Problem 1): Let $x, y, e, x^{-1} \in \mathcal{G}$ where $e \in \mathcal{G}$ is the identity element of $\mathcal{G}$ and x^{-1} is such that both $x\,x^{-1} = e = x^{-1}\,x$ and $y\,x^{-1} = e = x^{-1}\,y$ hold. Therefore we have

$$x\,x^{-1} = y\,x^{-1} \tag{1.1}$$

$$x\,x^{-1} = x^{-1}\,y \tag{1.2}$$

$$x^{-1}\,x = y\,x^{-1} \tag{1.3}$$

$$x^{-1}\,x = x^{-1}\,y \tag{1.4}$$

By applying the cancelation rule ($ab = ac \;\Rightarrow\; b = c$ for $a, b, c \in \mathbb{G}$ for any group $\mathbb{G}$) to the expression in 1.1 and 1.4, it is clear that we have

$$x = y \tag{1.5}$$

Since $\mathcal{G}$ is abelian, we may rewrite the expression in 1.2 as

$$x\,x^{-1} = x^{-1}\,x = x^{-1}\,y$$

or

$$x\,x^{-1} = y\,x^{-1} = x^{-1}\,y$$

From either expression, the application of the cancelation rule yields the same result as in expression 1.5. Similarly, we use the abelian property of $\mathcal{G}$ to rewrite the expression in 1.3 as

$$x^{-1}\,x = x\,x^{-1} = y\,x^{-1}$$

or

$$x^{-1}\,x = x^{-1}\,y = y\,x^{-1}$$

Again, applying the cancelation rule to either expression yields the same result as in 1.5. Therefore, every element in an abelian group must have a unique inverse.

□

Problem 2): Let $\mathcal{G}$ be a finite group and $g \in \mathcal{G}$. Now define $\langle g \rangle \equiv g^0, g^1, g^2, \ldots, g^k, \ldots$, where $k \in \mathbb{N}$. Beginning with the multiplicative case, let $m, n \in \mathbb{N}$ so that we have

$$g^m\,g^n = g^{m+n}$$

Since $m, n \in \mathbb{N}$ and $\mathbb{N}$ is closed under addition, $(m+n) \in \mathbb{N}$, it is clear that $g^{m+n} \in \langle g \rangle$. Therefore, $\langle g \rangle$ is closed under its operation. From our definition of $\langle g \rangle$, we know that $g^0 \in \langle g \rangle$. Additionally, $g^0 \equiv e = 1$; therefore $\langle g \rangle$ contains the identity element. Now, let $m \in \mathbb{Z}^+$ and write $g^{-m}\,g^m$. Using $g^{-m} \equiv \left(g^{-1}\right)^m$, this yields

$$g^{-m}\,g^m = \left(g^{-1}\right)^m\,g^m = \left(g^{-1}\,g\right)^m = (e)^m = e = 1$$

which implies the existence of an inverse for each element in $\langle g \rangle$. Finally, let $m, n, k \in \mathbb{N}$, then we have

$$g^m\,\left(g^n\,g^k\right) = g^m\,\left(g^{n+k}\right) = g^{m+(n+k)} \tag{2.1}$$

Since $\mathbb{N}$ is associative under addition, the expression in 2.1 may be rewritten as

$$g^{m+(n+k)} = g^{(m+n)+k} = \left(g^{m+n}\right)\,g^k = \left(g^m\,g^n\right)\,g^k$$

thereby demonstrating the associativity of operations in $\langle g \rangle$. Since $\mathcal{G}$ is finite, it has order $m = |\mathcal{G}|$. Therefore, the elements of $\langle g \rangle$ will be repeats of elements in $\mathcal{G}$ starting with g^{m+1}. Moreover, this means that $\langle g \rangle \subseteq \mathcal{G}$, thus satisfying the last condition for $\langle g \rangle$ to be a sub-group of $\mathcal{G}$.

Continuing with the additive case, let $m, n \in \mathbb{N}$ so that we have

$$m \times g\, n \times g = (m + n) \times g$$

Since $m, n \in \mathbb{N}$ and $\mathbb{N}$ is closed under addition, $(m + n) \in \mathbb{N}$, it is clear that $(m + n) \times g \in \langle g \rangle$. Therefore, $\langle g \rangle$ is closed under its operation. From our definition of $\langle g \rangle$, we know that $0 \times g \in \langle g \rangle$. Additionally, $0 \times g \equiv e = 0$; therefore $\langle g \rangle$ contains the identity element. Now, let $m \in \mathbb{Z}^+$ and write $(-m) \times g\, m \times g$. Using $(-m) \times g \equiv m \times (-g)^m$, this yields

$$(-m) \times g\, m \times g = m \times (-g)\ m \times g = m \times (-g\, g) = m \times (e) = e = 0$$

which implies the existence of an inverse for each element in $\langle g \rangle$. Finally, let $m, n, k \in \mathbb{N}$, then we have

$$m \times g\ (n \times g\, k \times g) = m \times g\ ((n + k) \times g) = (m + (n + k)) \times g \tag{2.2}$$

