Optimal Models for Resource Allocation in Classroom Teaching

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Abstract

Teaching is the transmission of knowledge from teacher to student. How should a school administrator allocate a fixed budget towards increasing the number of classrooms and increasing student assessment to best increase student learning? This paper investigates how stochastic models accounting for the inherent uncertainty in student beliefs and teacher communication in teacher-student dyads can help school administrators figure out how to maximize student learning by optimally allocating resources. We replicate existing results from education literature about the effect of class sizes, homogenous ability classrooms, and assessment on student learning using computer simulation. We also unify these different determinants of student learning into a more holistic model and report the tradeoffs that committing budget to these design features presents. We demonstrated that we can create a pareto frontier for the number of teachers and assessments against a fixed budget, and find an optimal allocation of the budget for increasing student learning. We also demonstrate how the effectiveness of class sizes and assessment is contingent upon the learning concept being taught. We believe that the insights about stu- dent learning in multi-classroom settings that we've found and can continue to surface are difficult to surface in real-world studies with human subjects.

Keywords: Teaching; learning; education; pragmatics; Bayesian modeling; social cognition

Education research has surfaced many insights into policies for designing schools to improve classroom learning for students. Very prominently, conventional wisdom makes independent claims that decreasing class sizes, increasing formative assessments, and increasing teaching periods *[citation needed for last one]* improves student learning outcomes. However, decreasing class sizes and developing formative assessments compete for a shared resource of money, while administering formative assessments and teaching lessons compete for a shared resource of student time. As such, it is important to understand the diminishing returns for each of these orthogonal design dimensions in order to find a unified policy that is optimal for student learning. We call the optimal policy **optimal school administration**.

Unfortunately, there exists many barriers to studying optimal school administration in real-world classrooms. Isolating the effects of a chosen school design policy would require controlling for student and teacher differences, and no two classrooms, teachers, and students are the same. Even if we could control for these sources of random error, the task of testing hundreds of permutations across multiple policy dimensions would be time- and cost-prohibitive, along with being ethically suspect in some extreme cases. As such, the task of determining optimal school administration lends itself well to computer modeling.

In this present paper, we turn to stochastic modeling to investigate how an optimal school administrator might allocate finite resources to produce maximal information gain. We elect to utilize stochasticity to build inherent student variability and error on assessments in the real world into our model. Taken together, this work describes a first-principles attempt at a framework for understanding resource allocation in classroom education.

Background

Even when students are motivated to learn, and teachers are motivated to teach, information transmission in classroom settings are imperfect. Education in classroom settings has two challenges that we seek to tackle. First, there exists a problem of *student variability*, where students within the same classroom may have different innate academic ability as well as upbringings. This means that a lesson that would be perfect for one child may be less accessible to another child. Secondly, there exists a problem of *imperfect teacher knowledge*. Teachers don't have perfect knowledge of each student's personal knowledge, and this uncertainty potentially results in choices of teaching exmaples that are not optimal for student learning.

Definitions

Because our work builds on some fields of research that have different vocabularies for similar concepts, we first clarify the language and meanings that will appear in this paper.

Types of knowledge Every student possesses their own malleable *personal knowledge* about any knowledge concept, which could be facts, skills, beliefs, or any other type of knowledge. Coloquially, personal knowledge is often referred to as ability or mastery level, and we use these terms interchangeably.

Teachers can identify a static *target knowledge* about any knowledge concept, and a student's personal knowledge can be close to or far from the target knowledge, representing accuracy or inaccuracy respectively. The goal of education, therefore, is to adjust the student's personal knowledge to be as close to the target knowledge as possible.

Teachers possess beliefs about what their students' personal knowledge is, which we will call *teacher beliefs*. These may not necessarily accurately reflect their students' actual personal knowledge.

Education processes Assessing, teaching and learning are the three key component processes of education. *Assessing* involve students providing information to teachers to *assessments* (e.g. assignments, quizzes, exams, classroom participation, etc.) based on their personal knowledge. The answers provided help teachers update their teacher beliefs.

Teaching involves a teacher providing *lessons* to students to help adjust their students' personal knowledge towards the target knowledge. The content of the lessons are determined by the teacher beliefs; in other words, the teacher chooses *teaching examples* that they believe are most suitable given their students' personal knowledge.

Finally, *learning* involves a student using the teaching examples provided by the teacher through teaching to update their own personal knowledge. Learning itself does not guarantee that a student will update their personal knowledge to be closer to the target knowledge; the student must rely on the teacher to pick effective teaching examples for the updates to be useful.

Example – TODO: Keep this section? A teacher wants to teach the effectiveness of nonviolent protest–the *target knowledge* is that nonviolent protests are somewhere between always successful and never successful. There is a student who holds the (inaccurate) *personal knowledge* that nonviolent protest is always unsuccessful. While there indeed are instances of failed nonviolent protests (e.g. Tianenmen Square, The White Rose), a teacher may elect to over-represent successful nonviolent protests in their *lessons*, using Martin Luther King Jr., Gandhi, and Nelson Mandela as *teaching examples*.

