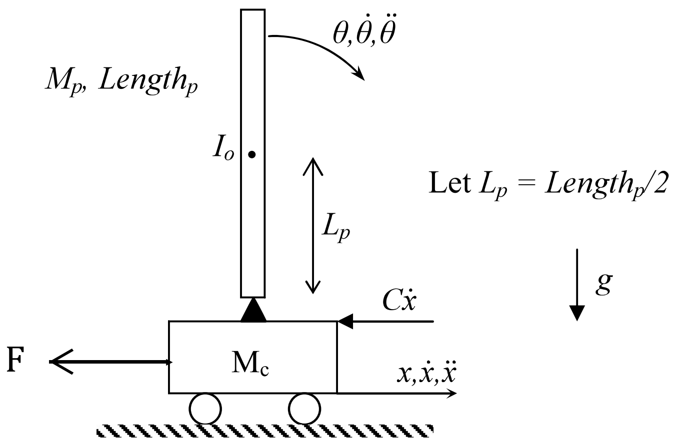
Free Body Diagram



Equations of Motion, Using Lagrange's Equations

Lagrange's Equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = Q_k \text{ where } L = T - V \qquad (1)$$

Two degree of freedom system defined by θ and x, therefore the generalised coordinate

$$\mathbf{q} = \begin{Bmatrix} x \\ \theta \end{Bmatrix} \text{ and } \dot{\mathbf{q}} = \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} \text{ etc}$$

- Potential Energy (only gravity)

$$V = -M_p g L_p (1 - \cos\theta)$$

  Using Taylor's expansion,

$$\cos\theta = 1 - \frac{\theta^2}{2!} + higher\ order\ terms$$

  Neglecting higher order terms, assuming only small rotations gives a linear model

$$V = \frac{-M_p g L_p \theta^2}{2} \qquad (2)$$

- Dissipation Energy (assume no damping on the pendulum air friction etc)

$$D = -\frac{1}{2} c \dot{x}^2 \qquad (3)$$

- Kinetic Energy

$$T = \frac{1}{2} M_c \dot{x}^2 + \frac{1}{2} M_p (\dot{x} + L_p \dot{\theta})^2 + \frac{1}{2} I_o \dot{\theta}^2$$

  Expanding gives

$$T = \frac{1}{2}\left\{ (M_c + M_p)\dot{x}^2 + 2 M_p L_p \dot{x}\dot{\theta} + (I_o + M_p L_p{}^2)\dot{\theta}^2 \right\} \qquad (4)$$

By observation assemble the symmetric mass, damping and stiffness matrices to form the EOM for the system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \hat{\mathbf{B}}\mathbf{F} \tag{5a}$$

$$\begin{bmatrix} (M_c + M_p) & M_p L_p \\ M_p L_p & (I_o + M_p L_p{}^2) \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} -C & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -M_p g L_p \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F \tag{5b}$$

State Space Form

Rearranging Equation 5a in terms of $\ddot{\mathbf{q}}$ gives

$$\ddot{\mathbf{q}} = -\mathbf{M}^{-1}\mathbf{C}\dot{\mathbf{q}} - \mathbf{M}^{-1}\mathbf{K}\mathbf{q} + \mathbf{M}^{-1}\hat{\mathbf{B}}\mathbf{F} \tag{6}$$

The state space equations

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}_1\mathbf{U} \tag{7a}$$

$$\mathbf{Y} = \mathbf{C}_1\mathbf{X} \tag{7b}$$

We have two second order equations, therefore need 4 states

Define the state vector

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{Bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix} \tag{8}$$