Since $\mathbb{N}$ is associative under addition, the expression in 2.2 may be rewritten as

$$(m + (n + k)) \times g = ((m + n) + k) \times g = (m + n) \times g\, k \times g = (m \times g\, n \times g)\ k \times g$$

thereby demonstrating the associativity of operations in $\langle g \rangle$. Since $\mathcal{G}$ is finite, it has order $m = |\mathcal{G}|$. Therefore, the elements of $\langle g \rangle$ will be repeats of elements in $\mathcal{G}$ starting with $(m + 1) \times g$. Moreover, this means that $\langle g \rangle \subseteq \mathcal{G}$, thus satisfying the last condition for $\langle g \rangle$ to be a sub-group of $\mathcal{G}$.

Problem 3): Since $\mathbb{Z}_{\mathfrak{p}}^{\star} \equiv \{a \in \{1, 2, \ldots, \mathfrak{p} - 1\} \,|\, \gcd(a, \mathfrak{p}) = 1\}$, for any $\mathfrak{p} \in \mathbb{Z}^{+}$, the set of possible elements for $\mathbb{Z}_{p^e}^{\star}$ is defined as

$$\mathbb{Z}_{p^e}^{\star} \subset \{1, 2, \ldots, p^e - 1\} \tag{3.1}$$

This implies the following relation between the cardinalities of these sets

$$|\mathbb{Z}_{p^e}^{\star}| < |\{1, 2, \ldots, p^e - 1\}|,$$

where $|\{1, 2, \ldots, p^e - 1\}|$ has the value $|\{1, 2, \ldots, p^e - 1\}| = (p^e - 1)$. It follows that the value of $|\mathbb{Z}_{p^e}^{\star}|$ can be obtained by determining the set of all values in $\{1, 2, \ldots, p^e - 1\}$ that do not satisfy the conition given in 3.1 and subtracting the cardinality of this set from $(p^e - 1)$. Since the common multiple is p, we will write this set in terms of be. Thus, the set of values in $\{1, 2, \ldots, p^e - 1\}$ that do not satisfy the condition in 3.1 may be defined as

$$\left\{p, 2p, 3p, \ldots, p\,p, 2p\,p, 3p\,p, \ldots, p^2\,p, \ldots, \left(p^{e-1} - 1\right)\,p\right\}$$

This definition arises becuase only multiples of p do not satistfy the condition in 3.1 and because $\left(p^{e-1} - 1\right)\,p = p^e - p$ is the largest element of $\{1, 2, \ldots, p^e - 1\}$ that does not satisfy the confition in 3.1. The cardinality of this set, $\left\{p, 2p, 3p, \ldots, p\,p, 2p\,p, 3p\,p, \ldots, p^2\,p, \ldots, \left(p^{e-1} - 1\right)\,p\right\}$ is clearly

$$|\left\{p, 2p, 3p, \ldots, p\,p, 2p\,p, 3p\,p, \ldots, p^2\,p, \ldots, \left(p^{e-1} - 1\right)\,p\right\}| = \left(p^{e-1} - 1\right)$$

Subtracting this value from $|\{1, 2, \ldots, p^e - 1\}| = (p^e - 1)$ finally yields

$$\phi(p^e) = (p^e - 1) - \left(p^{e-1} - 1\right) = p^e - 1 - p^{e-1} + 1 = p^e - p^{e-1} = p^{e-1}(p - 1)$$

as desired.

To show that

$$\phi\left(pq\right)=\phi\left(p\right)\,\phi\left(q\right)$$

holds for any relatively prime p and q, we apply a similarly strategy to the one used above. The number of possible elements of $\mathbb{Z}_{pq}^{\star}$ is $pq-1$. As before, we must take into account that some possible elements of $\mathbb{Z}_{pq}^{\star}$ will not satisfy the definition in 3.1. If we subtract the number of these elements, then we will have $\phi\left(pq\right)=|\mathbb{Z}_{pq}^{\star}|$. Since there are $p-1$ multiples of q that do not satisfy the condition in 3.1, we must subtract $p-1$ from $pq-1$. Similarly, since there are also $q-1$ multiples of p that do not satisfy the same condition, we must also subtract $q-1$ from $pq-1$. Carrying out these subtractions gives

$$\begin{aligned}\phi\left(pq\right)&=\left(pq-1\right)-\left(p-1\right)-\left(q-1\right)\\&=pq-1-p+1-q+1\\&=pq-p-q+1\\&=\left(p-1\right)\left(q-1\right)\\&=\phi\left(p\right)\,\phi\left(q\right)\end{aligned}$$

since $\phi\left(p\right)$ and $\phi\left(q\right)$ are defined as $\phi\left(p\right)=p-1$ and $\phi\left(q\right)=q-1$, respectively.

