

Problem 1): Let $x, y, e, x^{-1} \in \mathcal{G}$ where $e \in \mathcal{G}$ is the identity element of \mathcal{G} and x^{-1} is such that both $x x^{-1} = e = x^{-1} x$ and $y x^{-1} = e = x^{-1} y$ hold. Therefore we have

$$x x^{-1} = y x^{-1} \tag{1.1}$$

$$x x^{-1} = x^{-1} y \tag{1.2}$$

$$x^{-1} x = y x^{-1} \tag{1.3}$$

$$x^{-1} x = x^{-1} y \tag{1.4}$$

By applying the cancelation rule ($ab = ac \Rightarrow b = c$ for $a, b, c \in \mathbb{G}$ for any group \mathbb{G}) to the expression in 1.1 and 1.4, it is clear that we have

$$x = y \tag{1.5}$$

Since \mathcal{G} is abelian, we may rewrite the expression in 1.2 as

$$x x^{-1} = x^{-1} x = x^{-1} y$$

or

$$x x^{-1} = y x^{-1} = x^{-1} y$$

From either expression, the application of the cancelation rule yields the same result as in expression 1.5. Similarly, we use the abelian property of \mathcal{G} to rewrite the expression in 1.3 as

$$x^{-1} x = x x^{-1} = y x^{-1}$$

or

$$x^{-1}x = x^{-1}y = yx^{-1}$$

Again, applying the cancelation rule to either expression yields the same result as in 1.5.

Therefore, every element in an abelian group must have a unique inverse.

□