

**Problem 1):** Let  $x, y, e, x^{-1} \in \mathcal{G}$  where  $e \in \mathcal{G}$  is the identity element of  $\mathcal{G}$  and  $x^{-1}$  is such that both  $x x^{-1} = e = x^{-1} x$  and  $y x^{-1} = e = x^{-1} y$  hold. Therefore we have

$$x x^{-1} = y x^{-1} \quad (1.1)$$

$$x x^{-1} = x^{-1} y \quad (1.2)$$

$$x^{-1} x = y x^{-1} \quad (1.3)$$

$$x^{-1} x = x^{-1} y \quad (1.4)$$

By applying the cancelation rule ( $ab = ac \Rightarrow b = c$  for  $a, b, c \in \mathbb{G}$  for any group  $\mathbb{G}$ ) to the expression in 1.1 and 1.4, it is clear that we have

$$x = y \quad (1.5)$$

Since  $\mathcal{G}$  is abelian, we may rewrite the expression in 1.2 as

$$x x^{-1} = x^{-1} x = x^{-1} y$$

or

$$x x^{-1} = y x^{-1} = x^{-1} y$$

From either expression, the application of the cancelation rule yields the same result as in expression 1.5. Similarly, we use the abelian property of  $\mathcal{G}$  to rewrite the expression in 1.3 as

$$x^{-1} x = x x^{-1} = y x^{-1}$$

or

$$x^{-1} x = x^{-1} y = y x^{-1}$$

Again, applying the cancelation rule to either expression yields the same result as in 1.5. Therefore, every element in an abelian group must have a unique inverse.

□

**Problem 2):** Let  $\mathcal{G}$  be a finite group and  $g \in \mathcal{G}$ . Now define  $\langle g \rangle \equiv g^0, g^1, g^2, \dots, g^k, \dots$ , where  $k \in \mathbb{N}$ .

Beginning with the multiplicative case, let  $m, n \in \mathbb{N}$  so that we have

$$g^m g^n = g^{m+n}$$

Since  $m, n \in \mathbb{N}$  and  $\mathbb{N}$  is closed under addition,  $(m+n) \in \mathbb{N}$ , it is clear that  $g^{m+n} \in \langle g \rangle$ . Therefore,  $\langle g \rangle$  is closed under its operation. From our definition of  $\langle g \rangle$ , we know that  $g^0 \in \langle g \rangle$ . Additionally,  $g^0 \equiv e = 1$ ; therefore  $\langle g \rangle$  contains the identity element. Now, let  $m \in \mathbb{Z}^+$  and write  $g^{-m} g^m$ . Using  $g^{-m} \equiv (g^{-1})^m$ , this yields

$$g^{-m} g^m = (g^{-1})^m g^m = (g^{-1} g)^m = (e)^m = e = 1$$

which implies the existence of an inverse for each element in  $\langle g \rangle$ . Finally, let  $m, n, k \in \mathbb{N}$ , then we have

$$g^m (g^n g^k) = g^m (g^{n+k}) = g^{m+(n+k)} \tag{2.1}$$

Since  $\mathbb{N}$  is associative under addition, the expression in 2.1 may be rewritten as

$$g^{m+(n+k)} = g^{(m+n)+k} = (g^{m+n}) g^k = (g^m g^n) g^k$$

thereby demonstrating the associativity of operations in  $\langle g \rangle$ . Since  $\mathcal{G}$  is finite, it has order  $m = |\mathcal{G}|$ . Therefore, the elements of  $\langle g \rangle$  will be repeats of elements in  $\mathcal{G}$  starting with  $g^{m+1}$ . Moreover, this means that  $\langle g \rangle \subseteq \mathcal{G}$ , thus satisfying the last condition for  $\langle g \rangle$  to be a sub-group of  $\mathcal{G}$ .

