

Homework #1

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Information Theory

Problem 2.3): Let \mathbb{P}^n be the set of all n -dimensional probability vectors, with elements $\vec{p} \in \mathbb{P}^n$ defined as $\vec{p} = (p_1, p_2, \dots, p_i, \dots, p_n)$ for $i \in \mathbb{Z}^+ \ni i \leq n$. By the definition of a probability space, we must have

$$\vec{p} \cdot \vec{1} = \sum_{i=1}^n \{p_i\} = 1 \quad , \quad \forall \vec{p} \in \mathbb{P}^n, \quad (2.3-1)$$

where vector $\vec{1}$ is defined as $\vec{1} = (q_1, q_2, \dots, q_k, \dots, q_n) \in \mathbb{Z}^n$ with $q_k = 1, \forall k \in [1, n]$ (where the interval $[1, n]$ is defined such that $[1, n] \subseteq \mathbb{Z}^+$). Furthermore, the definition of a probability space also requires that, for any $\vec{p} \in \mathbb{P}^n$, the elements of \vec{p} (the $p_i \in \vec{p}$ such that $i \in \mathbb{Z}^+ \ni i \leq n$) satisfy the condition

$$p_i \geq 0 \quad (2.3-2)$$

for all $i \in \mathbb{Z}^+ \ni i \leq n$.

The expression in 2.3-1 guarantees that the p_i of any $\vec{p} \in \mathbb{P}^n$ satisfy the bound $0 \leq p_i \leq 1$ where $i \in \mathbb{Z}^+ \ni i \leq n$. Therefore, the relation

$$p_i \log_2 [p_i] \geq 0 \quad (2.3-3)$$

holds for all p_i of any $\vec{p} \in \mathbb{P}^n$. Moreover, for the cases where $p_i = 0$ or $p_i = 1$, it is clear that the expression in 2.3-3 reduces to equality. Specifically, the relation $p_i \log_2 [p_i]$ becomes

$$p_i \log_2 [p_i] = 0 \quad (2.3-4)$$

for the case where $p_i = 0$ or $p_i = 1$. Moreover, the relation in 2.3-4 also represents the **smallest** possible value/result for the expression $p_i \log_2 [p_i]$. That is to say, that when $p_i = 0$ or $p_i = 1$, then $p_i \log_2 [p_i]$ is at a minimum.

The result in 2.3-1, makes it is clear that only ONE p_i in each $\vec{p} \in \mathbb{P}^n$ may have the value $p_i = 1$; therefore the probability vectors $\vec{p} \in \mathbb{P}^n$ which result in a minimum value for $p_i \log_2 [p_i]$ all have exactly one non-zero element with the non-zero element having a value of one. This implies that there are only n such probability vectors, \vec{p}^* within any \mathbb{P}^n . Furthermore, the value of $H(X) = \sum_{i=1}^n \{p_i \log_2 [p_i]\}$ for any such \vec{p}^* is also zero.

Problem 2.4 a): Recall the chain-rule for conditional entropies of X given Y ,

$$H(X | Y) = H(X, Y) - H(Y) \quad (2.4-1)$$

We apply the expression in 2.4-1 to the case of $g(X)$ given X to obtain

$$H(g(X) | X) = H(g(X), X) - H(X) \quad (0.0.1)$$

by rearranging the expression in the previous result as follows,

$$H(g(X), X) = H(X) + H(g(X) | X) \quad (2.4-2)$$

we obtain the desired result.

Problem 2.4 b): For any given value of X , we automatically know $g(X)$. Therefore, the expression for $H(g(X), X)$ in 2.4-2 becomes

$$H(g(X), X) = H(X)$$

which is the desired result.

Problem 2.4 c): Recalling the expression for the conditional entropy chain rule in 2.4-1 and using it for the case where $X = X$ and $Y = g(X)$ yields the result

$$H(X | g(X)) = H(X, g(X)) - H(g(X))$$

rearranging the above expression yields

$$H(X, g(X)) = H(g(X)) + H(X | g(X)) \quad (2.4-3)$$

we obtain the desired result.

