Low-Rank 3D Tensor Completion

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Motivations

- Movies
- Signal processing
- Multiple parameters approximation

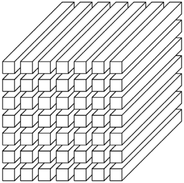
Matricization

- Generalization of Frobenius norm for $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$:

$$\|\mathcal{X}\| = \sqrt{\sum_{i=1}^{I_1} \sum_{j=1}^{I_2} \sum_{k=3}^{I_3} x_{i,j,k}^2}$$

- Unfolding $\mathcal{X}$ in the i^{th} dimension:

$$X_{(i)} \in \mathbb{R}^{I_i \times (I_j \cdot I_k)},\ j \neq i,\ k \neq i, j$$



- n-mode product of $\mathcal{X}$ with $U \in \mathbb{R}^{J \times I_n}$: $\mathcal{Y} \in \mathbb{R}^{I_1 \times \ldots \times I_{n-1} \times J \times I_{n+1} \ldots \times I_N}$

$$\mathcal{Y} = \mathcal{X} \times_n U \iff Y_{(n)} = UX_{(n)}$$

Tucker Decomposition

- n-rank of a tensor:

$$rank_n(\mathcal{X}) := \text{rank of } X_{(n)}$$

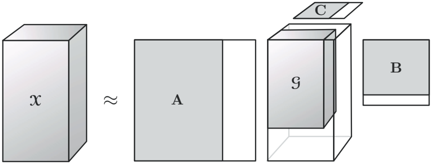
- Core tensor $\mathcal{G} \in \mathbb{R}^{R_1 \times R_2 \times R_3}$, principal components in each mode $A_1 \in \mathbb{R}^{I_1 \times R_1}, A_2 \in \mathbb{R}^{I_2 \times R_2}, A_3 \in \mathbb{R}^{I_3 \times R_3}$:

$$\mathcal{X} = \mathcal{G} \times_1 A_1 \times_2 A_2 \times_3 A_3$$

- Higher-Order SVD: Find a rank-$[R_1, R_2, R_3]$ core tensor and corresponding components that best capture the variation in mode-n, independently of other modes

Higher Order Singular Value Decomposition

procedure `HOSVD`($\mathcal{X}$,R_1,$R_2, \ldots, R_N$)
  for $n = 1, \ldots, N$ **do**
    $\mathbf{A}^{(n)} \leftarrow R_n$ leading left singular vectors of $\mathbf{X}_{(n)}$
  end for
  $\mathcal{G} \leftarrow \mathcal{X} \times_1 \mathbf{A}^{(1)\mathsf{T}} \times_2 \mathbf{A}^{(2)\mathsf{T}} \cdots \times_N \mathbf{A}^{(N)\mathsf{T}}$
  return $\mathcal{G}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \ldots, \mathbf{A}^{(N)}$
end procedure



Movie reconstruction and Image Inpainting

- Each frame is divided into Macro-Blocks (MBs) of size N
- For each incomplete Macro-Block P^0, look for $K-1$ complete MBs minimizing

$$S_i = \|P^0_\Omega - P^i_\Omega\|$$

- Stack all selected P^i into a tensor $\mathcal{X}$ with P^0 at the top.
- Ideally $\mathcal{X} = \mathcal{X}_I + \mathcal{E} = P \times_3 \mathbb{1} + \mathcal{E}$
- Tucker decomposition: $\mathcal{X}_I = \mathcal{G} \times_1 A_1 \times_2 A_2 \times_3 A_3$
  $\Rightarrow P = \mathcal{G} \times_1 A_1 \times_2 A_2$ and $A_3 = \mathbb{1}$
- Find $\mathcal{X}_I$ a $[R_1, R_2, R_3]$-rank approximation of $\mathcal{X}$, with R_3 small.
- Two algorithms: HOSVD with Alternating Least Square and GeomCG

