Comparison with Rosenthal et a. contrast effect size indexes

Supporting material for Standardized means difference effect size measures for planned comparisons, trend analysis and other applications of contrast analysis

mc & mp

luglio 02, 2018

Aim

Here we study some relations between the suggested way to scale the contrast Cohen's d and Rosenthal, Rosnow, and Rubin (2000) contrast effect size indexes.

Definitions

Rosenthal, Rosnow, and Rubin (2000) defined three contrast effect size indexes. The $r_{contrast}$, $r_{effectsize}$ and r_{alert}. Here we focus on $r_{contrast}$, later we show why the other two indexes are not related with δ-like measures. It is crucial to recall that Rosenthal, Rosnow, and Rubin (2000) effect size indexes are defined for the sample, not the population. We will denote with r those sample indexes, and with ρ their population counterparts. In the remaning we assume a design with k groups, no covariates, equal cell sizes n and equal within-group variance σ^2. Sample size is referred to as N, with $N = k \cdot n$. The design can be of any shape, but the contrasts are assumed to spread across all k cells. In the document, referenced equations are definitions or results, whereas derivations are not referenced.

Effect size index $r_{contrast}$

It is the partial correlation between each score and its associated contrast weight after we have adjusted for non contrast differences among treatment conditions (Rosenthal, Rosnow, and Rubin 2000, 38). It is originally defined in terms of F-tests.

$$r_{contrast} = \sqrt{\frac{F_{contrast}}{F_{contrast} + df_e}} \tag{1}$$

where df_e is the error degrees of freedom, which in our case is $df_e = N - k$.

Aim

Now we show that the population counterpart of $r_{contrast}$ is the partial eta-squared η_p^2. That is

$$\rho_{contrast} = \sqrt{\eta_p^2} \tag{2}$$

1

Derivation

Since

$$F_{contrast} = \frac{MS_{contrast}}{MS_{within}} = \frac{SS_{contrast}}{SS_{within}} \cdot N - k$$

the $r_{contrast}$ index can be written in terms of sum of squares.

$$r_{contrast} = \sqrt{\frac{\frac{SS_{contrast}}{SS_{within}} \cdot N - k}{\frac{SS_{contrast}}{SS_{within}} \cdot N - k + N - k}}$$

$$r_{contrast} = \sqrt{\frac{SS_{contrast} \cdot N - k}{SS_{contrast} \cdot (N - k) + (N - k)SS_{within}}}$$

which yields to the results we were looking for.

$$r_{contrast} = \sqrt{\frac{SS_{contrast}}{SS_{contrast} + SS_{within}}} \tag{3}$$

This is the defintion of the partial η_p^2, after square root. For the population, we have

$$\rho_{contrast} = \sqrt{\frac{\sigma^2_{contrast}}{\sigma^2_{contrast} + \sigma}} = \sqrt{\eta_p^2} \tag{4}$$

Relation between $\rho_{contrast}$ and δ-like measures

As a consequence of the fact that $\rho_{contrast} = \sqrt{\eta_p^2}$, we have that:

$$\rho_{contrast} = \frac{\delta_z}{\sqrt{\delta_z^2 + k}} \tag{5}$$

and

$$\rho_{contrast} = \frac{\delta_g}{\sqrt{\delta_g^2 + \frac{z^2}{g^2}k}} \tag{6}$$

Relation between $r_{contrast}$ and d-like measures (sample version)

For the sample indexes, we have

$$r_{contrast} = \frac{d_z}{\sqrt{d_z^2 + k \cdot \frac{N-k}{N}}}$$

$$r_{contrast} = \frac{d_z}{\sqrt{d_z^2 + k \cdot \frac{k(n-1)}{nk}}}$$

which yields

$$r_{contrast} = \frac{d_z}{\sqrt{d_z^2 + k \cdot \frac{(n-1)}{n}}} \tag{7}$$

Reversing:

$$\delta_z = \sqrt{k \cdot \frac{n-1}{n} \cdot \frac{r_{contrast}}{\sqrt{1 - r_{contrast}^2}}} \tag{8}$$

For the g-method, we have

$$r_{contrast} = \frac{d_g}{\sqrt{d_g^2 + \frac{g^2}{z^2} k \cdot \frac{(n-1)}{n}}} \tag{9}$$

$$\delta_z = \frac{g}{z} \sqrt{k \cdot \frac{n-1}{n} \cdot \frac{r_{contrast}}{\sqrt{1 - r_{contrast}^2}}} \tag{10}$$

Behavior

Comparison within the same design

The behavior of the three indexes, δ_g, δ_z and $\rho_{contrast}$ is quite obvious when they are compared within the same design (i.e. for a given k). Both δ_g, δ_z increase linearly with C_k, other things being equal, and thus any two effect sizes indexes can be compared independently of the size of their absolute values, because their difference depends only on the difference between contrast values C_k.