Continuing with the additive case, let  $m, n \in \mathbb{N}$  so that we have

$$m \times g n \times g = (m + n) \times g$$

Since  $m, n \in \mathbb{N}$  and  $\mathbb{N}$  is closed under addition,  $(m + n) \in \mathbb{N}$ , it is clear that  $(m + n) \times g \in \langle g \rangle$ . Therefore,  $\langle g \rangle$  is closed under its operation. From our definition of  $\langle g \rangle$ , we know that  $0 \times g \in \langle g \rangle$ . Additionally,  $0 \times g \equiv e = 0$ ; therefore  $\langle g \rangle$  contains the identity element. Now, let  $m \in \mathbb{Z}^+$  and write  $(-m) \times g m \times g$ . Using  $(-m) \times g \equiv m \times (-g)^m$ , this yields

$$(-m) \times g m \times g = m \times (-g) m \times g = m \times (-g g) = m \times (e) = e = 0$$

which implies the existence of an inverse for each element in  $\langle g \rangle$ . Finally, let  $m, n, k \in \mathbb{N}$ , then we have

$$m \times g (n \times g k \times g) = m \times g ((n + k) \times g) = (m + (n + k)) \times g \quad (2.2)$$

Since  $\mathbb{N}$  is associative under addition, the expression in 2.2 may be rewritten as

$$(m + (n + k)) \times g = ((m + n) + k) \times g = (m + n) \times g k \times g = (m \times g n \times g) k \times g$$

thereby demonstrating the associativity of operations in  $\langle g \rangle$ . Since  $\mathcal{G}$  is finite, it has order  $m = |\mathcal{G}|$ . Therefore, the elements of  $\langle g \rangle$  will be repeats of elements in  $\mathcal{G}$  starting with  $(m + 1) \times g$ . Moreover, this means that  $\langle g \rangle \subseteq \mathcal{G}$ , thus satisfying the last condition for  $\langle g \rangle$  to be a sub-group of  $\mathcal{G}$ .

**Problem 3):** Since  $\mathbb{Z}_p^* \equiv \{a \in \{1, 2, \dots, p-1\} \mid \gcd(a, p) = 1\}$ , for any  $p \in \mathbb{Z}^+$ , the set of possible elements for  $\mathbb{Z}_{p^e}^*$  is defined as

$$\mathbb{Z}_{p^e}^* \subset \{1, 2, \dots, p^e - 1\} \quad (3.1)$$

This implies the following relation between the cardinalities of these sets

$$|\mathbb{Z}_{p^e}^*| < |\{1, 2, \dots, p^e - 1\}|,$$

where  $|\{1, 2, \dots, p^e - 1\}|$  has the value  $|\{1, 2, \dots, p^e - 1\}| = (p^e - 1)$ . It follows that the value of  $|\mathbb{Z}_{p^e}^*|$  can be obtained by determining the set of all values in  $\{1, 2, \dots, p^e - 1\}$  that do not satisfy the condition given in 3.1 and subtracting the cardinality of this set from  $(p^e - 1)$ . Since the common multiple is  $p$ , we will write this set in terms of  $p$ . Thus, the set of values in  $\{1, 2, \dots, p^e - 1\}$  that do not satisfy the condition in 3.1 may be defined as

$$\{p, 2p, 3p, \dots, pp, 2pp, 3pp, \dots, p^2p, \dots, (p^{e-1} - 1)p\}$$

This definition arises because only multiples of  $p$  do not satisfy the condition in 3.1 and because  $(p^{e-1} - 1)p = p^e - p$  is the largest element of  $\{1, 2, \dots, p^e - 1\}$  that does not satisfy the condition in 3.1. The cardinality of this set,  $\{p, 2p, 3p, \dots, pp, 2pp, 3pp, \dots, p^2p, \dots, (p^{e-1} - 1)p\}$  is clearly

$$|\{p, 2p, 3p, \dots, pp, 2pp, 3pp, \dots, p^2p, \dots, (p^{e-1} - 1)p\}| = (p^{e-1} - 1)$$

Subtracting this value from  $|\{1, 2, \dots, p^e - 1\}| = (p^e - 1)$  finally yields

$$\phi(p^e) = (p^e - 1) - (p^{e-1} - 1) = p^e - 1 - p^{e-1} + 1 = p^e - p^{e-1} = p^{e-1}(p - 1)$$

as desired.