Problem 2.4 d): For any arbitrary function, $g(X)$, of a random variable X , the entropy $H(X | g(X))$ satisfies the condition

$$H(X | g(X)) \geq 0 \tag{2.4-4}$$

for the case where $g(X)$ is one-to-one, the relation in 2.4-4 simplifies to

$$H(X | g(X)) = 0$$

Applying the relation in 2.4-4 to the expression in 2.4-3 yields

$$\begin{aligned} H(X, g(X)) &= H(g(X)) + H(X | g(X)) \\ &\geq H(g(X)) + H(X | g(X)) - H(X | g(X)) \\ &\geq H(g(X)) \end{aligned}$$

Problem 2.9 a): Let $\rho(X, Y)$ be a function which is defined according to

$$\rho(X, Y) = H(X | Y) + H(Y | X) \quad (2.9-1)$$

for all x and y . Since conditional probabilities are always non-zero (for arbitrary X and Y we have $H(X | Y) \geq 0$), we can say that $H(X | Y)$ has the property

$$H(X | Y) \geq 0$$

and that $H(Y | X)$ has the property

$$H(Y | X) \geq 0$$

Applying these properties of $H(X | Y) \geq 0$ and $H(Y | X) \geq 0$ to the expression in 2.9-1 yields

$$\rho(X, Y) = H(X | Y) + H(Y | X) \geq 0 \quad (2.9-2)$$

which indicates that $\rho(X, Y)$ satisfies the first property of a metric over all x and y . By its definition in 2.9-1, we can say that $\rho(X, Y)$ is symmetric; therefore, we can additionally say that $\rho(X, Y)$ satisfies the second condition of a metric over all x and y .

Now, consider three random variables, X, Y , and Z . Then write

$$\rho(X, Y) = H(X | Y) + H(Y | X) \quad (2.9-3)$$

and

$$\rho(Y, Z) = H(Y | Z) + H(Y | Z) \quad (2.9-4)$$

and

$$\rho(X, Z) = H(X | Z) + H(Z | X) \quad (2.9-5)$$

We now add the expression in 2.9-3 and 2.9-4 to obtain

$$\begin{aligned} & H(X | Y) + H(Y | X) + H(Y | Z) + H(Z | Y) \\ & \left[H(X | Y) + H(Y | Z) \right] + \left[H(Z | Y) + H(Y | X) \right] \end{aligned} \quad (2.9-6)$$

By the chain rule for conditional entropies, we have $H(X | Y) + H(Y | Z) = H(X, Y | Z)$ and $H(Z | Y) + H(Y | X) = H(Z, Y | X)$ so the expression in 2.9-6 becomes

$$H(X, Y | Z) + H(Z, Y | X)$$

Again applying the chain rule for conditional entropies to the previous result, we have

$$\left[H(X | Z) + H(Y | X, Z) \right] + \left[H(Z | X) + H(Y | Z, X) \right]$$

Since conditional entropies are always greater or equal to zero, the $H(Y | X, Z)$ and $H(Y | Z, X)$ terms in the previous result satisfy $H(Y | X, Z) \geq 0$ and $H(Y | Z, X) \geq 0$. This allows us to rewrite the previous result as

$$\begin{aligned}
\rho(X, Y) + \rho(Y, Z) &= H(Y | Z) + H(Y | Z) + H(X | Y) + H(Y | X) \\
&= \left[H(X | Z) + H(Y | X, Z) \right] + \left[H(Z | X) + H(Y | Z, X) \right] \\
&\geq H(X | Z) + H(Z | X)
\end{aligned}$$

Using the definition from 2.9-5, the previous result becomes

$$\begin{aligned}
\rho(X, Y) + \rho(Y, Z) &= \left[H(X | Z) + H(Y | X, Z) \right] + \left[H(Z | X) + H(Y | Z, X) \right] \\
&\geq H(X | Z) + H(Z | X) \\
&\geq \rho(X, Z)
\end{aligned}$$

thereby indicating that $\rho(X, Y) + \rho(Y, Z) \geq \rho(X, Z)$ holds and, by extension, that the definition in 2.9-1 satisfies the fourth condition of a metric over all x and y .