The $\rho_{contrast}$ is bounded between -1 and 1, and thus its increment is not linearly related with C_k. Thus, the comparison between any two $\rho_{contrast}$ values depends on the difference in C_k and the absolute values of the $\rho_{contrast}$ indexes. In particular, the larger $\rho_{contrast}$, the smaller is the increment for a given increment of C_k.

Let C_k be the contrast value of a contrast 1, and $C_k + \epsilon$ the contrast value of contrast 2. Assume $g = 1$. The method-g effect size, for the two contrasts, is

$$\delta_{g1} = \frac{C_k}{\sigma} \quad ; \quad \delta_{g2} = \frac{C_k + \epsilon}{\sigma}$$

whose absolute difference is clearly $|\epsilon|/\sigma$. For $r_{contrast}$ we have

$$r_{g1} = \frac{C_k}{\sqrt{C_k^2 + k\sigma^2}} \quad ; \quad r_2 = \frac{C_k + \epsilon}{\sqrt{(C_k + \epsilon)^2 + k\sigma^2}}$$

whose difference

$$\frac{C_k^2}{C_k^2 + k\sigma^2} - \frac{(C_k + \epsilon)^2}{(C_k + \epsilon)^2 + k\sigma^2}$$

is related to ϵ and the magnitude of δ_{g1} as shown in Figure 1.

Clearly, the increment in ρ for a given ϵ depends on how large is ρ that is incremented.
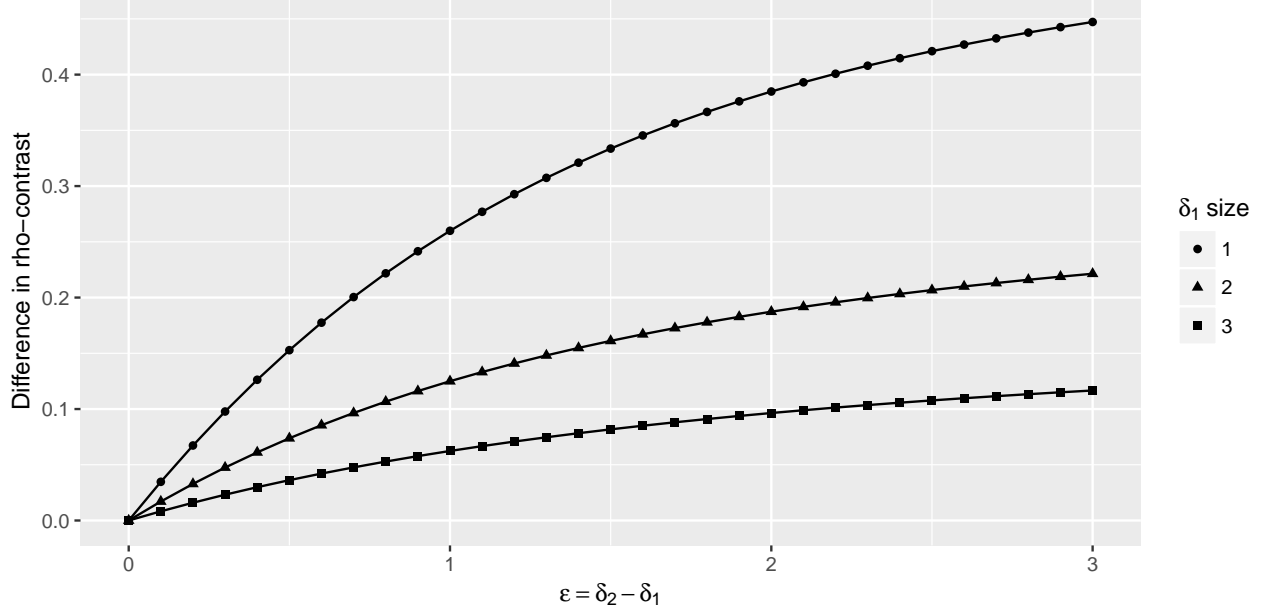
Figure 1: Differences in r-contrast as a function of difference in delta indexes for different size of delta

Comparison across different designs

It is interesting to show how those indexes relate in particular situations, where the intuition may lead to false expectations.

