Problem 1): Let $x, y, e, x^{-1} \in \mathcal{G}$ where $e \in \mathcal{G}$ is the identity element of $\mathcal{G}$ and x^{-1} is such that both $x\,x^{-1} = e = x^{-1}\,x$ and $y\,x^{-1} = e = x^{-1}\,y$ hold. Therefore we have

$$x\,x^{-1} = y\,x^{-1} \tag{1.1}$$

$$x\,x^{-1} = x^{-1}\,y \tag{1.2}$$

$$x^{-1}\,x = y\,x^{-1} \tag{1.3}$$

$$x^{-1}\,x = x^{-1}\,y \tag{1.4}$$

By applying the cancelation rule ($ab = ac \;\Rightarrow\; b = c$ for $a, b, c \in \mathbb{G}$ for any group $\mathbb{G}$) to the expression in 1.1 and 1.4, it is clear that we have

$$x = y \tag{1.5}$$

Since $\mathcal{G}$ is abelian, we may rewrite the expression in 1.2 as

$$x\,x^{-1} = x^{-1}\,x = x^{-1}\,y$$

or

$$x\,x^{-1} = y\,x^{-1} = x^{-1}\,y$$

From either expression, the application of the cancelation rule yields the same result as in expression 1.5. Similarly, we use the abelian property of $\mathcal{G}$ to rewrite the expression in 1.3 as

$$x^{-1}\,x = x\,x^{-1} = y\,x^{-1}$$

or

$$x^{-1}\,x = x^{-1}\,y = y\,x^{-1}$$

Again, applying the cancelation rule to either expression yields the same result as in 1.5. Therefore, every element in an abelian group must have a unique inverse.

□

Problem 2): Let $\mathcal{G}$ be a finite group and $g \in \mathcal{G}$. Now define $\langle g \rangle \equiv g^0, g^1, g^2, \ldots, g^k, \ldots$, where $k \in \mathbb{N}$. Next, let $m, n \in \mathbb{N}$ so that we have

$$g^m\,g^n = g^{m+n}$$

Since $m, n \in \mathbb{N}$ and $\mathbb{N}$ is closed under addition, $(m+n) \in \mathbb{N}$, it is clear that $g^{m+n} \in \langle g \rangle$. Therefore, $\langle g \rangle$ is closed under its operation. From our definition of $\langle g \rangle$, we know that $g^0 \in \langle g \rangle$. Additionally, $g^0 \equiv e$; therefore $\langle g \rangle$ contains the identity element. Now, let $m \in \mathbb{Z}^+$ and write $g^{-m}\,g^m$. Using $g^{-m} \equiv \left(g^{-1}\right)^m$, this yields

$$g^{-m}\,g^m = \left(g^{-1}\right)^m\,g^m = \left(g^{-1}\,g\right)^m = (e)^m = e$$

which implies the existence of an inverse for each element in $\langle g \rangle$. Finally, let $m, n, k \in \mathbb{N}$, then we have

$$g^m\,\left(g^n\,g^k\right) = g^m\,\left(g^{n+k}\right) = g^{m+(n+k)} \tag{2.1}$$

Since $\mathbb{N}$ is associative under addition, the expression in 2.1 may be rewritten as

$$g^{m+(n+k)} = g^{(m+n)+k} = \left(g^{m+n}\right)\,g^k = \left(g^m\,g^n\right)\,g^k$$

thereby demonstrating the associativity of operations in $\langle g \rangle$. Since $\mathcal{G}$ is finite, it has order $m = |\mathcal{G}|$.

Therefore, the elements of $\langle g \rangle$ will be repeats of elements in $\mathcal{G}$ starting with g^{m+1}. Moreover, this means that $\langle g \rangle \subseteq \mathcal{G}$, thus satisfying the last condition for $\langle g \rangle$ to be a sub-group of $\mathcal{G}$.