Then as per Equation 7a

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{2x2} & \mathbf{I}_{2x2} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{2x1} \\ \mathbf{M}^{-1}\hat{\mathbf{B}} \end{bmatrix} \mathbf{U} \tag{9a}$$

where U=F and 7b is defined:

$$\mathbf{Y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{X} \tag{9b}$$

Writing Equation 9a in full the state space equation

$$\dot{\mathbf{X}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{-M_p{}^2 L_p{}^2 g}{(M_c + M_p) I_o + M_c M_p L_p{}^2} & \dfrac{(I_o + M_p L_p{}^2) C}{(M_c + M_p) I_o + M_c M_p L_p{}^2} & 0 \\ 0 & \dfrac{(M_c + M_p) M_p L_p g}{(M_c + M_p) I_o + M_c M_p L_p{}^2} & \dfrac{-M_p L_p C}{(M_c + M_p) I_o + M_c M_p L_p{}^2} & 0 \end{bmatrix} \begin{Bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{I_o + M_p L_p{}^2}{(M_c + M_p) I_o + M_c M_p L_p{}^2} \\ \dfrac{-M_p L_p}{(M_c + M_p) I_o + M_c M_p L_p{}^2} \end{bmatrix} F$$

(10)

But the force applied to the cart can be written in terms of the voltage applied,

$$F = \frac{-k_m k_g}{Rr} V + \frac{k_m{}^2 k_g{}^2}{Rr^2} \dot{x}$$

Where R = resistance of motor armature

k_m = back EMF constant

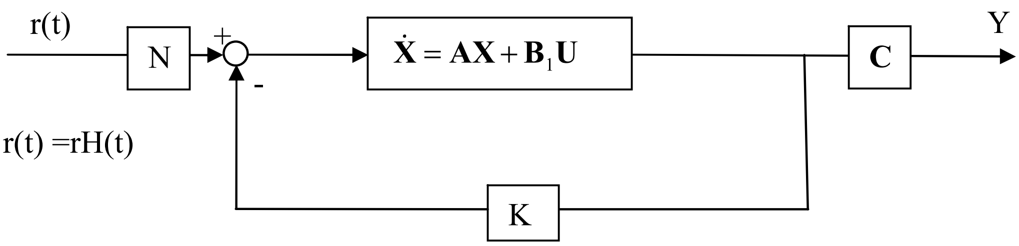
kg = gear ration in motor gearbox

r = radius of gear pinion that meshes with the track

Thus Equation 10 is rewritten as

$$\dot{\mathbf{X}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{-M_p^2 L_p^2 g}{(M_c+M_p)I_o + M_c M_p L_p^2} & \dfrac{(I_o + M_p L_p^2)(CRr^2 + k_m^2 k_g^2)}{\left[(M_c+M_p)I_o + M_c M_p L_p^2\right]Rr^2} & 0 \\ 0 & \dfrac{(M_c+M_p)M_p L_p g}{(M_c+M_p)I_o + M_c M_p L_p^2} & \dfrac{-M_p L_p(CRr^2 + k_m^2 k_g^2)}{\left[(M_c+M_p)I_o + M_c M_p L_p^2\right]Rr^2} & 0 \end{bmatrix} \begin{Bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{-(I_o + M_p L_p^2)k_m k_g}{\left[(M_c+M_p)I_o + M_c M_p L_p^2\right]Rr} \\ \dfrac{M_p L_p k_m k_g}{\left[(M_c+M_p)I_o + M_c M_p L_p^2\right]Rr} \end{bmatrix} V$$

The Tracking Problem



$$\mathbf{U}(t) = Nr(t) - \mathbf{KX}(t)$$

N is a scalar, find it so that as $t \to \infty$ the Y(t) $\to$ r, the reference input

Using derivation on page D8 of notes, N is given by

$$N = -\left[\mathbf{C}(\mathbf{A} - \mathbf{B}_1\mathbf{K})^{-1}\mathbf{B}\right]^{-1}$$

For the purposes of calculation N for this model let $\mathbf{C} = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}$, ie the only state of relevance is the cart position.

Remember:

- *r(t) = r*H(t) = 0.1*H(t)*, therefore
- **B****N*r(t) = 0.1****B****N*H(t)*, thus in using the step command you must use…
- **B** ➔ *0.1****B** if you want to get the correct results.

If you write out the equations with these notes above for the closed loop state feedback system you will see why this is (note especially the second equation). This is also something likely to appear as necessary in your lab write up!