Example 1: Same mean difference across different designs

Consider the case where the same contrast value $C_k = \sum c_i \cdot \mu_i$ is expected across designs with different k. Let $c_1 = 1$ and $c_i = -1/(k-1)$ for $1 > i \geq k$

To give an example, assume $\mu_1 = 10$ and $\mu_i = 5$ for $1 > i \geq k$, and $\sigma = 20$. Four possible sets of expected means for different k is shown in Table 1.

	m1	m2	m3	m4	m5	m6
k=2	10	5				
k=4	10	5	5	5		
k=6	10	5	5	5	5	5

Table 1: Example 1: same mean comparison spread across different number of groups

For the g-method we have that $g = \frac{2}{(1+(k-1)\cdot(1/(k-1))} = 1$, thus gives:

$$\delta_{gk} = \frac{C_k}{\sigma} \tag{11}$$

For any k. The z-method gives:

$$\delta_{zk} = \frac{C_k}{\sqrt{\sum c_i^2} \cdot \sigma} = \frac{C_k}{\sqrt{1 + \frac{k-1}{(k-1)^2}} \cdot \sigma}$$

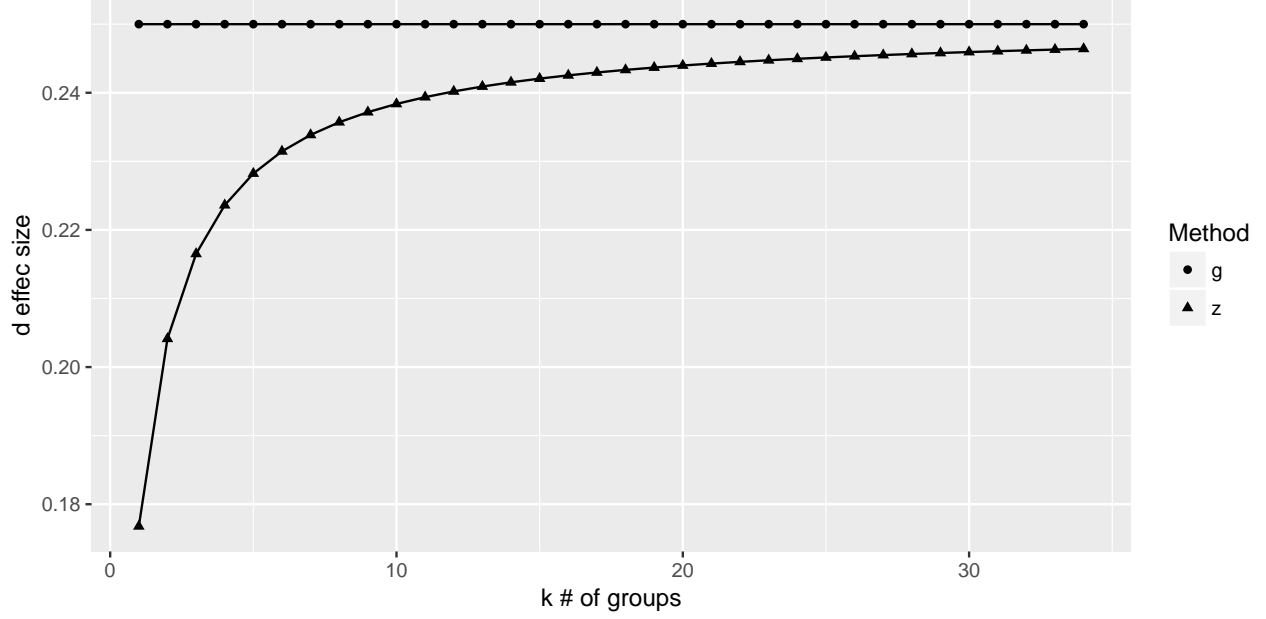$$\delta_{zk} = \frac{C_k}{\sqrt{\frac{(k-1)+1}{(k-1)}} \cdot \sigma}$$