To show that

$$\phi(pq) = \phi(p) \phi(q)$$

holds for any relatively prime  $p$  and  $q$ , we apply a similarly strategy to the one used above. The number of possible elements of  $\mathbb{Z}_{pq}^*$  is  $pq - 1$ . As before, we must take into account that some possible elements of  $\mathbb{Z}_{pq}^*$  will not satisfy the definition in 3.1. If we subtract the number of these elements, then we will have  $\phi(pq) = |\mathbb{Z}_{pq}^*|$ . Since there are  $p - 1$  multiples of  $q$  that do not satisfy the condition in 3.1, we must subtract  $p - 1$  from  $pq - 1$ . Similarly, since there are also  $q - 1$  multiples of  $p$  that do not satisfy the same condition, we must also subtract  $q - 1$  from  $pq - 1$ . Carrying out these subtractions gives

$$\begin{aligned} \phi(pq) &= (pq - 1) - (p - 1) - (q - 1) \\ &= pq - 1 - p + 1 - q + 1 \\ &= pq - p - q + 1 \\ &= (p - 1)(q - 1) \\ &= \phi(p) \phi(q) \end{aligned}$$

since  $\phi(p)$  and  $\phi(q)$  are defined as  $\phi(p) = p - 1$  and  $\phi(q) = q - 1$ , respectively.

We will now use the previous result to show that, for an integer  $N = \prod_i \{p_i^{e_i}\}$  and  $p_i$  distinct primes, we have

$$\phi(N) = \prod_i \{p_i^{e_i-1} (p_i - 1)\}$$

To begin, we substitute  $N = \prod_i \{p_i^{e_i}\}$  for  $N$  in the previous expression. This gives

$$\phi(N) = \phi\left(\prod_i \{p_i^{e_i}\}\right)$$

Using the result  $\phi(pq) = \phi(p) \phi(q)$ , we have

$$\phi(N) = \prod_i \{\phi(p_i^{e_i})\}$$

Finally, we apply the result  $\phi(p^e) = p^{e-1}(p-1)$  to obtain

$$\phi(N) = \prod_i \{p_i^{e_i-1}(p_i-1)\}$$

as expected.

**Problem 4):** We denote the cross product of groups  $\mathcal{G}$  and  $\mathcal{H}$  as  $\mathcal{G} \times \mathcal{H}$  and define it by

$$(g, h) \circ (g', h') \equiv (g \circ_{\mathcal{G}} g', h \circ_{\mathcal{H}} h') \quad (4.1)$$

To show that  $\mathcal{G} \times \mathcal{H}$  is a group, we begin by proving closure under its operation. Since  $\mathcal{G}$  and  $\mathcal{H}$  are groups, then we have  $(g \circ_{\mathcal{G}} g') \in \mathcal{G}$  and  $(h \circ_{\mathcal{H}} h') \in \mathcal{H}$ . Thus  $\mathcal{G} \times \mathcal{H}$  is closed under its operation. Next, we must show the existence of an identity in  $\mathcal{G} \times \mathcal{H}$ . If we modify the expression in 4.1 so that  $g' = e_{\mathcal{G}}$  and  $h' = e_{\mathcal{H}}$ , then we have

$$\begin{aligned} (g, h) \circ (e_{\mathcal{G}}, e_{\mathcal{H}}) &= (g \circ_{\mathcal{G}} e_{\mathcal{G}}, h \circ_{\mathcal{H}} e_{\mathcal{H}}) \\ &= (g, h) \end{aligned}$$

Therefore,  $\mathcal{G} \times \mathcal{H}$  contains an identity element and it is defined as  $(e_{\mathcal{G}}, e_{\mathcal{H}})$ . Next, we must demonstrate

the existence of inverses in  $\mathcal{G} \times \mathcal{H}$ . To do this, we again modify the expression in 4.1. This time we substitute  $g' = g^{-1}$  and  $h' = h^{-1}$ . Applying this substitution to the expression in 4.1 gives

$$\begin{aligned}(g, h) \circ (g^{-1}, h^{-1}) &= (g \circ_{\mathcal{G}} g^{-1}, h \circ_{\mathcal{H}} h^{-1}) \\ &= (e_{\mathcal{G}}, e_{\mathcal{H}})\end{aligned}$$

