

## Problem 4 (a)

First, import the **math** library

```
In [1]: import math
import scipy.special
```

Then move on to the formal math

### Define R.V.s

We define the random variables  $X$  and  $Y$  as

- $X$  - The outcome of a series
- $Y$  - The number of games in a series

The R.V.  $X$  has the probabilities

- **A** wins in 5 :  $(0.5)^5 = 0.03125$  with multiplicity 1
- **A** wins in 6 :  $(0.5)^5(0.5)^1 = 0.015625$  with multiplicity  $\binom{5}{4}$
- **A** wins in 7 :  $(0.5)^5(0.5)^2 = 0.0078125$  with multiplicity  $\binom{6}{4}$
- **A** wins in 8 :  $(0.5)^5(0.5)^3 = 0.00390625$  with multiplicity  $\binom{7}{4}$
- **A** wins in 9 :  $(0.5)^5(0.5)^4 = 0.00195313$  with multiplicity  $\binom{8}{4}$
- **B** wins in 5 :  $(0.5)^5 = 0.03125$  with multiplicity 1
- **B** wins in 6 :  $(0.5)^5(0.5)^1 = 0.015625$  with multiplicity  $\binom{5}{4}$
- **B** wins in 7 :  $(0.5)^5(0.5)^2 = 0.0078125$  with multiplicity  $\binom{6}{4}$
- **B** wins in 8 :  $(0.5)^5(0.5)^3 = 0.00390625$  with multiplicity  $\binom{7}{4}$
- **B** wins in 9 :  $(0.5)^5(0.5)^4 = 0.00195313$  with multiplicity  $\binom{8}{4}$

Thus, we define the probability and multiplicity arrays for  $X$

```
In [2]: probX = []
probX.append((0.5)**5)
probX.append(((0.5)**5)*((0.5)**1))
probX.append(((0.5)**5)*((0.5)**2))
probX.append(((0.5)**5)*((0.5)**3))
probX.append(((0.5)**5)*((0.5)**4))
probX.append((0.5)**5)
probX.append(((0.5)**5)*((0.5)**1))
probX.append(((0.5)**5)*((0.5)**2))
probX.append(((0.5)**5)*((0.5)**3))
probX.append(((0.5)**5)*((0.5)**4))

print(probX)
print(len(probX))

[0.03125, 0.015625, 0.0078125, 0.00390625, 0.001953125, 0.03125, 0.015625, 0.0078125, 0.00390625, 0.001953125]
10
```

```
In [3]: multX = []
        multX.append(1)
        multX.append(scipy.special.binom(5, 4))
        multX.append(scipy.special.binom(6, 4))
        multX.append(scipy.special.binom(7, 4))
        multX.append(scipy.special.binom(8, 4))
        multX.append(1)
        multX.append(scipy.special.binom(5, 4))
        multX.append(scipy.special.binom(6, 4))
        multX.append(scipy.special.binom(7, 4))
        multX.append(scipy.special.binom(8, 4))

        print(multX)
        print(len(multX))

[1, 5.0, 15.0, 35.0, 70.0, 1, 5.0, 15.0, 35.0, 70.0]
10
```

Now, the R.V.  $Y$  has the probabilities

- Series lasts 5 :  $2(0.5)^5 = 0.0625$
- Series lasts 6 :  $2\binom{5}{4}(0.5)^5(0.5)^1 = 0.15625$
- Series lasts 7 :  $2\binom{5}{4}(0.5)^5(0.5)^2 = 0.234375$
- Series lasts 8 :  $2\binom{5}{4}(0.5)^5(0.5)^3 = 0.273438$
- Series lasts 9 :  $2\binom{5}{4}(0.5)^5(0.5)^4 = 0.273438$

Thus, we define the probability array for  $Y$

```
In [4]: probY = []
        probY.append(2*probX[0]*multX[0])
        probY.append(2*probX[1]*multX[1])
        probY.append(2*probX[2]*multX[2])
        probY.append(2*probX[3]*multX[3])
        probY.append(2*probX[4]*multX[4])

        print(probY)
        print(len(probY))

[0.0625, 0.15625, 0.234375, 0.2734375, 0.2734375]
5
```

Before computing the entropies  $H(x)$  and  $H(y)$ , we check that the probabilities we calculated are normal

```
In [5]: xSum = 0
        ySum = 0
        for i in range(5):
            xSum += 2 * probX[i] * multX[i]
            ySum += probY[i]

        print(xSum)
        print(ySum)

1.0
1.0
```

