3D Tensor Completion

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1 Introduction

Error concealment is a domain of image processing which deals with

Error typically in form of patches

This report presents an error concealment algorithm, called Alternating Least Square, that reconstructs the corrupted patches by exploiting the low-rank property of a stack of similar patches. Results of the application of this algorithm to movie reconstruction and inpainting are presented here as well.

Moreover, for comparison purpose, the same movies and images were reconstructed using the algorithm GeomCG for low-rank approximation.

2 Basics of tensor algebra

Before looking at the error concealment algorithm in details, a few basic tools of tensor algebra need to be defined. This section is based on the paper by Kolda and Bader [3], which should be referred back to for further details.

It is sufficient for our purpose to consider three-dimensional tensors only, but all algebraic concepts presented in this section can be generalised to higher dimensions.

Let $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$. The ith matricisation of $\mathcal{X}$ is a reordering of its entries into a matrix $X_{(i)} \in \mathbb{R}^{I_i \times \prod_{j \neq i} I_j}$ such that the indices of $\mathcal{X}$ in the ith dimension become the row indices of $X_{(i)}$ and the indices of $\mathcal{X}$ in the other dimensions are rearranged in lexicographical order to become column indices of X_i.

The $i-$rank of $\mathcal{X}$ is defined as the rank of its ith matricisation. The multilinear rank of $\mathcal{X}$ is a 3-tuple grouping the i-rank of $\mathcal{X}$ in each of the three dimensions:

$$\operatorname{rank}(\mathcal{X}) := \Big[\operatorname{rank}(X_{(1)}), \operatorname{rank}(X_{(2)}), \operatorname{rank}(X_{(3)})\Big]$$

The ith-mode product of $\mathcal{X}$ with a matrix $M \in \mathbb{R}^{m \times I_i}$ is a tensor $\mathcal{Y} \in \mathbb{R}^{J_1 \times J_2 \times J_3}$, with $J_j = I_j$ for $j \neq i$, and $J_i = m$. $\mathcal{Y}$ is defined by means of the matricisation as follows:

$$\mathcal{Y} = \mathcal{X} \times_i M \iff Y_{(i)} = M X_{(i)}$$

Note that the matricisation operation is a bijection from $\mathbb{R}^{I_1 \times I_2 \times I_3}$ into $\mathbb{R}^{I_i \times \prod_{j \neq i} I_j}$. The product $\mathcal{Y}$ is thus uniquely determined by any of its matricisation.

The inner product of two tensors $\mathcal{X}, \mathcal{Y} in \mathbb{R}^{I_1 \times I_2 \times I_3}$ is the sum of the product of their entries, that is to say:

$$< \mathcal{X}, \mathcal{Y} > := \sqrt{\sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \sum_{i_3=1}^{I_3} x_{i_1,i_2,i_3} y_{i_1,i_2,i_3}}$$

The induced norm $\|\mathcal{X}\| := \sqrt{< \mathcal{X}, \mathcal{X} >}$ is thus a generalisation of the matrix Frobenius norm to 3 dimensions.

Any tensor $\mathcal{X}$ of multilinear rank $\operatorname{rank}(\mathcal{X}) = [r_1, r_2, r_3]$ can be decomposed into a core tensor $\mathcal{C} \in \mathbb{R}^{r_1 \times r_2 \times r_3}$ and three orthogonal basis matrices $U_i \in \mathbb{R}^{I_i \times r_i}$, such that

$$\mathcal{X} = \mathcal{C} \times_1 U_1 \times_2 U_2 \times_3 U_3$$

This is called the Tucker Decomposition and is the basis of the High-Order Singular Value Decomposition (HOSVD) algorithm.

HOSVD aims at computing a rank-$[R_1, R_2, R_3]$ approximation of a tensor $\mathcal{X}$ of higher rank in the Tucker format. This is done by matricising $\mathcal{X}$ in each dimension $i = 1, 2, 3$ and using the R_i principal left singular vectors as basis U_i of the Tucker decomposition. The pseudo-code for the HOSVD procedure is given in Algorithm 1.

Algorithm 1 Higher-Order Sinngular Value Decomposition

```
1: procedure HOSVD (𝒳, R_1, R_2, R_3)
2:     for i = 1, 2, 3 do
3:         U_i ← R_i leading left singular vectors of X_(i)
4:     𝒢 ← 𝒳 ×_1 U_1^⊤ ×_2 U_2^⊤ ×_3 U_3^⊤
5:     return 𝒢, U_1, U_2, U_3
```

This procedure can be viewed as a generalisation of the matrix Singular Value Decomposition (SVD) as a truncated SVD is performed on each matricisation of $\mathcal{X}$ to obtain a lower rank approximation. However, in contrast to the 2-dimensional case, the HOSVD does not lead to the best rank-$[R_1, R_2, R_3]$ approximation of $\mathcal{X}$. Let us denote the subspace of all 3-dimensional tensors of rank $r = [R_1, R_2, R_3]$ by $\mathcal{M}_r := \{\mathcal{T} \in \mathbb{R}^{I_1 \times I_2 \times I_3} | \text{rank}(\mathcal{T}) = [R_1, R_2, R_3]\}$ and let $P_{\mathcal{M}_r}\mathcal{X}$ be a best approximation of $\mathcal{X}$ in the subspace $\mathcal{M}_r$ relatively to the norm defined above. If $P_{HO}\mathcal{X}$ represents the result of the HOSVD procedure applied to $\mathcal{X}$ then the following results holds [2]:

$$\|\mathcal{X} - P_{HO}\mathcal{X}\| \leqslant \sqrt{3}\|\mathcal{X} - P_{\mathcal{M}_r}\mathcal{X}\|$$

This means that a best low-rank approximation of $\mathcal{X}$ is only better by a factor $\frac{1}{\sqrt{3}}$ than what is computed by means of the HOSVD procedure.

3 Selection of samples

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5 GeomCG

6 Results

6.1 Bus Sequence

6.2 Inpainting

7 Conclusion

8 Suggestion for further research

References

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