Thus,  $\mathcal{G} \times \mathcal{H}$  contains inverses for each of its elements. Lastly, we show that associativity holds in  $\mathcal{G} \times \mathcal{H}$ . We begin with

$$\begin{aligned}((g_1, h_1) \circ (g_2, h_2)) \circ (g_3, h_3) &= (g_1 \circ_{\mathcal{G}} g_2, h_1 \circ_{\mathcal{H}} h_2) \circ (g_3, h_3) \\ &= ((g_1 \circ_{\mathcal{G}} g_2) \circ_{\mathcal{G}} g_3, (h_1 \circ_{\mathcal{H}} h_2) \circ_{\mathcal{H}} h_3)\end{aligned}\tag{4.2}$$

Using the associativity of  $\mathcal{G}$  and  $\mathcal{H}$ , we have

$$((g_1 \circ_{\mathcal{G}} g_2) \circ_{\mathcal{G}} g_3, (h_1 \circ_{\mathcal{H}} h_2) \circ_{\mathcal{H}} h_3) = (g_1 \circ_{\mathcal{G}} (g_2 \circ_{\mathcal{G}} g_3), h_1 \circ_{\mathcal{H}} (h_2 \circ_{\mathcal{H}} h_3))$$

Thus, the expression in 4.2 becomes

$$\begin{aligned}((g_1, h_1) \circ (g_2, h_2)) \circ (g_3, h_3) &= ((g_1 \circ_{\mathcal{G}} g_2) \circ_{\mathcal{G}} g_3, (h_1 \circ_{\mathcal{H}} h_2) \circ_{\mathcal{H}} h_3) \\ &= (g_1 \circ_{\mathcal{G}} (g_2 \circ_{\mathcal{G}} g_3), h_1 \circ_{\mathcal{H}} (h_2 \circ_{\mathcal{H}} h_3))\end{aligned}$$

which implies that associativity holds for  $\mathcal{G} \times \mathcal{H}$ .

□

**Problem 5):** First, we will show that if  $x \in \mathbb{Z}_N$ , then  $\forall x \in \mathbb{Z}_N^*$ ,  $f(x) = (x_p, x_q)$  where  $x_p \in \mathbb{Z}_p$  &  $x_q \in \mathbb{Z}_q$  and  $x_p \in \mathbb{Z}_p^*$  &  $x_q \in \mathbb{Z}_q^*$ . To do this, we assume, to the contrary, that  $x_p \notin \mathbb{Z}_p^*$ . This assumption implies that  $\gcd([x \bmod p], p) \neq 1$  and, by extension, that  $\gcd(x, p) \neq 1$ . Moreover, this leads to the conclusion that

$\gcd(x, N) \neq 1$ . This cannot be, otherwise we would have  $z \notin \mathbb{Z}_N^*$ , violating the definition of  $\mathbb{Z}_N^*$  we started with. Therefore,  $x_p \in \mathbb{Z}_p^*$  *must* hold. To show that  $x_q \in \mathbb{Z}_q^*$  *must* also hold, we make the similar contrary assumption (that  $x_q \notin \mathbb{Z}_q^*$ ) and arrive at a similar contradiction, thereby requiring that  $x_q \in \mathbb{Z}_q^*$ .

Next, we will show that  $f$  is an isomorphism. We begin by showing that  $f$  is one-to-one. To begin, let

$$f(x) = (x_p, x_q) = f(x')$$

Then, we let

$$x = x_p = x' \pmod{p}$$

and

$$x = x_q = x' \pmod{q}$$

This implies that  $(x - x')$  is divisible by both  $p$  and  $q$ . However, since  $p|N$  &  $q|N$  and  $\gcd(p, q) = 1$ , we must have  $(x - x')$  divisible by  $pq = N$ . This implies that  $x = x' \pmod{N}$  and  $x' = x \pmod{N}$ . Moreover, since  $x, x' \in \mathbb{Z}_N$ , we must have  $x = x'$  so  $f$  *must* also be **one-to-one**.

Continuing, since  $|\mathbb{Z}_p| = p$  and  $|\mathbb{Z}_q| = q$ , we must have

$$|\mathbb{Z}_p \times \mathbb{Z}_q| = |\mathbb{Z}_p| \cdot |\mathbb{Z}_q| = pq \tag{5.1}$$

Now, we have  $N = pq$  and  $|\mathbb{Z}_N| = N$ , so the expression in 5.1 becomes



$$\begin{aligned}
|\mathbb{Z}_p \times \mathbb{Z}_q| &= pq \\
&= N \\
&= |\mathbb{Z}_N|
\end{aligned}$$

Therefore,  $f$  must also be onto and, by extension, bijective.