Since the probabilities are normal, we can calculate  $H(x)$  as

```
In [6]: Hx = 0
for i in range(10):
    Hx += multX[i] * (- probX[i] * math.log2( probX[i] ))

print(Hx)

7.5390625
```

and  $H(y)$  as

```
In [7]: Hy = 0
for i in range(5):
    Hy += - probY[i] * math.log2( probY[i] )

print(Hy)

2.18206960194
```

Now, since  $p(x, y)$  is

```
In [8]: probXY = []
for i in range(10):
    xyRow = []
    for j in range(5):
        if i == j:
            xyRow.append(1/2)
        elif i % 5 == j:
            xyRow.append(1/2)
        else:
            xyRow.append(0)

    probXY.append(xyRow)

print(probXY)

[[0.5, 0, 0, 0, 0], [0, 0.5, 0, 0, 0], [0, 0, 0.5, 0, 0], [0, 0, 0, 0.5, 0], [0,
0, 0, 0, 0.5], [0.5, 0, 0, 0, 0], [0, 0.5, 0, 0, 0], [0, 0, 0.5, 0, 0], [0, 0, 0
, 0.5, 0], [0, 0, 0, 0, 0.5]]
```

and  $H(x, y)$  as

```
In [9]: Hxy = 0
for i in range(10):
    for j in range(5):
        tmpArr = probXY[i]
        tmp = tmpArr[j]

        if tmp == 0:
            Hxy = Hxy
        else:
            Hxy += - tmp * math.log2(tmp)

print(Hxy)

5.0
```

We also have the conditional probability  $H(x | y)$  as

```
In [10]: HxGIVy = 0
         for i in range(5):
             # From player A winning in (i+5) games
             tmpA = multX[i] * (- (probX[i]*probY[i]) * math.log2(probX[i]*probY[i]))

             # From player B winning in (i+5) games
             tmpB = multX[i+5] * (- (probX[i+5]*probY[i]) * math.log2(probX[i]*probY[i]))

             tmpTOT = tmpA + tmpB
             HxGIVy += tmpTOT

         print(HxGIVy)

2.29731956998
```

As well as the conditional probability  $H(y | x)$  as

```
In [11]: HyGIVx = 0
         for i in range(10):
             HyGIVx += multX[i] * (- probY[i % 5]*probX[i] * math.log2(probY[i % 5]*probX[i]
             )))

         print(HyGIVx)

2.29731956998
```

Using the relation  $I(X; Y) = H(X) + H(Y) - H(X, Y)$ , we have  $I(X; Y)$  as

```
In [12]: Ixy = Hx + Hy - Hxy
         print(Ixy)

4.72113210194
```

Next, we need to define the distributions  $p_A$  (for  $X$  when  $A$  wins) and  $q_A$  (for  $Y$  when  $A$  wins). The distribution  $p_A$  can be represented by the matrix

```
In [13]: pA = []
         for i in range(5):
             tmp = multX[i] * (probX[i] + probX[i+5])
             pA.append(tmp)

         print(pA)

[0.0625, 0.15625, 0.234375, 0.2734375, 0.2734375]
```

Similarly, the distribution  $q_A$  can be represented by the matrix

```
In [14]: qA = []
         for i in range(5):
             qA.append(probY[i])

         print(qA)

[0.0625, 0.15625, 0.234375, 0.2734375, 0.2734375]
```

After checking these new distributions for normality, we proceed.

```
In [15]: pAsum = 0
         qAsum = 0
         for i in range(5):
             pAsum += pA[i]
             qAsum += qA[i]

         print(pAsum)
         print(qAsum)

1.0
1.0
```

For  $D(pA \parallel X)$  we have

```
In [16]: dPaX = 0
         for i in range(5):
             dPaX += pA[i] * math.log2(pA[i] / probX[i])

         print(dPaX)

5.35699289806
```

```
In [17]: dPaX = 0
         for i in range(5):
             dPaX += pA[i] * math.log2(pA[i] / (multX[i]*probX[i]))

         print(dPaX)

1.0
```

and for  $D(qA \parallel Y)$  we have

```
In [18]: dQaY = 0
         for i in range(5):
             dQaY += qA[i] * math.log2(qA[i] / probY[i])

         print(dQaY)

0.0
```