Related Work

Education Research Educators have a variety of strategies to address the problems of student variability and imperfect teacher knowledge. Formative assessments can help teachers monitor their students' personal knowledge. When students complete formative assessments, teacher gain certainty on their teacher beliefs about their students' personal knowledge (e.g., L. S. Fuchs & Fuchs, 1986; Sadler, 1989). If there is large variation in students' ability levels, students can be sorted into groups by teacher beliefs about their personal knowledge and taught different lessons (e.g. Slavin, 1987; Tomlinson, 1999). Finally, decreasing class sizes can help minimize the average variation that each teacher has to deal with in their classroom (e.g., Glass & Smith, 1979; Slavin, 1989). Each of these strategies are considered effective ways to improve student learning.

Classroom Modeling In our previous work, we conceptualized the teacher's task as one of optimal communication (Frank, 2014). Following the models of pragmatic reasoning in language comprehension (Frank & Goodman, 2012; Goodman & Frank, 2016), we can formalize teaching and learning as inferential cognitive processes of rational agents. Teachers reason about what evidence would best change students' personal knowledge to more closely correspond to a target knowledge. Teachers – each with perfect teacher knowledge about each of their students in our prior work – would then choose the teaching examples that maximized information gain across their students. Using this conceptualization, we were able to derive a number of results through simulation. For example, we found that individual student outcomes were inversely related to class size, since in smaller classes, teachers could customize their teaching better to the idiosynrasies of their particular group of students' personal knowledge.[1]

In that previous work and the current work, the fundamental unit of analysis is a teaching game. In each teacher-class unit, a teacher tries to guide the students to discover a particular target knowledge by presenting teaching examples. We use a very simple concept: the weight of a biased coin. The target knowledge is a particular fixed weight, and the students' personal knowledge was their individual beliefs about what the weigh tof the coin is. Teachers must choose a particular set of teaching examples of heads and tails that will alter the personal knowledge of the students in the class so that their updated estimate of the coin weight is closer to the teacher's target knowledge value.

The Present Study

In the current work, we consider issues of student variability and imperfect teacher knowlege through the lens of *optimal school administration*. We describe a generalization of the model presented in Frank (2014) and use this framework to investigate how an optimal adminstrator might make decisions. Our first two simulations replicate and extend results from the previous paper. Then our next two generalize the model to the case of imperfect information about students and finite resources for hiring teachers and conducting assessments. Our final simulation maps out a Pareto frontier for allocation of instructional time and teaching resources.

Model

We model three types of agents: *students*, *teachers*, and *administrators*. A school consists of an administrator, at least one teacher, and at least one student. Every teacher has at least one student (so there are always at least as many students as teachers). Each agent's functions are described below.

The general teaching game that we analyze is one in which learners must estimate the parameter of a Bernoulli distribution. Teaching lessons are the results of individual coin flips, which provide evidence about the coin's weight. Beliefs can then be represented as the parameters of a Beta distribution. For example, a student who has a weak belief that a coin is fair (e.g., $Beta(1,1)$) can be pursuaded that it is actually biased towards heads by seeing the examples $E = \{H,H,H,H,H\}$.

Student

Each student is an optimal Bayesian learner, using a standard conjugate Beta-Bernoulli model. Students have a *prior belief* about the bias of the coin, represented as $Beta(\alpha,\beta)$, where α and β are "pseudo-counts" (they can be interpreted as H=heads=1 and T=tails=0 coin flips that the student has

[1] Ability grouping has a complicated history in education (e.g., Slavin, 1990), and we return to motivational issues related to this finding in the General Discussion.

previously seen). Student beliefs are drawn by sampling $\alpha \sim Unif(1,10)$ and then setting $\beta = 11 - \alpha$ (so that total pseudocounts sum to 11). Students' learning is then modeled as updating this distribution by adding observed counts to their priors, e.g. after observing x heads and y tails, their updated knowledge state is $Beta(\alpha + x, \beta + y)$. Note that for simplicity, student learning is sequence-independent (i.e., seeing $\{H,T,T\}$ is the same as seeing $\{T,T,H\}$).

Teacher

Each teacher is assigned a classroom of students and a target concept (i.e., a particular coin weight) to teach. The goal of the teacher is to provide the set of examples that maximize the student's information gain (IG; defined below). This goal is accomplished by evaluating the information gain for each student for each possible example set and choosing the one that produces the largest total information gain for the class.[2] Since information gain is computed over the conjugate posterior representation of student knowledge, choosing an action relative to IG constitutes full posterior inference. With only a single teaching example, this choice is simply the ratio of the IG for H to the IG for T.

In Simulations 1 and 2, teachers have *perfect knowledge* of student beliefs. Teachers with perfect knowledge infer their choice of examples using the exact parameters of student distributions. In contrast, the teachers in the remaining simulations have *uncertain knowledge*. Teachers with uncertain knowledge initially represent students as having beliefs in the form of $Beta(1,1)$ – weak uniform distributions over possible parameter values. They update these representations based on *assessments*. Assessments are sampled examples from each student's mean parameter estimate, using $\mu = \frac{\alpha}{\alpha+\beta}$. Teachers integrate the samples from these assessments into their distributional estimate for each student. For example, if a student was given three assessments and produced $\{H,T,H\}$, the teacher would represent that student as $Beta(1+2,1+1) = Beta(3,2)$ and choose examples to accordingly.

