Summer Research Project

On

Rattlebacks

Done At

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Under The Guidance of

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Contents

Abstract

We begin by examining the dynamics of a single rattleback. Then the paper by Tsai et.al (2005) is described briefly based upon which the equations of the system for a mixture of chiral objects and their mirror objects were worked out (section (3)). The theme of the report is to observe how the *chirality* of the object induces a spin preference to the object. Be it a rattleback which has a complex shape or a simple wire bent in two places to make it chiral.

1 Rattleback (*celt/wobblestone*)

1.1 Description

Before[1] reading further on, I would recommend you to watch the video

`http://www.youtube.com/watch?v=CJzRuprW_cc`

showing how a celt behaves. Otherwise, I am describing below as to how it likes to move:

1. Keep the celt with the tip of it's curved surface touching the smooth horizontal surface of the table, and now press the tip of the celt. The celt vertically oscillates and also starts spinning in it's preferred direction while the oscillation simultaneously keeps damping with time.
2. If a stationary celt is made to spin in the direction it does not prefer, the celt rotates for a small time in the forced direction and simultaneously builds up the amplitude of the vertical oscillations, stops, and then starts spinning in the direction it prefers and simultaneously damps the vertical oscillation.

1.2 Theory

The basic theory is described below in a relatively detailed manner in comparison to the current literature. The equations are as given in Franti[1] who uses Bondi's[2] model in his study.

1.2.1 Notations and diagram

- $\vec{s}$ =position of center of mass
- $\frac{d\vec{s}}{dt} = \vec{v}$ =velocity of center of mass
- $\vec{r}$ =vector from the center of mass to the point of contact
- $\vec{F}$ =force exerted to the rattleback (celt) by the table
- $\vec{\omega}$ =angular velocity of the celt
- $\vec{h}$ =angular momentum of the celt

[1]The knowledge of the theory of this section is unnecessary for understanding sections (2) and (3).

- M =mass of the celt
- $\vec{u}$ =normal vector to the table, pointing upwards.

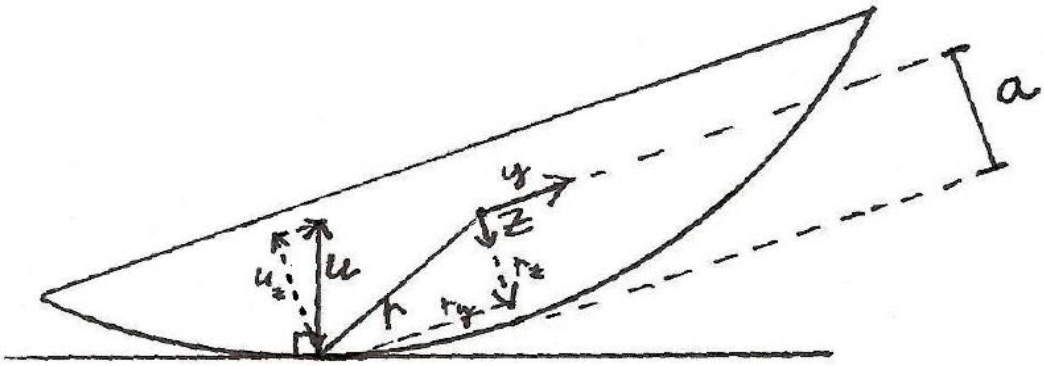

Figure 1: Non-equilibrium position [1]

1.2.2 The Newtonian equations

Writing the Newton's law to find the equation of motion of the center of mass,

$$M\frac{d\vec{v}}{dt} = \vec{F} - Mg\vec{u} \tag{1}$$

$$\vec{v} + (\vec{\omega} \times \vec{r}) = 0 \tag{2}$$

The convention here is different from the convention I was used to, $\vec{v} = \vec{\omega} \times \vec{r}$.
Now, considering the torque about the center of mass of celt,

$$\vec{\tau} = \frac{d\vec{h}}{dt} = \vec{r} \times \vec{F} \tag{3}$$

Here, the gravitational force $-Mg\vec{u}$ does not provide any torque as this force passes through the center of mass. Substituting equation (1) in equation (3)

$$\frac{d\vec{h}}{dt} = M\vec{r} \times \left[\frac{d\vec{v}}{dt} + g\vec{u}\right] \tag{4}$$

Taking the dot product of equation (4) with $\vec{u}$ on both sides,

$$\vec{u}.\frac{d\vec{h}}{dt} = \vec{u}.\left[M\vec{r} \times \left\{\frac{d\vec{v}}{dt} + g\vec{u}\right\}\right]$$

Also, in general, the scalar product,

$$\vec{A}.(\vec{B} \times \vec{C}) = \vec{B}.(\vec{C} \times \vec{A}) = \vec{C}.(\vec{A} \times \vec{B}) \tag{5}$$

As $\vec{u}$ is time independent, we can now write

$$\frac{d}{dt}(\vec{u}.\vec{h}) = M(\vec{u} \times \vec{r}).\frac{d\vec{v}}{dt} \tag{6}$$

In this model the dissipation of energy from the rattleback to the surroundings is ignored and the system is taken to be conservative. We also assume pure rolling, i.e. rolling without slipping which is to be provided by sufficient friction.