Finally, we consider the case where $X = Y$ via a one-to-one mapping. Since $H(X, Y) = 0$ iff X is a function of Y and $H(Y, X) = 0$ iff Y is a function of X , $\rho(X, Y)$ can only equal zero if and only if $X = Y$. This satisfies the third and only remaining condition of a metric over all x and y ; therefore, for cases where X and Y are related by a one-to-one mapping, $\rho(X, Y)$ is a metric over all x and y .

Problem 2.9 b): Starting with the expression $I(X; Y) = H(X) - H(X | Y)$ we rearrange to obtain

$$H(X | Y) = H(X) - I(X; Y) \quad (2.9-7)$$

We also obtain

$$H(Y | X) = H(Y) - I(Y; X) \quad (2.9-8)$$

similarly. We now apply the expressions in 2.9-7 and 2.9-8 to the definition of $\rho(X, Y)$ in 2.9-1 to obtain the result

$$\begin{aligned} \rho(X, Y) &= H(X | Y) + H(Y | X) \\ &= H(X) - I(X; Y) + H(Y) - I(Y; X) \end{aligned}$$

Noting that $I(X; Y) = I(Y; X)$ the previous result can be simplified to the expression

$$\begin{aligned} \rho(X, Y) &= H(X) - I(X; Y) + H(Y) - I(Y; X) \\ &= H(X) - I(X; Y) + H(Y) - I(X; Y) \\ &= H(X) + H(Y) - 2I(X; Y) \end{aligned} \quad (2.9-9)$$

which proves the first line of the problem. Next, we note that $I(X; Y) = H(X) + H(Y) - H(X, Y)$ and apply this relation to the expression in 2.9-9 so that we obtain

$$\begin{aligned} \rho(X, Y) &= H(X) + H(Y) - 2I(X; Y) \\ &= H(X) + H(Y) - I(X; Y) - I(X; Y) \\ &= H(X) + H(Y) - I(X; Y) - [H(X) + H(Y) - H(X, Y)] \\ &= H(X, Y) - I(X; Y) \end{aligned} \quad (2.9-10)$$

as our result and proving the second line of the problem. Finally, we again note

$I(X; Y) = H(X) + H(Y) - H(X, Y)$ and then apply it to the expression in 2.9-10 to yield the result

$$\begin{aligned}
 \rho(X, Y) &= H(X, Y) - I(X; Y) \\
 &= H(X, Y) - [H(X) + H(Y) - H(X, Y)] \\
 &= 2H(X, Y) - H(X) - H(Y)
 \end{aligned}$$

which proves the third and final line of the problem.

Problem 2.10 a): Let X_1 and X_2 be discrete R.V.s having PMFs $p_1(\cdot)$ and $p_2(\cdot)$, respectively.

Furthermore, let the alphabets of X_1 and X_2 be denoted as \mathcal{X}_1 and \mathcal{X}_2 , respectively. These alphabets are defined $\mathcal{X}_1 = \{1, 2, \dots, m\}$ and $\mathcal{X}_2 = \{m+1, \dots, n\}$, where $m, n \in \mathbb{Z}^+$ and $m < n$. Now define another R.V. X such that

$$X = \begin{cases} X_1, & \text{with probability } \alpha \\ X_2, & \text{with probability } 1 - \alpha \end{cases}$$

In order to find $H(X)$ in terms of $H(X_1)$, $H(X_2)$, and α , we must first define the function, of X , $\theta = f(X)$, as follows

$$f(X) = \begin{cases} 1, & \text{when } X = X_1 \\ 2, & \text{when } X = X_2 \end{cases}$$

Noting that $H(X) = H(X, f(X)) = H(f(X)) + H(X | f(X))$, we can write

$$\begin{aligned}
H(X) &= H(X, f(X)) \\
&= H(f(X)) + H(X | f(X)) \\
&= H(f(X)) + f(f(X) = 1) H(X | f(X) = 1) + f(f(X) = 2) H(X | f(X) = 2) \\
&= \boxed{H(\alpha) + \alpha H(X_1) + (1 - \alpha) H(X_2)} \tag{2.10-1}
\end{aligned}$$

which gives $H(X)$ in terms of $H(X_1)$, $H(X_2)$, and α , where $H(\alpha)$ is the binary entropy function written, in terms of α , as $H(\alpha) = -\alpha \log_2 [\alpha] - (1 - \alpha) \log_2 [1 - \alpha]$.