Administrator

The objective of the administrator is to maximize the information gain of all students in the school. Across our simulations, the administrator can decide: 1) how many teachers to hire, 2) how many assessments to give, and 3) whether to sort students into classrooms by their knowledge. We also vary (for purposes of comparison) whether the teachers have perfect or uncertain knowledge of the students. The administrator weights the various policies that they simulate based on the aggregate information gain of the students in the entire school (compared to just each classroom for each teacher's inference), and is able to infer the most effective school design within fixed constraints.

In practice, because our results demonstrate near-strict dominance of some design choices over others, the administrator's inference is often uninteresting and corresponds to the best choice. Thus, in reporting our simulations, we report school-wide information gain (the decision-making metric for the administrator), often with respect to some meaningful baseline.

Knowledge about students As in the case of teachers, in Simulations 1 and 2, administrators also have perfect knowledge of their students. In contrast, in Simulations 3 and 4, administrators sort students based on the results of their assessmetns. This noisy sorting may produce sorting errors, since students who produce similar responses in the assessment phase (a stochastic sampling process) would be grouped into the same classroom even if there were students with more similar prior beliefs.

Assessment The administrator's main constraint is on assessment: each assessment takes the same "time" as a teaching example. This constraint represents the idea that there is a cost in terms of instruction time for getting information about students via assessment. Of course, the exact ratio of costs is an arbitrary parameter that we fix at 1 (each *time step* permits either one example taught by the teacher or one assessment sample from the student) for the sake of simplicity.

Limited Resources

In our model, two types of resources restrict the behavior of the different agents.

Time In real-world education settings, students are in a school environment for a (roughly) fixed amount of time, and it is up to the teachers to decide what amount of that time they want to dedicate to assessing (i.e. formative assessments) and teaching (i.e. showing teaching examples). In Analysis 3 and 4, where we use "noisy" students, teachers have the option to assess students to improve their imperfect teacher beliefs about the students' personal knowledge. We model this using a currency we call fixed *time steps*. The sum of the number of *assessment periods* and *teaching periods* is constant, and increasing assessments comes at the expense of opportunities to show examples. The administrator commits to a particular allocation of time steps towards assessments and teaching lessons school-wide, so every teaacher presents the same number of teaching examples.

Money Certain educational policies that may be most effective are simply too monetarily expensive to implement. For instance, while one-to-one instruction (i.e. class sizes of 1) may be most beneficial for student learning outcomes (e.g., Cohen, Kulik, & Kulik, 1982), it is unrealistic to expect schools to be able afford hiring as many teachers as they have students. To account for this kind of limitation, we introduce the currency of money. In Analysis 4, the administrator is

[2] A fruitful direction for future work would be to investigate different classroom rules for information gain. For example, a teacher following a remedial policy could try to find the set of examples that maximized the performance of the lowest-performing students or that minimized loss with respect to some threshold.

constrained by a fixed budget. This budget can be used to hire teachers at a one-time expense C_T per teacher. We then assume an additional monetary cost per assessment C_A, corresponding to the costs of implementing a school-wide assessment. We report the optimal allocation of money at varying budget levels towards teachers and assessments.

Information gain

Following Frank (2014), we use information gain to quantify student learning. Formally, we assess the Kullback-Leibler divergence (Cover & Thomas, 2012) between student knowledge (B_S) and the teacher's target distribution (B_T) both before and after teaching. The difference between these quantities gives the student's information gain for a single example e:

$$IG(e) = D_{KL}(B_T||B_S) - D_{KL}(B_T||B_{S+e})$$

We derived a closed form expression for information gain that generalizes to any number of examples in an example set e. Derivation details can be found in our linked repository. The final form for h heads and t tails is:

$$\begin{aligned} IG(E) = &\sum_{k=0}^{h+t-1} log(\alpha_S + \beta_S + k) - \\ &\sum_{i=0}^{h-1} log(\alpha_S + i) - \sum_{j=0}^{t-1} log(\beta_S + j) + \\ &\psi(\alpha_T)h + \psi(\beta_T)t + \psi(\alpha_T + \beta_T)(h+t). \end{aligned} \quad (1)$$

where ψ represents the digamma function, and α_S, β_S, α_T, and β_T are student and teacher priors respectively.

Simulation details

For simplicity, we adopt a set of parameters uniformly across our simulations. While changes to these parameters will have some impact on the effect sizes we recover, to our knowledge all qualitative results are general across parameter sets. Teacher target concepts are selected via pseudocounts on a Beta distribution such that $\alpha + \beta = 10$ (thus target concepts can be .1, .2, .3, etc.).[3] Each student has 12 learning opportunities (also referred to in this paper as *time steps*) that can be used for showing a teaching example or for assessment in later simulations.

In each simulation, we report the total information gain over baseline for 100 students given any particular school design. The particular design parameters (number of teachers, number of assessments, teaching concept, perfect vs. imperfect teacher knowledge, and sorting) and baseline configuration differs in each simulation. Each simulation is tested on 100 random sets (*trials*) of 100 students and performance is averaged across these trials. All simulations were conducted using the probabilistic programming language `webppl` (Goodman & Stuhlmller, n.d.); code is available at [`http://github.com/mcfrank/teaching`].

