Problem 2): Let \mathcal{G} be a finite group and $g \in \mathcal{G}$. Now define $\langle g \rangle \equiv g^0, g^1, g^2, \dots, g^k, \dots$, where $k \in \mathbb{N}$. Beginning with the multiplicative case, let $m, n \in \mathbb{N}$ so that we have

$$g^m g^n = g^{m+n}$$

Since $m, n \in \mathbb{N}$ and \mathbb{N} is closed under addition, $(m+n) \in \mathbb{N}$, it is clear that $g^{m+n} \in \langle g \rangle$. Therefore, $\langle g \rangle$ is closed under its operation. From our definition of $\langle g \rangle$, we know that $g^0 \in \langle g \rangle$. Additionally, $g^0 \equiv e = 1$; therefore $\langle g \rangle$ contains the identity element. Now, let $m \in \mathbb{Z}^+$ and write $g^{-m} g^m$. Using $g^{-m} \equiv (g^{-1})^m$, this yields

$$g^{-m} g^m = (g^{-1})^m g^m = (g^{-1} g)^m = (e)^m = e = 1$$

which implies the existence of an inverse for each element in $\langle g \rangle$. Finally, let $m, n, k \in \mathbb{N}$, then we have

$$g^{m}(g^{n}g^{k}) = g^{m}(g^{n+k}) = g^{m+(n+k)}$$
 (2.1)

Since $\mathbb N$ is associative under addition, the expression in 2.1 may be rewritten as

$$g^{m+(n+k)} = g^{(m+n)+k} = (g^{m+n}) g^k = (g^m g^n) g^k$$

thereby demonstrating the associativity of operations in $\langle g \rangle$. Since $\mathcal G$ is finite, it has order $m=|\mathcal G|$. Therefore, the elements of $\langle g \rangle$ will be repeats of elements in $\mathcal G$ starting with g^{m+1} . Moreover, this means that $\langle g \rangle \subseteq \mathcal G$, thus satisfying the last condition for $\langle g \rangle$ to be a sub-group of $\mathcal G$.

Continuing with the additive case, let $m, n \in \mathbb{N}$ so that we have

$$m \times g \, n \times g = (m+n) \times g$$

Since $m, n \in \mathbb{N}$ and \mathbb{N} is closed under addition, $(m+n) \in \mathbb{N}$, it is clear that $(m+n) \times g \in \langle g \rangle$. Therefore, $\langle g \rangle$ is closed under its operation. From our definition of $\langle g \rangle$, we know that $0 \times g \in \langle g \rangle$. Additionally, $0 \times g \equiv e = 0$; therefore $\langle g \rangle$ contains the identity element. Now, let $m \in \mathbb{Z}^+$ and write $(-m) \times g \times g$. Using $(-m) \times g \equiv m \times (-g)^m$, this yields

$$(-m) \times g \, m \times g = m \times (-g) \, m \times g = m \times (-g \, g) = m \times (e) = e = 0$$

which implies the existence of an inverse for each element in $\langle g \rangle$. Finally, let $m, n, k \in \mathbb{N}$, then we have

$$m \times g \ (n \times g \ k \times g) = m \times g \ ((n+k) \times g) = (m+(n+k)) \times g \tag{2.2}$$

Since $\mathbb N$ is associative under addition, the expression in 2.2 may be rewritten as

$$(m+(n+k))\times g=((m+n)+k)\times g=(m+n)\times g\,k\times g=(m\times g\,n\times g)\,\,k\times g$$

thereby demonstrating the associativity of operations in $\langle g \rangle$. Since $\mathcal G$ is finite, it has order $m=|\mathcal G|$. Therefore, the elements of $\langle g \rangle$ will be repeats of elements in $\mathcal G$ starting with $(m+1)\times g$. Moreover, this means that $\langle g \rangle \subseteq \mathcal G$, thus satisfying the last condition for $\langle g \rangle$ to be a sub-group of $\mathcal G$.