Finally, we must show that

$$\begin{aligned}
f((a+b) \bmod N) &= [(a+b) \bmod p] \circ_{\mathbb{Z}_N} [(a+b) \bmod q] \\
&= [(a+b) \bmod p] \boxplus [(a+b) \bmod q] \\
&= f(a) \boxplus f(b)
\end{aligned}$$

Since we have defined  $f(x) \equiv ([x \bmod p], [x \bmod q])$ , we may write  $f((a+b) \bmod N)$  as

$$f((a+b) \bmod N) = ([[(a+b) \bmod N] \bmod p], [[(a+b) \bmod N] \bmod q]) \quad (5.2)$$

Now, since  $p|N$  and  $q|N$ , we have

$$\begin{aligned}
[[X \bmod N] \bmod p] &= [[X \bmod p] \bmod p] \\
&= [X \bmod p]
\end{aligned}$$

and

$$\begin{aligned}
[[X \bmod N] \bmod q] &= [[X \bmod q] \bmod q] \\
&= [X \bmod q]
\end{aligned}$$

Therefore, the expression in 5.2 becomes

$$\begin{aligned}
 f((a+b) \bmod N) &= ([[(a+b) \bmod N] \bmod p], [(a+b) \bmod N] \bmod q]) \\
 &= ([[(a+b) \bmod p] \bmod p], [(a+b) \bmod q] \bmod q]) \\
 &= ([(a+b) \bmod p], [(a+b) \bmod q])
 \end{aligned}$$

Separating this result according to  $a$  and  $b$  gives

$$\begin{aligned}
 ([ (a+b) \bmod p], [(a+b) \bmod q]) &= ([a \bmod p], [a \bmod q]) \boxplus ([b \bmod p], [b \bmod q]) \\
 &= f(a) \boxplus f(b)
 \end{aligned}$$

as desired.

□

**Problem 6):** We begin by applying the function mapping  $\mathbb{Z}_N$  to  $\mathbb{Z}_p \times \mathbb{Z}_q$  (denoted  $f$ ) to  $(x^e)^d$ . This gives us

$$\begin{aligned}
 f((x^e)^d) &= ([ (x_p^e)^d \bmod p], [(x_q^e)^d \bmod q]) \\
 &= ([ (x_p^{ed}) \bmod p], [(x_q^{ed}) \bmod q])
 \end{aligned}$$

We now substitute  $ed = 1 \bmod \phi(N)$  into our previous result to obtain

$$\begin{aligned}
 f((x^e)^d) &= ([ (x_p^{ed}) \bmod p], [(x_q^{ed}) \bmod q]) \\
 &= ([ (x_p^{1 \bmod \phi(N)}) \bmod p], [(x_q^{1 \bmod \phi(N)}) \bmod q])
 \end{aligned}$$

Using the definition of  $\phi(\dots)$  as well as the fact that  $N = pq$ , where  $p$  and  $q$  are distinct primes, we note that  $\phi(N) = \phi(p) \phi(q)$ . Therefore, our previous result can be rewritten as

$$\begin{aligned}
f\left((x^e)^d\right) &= \left(\left[\left(x_p^1 \bmod \phi(N)\right) \bmod p\right], \left[\left(x_q^1 \bmod \phi(N)\right) \bmod q\right]\right) \\
&= \left(\left[\left(x_p^1 \bmod \phi(p)\phi(q)\right) \bmod p\right], \left[\left(x_q^1 \bmod \phi(p)\phi(q)\right) \bmod q\right]\right)
\end{aligned} \tag{6.1}$$

Using the relation  $a^{\phi(N)} = 1 \bmod N$ , we see that  $\phi(q)$  will cancel from the exponent in the first part of the left-hand-term in 6.1. Similarly,  $\phi(p)$  will also cancel from the second part of the left-hand-term in 6.1. Therefore, our expression in 6.1 can be simplified to give

$$\begin{aligned}
f\left((x^e)^d\right) &= \left(\left[\left(x_p^1 \bmod \phi(p)\phi(q)\right) \bmod p\right], \left[\left(x_q^1 \bmod \phi(p)\phi(q)\right) \bmod q\right]\right) \\
&= \left(\left[\left(x_p^1 \bmod \phi(q)\right) \bmod p\right], \left[\left(q^1 \bmod \phi(p)\right) \bmod q\right]\right)
\end{aligned}$$

