

Low-Rank 3D Tensor Completion

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Motivations

- Movies
- Signal processing
- Multiple parameters approximation

Norm and matricization

- Generalization of Frobenius norm for $\mathcal{X} \in \mathbb{R}^{l_1 \times l_2 \times l_3}$:

$$\|\mathcal{X}\| = \sqrt{\sum_{i=1}^{l_1} \sum_{j=1}^{l_2} \sum_{k=1}^{l_3} x_{i,j,k}^2}$$

- Unfolding \mathcal{X} in the i^{th} dimension:

$$X_{(i)} \in \mathbb{R}^{l_i \times (l_j \cdot l_k)}, j \neq i, k \neq i, j$$

- n -mode product:

$$\mathcal{Y} = \mathcal{X} \times_n U \iff Y_{(n)} = UX_{(n)}$$

- n -rank of a tensor:

$$\text{rank}_n(\mathcal{X}) := \text{rank of } X_{(n)}$$

Tucker Decomposition

- Core tensor $\mathcal{G} \in \mathbb{R}^{R_1 \times R_2 \times R_3}$, principal components in each mode $A_1 \in \mathbb{R}^{I_1 \times R_1}$, $A_2 \in \mathbb{R}^{I_2 \times R_2}$, $A_3 \in \mathbb{R}^{I_3 \times R_3}$:

$$\mathcal{X} = \mathcal{G} \times_1 A_1 \times_2 A_2 \times_3 A_3$$

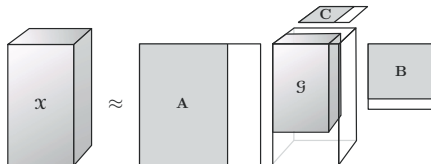
- Higher-Order SVD: Find a rank- $[R_1, R_2, R_3]$ core tensor and corresponding components that best capture the variation in mode- n , independently of other modes

Higher Order Singular Value Decomposition

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procedure HOSVD( $\mathcal{X}, R_1, R_2, \dots, R_N$ )
  for  $n = 1, \dots, N$  do
     $\mathbf{A}^{(n)} \leftarrow R_n$  leading left singular vectors of  $\mathbf{X}_{(n)}$ 
  end for
   $\mathcal{G} \leftarrow \mathcal{X} \times_1 \mathbf{A}^{(1)\top} \times_2 \mathbf{A}^{(2)\top} \dots \times_N \mathbf{A}^{(N)\top}$ 
  return  $\mathcal{G}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \dots, \mathbf{A}^{(N)}$ 
end procedure

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Movie reconstruction and Image Inpainting

- Each frame is divided into Macro-Blocks (MBs) of size N
- For each incomplete Macro-Block P^0 , look for $K - 1$ complete MBs minimizing

$$S_i = \|P_{\Omega}^0 - P_{\Omega}^i\|$$

- Stack all selected P^i into a tensor \mathcal{X} with P^0 at the top.
- Ideally $\mathcal{X} = \mathcal{X}_I + \mathcal{E} = P \times_3 \mathbb{1} + \mathcal{E}$
- Tucker decomposition: $\mathcal{X}_I = \mathcal{G} \times_1 A_1 \times_2 A_2 \times_3 A_3$
 $\Rightarrow P = \mathcal{G} \times_1 A_1 \times_2 A_2$ and $A_3 = \mathbb{1}$
- Find \mathcal{X}_I a $[R_1, R_2, R_3]$ -rank approximation of \mathcal{X} , with R_3 small.
- Two algorithms: HOSVD with Alternating Least Square and GeomCG

Alternating Least Square Algorithm

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1. Form \mathcal{X} from $\mathbf{P}^0, \dots, \mathbf{P}^{K-1}$ using Block Matching criteria
 2. $(\mathcal{X}(:, :, 1))_{\overline{\Omega}} = \left(\frac{1}{K-1} \sum_{i=1}^{K-1} \mathbf{P}^i \right)_{\overline{\Omega}}$
 3. Choose mode ranks $\{R_1, R_2, R_3\}$, tolerance σ ;
Initialize $\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \mathbf{A}^{(3)}$
 4. $\mathbf{A}^{(3)}(:, 1) = [1, \dots, 1]^T / K$
 5. for $n = 1, 2, 3$
 - $\mathcal{Y} = \mathcal{X} \times_1 \dots \times_{n-1} \mathbf{A}^{(n-1)T} \times_{n+1} \mathbf{A}^{(n+1)T} \dots$
 - $\mathbf{Y}_n \leftarrow$ unfold \mathcal{Y} in mode n
 - $\mathbf{A}^{(n)} \leftarrow$ first R_n principal component of \mathbf{Y}_n
 - end
 6. $\mathcal{G} = \mathcal{X} \times_1 \mathbf{A}^{(1)T} \times_2 \mathbf{A}^{(2)T} \times_3 \mathbf{A}^{(3)T}$
 7. $\mathcal{X}_l = \mathcal{G} \times_1 \mathbf{A}^{(1)} \times_2 \mathbf{A}^{(2)} \times_3 \mathbf{A}^{(3)}$
 8. If $\|\mathcal{X}_l - \mathcal{X}\|_F \leq \sigma$ STOP, otherwise return to Step 4.
 9. Recover missing area in \mathbf{P}^0 : $(\mathbf{P}^0)_{\overline{\Omega}} = (\mathcal{X}_l(:, :, 1))_{\overline{\Omega}}$
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GeomCG

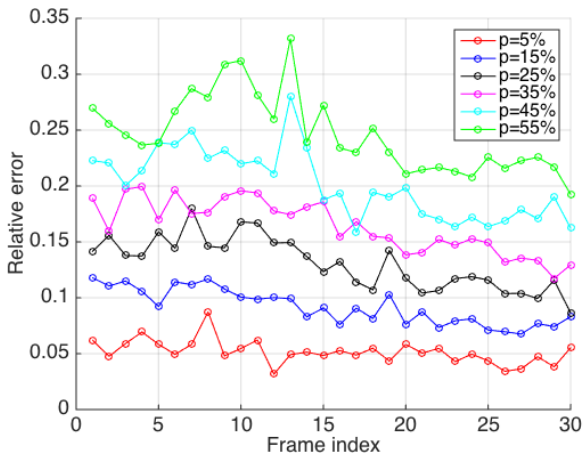
Optimization on the manifold

Movie reconstruction

Macro-Block size: $N=8$

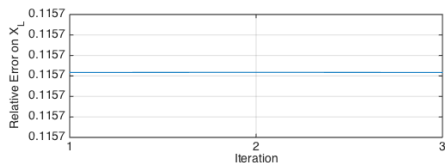


$$\text{Relative error} = \frac{\|F_{\text{original}} - F_{\text{restored}}\|}{\|F_{\text{original}}\|}$$

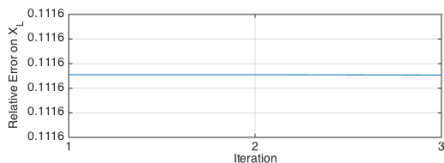


Importance of initial guess

Educated guess



Random guess



Inpainting



1



2

¹ <http://www.briqueterie-chimot.fr/wpcproduct/la-brique-rouge-chimot/>

² <http://images.forwallpaper.com/files/images/0/0249/0249d4f1/113975/winter-park-snow-bench.jpg>

Comparison with GeomCG

Compare results for rank 1



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