the abelian property of \mathcal{G} to rewrite the expression in 1.3 as

$$x^{-1} x = x x^{-1} = y x^{-1}$$

or

$$x^{-1} x = x^{-1} y = y x^{-1}$$

Again, applying the cancelation rule to either expression yields the same result as in 1.5. Therefore, every element in an abelian group must have a unique inverse.

□

Problem 2): Let \mathcal{G} be a finite group and $g \in \mathcal{G}$. Now define $\langle g \rangle \equiv g^0, g^1, g^2, \dots, g^k, \dots$, where $k \in \mathbb{N}$. Next, let $m, n \in \mathbb{N}$ so that we have

$$g^m g^n = g^{m+n}$$

Since $m, n \in \mathbb{N}$ and \mathbb{N} is closed under addition, $(m+n) \in \mathbb{N}$, it is clear that $g^{m+n} \in \langle g \rangle$. Therefore, $\langle g \rangle$ is closed under its operation. From our definition of $\langle g \rangle$, we know that $g^0 \in \langle g \rangle$. Additionally, $g^0 \equiv e$; therefore $\langle g \rangle$ contains the identity element. Now, let $m \in \mathbb{Z}^+$ and write $g^{-m} g^m$. This yields

$$g^{-m} g^m = g^{-m+m} = g^0 = e$$