Noting that  $b \bmod p = [b \bmod c] \bmod c$ , we modify our previous result to give

$$\begin{aligned}
f\left((x^e)^d\right) &= \left(\left[\left(x_p^1 \bmod \phi(q)\right) \bmod p\right], \left[\left(x_q^1 \bmod \phi(p)\right) \bmod q\right]\right) \\
&= \left(\left[\left(\left(x_p^1 \bmod \phi(q)\right) \bmod p\right) \bmod p\right], \left[\left(\left(x_q^1 \bmod \phi(p)\right) \bmod q\right) \bmod q\right]\right)
\end{aligned}$$

Again using the relation  $a^{\phi(N)} = 1 \bmod N$ , we are able to simplify our previous result as

$$\begin{aligned}
f\left((x^e)^d\right) &= \left(\left[\left(\left(x^1 \bmod \phi(q)\right) \bmod p\right) \bmod p\right], \left[\left(\left(x^1 \bmod \phi(p)\right) \bmod q\right) \bmod q\right]\right) \\
&= \left(\left[\left(x_p \bmod (pq)\right) \bmod p\right], \left[\left(x_q \bmod (qp)\right) \bmod q\right]\right)
\end{aligned} \tag{6.2}$$

Finally, we note that  $pq = qp = N$  and recall the definition of  $f(\dots)$ . These relations allow us to rewrite our result in expression 6.2 to give

$$\begin{aligned}
f\left((x^e)^d\right) &= ([ (x_p \bmod (pq)) \bmod p ], [ (x_q \bmod (qp)) \bmod q ]) \\
&= ([ (x_p \bmod N) \bmod p ], [ (x_q \bmod N) \bmod q ]) \\
&= f(x \bmod N)
\end{aligned}$$

We then take the inverse of  $f(\dots)$  to ultimately give

$$f\left((x^e)^d\right) = f(x \bmod N) \implies (x^e)^d = x \bmod N$$

as desired.

□

**Problem 7):** We start with values for  $N$  and  $\phi(N)$ . For clarity, we will denote the numerical value for  $\phi(N)$  by the symbol  $\Phi_N$ . Further, we know both that  $N = pq$  and

$$\begin{aligned}
\phi(N) &= \phi(p) \phi(q) \\
&= (p-1)(q-1) \\
&= pq - p - q + 1 = \Phi_N
\end{aligned} \tag{7.1}$$

Additionally, note that the result in 7.1 was obtained using the relations  $\phi(p) = p-1$  and  $\phi(q) = q-1$ . The result in 7.1, along with  $N = pq$ , means that we have the system of equations

$$\Phi_N = pq - p - q + 1 \tag{7.1}$$

and

$$N = p q \quad (7.2)$$

Rewriting the expression in 7.2 as  $N = p q \Rightarrow q = N/p$  and applying the result, along with  $N = p q$  to the expression in 7.1, we have

$$\begin{aligned} \Phi_N &= N - p - \frac{N}{p} + 1 \\ p \Phi_N &= p N - p^2 - N + p \\ 0 &= p^2 + (\Phi_N - N - 1)p + N \end{aligned} \quad (7.3)$$

which is solvable for  $p$  in polynomial time (using the quadratic formula). Applying the result from solving 7.3 for  $p$  to the expression in 7.2 yields a value for  $q$  in polynomial time as well.