4

Figure 2: Example 1: d-like effect size as a function of number of groups

$$\delta_{zk} = \frac{C_k}{\sqrt{\frac{k}{(k-1)}} \cdot \sigma} \tag{12}$$

The behavior of δ_g as a function of the number of group is clear: it does not change (Figure 1). The δ_z differs because of $k/(k-1)$ in the denominator. Becuase $k/(k-1)$ becomes smaller with increasing k, δ_z is smaller than δ_g for small number of groups, increases with k, and converges to δ_g as the number of group becomes very large (cf. Figure 1).

The corresponding $\rho_{contrasts}$ can be derived as:

$$\rho_k^2 = \frac{\frac{C_k^2}{\frac{k}{(k-1)} \cdot \sigma^2}}{\frac{C_k^2}{\frac{k}{(k-1)} \cdot \sigma^2} + k}$$

$$\rho_k^2 = \frac{C_k^2}{C_k^2 \cdot + \frac{k^2}{k-1}\sigma^2}$$

$$\rho_k = \frac{C_k}{\sqrt{C_k^2 \cdot + \frac{k^2}{k-1}\sigma^2}} \tag{13}$$

It can be noticed that C_k does not change with k, but the weight of the error variance, $k^2/(k-1)$ becomes larger and larger whith increasing k, making the coefficient to decrease with k.

Its behavior for the present case is shown in Figure 2:

What happens is that as the number of groups increases, the error sum of squares increases without increasing the contrast sum of squares, thus the effect size decreases.
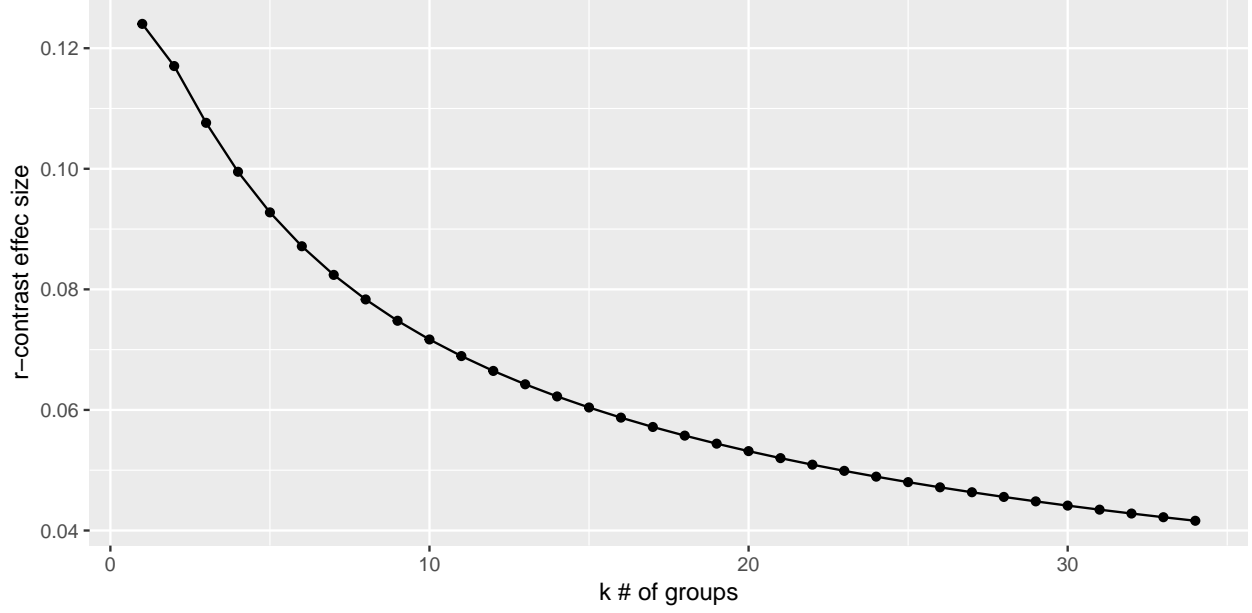
Figure 3: Example 1: Population r-contrast effect size as a function of number of groups

