Problem 4): We denote the cross product of groups \mathcal{G} and \mathcal{H} as $\mathcal{G} \times \mathcal{H}$ and define it by

$$(g,h) \circ (g',h') \equiv (g \circ_{\mathcal{G}} g', h \circ_{\mathcal{H}} h') \tag{4.1}$$

To show that $\mathcal{G} \times \mathcal{H}$ is a group, we begin by proving closure under its operation. Since \mathcal{G} and \mathcal{H} are groups, the we have $(g \circ_{\mathcal{G}} g') \in \mathcal{G}$ and $(h \circ_{\mathcal{H}} h') \in \langle$. Thus $\mathcal{G} \times \mathcal{H}$ is closed under its operation. Next, we must show the existence of an identity in $\mathcal{G} \times \mathcal{H}$. If we modify the expression in 4.1 so that $g' = e_{\mathcal{G}}$ and $h' = e_{\mathcal{H}}$, then we have

$$(g,h) \circ (e_{\mathcal{G}}, e_{\mathcal{H}}) = (g \circ_{\mathcal{G}} e_{\mathcal{G}}, h \circ_{\mathcal{H}} e_{\mathcal{H}})$$

= (g,h)

Therefore, $\mathcal{G} \times \mathcal{H}$ contains an identity element and it is defined as $(e_{\mathcal{G}}, e_{\mathcal{H}})$. Next, we must demonstrate the existence of inversed in $\mathcal{G} \times \mathcal{H}$. To do this, we again modify the expression in 4.1. This time we substitute $g' = g^{-1}$ and $h' = h^{-1}$. Applying this substitution to the expression in 4.1 gives

$$(g,h) \circ (g^{-1}, h^{-1}) = (g \circ_{\mathcal{G}} g^{-1}, h \circ_{\mathcal{H}} h^{-1})$$
$$= (e_{\mathcal{G}}, e_{\mathcal{H}})$$

Thus, $\mathcal{G} \times \mathcal{H}$ contains inverses for each of its elements. Lastly, we show that associativity holds in $\mathcal{G} \times \mathcal{H}$. We begin with

$$((g_1, h_1) \circ (g_2, h_2)) \circ (g_3, h_3) = (g_1 \circ_{\mathcal{G}} g_2, h_1 \circ_{\mathcal{H}} h_2) \circ (g_3, h_3)$$

$$= ((g_1 \circ_{\mathcal{G}} g_2) \circ_{\mathcal{G}} g_3, (h_1 \circ_{\mathcal{H}} h_2) \circ_{\mathcal{H}} h_3)$$

$$(4.2)$$

Using the associativity of \mathcal{G} and \mathcal{H} , we have

$$((g_1 \circ_{\mathcal{G}} g_2) \circ_{\mathcal{G}} g_3, (h_1 \circ_{\mathcal{H}} h_2) \circ_{\mathcal{H}} h_3) = (g_1 \circ_{\mathcal{G}} (g_2 \circ_{\mathcal{G}} g_3), h_1 \circ_{\mathcal{H}} (h_2 \circ_{\mathcal{H}} h_3))$$

Thus, the expression in 4.2 becomes

$$((g_1, h_1) \circ (g_2, h_2)) \circ (g_3, h_3) = ((g_1 \circ_{\mathcal{G}} g_2) \circ_{\mathcal{G}} g_3, (h_1 \circ_{\mathcal{H}} h_2) \circ_{\mathcal{H}} h_3)$$
$$= (g_1 \circ_{\mathcal{G}} (g_2 \circ_{\mathcal{G}} g_3), h_1 \circ_{\mathcal{H}} (h_2 \circ_{\mathcal{H}} h_3))$$

which implies that associativity holds for $\mathcal{G} \times \mathcal{H}$.