We will now use the previous result to show that, for an integer $N=\prod_i\left\{p_i^{e_i}\right\}$ and p_i distinct primes, we have

$$\phi\left(N\right)=\prod_i\left\{p_i^{e_i-1}\left(p_i-1\right)\right\}$$

To begin, we substitute $N=\prod_i\left\{p_i^{e_i}\right\}$ for N in the previous expression. This gives

$$\phi(N) = \phi\left(\prod_i \{p_i^{e_i}\}\right)$$

Using the result $\phi(pq) = \phi(p)\ \phi(q)$, we have

$$\phi(N) = \prod_i \{\phi(p_i^{e_i})\}$$

Finally, we apply the result $\phi(p^e) = p^{e-1}(p-1)$ to obtain

$$\phi(N) = \prod_i \left\{p_i^{e_i - 1}(p_i - 1)\right\}$$

as expected.

Problem 4): We denote the cross product of groups $\mathcal{G}$ and $\mathcal{H}$ as $\mathcal{G} \times \mathcal{H}$ and define it by

$$(g, h) \circ (g', h') \equiv (g \circ_{\mathcal{G}} g'\ ,\ h \circ_{\mathcal{H}} h') \tag{4.1}$$

To show that $\mathcal{G} \times \mathcal{H}$ is a group, we begin by proving closure under its operation. Since $\mathcal{G}$ and $\mathcal{H}$ are groups, the we have $(g \circ_{\mathcal{G}} g') \in \mathcal{G}$ and $(h \circ_{\mathcal{H}} h') \in \langle$. Thus $\mathcal{G} \times \mathcal{H}$ is closed under its operation. Next, we must show the existence of an identity in $\mathcal{G} \times \mathcal{H}$. If we modify the expression in 4.1 so that $g' = e_{\mathcal{G}}$ and $h' = e_{\mathcal{H}}$, then we have

$$\begin{aligned}(g, h) \circ (e_{\mathcal{G}}, e_{\mathcal{H}}) &= (g \circ_{\mathcal{G}} e_{\mathcal{G}}\ ,\ h \circ_{\mathcal{H}} e_{\mathcal{H}}) \\ &= (g, h)\end{aligned}$$

Therefore, $\mathcal{G} \times \mathcal{H}$ contains an identity element and it is defined as $(e_{\mathcal{G}}, e_{\mathcal{H}})$. Next, we must demonstrate the existence of inversed in $\mathcal{G} \times \mathcal{H}$. To do this, we again modify the expression in 4.1. This time we substitute $g' = g^{-1}$ and $h' = h^{-1}$. Applying this substitution to the expression in 4.1 gives

$$\begin{aligned}(g,h) \circ \left(g^{-1}, h^{-1}\right) &= \left(g \circ_{\mathcal{G}} g^{-1}\ ,\ h \circ_{\mathcal{H}} h^{-1}\right) \\ &= (e_{\mathcal{G}}, e_{\mathcal{H}})\end{aligned}$$

Thus, $\mathcal{G} \times \mathcal{H}$ contains inverses for each of its elements. Lastly, we show that associativity holds in $\mathcal{G} \times \mathcal{H}$. We begin with

$$\begin{aligned}((g_1,h_1) \circ (g_2,h_2)) \circ (g_3,h_3) &= (g_1 \circ_{\mathcal{G}} g_2, h_1 \circ_{\mathcal{H}} h_2) \circ (g_3,h_3) \\ &= ((g_1 \circ_{\mathcal{G}} g_2) \circ_{\mathcal{G}} g_3, (h_1 \circ_{\mathcal{H}} h_2) \circ_{\mathcal{H}} h_3) \qquad (4.2)\end{aligned}$$

Using the associativity of $\mathcal{G}$ and $\mathcal{H}$, we have

$$((g_1 \circ_{\mathcal{G}} g_2) \circ_{\mathcal{G}} g_3, (h_1 \circ_{\mathcal{H}} h_2) \circ_{\mathcal{H}} h_3) = (g_1 \circ_{\mathcal{G}} (g_2 \circ_{\mathcal{G}} g_3), h_1 \circ_{\mathcal{H}} (h_2 \circ_{\mathcal{H}} h_3))$$