Problem 2.10 b): Since the function $H(\alpha) = -\alpha \log_2 [\alpha] - (1 - \alpha) \log_2 [1 - \alpha]$ is greater than or equal to zero for all $\alpha \in [0, 1]$, we may rewrite the expression in 2.10-1 as

$$\begin{aligned}
H(X) &= H(\alpha) + \alpha H(X_1) + (1 - \alpha) H(X_2) \\
H(X) - H(\alpha) &= \alpha H(X_1) + (1 - \alpha) H(X_2) \\
H(X) &\leq H(X_1) + H(X_2) \tag{2.10-2}
\end{aligned}$$

by also noting that $\alpha H(X_1) \leq H(X_1)$ and $(1 - \alpha) H(X_2) \leq H(X_2)$. Taking the expression in 2.10-2 to the power 2 gives

$$2^{H(X)} \leq 2^{H(X_1) + H(X_2)} \tag{0.0.2}$$

We can interpret the expression in 0.0.2 as indicating that, while X_1 and X_2 may have disjoint alphabets, it is possible that some combinations of elements from these alphabets may yield identical entropies.

Problem 2.11 a): Starting with

$$\rho = 1 - \frac{H(X_2 | X_1)}{H(X_1)} \quad (2.11-1)$$

We obtain a common denominator and note that, since X_1 and X_2 are identically distributed, $H(X_1) = H(X_2)$ to yield the result

$$\begin{aligned} \rho &= 1 - \frac{H(X_2 | X_1)}{H(X_1)} \\ &= \frac{H(X_1) - H(X_2 | X_1)}{H(X_1)} \\ &= \frac{H(X_2) - H(X_2 | X_1)}{H(X_1)} \end{aligned}$$

Now, since $I(X_1; X_2) = H(X_1) - H(X_1 | X_2) = H(X_2) - H(X_2 | X_1)$, the previous result finally becomes

$$\begin{aligned} \rho &= \frac{H(X_2) - H(X_2 | X_1)}{H(X_1)} \\ &= \frac{I(X_2; X_1)}{H(X_1)} = \boxed{\frac{I(X_1; X_2)}{H(X_1)}} \end{aligned} \quad (2.11-2)$$

since $I(X_2; X_1) = I(X_1; X_2)$, by the definition of mutual information.

Problem 2.11 b): Note that, by the definitions of both entropy and conditional entropy, the expressions

$$0 \leq H(X_2 | X_1) \leq (H(X_2) = H(X_1)) \quad (2.11-3)$$

and

$$0 \leq H(X_1 | X_2) \leq (H(X_1) = H(X_2)) \quad (2.11-4)$$

are both valid. Dividing 2.11-3 by $H(X_1)$ and 2.11-4 by $H(X_2)$, we obtain

$$0 \leq \frac{H(X_2 | X_1)}{H(X_1)} \leq 1$$

and

$$0 \leq \frac{H(X_1 | X_2)}{H(X_2)} \leq 1$$

Since these quantities are bounded by $[0, 1]$, the expression in 2.11-1 then implies that $0 \leq \rho \leq 1$.

Problem 2.11 c): Recalling the result in 2.11-2, it is clear that the only way for $\rho = 0$ to hold, is for $I(X_1; X_2) = 0$ to also hold. Since $I(X_1; X_2) = 0$ only holds for independent X_1 and X_2 , $\rho = 0$ can hold iff X_1 and X_2 are independent.

Problem 2.11 d): Again, from the expression in 2.11-1, it is clear that $\rho = 1$ only if $H(X_2 | X_1) = 0$. For this to be the case, X_2 **must** be a function of X_1 . Additionally, that function must be **one-to-one**, since the relation in 2.10-2 implies symmetry which means that X_1 can also be represented as a function of X_2 .