Example 2: Two means comparison embedded in different designs

Another interesting case is when the researcher is interested in the same comparison $C_k = \mu_1 - \mu_2$, but the comparison is embedded in a larger design with k groups. Thus, for any k, we have $C_k = \mu_1 - \mu_2 + \sum 0 \cdot \mu_i$, with $2 > i \geq k$. It is simple to verify that:

$$\delta_{zk} = \frac{C_k}{\sqrt{2} \cdot \sigma} \tag{14}$$

and

$$\delta_{gk} = \frac{C_k}{\sigma} \tag{15}$$

for any k.

However, the Rosenthal's contrast effect size will change as the number of groups increases. In particular

$$\rho_{contrast} = \frac{\delta_z}{\sqrt{\delta_z^2 + k}}$$

$$\rho_{contrast}^2 = \frac{\frac{C_k^2}{2\sigma^2}}{\frac{C_k^2}{2\sigma^2} + k}$$

$$\rho_{contrast} = \sqrt{\frac{C_k^2}{C_k^2 + 2k\sigma^2}} \tag{16}$$

which become smaller and smaller as the number of groups increases. Note that this does not imply any deficiency in $\rho_{contrast}$. It is a measure of variance explained, thus it is correct that a single comparison should explain less and less variance as the group number increases. It is a different index as compared with the δ-like indexes, and conveys different information.
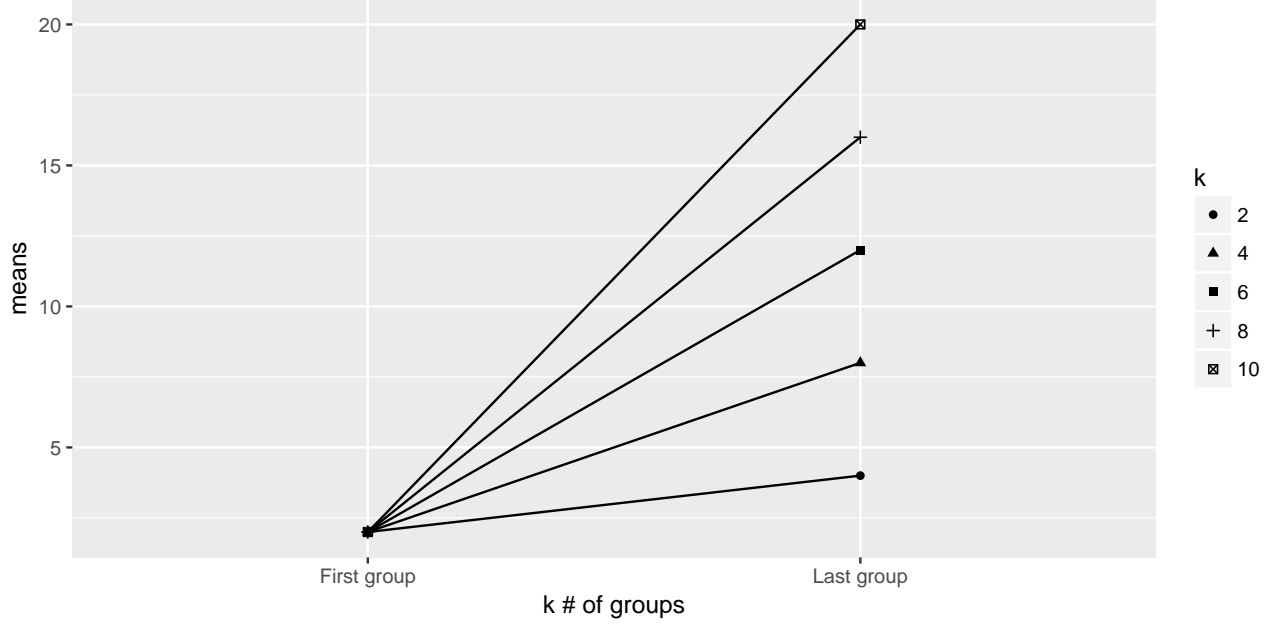
6

Figure 4: Example 3: extreme groups trend as a function of number of groups

Example 3: Same linear trend

We now consider another example, where the same linear trend is compared across differnt designs. Let assume we have k group means μ_i, such that any two subsequent means are ϕ units apart, that is, $\mu_i = i \cdot \phi$. Contrast weights are linear trend weights, $c_i = i - (1 + k)/2$.