To exhibit the properties of a celt, the axes of inertia of the object should differ from the axes of symmetry and the chirality of the object is necessary. This will become clear as soon as the necessary equations are written down. The origin of the co-ordinate system is chosen to be at the center of mass and the co-ordinate axes are aligned along the axes of inertia. The positive z- axis points downwards from the center of mass. Also all the principal moments of inertial A, B, C should be different and $A > B$.

1.2.3 Describing the shape of the celt

The equation of the curved surface of the celt

$$z = a\left[1 - \frac{1}{2}p\left(\frac{x}{a}\right)^2 - q\frac{xy}{a^2} - \frac{1}{2}s\left(\frac{y}{a}\right)^2\right] \tag{7}$$

a is the equilibrium distance from center of mass to point of contact.
This is the form of the curved surface of the celt only till second order, near the equilibrium. As the point of contact at equilibrium is the maximum[2], it is necessary that the first order terms be avoided in equation (7). Otherwise the first partial derivatives of z with respect to x, y will not be zero at $(0, 0, a)$.
It should also be noted that in equation (7), the 'chirality'[3] of the object is ensured. This surface does not have any symmetries[4].
For a fixed $y \neq 0$, for $x = k$ and $x = -k$ where k is any number, z is found to be different. The co-efficient q, ensures this chirality.
This surface has to be concave. So, the Hessian matrix should be negative definite.

$$\begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix}$$

$$\frac{\partial^2 z}{\partial x^2} < 0 \Rightarrow p > 0$$

$$\frac{\partial^2 z}{\partial y^2} < 0 \Rightarrow s > 0$$

The determinant should be $> 0 \Rightarrow ps > 4q^2$
Also,

$$p < 1,\ s < 1,\ q^2 < (1-p)(1-s) \tag{8}$$

I will explain the reason for this condition in section (1.2.12)

[2]Not minimum. Remember that our positive z-axis points downwards.

[3]The non-super imposablility of the mirror image of the object on the object.

[4]Example: The surface is not symmetric under reflection about x, y axis, neither it has any rotational symmetry, etc.

1.2.4 The co-ordinate system

The celt is analyzed using a co-ordinate system *fixed* to the celt.
$\vec{r} = (x, y, z)$ is the position vector of the point of contact. z is given by equation (7).

The negative of the gradient of the surface, equation (7) should point along $\vec{u}$.

$$\vec{u} = -\frac{\frac{px+qy}{a}\,\hat{x} + \frac{qx+sy}{a}\,\hat{y} + \hat{z}}{\sqrt{\left(\frac{px+qy}{a}\right)^2 + \left(\frac{qx+sy}{a}\right)^2 + 1}} \tag{9}$$

The denominator is just the normalization factor (w). At equilibrium $(0, 0, a)$, the gradient is opposite $\vec{u}$ as expected.

1.2.5 Angular velocity

For any general $\vec{p}$,

$$\frac{d\vec{p}}{dt} = \dot{\vec{p}} + (\vec{\omega} \times \vec{p}) \tag{10}$$

holds. $\frac{d\vec{p}}{dt}$ is the rate of change of $\vec{p}$ in the stationary co-ordinate system and $\dot{\vec{p}}$ is the rate of change of $\vec{p}$ in a rotating frame (with uniform angular velocity).
Likewise,

$$\frac{d\vec{u}}{dt} = \dot{\vec{u}} + (\omega \times \vec{u}) = 0 \tag{11}$$

Taking cross product with $\vec{u}$ on both sides,

$$(\vec{u} \times \dot{\vec{u}}) + \vec{u} \times (\vec{\omega} \times \vec{u}) = (\vec{u} \times \dot{\vec{u}}) + \vec{\omega}(\vec{u}.\vec{u}) - \vec{u}(\vec{u}.\vec{w}) = 0$$

$$\Rightarrow (\vec{u} \times \dot{\vec{u}}) + \vec{\omega} - n\vec{u} = 0$$

$$\vec{\omega} = n\vec{u} + (\dot{\vec{u}} \times \vec{u}) \tag{12}$$

n is the spin/vertical componet of the angular momentum ω. Taking $x, y << 1$ and expanding ω till first order of x, y

$$\omega_1 = nu_1 + \frac{w^2}{a}(q\dot{x} + s\dot{y}) = -\frac{nw}{a}(px + qy) + \frac{w^2}{a}(q\dot{x} + s\dot{y})$$