[3]These pseudocounts are slightly offset from the pseudocount of the prior student belief distributions, 11, to avoid edge cases where a student belief distribution perfectly matches the teaching concept.

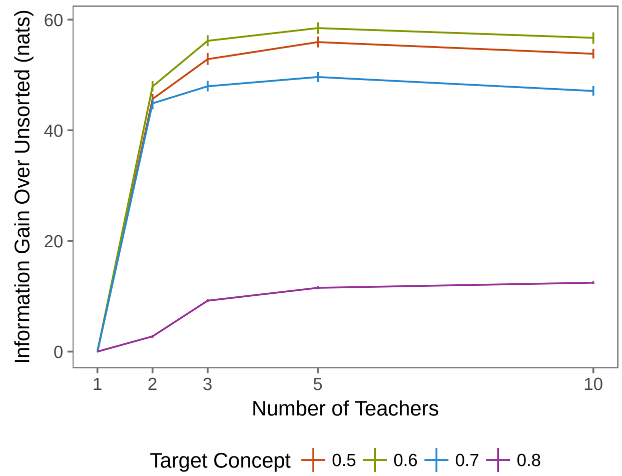


Figure 1: Student information gain, plotted by target concept and number of teachers in the school. Information gain represents the gain when students are sorted into classrooms based on knowledge compared with an unsorted baseline. Error bars show 95% confidence intervals by non-parametric bootstrap.

Simulations

We next present a set of four analyses of simulation data examining the effects of factors including grouping students into classess by their prior beliefs, changing class sizes, and the use of assessments to help teachers tailor their teaching to their specific students. In our first two analyses, we focus on the *perfect knowledge* case, in which teachers and administrators have full knowledge of students' knowledge state. In the subsequent two analyses, we relax this assumption and explore .

Analysis 1: Grouping students

In our first analysis, we explored the effects of grouping students by their *true prior beliefs* (admins with perfect knowledge), so that teachers can tailor the examples they teach to that particular group. This analysis is a replication of results reported in Frank (2014) using the multi-classroom model developed here. We hypothesized that sorting students by their true beliefs would increase information gain compared to random classroom assignment. Our baseline for this analysis was the unsorted information gain under the same parameters. For instance, the sorted aggregate IG of a school with 5 teachers and a target bias of .6 is compared to the unsorted aggregate IG for a school with 5 teachers and a target bias of .6 using the same school of students.

Results are shown in Figure 1. Sorted students show greater information gain than if the same set of students are distributed into unsorted classrooms. This effect is present for all target concepts but is most pronounced for less-extreme

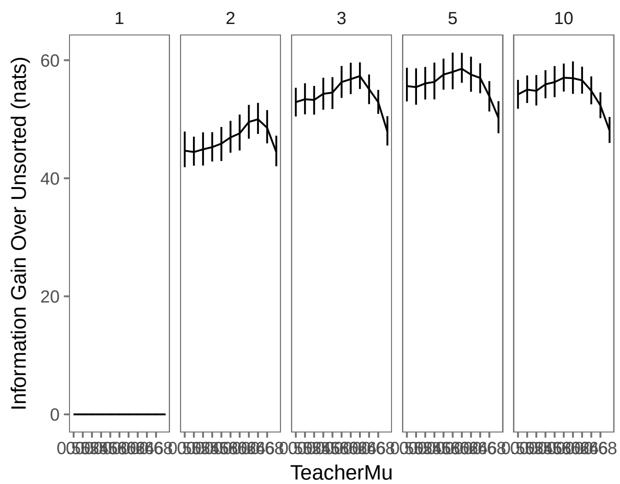


Figure 2: Student information gain, plotted by target concept and number of teachers in the school. Information gain represents the gain when students are sorted into classrooms based on knowledge compared with an unsorted baseline. Error bars show 95% confidence intervals by non-parametric bootstrap.

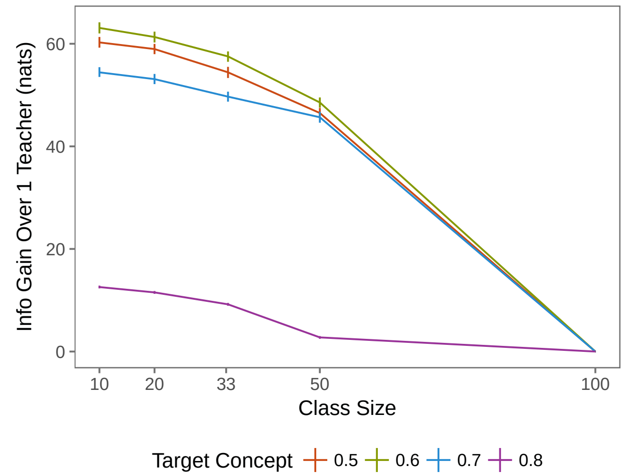


Figure 3: Student information gain, plotted by target concept and class size. Information gain represents the gain when students are sorted into classrooms based on knowledge compared with a single-classroom baseline. Error bars show 95% confidence intervals by non-parametric bootstrap.

concepts – for extreme value concepts (e.g., a target of .9), almost all students will benefit from seeing the same examples anyway, rendering sorting irrelevant. Thus, an optimal school administrator with perfect student knowledge should consistently opt to sort students into classrooms by their prior beliefs over random assignment.