Altough it may not seem intuitive at first glance, this set-up implies an increasingly stronger effect as the number of the groups increases. This is because as the number of groups increases, the linear trend defines a larger difference between the first and the last group. This can be seen (Figure 4) by ploting the expected means, for increasingly largeer designs, only for the two extreme groups.

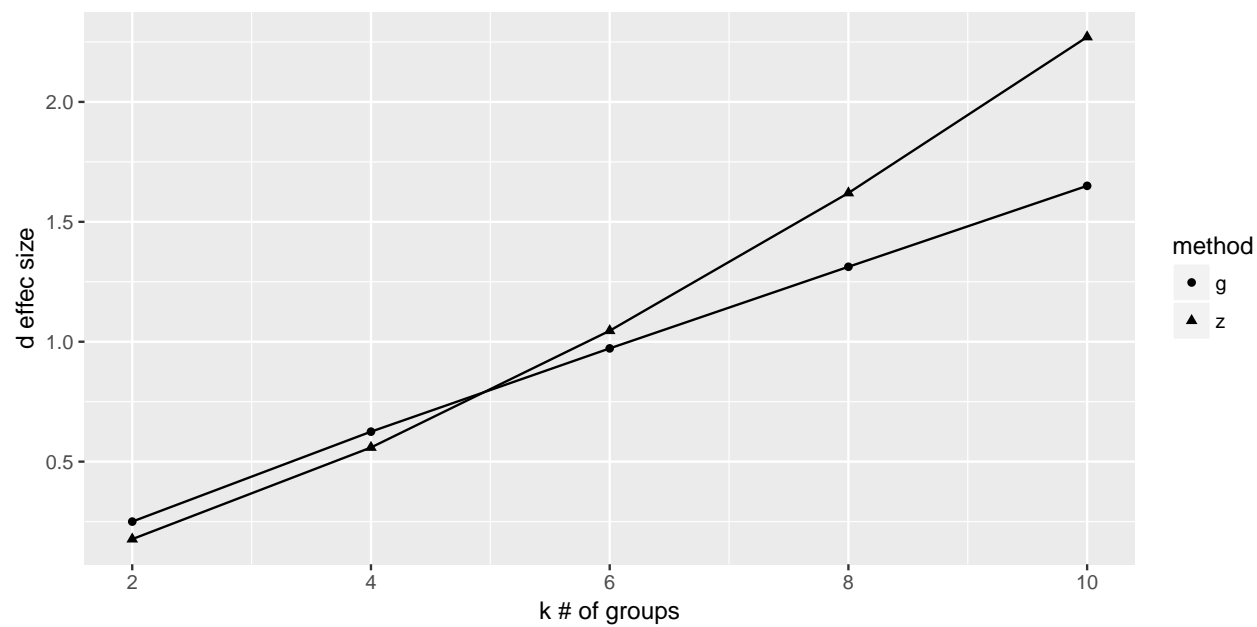Accordingly, all the effect size indexes increase with k.