Problem 2.24 a): Consider a choice of four unique objects and specify a specific, arbitrarily picked object to be "special". We can model this situation using two random variables X and Y . We define X to be the random variable describing whether the "special" object was picked, so $\mathcal{X} = \{0, 1\}$ with $X = 0$ if the "special" object was picked and $X = 1$ otherwise. The entropy of this random variable, $H(X)$, is the entropy we seek to find, namely $H(1/4)$. Additionally, we define Y to be the random variable describing which of the three "non-special" items was picked, thus $\mathcal{Y} = \{0, 1, 2\}$. We define $Y = 0$ if the first "non-special" object was picked, $Y = 1$ for the second, and $Y = 2$ for the third.

We can also define a third random variable Z which describes choosing one of four unique objects, thus $\mathcal{Z} = \{0, 1, 2, 3\}$ and $\Pr[Z = z] = 1/4$ for all $z \in \mathcal{Z}$. Furthermore, we can equate $Z = XY$, thus we can say

$$\begin{aligned} H(Z) &= H(X, Y) \\ &= H(X) + H(Y | X) \end{aligned}$$

by also using the chain rule $H(X, Y) = H(X) + H(Y | X)$. Without loss of generality, we also say that the "special" object is represented by $Z = 0$. That is to say that $H(X) = H(1/4) = H(Z = 0)$.

We continue by applying the relation $H(X) = H(1/4)$ to the previous result gives us

$$\begin{aligned} H(Z) &= H(X) + H(Y | X) \\ &= H(1/4) + H(Y | X) \end{aligned}$$

which becomes

$$\begin{aligned}
&\longrightarrow H(1/4) = H(Z) - H(Y | X) \\
&= H(Z) + \sum_{\substack{x,y \\ x=1 \\ y \in \mathcal{Y}}} \left\{ p(x,y) \log_2 [p(y|x)] \right\} \\
&= H(Z) + \sum_{\substack{x,y \\ x=1 \\ y \in \mathcal{Y}}} \left\{ \sum_z \{p(z) p(y | x)\} \log_2 [p(y|x)] \right\} \\
&= H(Z) + \sum_{\substack{x,y \\ x=1 \\ y \in \mathcal{Y}}} \left\{ \sum_z \{p(z)\} p(y | x) \log_2 [p(y|x)] \right\} \\
&= H(Z) + \sum_{\substack{x,y \\ x=1 \\ y \in \mathcal{Y}}} \left\{ \sum_z \{p(z)\} p(y | x) \log_2 [p(y|x)] \right\} \\
&= H(Z) + \sum_{\substack{z \\ z \neq 0}} \left\{ \sum_{\substack{x,y \\ x=1 \\ y \in \mathcal{Y}}} \{p(z) p(y | x) \log_2 [p(y|x)]\} \right\} \\
&= H(Z) + \sum_{\substack{z \\ z \neq 0}} \left\{ p(z) \left(\sum_{\substack{x,y \\ x=1 \\ y \in \mathcal{Y}}} \{p(y | x) \log_2 [p(y|x)]\} \right) \right\} \tag{2.24-1}
\end{aligned}$$

through some simple rearranging along with noting the definition of $H(Y | X)$ and the fact that $Z = XY$ implies that $p(z) = p(x, y)$. Now, since $\Pr[Z = z] = 1/4$ for all $z \in \mathcal{Z}$ we can compute the value of $H(Z)$ to be

$$H(Z) = \sum_{i=1}^4 \left\{ \frac{1}{4} \log_2 \left[\frac{1}{4} \right] \right\} = 2 \tag{2.24-2}$$

Moreover, we can also evaluate the summation over z in the second term of 2.24-1 to obtain

$$\sum_{\substack{z \\ z \neq 0}} \left\{ p(z) \left(\sum_{\substack{x,y \\ x=1 \\ y \in \mathcal{Y}}} \{p(y | x) \log_2 [p(y|x)]\} \right) \right\} = \frac{3}{4} \sum_{\substack{x,y \\ x=1 \\ y \in \mathcal{Y}}} \left\{ p(y | x) \log_2 [p(y|x)] \right\}$$