Simulation 2: Class size

In our second analysis, again a replication of our prior work, we explored the effects of adding teachers to the simulated school, leading to lower class sizes. We hypothesized that increasing the number of teachers would strictly improve student learning rate (again assuming perfect knowledge about students): more teachers in a school means that students are in smaller classes and hence receive better customized sets of examples from the teacher that appropriately helps students calibrate their prior beliefs towards the target concept. Our baseline for this analysis was the sorted information gain under the same target bias parameters. We observed the predicted pattern (Figure 2), although there were diminishing returns. After a certain number of teachers, additional lesson customization becomes less helpful.

Analysis 3: Noisy students and assessments

In our third analysis, we relax the assumption that teachers have perfect knowledge about students. In the real world, neither the admin nor the teacher is an omniscient being that knows the true prior student belief parameters. Instead, they diagnose student beliefs by administering assessments (e.g. placement exams). Using the outcome of these assessments, school faculty can estimate the existing beliefs that students hold, and teach to those beliefs.

Simulation details

Student noisiness To model these agent behaviors, we assume that the admin and teachers start with a na"ive, uniform representation of each student (i.e. $Beta(1,1)$) and learn about the student's prior beliefs via the administration of assessments. In assessments, students are called upon to demonstrate their knowledge by sampling from their own true distribution. These samples then serve to update the teacher's estimate about student knowledge. In this sense, the teachers and administrator are modeled as a perfect Bayesian agent that updates its beliefs about student knowledge based on evidence it sees from student performance on assessments.

For instance, a student may have a true prior belief represented by $Beta(9,2)$. When assessed, they are extremely likely to respond with H (or 1) to each question on the assessment. If they are asked 6 questions, the most common outcome will be H,H,H,H,H,T. The teachers and administrator starts with the naive $Beta(1,1)$ representation of the student beliefs, but then updates it for seeing 5 heads and 1 tail in the assessment phase. Their *guessed belief* of student learning becomes $Beta(6,2)$ after the Bayesian update. By construction, our uncertain teachers have very weak and inaccurate beliefs about student knowledge, captured by the low magnitude of the initial Beta parameters, and we assume that increasing assessments will help them improve the accuracy of their representation of student knowledge.

Effects of sorting in a noisy setting

Currency of time We hypothesize that increasing the number of assessments monotonically improves student learning on average, though with diminishing returns. As such, simply measuring information gain as the number of assessments increases is rather trivial. In this simulation, we introduce the currency of time–each student spends a fixed amount of 12 time steps in the school system, and any single time step can be devoted to assessing the student (3 samples from the student's prior distribution about the learning concept) or a teacher showing the student one piece evidence (a heads or a tails). By modeling the tradeoff between increasing assessments and increasing teaching opportunities, we attempt to identify a tipping point at which giving teachers more opportunities to show examples outweight the diminishing returns on information gain of increasing assessments.

Baseline We have a baseline model that uses a non-inferential admin and teacher. This admin and teacher does not take their students' prior beliefs into consideration, and simply selects a set of examples that they believe is an accurate representation of the target concept. **TODO: ELABORATE HERE**

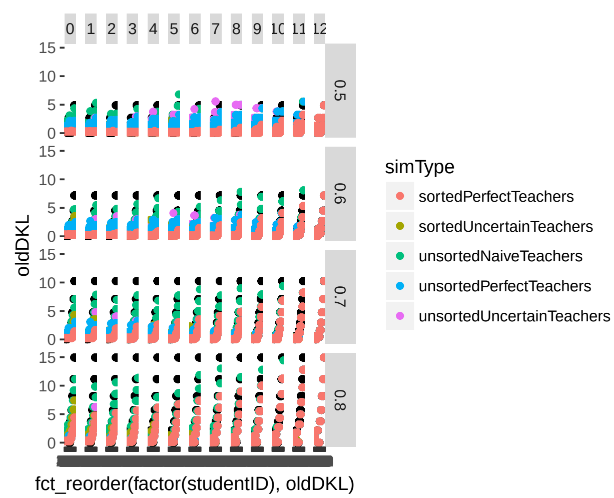
For each trial, we test students in two conditions of faculty knowledge: omniscient teachers with perfect knowledge vs. uncertain teachers with noisy knowledge. The full set of control parameters spans 2 (sorted vs. unsorted students) x 5 (target concepts) x 5 (number of teachers) x 13 (number of assessments = 12 - number of teaching periods) = 650 different regimes.

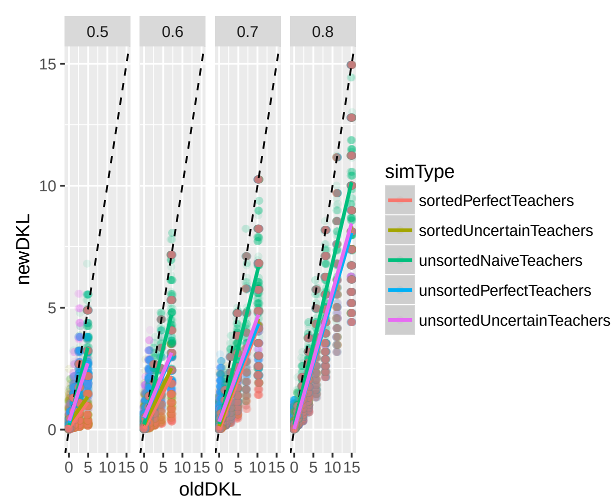
teachers and administrators who have perfect knowledge of students and sort them into classrooms via this knowledge teachers who have perfect knowledge, but with students randomly assigned to their classroom teachers and administrators both have uncertainty about student knowledge, with administrators sorting students and teachers selecting examples based on assessment results, and teachers who have uncertainty about student knowledge and select examples based on assessment results, with students randomly assigned to their classroom