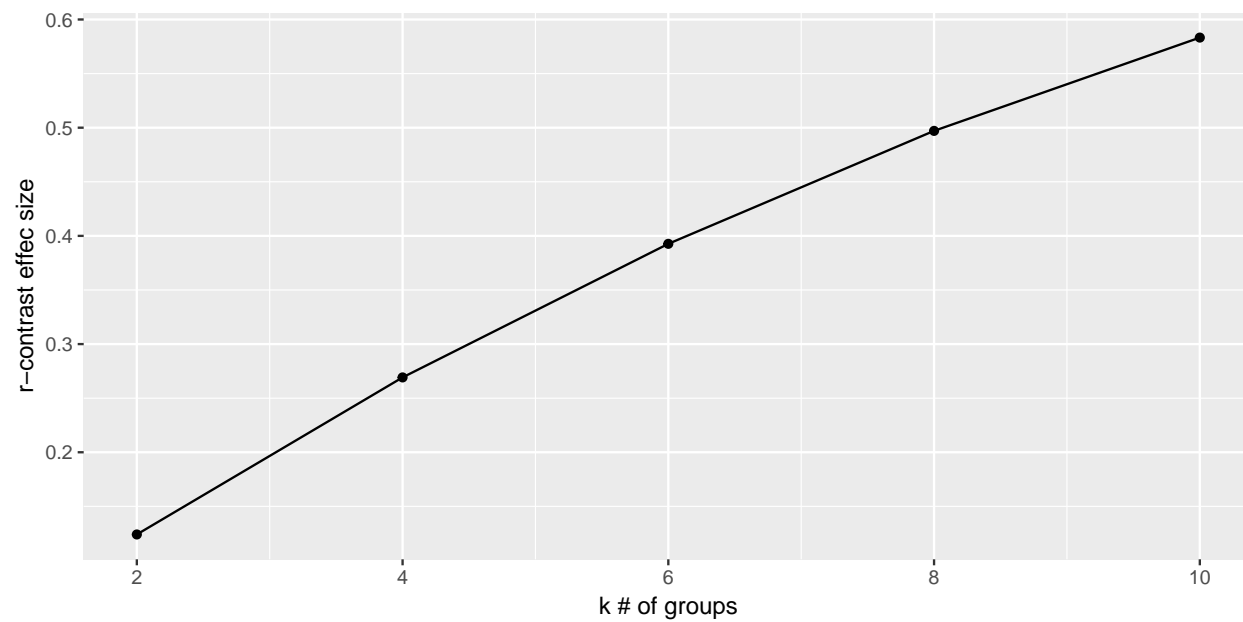Figure 5: d-like effect size as a function of number of groups

Figure 6: r-constrast effect size as a function of number of groups

Formal derivation

We can now reach the same results a bit more formally. The contrast value C_k is:

$$C_k = \sum [(i - (1+k)/2) \cdot i \cdot \phi] = \frac{\phi}{2} \cdot [\sum (2i - 1 - k) \cdot i]$$

$$C_k = \frac{\phi}{2} \cdot [\sum (2i^2 - i - ki)]$$

$$C_k = \frac{\phi}{2} \cdot [2 \sum (i^2) - \sum (i) - k \sum (i)]$$

$$C_k = \frac{\phi}{2} \cdot [2 \sum (i^2) - (1 - k) \sum (i)]$$

now, because $\sum i^2 = \frac{k(k+1)(2k+1)}{6}$ and $\sum i = \frac{k(k+1)}{2}$ we have:

$$C_k = \frac{\phi}{2} \cdot [2 \frac{k(k+1)(2k+1)}{6} - \frac{(1+k)k(k+1)}{2}]$$

$$C_k = \frac{\phi}{2} \cdot k(k+1)[\frac{(2k+1)}{3} - \frac{(1+k)}{2}]$$

$$C_k = \frac{\phi}{2} \cdot k(k+1)[\frac{(4k+2-3-3k)}{6}]$$

$$C_k = \frac{\phi}{2} \cdot k(k+1)[\frac{(k-1)}{6}]$$

$$C_k = \frac{\phi}{12} \cdot k(k+1)(k-1) \tag{17}$$

For the g-method, we need the sum of the obsolute contrasts weights. Consider k as even (when it is odd similar derivation can be otained showing the same properties), and note that for the first half of the weights, their sum is

$$\sum_{i=1}^{k/2} (i - (1+k)/2) = \frac{1}{2} \sum_{i=1}^{k/2} (2i) - k/2 - k^2/2$$

Recall that the sum of the first $k/2$ i is:

$$\frac{1}{2} \cdot \frac{k}{2} \cdot \frac{k-2}{2} = \frac{k(k-2)}{8}$$

than, the sum of the first half contrast weights becomes:

$$\sum (i - (1+k)/2) = \frac{1}{2} [2 \cdot \frac{k(k-2)}{8} - \frac{k}{2} - \frac{k^2}{2}]$$

$$\sum (i - (1+k)/2) = \frac{1}{2} [k \cdot \frac{k-2-2-2k}{4}]$$

$$\sum (i - (1+k)/2) = -\frac{k}{2} \frac{k}{4} = \frac{k^2}{8}$$

Obviously, the second half has the same sum with oppositve sign. Being the sum carried over the absolute values, this gives:

$$\sum |(i - (1 + k)/2)| = \frac{k^2}{4}$$

Thus, the g multiplier is $g = 8/k^2$. and the effect size is:

$$\delta_g = \frac{8 \cdot \phi \cdot k(k+1)(k-1)}{12 \cdot k^2 \cdot \sigma}$$

$$\delta_g = \frac{2 \cdot \phi \cdot (k+1)(k-1)}{3 \cdot k \cdot \sigma} \tag{18}$$