Additionally, we can say that, when $X = 1$, $p(y|x) = 1/3$ for all $y \in \mathcal{Y}$. Therefore, we can evaluate the

summation over x and y in the previous result

$$\begin{aligned}
 \sum_{\substack{z \\ z \neq 0}} \left\{ p(z) \left(\sum_{\substack{x,y \\ x=1 \\ y \in \mathcal{Y}}} \{ p(y | x) \log_2 [p(y|x)] \} \right) \right\} &= \frac{3}{4} \sum_{\substack{x,y \\ x=1 \\ y \in \mathcal{Y}}} \left\{ p(y | x) \log_2 [p(y|x)] \right\} \\
 &= \frac{3}{4} \left(\beta \left(\frac{1}{\beta} \log_2 \left[\frac{1}{3} \right] \right) \right) \\
 &= \frac{3}{4} \log_2 \left[\frac{1}{3} \right] \\
 &= -\frac{3}{4} \log_2 [3] \tag{2.24-3}
 \end{aligned}$$

We can now apply the values from 2.24-2 and 2.24-3 to the expression in 2.24-1 to obtain

$$\begin{aligned}
 H(1/4) &= H(Z) + \sum_{\substack{z \\ z \neq 0}} \left\{ p(z) \left(\sum_{\substack{x,y \\ x=1 \\ y \in \mathcal{Y}}} \{ p(y | x) \log_2 [p(y|x)] \} \right) \right\} \\
 &= 2 + \left(-\frac{3}{4} \log_2 [3] \right) \\
 &= \boxed{2 - \frac{3}{4} \log_2 [3] \approx 0.811278 \text{ bits}}
 \end{aligned}$$

as our result and final answer.

Problem 2.24 b): Let $f(x)$ be an arbitrary function which maps $f : x \in \mathbb{R} \rightarrow y \in \mathbb{R}$ where $y = f(x) \in \mathbb{R}$ for $x, y \in \mathbb{R}$. The average value of any function such as $f(x)$ over any interval covered by the function, $[a, b] \subseteq \mathbb{R} \ni a < b$, is defined according to the integral

$$\bar{f}(x \in [a, b]) = \frac{1}{b-a} \int_a^b f(x) dx$$

so that the average value can be determined by evaluating the integral expression.

When we consider the average entropy, $\overline{H}(p)$ of a system having uniform distribution, our bounds are clearly $[0, 1]$. Thus, since $H(p) = -p \log_2 [p] - (1-p) \log_2 [1-p]$, the above integral expression for the average value of a function becomes

$$\begin{aligned}\overline{H}(p \in [0, 1]) &= \frac{1}{1-0} \int_0^1 H(p) dp \\ &= \int_0^1 \left(-p \log_2 [p] - (1-p) \log_2 [1-p] \right) dp\end{aligned}\tag{2.24-4}$$

Using Mathematica to evaluate the integral in 2.24-4, we obtain

$$\begin{aligned}\overline{H}(p \in [0, 1]) &= \frac{1}{1-0} \int_0^1 H(p) dp \\ &= \int_0^1 \left(-p \log_2 [p] - (1-p) \log_2 [1-p] \right) dp \\ &= \boxed{0.721}\end{aligned}$$

Problem 2.29 a) Starting with $H(X, Y | Z)$, we apply the chain rule for entropies to obtain

$$H(X, Y | Z) = H(X | Z) + H(Y | X, Z)$$

Since all entropies are greater than or equal to zero (in this case $H(Y | X, Z)$), the previous result can be rewritten as

$$\begin{aligned}H(X, Y | Z) &= H(X | Z) + H(Y | X, Z) \\ &\geq H(X | Z)\end{aligned}$$

Problem 2.29 b):

Problem 2.29 c):

Problem 2.29 d):

Problem 2.31):

Problem 2.35):

Problem 2.42 a):

Problem 2.42 b):

Problem 2.42 c):

Problem 2.42 d):