We hypothesized that with uncertainty about student knowledge, teachers should perform strictly worse in any given regimes of control parameters than if they had omniscient, perfect knowledge about the student prior beliefs. We further hypothesized that there would be a non-linear effect of number of assessments on information gain. Having too few assessments would be non-optimal because teachers and administrators cannot accurately gauge student learning, causing administrators to make more errors in sorting students by ability level and teachers to select examples that don't maximize student learning within their classrooms. Having too many assessments would also be non-optimal because teachers do not have enough opportunities to precisely teach the examples they would want. In other words, there are diminishing returns to the number of assessments performed.

As expected, we found that the uncertain teachers performed strictly worse than the omniscient teachers when controlling for every other feature (sorting, target concept, number of teachers, number of assessments). Furthermore, consistent with Analysis 1, sorting the students produced greater gains for student learning relative to not sorting the closer the teacher mu is to 0.5. Even with the uncertainty in teacher beliefs about student knowledge, students learned more when the classrooms were sorted by ability than when the teachers had perfect knowledge in unsorted classrooms. Consistent with Analysis 2, decreasing class size also improves student learning, though also with diminishing returns.

Additionally, we found a concave down shape in the student IG based on number of assessments that is consistent with our non-linearity hypotheses. Between teacher mus of 0.5 and 0.7, the optimal number of assessments is greater than 0 and less than 12. As teacher mu approached 0.5, i.e. as greater variation in student beliefs relative to the true learning concept increased, the higher the number of optimal number of assessments. The optimal number of assessments also decreases as the number of teachers increase. In both cases, it appears that the optimal number of assessments increases when there exists a possibility of greater intra-classroom variation on student prior beliefs.





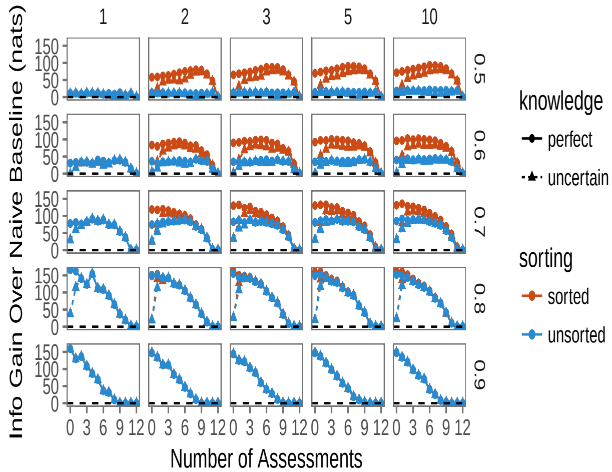
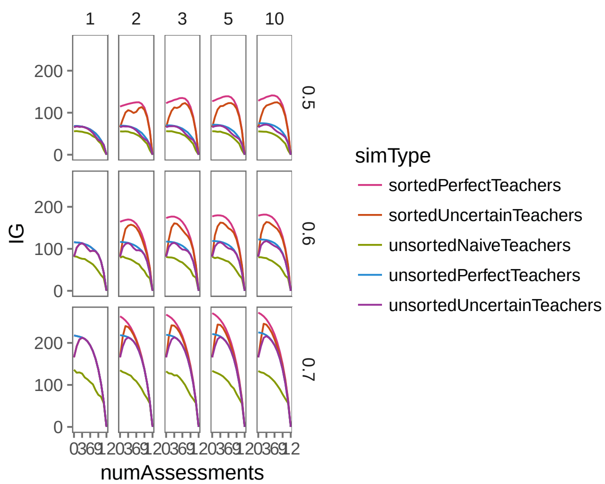
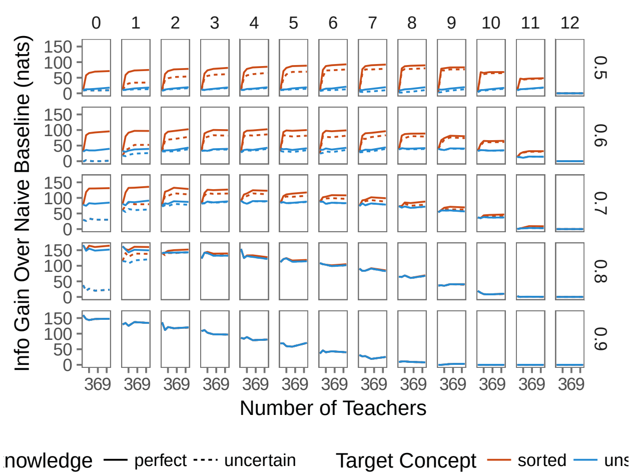


Figure 4: Information gain plotted by number of assessments (out of 12) for teachers with perfect and uncertain student knowledge. Results shown are for 5 teachers.





