

**Problem 1):** Let  $x, y, e, x^{-1} \in \mathcal{G}$  where  $e \in \mathcal{G}$  is the identity element of  $\mathcal{G}$  and  $x^{-1}$  is such that both  $x x^{-1} = e = x^{-1} x$  and  $y x^{-1} = e = x^{-1} y$  hold. Therefore we have

$$x x^{-1} = y x^{-1} \tag{1.1}$$

$$x x^{-1} = x^{-1} y \tag{1.2}$$

$$x^{-1} x = y x^{-1} \tag{1.3}$$

$$x^{-1} x = x^{-1} y \tag{1.4}$$

By applying the cancelation rule ( $ab = ac \Rightarrow b = c$  for  $a, b, c \in \mathbb{G}$  for any group  $\mathbb{G}$ ) to the expression in 1.1 and 1.4, it is clear that we have

$$x = y \tag{1.5}$$

Since  $\mathcal{G}$  is abelian, we may rewrite the expression in 1.2 as

$$x x^{-1} = x^{-1} x = x^{-1} y$$

or

$$x x^{-1} = y x^{-1} = x^{-1} y$$

From either expression, the application of the cancelation rule yields the same result as in expression 1.5. Similarly, we use the abelian property of  $\mathcal{G}$  to rewrite the expression in 1.3 as

$$x^{-1} x = x x^{-1} = y x^{-1}$$

or

$$x^{-1}x = x^{-1}y = yx^{-1}$$

Again, applying the cancelation rule to either expression yields the same result as in 1.5.

Therefore, every element in an abelian group must have a unique inverse.

□