**Problem 8):** For a public key encryption scheme  $\Pi = (\text{Gen}, \text{Enc}, \text{Dec})$ , we define **CPA** security according to the probability obtaining a secure result, as defined in the privacy experiment  $\text{PubK}_{\mathcal{A}, \Pi}^{\text{LR-cpa}}$ . This experiment goes as follows

**The LR-oracle experiment**  $\text{PubK}_{\mathcal{A}, \Pi}^{\text{LR-cpa}}(n)$

1.  $\text{Gen}(1^n)$  is run to obtain keys  $(pk, sk)$ .
2. A uniform bit  $b \in \{0, 1\}$  is chosen.
3. The adversary  $\mathcal{A}$  is given input  $pk$  and oracle access to  $\text{LR}_{pk, b}(\cdot, \cdot)$ .
4. The adversary  $\mathcal{A}$  outputs a bit  $b'$ .
5. The adversary  $\mathcal{A}$  is defined to be 1 if  $b' = b$ , and 0 otherwise. If  $\text{PubK}_{\mathcal{A}, \Pi}^{\text{LR-cpa}}(n) = 1$ , we say that  $\mathcal{A}$  **succeeds**.

Using this definition for the experiment  $\text{PubK}_{\mathcal{A}, \Pi}^{\text{LR-cpa}}$ , we say that the encryption scheme  $\Pi$  is secure if the probability of  $\mathcal{A}$  succeeding,  $\Pr[\text{PubK}_{\mathcal{A}, \Pi}^{\text{LR-cpa}}(n) = 1]$  satisfies the condition

$$\Pr \left[ \text{PubK}_{\mathcal{A}, \Pi}^{\text{LR-cpa}}(n) = 1 \right] \leq \frac{1}{2} + \text{negl}(n) \quad (8.1)$$

where  $\text{negl}(n)$  is a function/value which is negligible on the order of  $n$ .

In detail, what we are seeking is indistinguishability of multiple encryptions. That is to say, if we have the plain-text of two different messages (*denote them*  $m_1$  *and*  $m_2$ ), which we encrypt using a public key (*denote it*  $pk$ ), then an adversary  $\mathcal{A}$  having access to the cipher-text of both messages **and** the public key should not be able to distinguish the cipher-text of the messages under any circumstances. Using  $pk$ , the encryption algorithm (*denoted*  $\text{Enc}_{pk}$ ) generates cipher-text from messages  $m_1$  and  $m_2$ . We use

$$\text{Enc}_{pk}(m_1) \quad \textbf{and} \quad \text{Enc}_{pk}(m_2)$$

to denote the cipher-text generated for these messages, respectively.

We denote both the information ( $pk$ ,  $\text{Enc}_{pk}(m_1)$ , &  $\text{Enc}_{pk}(m_2)$ ) available/provided to the adversary  $\mathcal{A}$  by

$$\mathcal{A}(pk, \text{Enc}_{pk}(m_1), \text{Enc}_{pk}(m_2)) \quad (8.2)$$

furthermore, we also use this notation to represent the outcome of running PubK on  $\mathcal{A}$ . When  $\mathcal{A}$  succeeds, then the expression in 8.2 yields the result

$$\mathcal{A}(pk, \text{Enc}_{pk}(m_1), \text{Enc}_{pk}(m_2)) = 1 \quad (8.3)$$

The expression in 8.2 yields

$$\mathcal{A}(pk, \text{Enc}_{pk}(m_1), \text{Enc}_{pk}(m_2)) = 0 \quad (8.4)$$

otherwise.

Since **CPA** security requires security over multiple encryptions using the same public key, we will formally define this security using **two** pairs of messages that are all being encrypted using the same public key. We denote the first pair of messages by  $m_{1,0}$  and  $m_{2,0}$ . Similarly, the second pair of messages are denoted by  $m_{1,1}$  and  $m_{2,1}$ . We now use the same notation as in 8.2 with these message pairs (*and their associated public key  $pk$* ) to represent the attack by  $\mathcal{A}$ . This gives

$$\mathcal{A}(pk, \text{Enc}_{pk}(m_{1,0}), \text{Enc}_{pk}(m_{2,0})), \quad (8.2 \text{ a})$$

for the first message pair; and

$$\mathcal{A}(pk, \text{Enc}_{pk}(m_{1,1}), \text{Enc}_{pk}(m_{2,1})), \quad (8.2 \text{ b})$$

for the second message pair.