The admin uses these estimates to sort students into classrooms, and teachers use these estimates to customize the set of teaching lessons to a particular student group.

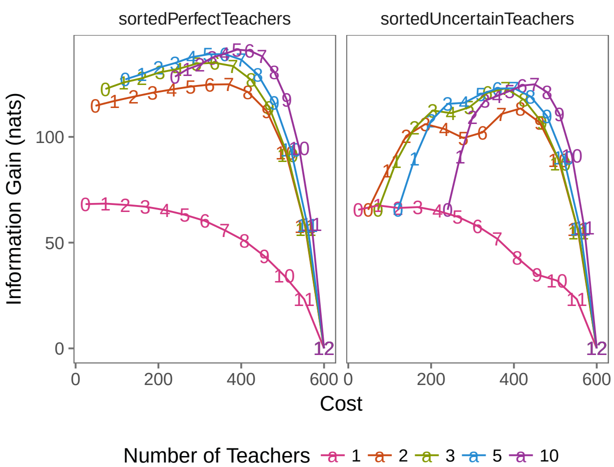
In the simulation, the students have a limited number of learning opportunities. We found that increasing aggregate IG converges quick after 10 time steps (noticeable in Figure 5), so we report results with an arbitrarily chosen 12 time steps. Each assessment or lesson takes 1 time step, and all assessments come before all lessons.

We hypothesized that the noisy students will produce less information gain than if the faculty had perfect knowledge of student beliefs. We suspect the school design will differ from the optimal configurations at two stages: the sorting stage (classrooms will not have maximal homogeneity), and the teaching stage (the examples selected by any given teacher will not maximize IG for the students assigned to their class). Nevertheless, we still believe the improvements to IG by increasing the number of teachers and reducing class sizes would persist for noisy students.

We further hypothesized that the optimal use of the allotted time steps would involve a small or moderate number of assessments and a moderate to large number of lessons. We thought that a few assessments would get the estimated beliefs about student knowledge close enough to the students' true beliefs to minimize sorting errors, leaving teachers with enough time to teach lessons to move the students' actual beliefs towards the target concept. Too many assessments would reduce the amount of time teachers have to teach regardless of how precise their estimates of student knowledge are; too few assessments would prevent teachers from selecting helpful examples in their lessons.

Indeed, we found a quadratic shape in the relationship between number of assessments and aggregate IG. Assessing students helps improve learning rate by giving teachers better estimates of their beliefs, but they still need time to subsequently teach towards the target concept. Additionally, like previous simulations, this effect was most pronounced in moderate target concepts. These results make intuitive sense. Teaching concepts where there may be greater variance in student knowledge relative to each other (e.g. teaching concepts of .5) demand more customization in lesson plans with students exhibiting different levels of mastery.

Simulation 4: Pareto frontier with imposed budget constraints



We hypothesized that for a particular budget constraint and fixed costs of teachers and assessment, there will be a Pareto frontier of possible allocations of that budget towards teachers and assessments on which there will be an optimal allocation that maximizes student learning. Furthermore, we hypothesized that the individual results about number of teachers number of assessments would hold within levels of the other. This is the unification of all previous simulations into one model that allows us to introduce a real-world constraint of budget.

In this simulation, we test all integer combinations of number of teachers between 1 and 10 with number of assessments between 0 and 5. We assigned the cost of each teacher to be CT = \$10 and the cost of each assessment CA = \$20.

Our baselines were the worst-performing within-target setup. This always happened to be the 1-teacher 0-assessment setup, which is consistent with existing education literature, since a single teacher cannot significantly differentiate learning, and still has a lot of uncertainty about student beliefs because no assessments are performed.

Our simulated findings support our hypothesis. There is con- siderable support for an interaction model between teachers, assessments, and the target bias, $F(7, 112) = 48.61$, $p<.001$. To visualize the pareto frontier, we draw a heatmap of scores

above baseline along teacher and assessment axes. We continue to see that the effects continue to be weaker for more extreme learning concepts, $t(294) = -11.09$, $p<.001$. This result can be seen in Figure 5.

Since real-world learning concepts are essentially non- negotiable, the effectiveness of a particular allocation of budget should be judged within-target, i.e. an optimal setup within the target bias = 0.8 level should not be penalized for being ineffective in comparison to the optimal setup for teaching a target bias of 0.5. We normalized the information gain above baseline within each target bias level. The results can be seen in 6

Consistent with Simulation 3, increasing the number of assess- ments on average improves learning rate up until you cannot afford any more assessments, across all levels of teachers and target biases. Consistent with Simulation 2, increasing the number of assessments improves learning rate up until you cannot afford any more assessments, across all levels of assessments and target biases of 0.5, 0.6, and 0.7.

An interesting result, however, is that having noisy students diminishes the effectiveness of increasing the number of teach- ers at certain a target bias of 0.8. While we aren't sure of the specific mechanism by which this happens, we suspect that having more teachers introduces additional risk of wrongly sorting students' prior beliefs, which would increase the het- erogeneity of classrooms. Since a target belief of 0.8 is a possible target belief to show precisely with 5 examples (the basis of our simulations) but also extreme enough that most students's prior belief will fall on one side of that target, in- creasing the number of teachers might actually reduce the effectiveness of the teacher's choice of examples. A novel finding that emerged not described in existing education literature is that there are different patterns of optimality along the pareto frontiers. As the extremity of the target bias moves from 0.5 to 0.7, we see that the IG-optimal allocation of budget shifts from about 5-6 teachers and 2 assessments towards more assessments at the expense of the number of teachers.