Alternating Least Square Algorithm

1. Form $\boldsymbol{\mathcal{X}}$ from $\boldsymbol{P}^{0,\ldots,K-1}$ using Block Matching criteria
2. $(\boldsymbol{\mathcal{X}}(:,:,1))_{\overline{\Omega}} = \left(\frac{1}{K-1}\sum_{i=1}^{K-1}\boldsymbol{P}^i\right)_{\overline{\Omega}}$
3. Choose mode ranks $\{\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3\}$, tolerance σ;
   Initialize $\boldsymbol{A}^{(1)}$, $\boldsymbol{A}^{(2)}$, $\boldsymbol{A}^{(3)}$
4. $\boldsymbol{A}^{(3)}(:,1) = [1, \ldots, 1]^T / K$
5. for n = 1,2,3
   $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}} \times_1 \ldots \times_{n-1} \boldsymbol{A}^{(n-1)T} \times_{n+1} \boldsymbol{A}^{(n+1)T} \ldots$
   $\boldsymbol{Y}_n \leftarrow$ unfold $\boldsymbol{\mathcal{Y}}$ in mode n
   $\boldsymbol{A}^{(n)} \leftarrow$ first R_n principal component of $\boldsymbol{Y}_n$
   end
6. $\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{X}} \times_1 \boldsymbol{A}^{(1)T} \times_2 \boldsymbol{A}^{(2)T} \times_3 \boldsymbol{A}^{(3)T}$
7. $\boldsymbol{\mathcal{X}}_l = \boldsymbol{\mathcal{G}} \times_1 \boldsymbol{A}^{(1)} \times_2 \boldsymbol{A}^{(2)} \times_3 \boldsymbol{A}^{(3)}$
8. If $\|\boldsymbol{\mathcal{X}}_l - \boldsymbol{\mathcal{X}}\|_F \leq \sigma$ STOP, otherwise return to Step 4.
9. Recover missing area in $\boldsymbol{P}^0$: $\left(\boldsymbol{P}^0\right)_{\overline{\Omega}} = (\boldsymbol{\mathcal{X}}_l(:,:,1))_{\overline{\Omega}}$

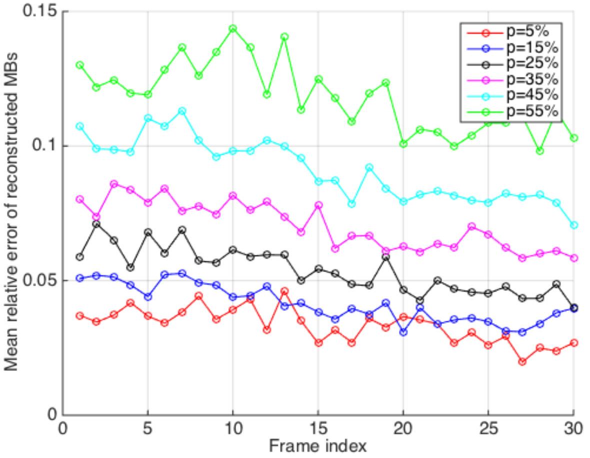
GeomCG

- Look for $\min_{\mathcal{T}} \frac{1}{2}\|P_\Omega \mathcal{T} - P_\Omega \mathcal{X}\|$ with
$$\mathcal{T} \in \mathcal{M}_r := \{\mathcal{T} \in \mathbb{R}^{I_1 \times I_2 \times I_3} | rank(\mathcal{T}) = [r_1, r_2, r_3]\}$$
- $\mathcal{M}_r$ is a smooth manifold
- Riemannian optimization on the manifold

Movie reconstruction

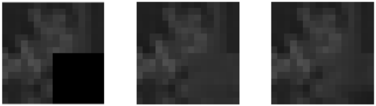
Macro-Block size: N=8

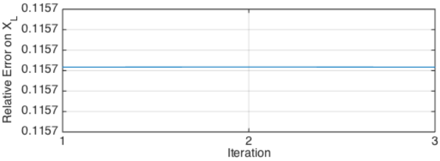
$$\text{Mean relative error} = \frac{1}{M}\sum_{i=1}^{M} \frac{\|P^i_{original} - P^i_{restored}\|}{\|P^i_{original}\|}$$



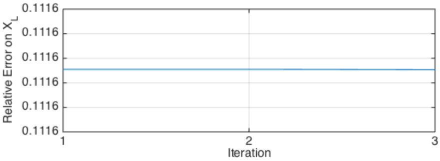
Importance of initial guess

Educated guess





Random guess



Inpainting

[1]

[2]

[1] http://www.briqueterie-chimot.fr/wpcproduct/la-brique-rouge-chimot/

[2] http://images.forwallpaper.com/files/images/0/0249/0249d4f1/113975/winter-park-snow-bench.jpg

Comparison with GeomCG

Compare results for rank 1

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