Before proceeding, we point out that we can equivalently use the expression from 8.3 in place of the  $\text{PubK}_{\mathcal{A}, \Pi}^{\text{LR-cpa}}(n) = 1$  term from 8.1. More clearly, we may formally write this equivalence as

$$\text{PubK}_{\mathcal{A}, \Pi}^{\text{LR-cpa}}(n) = 1 \quad \longleftrightarrow \quad \mathcal{A}(pk, \text{Enc}_{pk}(m_1), \text{Enc}_{pk}(m_2)) = 1$$

This allows us to write a version of 8.1 for both and . For the first message pair (*represented in* ), this gives the result

$$\Pr[\mathcal{A}(pk, \text{Enc}_{pk}(m_{1,0}), \text{Enc}_{pk}(m_{2,0})) = 1] \leq \frac{1}{2} + \text{negl}_0(n), \quad (8.3)$$

where  $\text{negl}_0$  represents the negligible function required to satisfy this expression as applied to this

message pair (*we are making allowances in case the results in and use different  $\text{negl}$  functions*).

Writing our expression for the second message pair In a similar fashion yields

$$\Pr [\mathcal{A}(pk, \text{Enc}_{pk}(m_{1,1}), \text{Enc}_{pk}(m_{2,1})) = 1] \leq \frac{1}{2} + \text{negl}_1(n) \quad (8.4)$$

where  $\text{negl}_1$  represents the negligible function required to satisfy this expression as applied to this message pair just as before (*we will see later that any difference between these  $\text{negl}$  functions is inconsequential; however differentiating between the  $\text{negl}$  functions used in either case is required for mathematical rigor*).

To continue the equation in 8.4 is subtracted from the equation in 8.3, after which the result *difference* will be simplified, thereby allowing us to obtain the following expressions for the initial and then the simplified results

$$\begin{aligned} & \left\{ \Pr [\mathcal{A}(pk, \text{Enc}_{pk}(m_{1,0}), \text{Enc}_{pk}(m_{2,0})) = 1] - \right. \\ & \quad \left. - \Pr [\mathcal{A}(pk, \text{Enc}_{pk}(m_{1,1}), \text{Enc}_{pk}(m_{2,1})) = 1] \right\} \leq \left( \frac{1}{2} + \text{negl}_0(n) \right) - \left( \frac{1}{2} + \text{negl}_1(n) \right) \\ & \left\{ \Pr [\mathcal{A}(pk, \text{Enc}_{pk}(m_{1,0}), \text{Enc}_{pk}(m_{2,0})) = 1] - \right. \\ & \quad \left. - \Pr [\mathcal{A}(pk, \text{Enc}_{pk}(m_{1,1}), \text{Enc}_{pk}(m_{2,1})) = 1] \right\} \leq \text{negl}_0(n) - \text{negl}_1(n) \end{aligned}$$

Taking the absolute value of this simplified expression allows us to obtain the result

$$\begin{aligned} & \left| \Pr [\mathcal{A}(pk, \text{Enc}_{pk}(m_{1,0}), \text{Enc}_{pk}(m_{2,0})) = 1] - \right. \\ & \quad \left. - \Pr [\mathcal{A}(pk, \text{Enc}_{pk}(m_{1,1}), \text{Enc}_{pk}(m_{2,1})) = 1] \right| \leq \left| \text{negl}_0(n) - \text{negl}_1(n) \right| \quad (8.5) \end{aligned}$$

Considering the right-hand-side of 8.5, we see that  $\left| \text{negl}_0(n) - \text{negl}_1(n) \right|$  also negligible itself. Therefore, we may define another negligible function, of order  $n$ , that satisfies the relation



$$\left| \text{negl}_0(n) - \text{negl}_1(n) \right| = \text{negl}(n),$$

where  $\text{negl}$  is another negligible function, of order  $n$ . Applying this to the expression in 8.5, we obtain the final result

$$\left| \Pr[\mathcal{A}(pk, \text{Enc}_{pk}(m_{1,0}), \text{Enc}_{pk}(m_{2,0})) = 1] - \Pr[\mathcal{A}(pk, \text{Enc}_{pk}(m_{1,1}), \text{Enc}_{pk}(m_{2,1})) = 1] \right| \leq \text{negl}(n) \quad (8.6)$$

which provides a formal mathematical definition for **CPA** security.

**Problem 9):**

**Problem 10):**

**Problem 11):**

**Problem 12):**

**Problem 14):**

**Problem 15):**

**Problem 16):**



**Problem 17):**

**Problem 18):**