We can see that δ_g increases linearly with k, as the example in the Figure 4 shows. For the z-method, we need $\sqrt{\sum c^2}$, that is:

$$\sqrt{\sum (i - (1 + k)/2)^2} = \sqrt{\sum (i^2) - \frac{k(1+k)^2}{4}}$$

because $\sum i^2 = \frac{k(k+1)(2k+1)}{6}$ and $\sum i = \frac{k(k+1)}{2}$ we have:

$$= \sqrt{\frac{k(k+1)(2k+1)}{6} - \frac{k(1+k)^2}{4}}$$

$$= \sqrt{\frac{2k(k+1)(2k+1) - 3k(1+k)^2}{12}}$$

$$= \sqrt{k(1+k)\frac{2(2k+1) - 3(1+k)}{12}}$$

$$= \sqrt{k(1+k)\frac{4k+2-3-3k}{12}}$$

$$\sqrt{\sum c_i^2} = \sqrt{\frac{k(1+k)(k-1)}{12}}$$

therefore:

$$\delta_z = \frac{\phi \cdot k(k+1)(k-1)}{\sigma \cdot 12\sqrt{\frac{k(k+1)(k-1)}{12}}}$$

$$\delta_z = \frac{\phi \cdot k(k+1)(k-1)}{\sigma \cdot \sqrt{12 \cdot k(k+1)(k-1)}}$$

$$\delta_z = \frac{\phi}{\sigma}\sqrt{\frac{k(k+1)(k-1)}{12}} \tag{19}$$

which increases non-linearly with k, as shown in Figure 4.

Finally, the $\rho_{contrast}$ can be obtained as

$$\rho_{contrast}^2 = \frac{\frac{\phi^2}{\sigma^2}\frac{k(k+1)(k-1)}{12}}{\frac{\phi^2}{\sigma^2}\frac{k(k+1)(k-1)}{12} + k}$$

$$\rho_{contrast}^2 = \frac{\phi^2 k(k+1)(k-1)}{\phi^2 k(k+1)(k-1) + 12k\sigma^2}$$

$$\rho_{contrast}^2 = \frac{\phi^2(k+1)(k-1)}{\phi^2(k+1)(k-1) + 12\sigma^2}$$

$$\rho_{contrast} = \frac{\phi\sqrt{k(k+1)(k-1)}}{\sqrt{\phi^2 k(k+1)(k-1) + 12k\sigma^2}}$$

which increases with k. However it does that not increase linearly because is bounded to 1.

Other effect size indexes defined by Rosenthal et al.

$r_{effectsize}$ does not remove the variance explained by other sources other than the contrast, thus its definition is:

$$r_{effectsize} = \sqrt{\frac{F_{contrast}}{F_{contrast} + F_{noncontrast}df_{noncontrast} + df_e}}$$

by definition the numerator is the overal variance of the dependen variable, i.e. SS_{tot}. Thus

$$r_{effectsize} = \sqrt{\frac{SS_{contrast}}{SS_{tot}}}$$

Now, this effect size does not have a neat correspondence with any d measure, besides for the trivial case where $k = 2$, in which we have:

$$r_{effectsize} = \frac{d_z}{\sqrt{d_z^2 + 2}}$$

The reason is that any Cohen's d considers only the within subject variance, thus the scale of the effect size cannot make it equivalent to the total variance.

The last effect size index defined by Rosenthal et al is the r_{alert}

$$r_{alert} = \sqrt{\frac{SS_{contrast}}{SS_{between}}}$$

which is clealy not related with the within-subject variability, and thus has no relation with δ-like measures.

References

Rosenthal, Robert, Ralph L Rosnow, and Donald B Rubin. 2000. *Contrasts and Effect Sizes in Behavioral Research: A Correlational Approach*. Cambridge University Press.