Site percolation on a Bethe lattice

Bethe lattices do not have loops, i.e. the only way to return to a lattice site which has been previously traversed, is by retracing your path. Not having loops makes the connectivity percolation problem analytically tractable [1, 2].

A Bethe lattice with $z = 3$ can be constructed by choosing a point to be a site in the lattice, drawing three bonds emanating from the site, drawing two more bonds from each of the three sites and so on. The number of sites in the lattice increases exponentially with the number of such generations of bonds. The last generation of bonds are the dangling bonds whose outer sites define the surface of the Bethe lattice.

A probability p can be associated with every lattice site being active (in figure: green - active, red - inactive) [3]. If two neighbouring bonds are active, then they are said to be connected. The problem of connectivity percolation can now be posed as, what is the critical value p_{cr} at which a site in the interior of the lattice gets connected to the surface of the lattice.

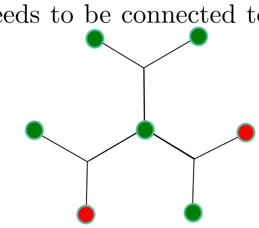
Starting a journey from a given site, lets call it the origin site, the average number of neighbours which are connected to the site is $(z-1)p$. The probability that you can traverse further generations of the lattice in your expedition to the surface of the lattice, goes as $[(z-1)p]^g$ where g is the number of generations of bonds or the number of steps in your walk. Notice that calculation of such a probability is not within the scope of say, a square lattice, since the number of paths between any two points is infinite for an infinite lattice and not easily calculable for a finite lattice either. In other words, the presence of 'loops' (any two paths connecting two given points make a loop) makes the problem analytically intractable. Carrying on, if $(z-1)p < 1$, then the probability you reach the surface tends to zero. The critical probability p_{cr} can then be calculated by setting $(z-1)p_{cr}$ to one, giving

$$p_{cr} = \frac{1}{z-1}. \tag{1}$$

However, we have made an assumption here that the neighbors of the origin site we have picked for our journey to the surface to be active as well. That very well need not be the case. Thus $p > p_r$ doesn't guarantee the site to be a part of the cluster. Let P be the probability that a given site be a part of a macroscopically spanning infinite cluster of connected active sites. Let Q be the probability that the given site is not connected to the spanning cluster *through* a given bond. Then,

$$P = p(1-Q)^3 \tag{2}$$

The factor p above is that the site itself must be active to be a part of the spanning cluster. The cubic factor comes since the site needs to be connected to every neighbour



it has to be a part of the connected spanning cluster (see Figure, if you miss a neighbour, you miss the whole of the cluster behind the neighbour!).

Q can be found by a self-consistency equation.

$$Q = 1 - p + pQ^2 \tag{3}$$

The site itself is not active by a probability $1-p$. Also through a given bond, the neighbours of the site at the other end of the bond will not be a part of the spanning cluster with the same probability Q. $1, (1-p)/p$ are the trivial and non-trivial solutions to the above equation respectively. The non-trivial solution can then be used to determine P by using Eq. (2),

$$P = 8\,\frac{(p-0.5)^3}{p^2}. \tag{4}$$

Evaluating for P at p close to $p_{cr} = 0.5$, we get $P \sim (p-0.5)$ showing the critical exponent of this continuous transition to be one. For Bethe lattices with higher z, one needs to solve a polynomial equation of a higher degree to obtain Q in Eq. (3).

[1] K. Christensen and N. R. Moloney, *Complexity and criticality*, Vol. 1 (World Scientific Publishing Company, 2005).

[2] D. Stauffer and A. Aharony, *Introduction to percolation theory* (CRC press, 2018).

[3] D. Beysens and G. Forgacs, *Dynamical Networks in Physics and Biology: At the Frontier of Physics and Biology Les Houches Workshop, March 17–21, 1997*, Vol. 10 (Springer Science & Business Media, 2013).