Discussion

Summary

In our simulations, we refer to the entire group of students as a "school," broken into "classrooms," and students are given "assessments" and "lessons." These mappings are inexact, however, and the analogical correspondence can be made at a lower level. For example, the same set of ideas could be applied *within a classroom*, for example to the decision whether to split the class into smaller groups and divide the teachers time amongst supervising them. And similarly, what we describe "assessments" and "lessons" could be either multi-item tests and then several days of instruction or – again at a more micro scale – a single question posed to the class followed by several pieces of related content within a single class session. Our goal is to elucidate general principles that govern the process of teaching rather than to claim a correspondence with specific phenomena or teaching methodology.

Overall, we were able to replicate many real-world findings from education literature. We found that sorting students into ability level classrooms, increasing assessment, and decreasing class size all increase information gain.

One of the more interesting findings was the influence of the target bias on the effects we were finding. Based on our results, target bias significantly moderates each of the effects we found. When teaching an extremely difficult (or easy, given symmetry) learning concept such that all student prior beliefs are on one side of the concept, the payoffs to information gain that sorting, class sizes, and assessment have diminish because regardless of the level of noise, teachers have relative certainty about the students' relative level of understanding. Similarly, target biases that are very close to central (around 0.5) have a lot to gain from diminishing noise in the system.

Finally, we found that the relative effectiveness of number of teachers and and the number of assessments varies depending on the target learning concept. We saw that the most optimal distribution of budget on the pareto frontier shifted towards reducing student noise when the target bias was more extreme–this seemed consistent with intuition, since having more teachers doesn't allow them to better select examples if the administrator produces greater sorting errors, resulting in heterogeneous classrooms where each teacher has less certainty about whether their examples will be useful for any particular student or not.

While a lot of aspects of school learning (peer effects, social dynamics, etc.) are not built into our model and would be difficult to capture algorithmically, we believe that there is a lot to be gained from simulating classroom dynamics. Using stochastic modeling is particularly useful when exploring the effectiveness of education policy measures for a handful of reasons. Fundamentally, the merits of stochastic modeling arise from the use of synthetic data. Since the studies are computer-simulated, there are no human subjects in the studies, and all of the data is synthetic. This allows researchers to take greater liberties in experimental design because there are no ethical concerns. For instance, we can execute a simulation that will intentionally misclassify synthetic students into classrooms that are not appropriate for their achievement level to understand the negative impact doing so has on both the misclassified student and his peers. Such a study would not be ethically permissible in an actual school.

Furthermore, no two schools are the same, and the use of mod- eling and synthetic data enables rapid iteration over different simulated schools with different simulated students. We can quickly test whether the purported benefit to student learning of various education interventions holds with, for example, a limited budget, or highly varied student achievement levels, or immensely large class sizes. There is also no risk of student subject attrition that may jeopardize real-world school studies; synthetic students do not exhibit unpredictable absenteeism, transfer schools, take sick days, etc. As such, we can cus- tomize our interventions based on school parameters, validate the robustness of our results, and better infer generalized find- ings about education policy without incurring astronomical experimental monetary costs nor require lengthy longitudinal study. With a complete stochastic model, we can quickly scale the number of hypotheses we test and the variety of schools we test them on. Our series of simulations are designed to replicate real-world results from education literature in a quantitative, simulated fashion. The first few simulations looked at a lot of existing educational theory in isolation, while our final simulation attempts to unify these different design aspects of a school system into a more holistic view. We found that increasing the number of teachers (and thus decreasing class size) and increasing the amount of assessment to improve precision about student beliefs increases student learning, as predicted in real-world school studies.

Importantly, we were able to generate pareto frontiers allocating a fixed shared resource of budget into the dimensions of teachers and sorting. The ability to make sense of the tradeoffs that school administrators and policymakers need to make when designing their schools can help improve education. This would be particularly impactful in communities of low socioeconomic status where academic achievement tends to be lower because budgets are particularly constrained and attracting teaching staff is difficult.

Some of the limitations of the present work is in the assump- tions that we make to build our model. Most obvious is the simplicity of the learning task we described during the intro- duction. Most real-world learning tasks are more complicated and may have multiple objectives. Furthermore learning in the real-world is affected by a lot of more abstract facets: peer learning and social environment, parenting styles, stereotype threat, among other things. These are not captured in the model, and would be difficult to capture in any model. The agnostic prior beliefs we 1) use for teachers to generate examples from; 2) generate student prior beliefs from; and 3) create target bias distributions with are all uniform distributions. These are not necessarily the case. Respectively, 1) teachers may have prior beliefs about which examples are more effective, akin to pedagogical knowledge or pedagogical content knowledge (Cochran, 1991); 2) students may tend towards certain common prior beliefs/misconceptions about a particular learning concept; 3) more extreme learning concepts may be more rare, and thus the admin should weight more heavily the pareto frontier of moderate target bias values.

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