Taking $w \approx 1$

$$\omega_1 = \frac{1}{a}[q\dot{x} + s\dot{y} - n(px + qy)] \tag{13}$$

Similarly,

$$\omega_2 = \frac{1}{a}[-p\dot{x} - q\dot{y} - n(qx + sy)] \tag{14}$$

$$\omega_3 = \frac{1}{a^2}(ps - q^2)(\dot{x}y - \dot{y}x) - n\left[1 - \frac{1}{2}\left(\frac{px + qy}{a}\right)^2 - \frac{1}{2}\left(\frac{qx + sy}{a}\right)^2\right] \tag{15}$$

1.2.6 Velocity of contact point

As the co-ordinate axes (fixed to the celt) are the principal axes of inertia,

$$h_1 = A\omega_1, h_2 = B\omega_2, h_3 = C\omega_3 \tag{16}$$

$$\vec{v} = \vec{r} \times \vec{\omega}$$

$$v_1 = y\omega_3 - z\omega_2$$

$$= y\left[\frac{1}{a^2}(ps-q^2)(x\dot{y}-\dot{y}x) - n\left\{1-\frac{1}{2}\left(\frac{px+qy}{a}\right)^2 - \frac{1}{2}\left(\frac{qx+sy}{a}\right)^2\right\}\right] -$$

$$\frac{1}{a}\left\{-p\dot{x}-q\dot{y}-n(qx+sy)\right\} a\left\{1-\frac{1}{2}p\left(\frac{x}{a}\right)^2 - q\frac{xy}{a^2} - \frac{1}{2}s\left(\frac{y}{a}\right)^2\right\}$$

where I have used equation (7) as z.
Ignoring all the second order and higher terms,

$$v_1 = p\dot{x} + q\dot{y} + n(qx - (1-s)y) \tag{17}$$

Similarly,

$$v_2 = q\dot{x} + s\dot{y} + n(1-p)x + qy \tag{18}$$

$$v_3 \to \text{second order} \tag{19}$$

1.2.7 Rate of change of angular momentum

Now,

$$\frac{d\vec{h}}{dt} = \dot{\vec{h}} + (\vec{\omega} \times \vec{h}) \tag{20}$$

$$\frac{dh_1}{dt} = \dot{h_1} + (\vec{\omega} \times \vec{h})_1 = \dot{h_1} + (\omega_2 h_3 - \omega_3 h_2)$$

$$\frac{dh_1}{dt} = A\dot{\omega_1} + \omega_2 C\omega_3 - \omega_3 B\omega_2 = A\dot{\omega_1} - (B-C)\omega_2\omega_3 \tag{21}$$

By using equations (14, 15, 13),

$$\frac{dh_1}{dt} = \frac{A}{a}[q\ddot{x} + s\ddot{y} - n(p\dot{x} + q\dot{y}) - \dot{n}(px+qy)] - \frac{n}{a}(B-C)[p\dot{x} + q\dot{y} + n(qx+sy)] \tag{22}$$

Consider,

$$\frac{d\vec{v}}{dt} = \frac{d(\vec{r} \times \vec{\omega})}{dt} = \left(\frac{d\vec{r}}{dt} \times \vec{\omega}\right) + \left(\vec{r} \times \frac{d\vec{\omega}}{dt}\right) \tag{23}$$

Thus,

$$\frac{dv_1}{dt} = \dot{y}\omega_3 - \dot{z}\omega_2 + y\dot{\omega_3} - z\dot{\omega_2}$$

$$\frac{dv_1}{dt} = v_2\omega_3 - v_3\omega_2 + y\dot{\omega_3} - z\dot{\omega_2} \tag{24}$$

$$\frac{d\vec{v}}{dt} + g\vec{u} = \left(\dot{v_1} - g\left(\frac{px+qy}{a}\right), \dot{v_2} - g\left(\frac{qx+sy}{a}\right), \dot{v_3} - g\right) \tag{25}$$

The above result is got by using equation (9). Now, that we have this result, we can compute equation (4) which comes out to be,

$$\begin{aligned}\frac{dh_1}{dt} = &-Ma[q\ddot{x} + s\ddot{y} + n(1-2p)\dot{x} - 2nq\dot{y} + \dot{n}\{(1-p)x - qy\} \\ &-n^2\{qx - (1-s)y\}] + Mga[qx - (1-s)y]\end{aligned} \tag{26}$$

So, we have two equalities for $\frac{dh_1}{dt}$, equation (22) and equation(26). This I am summing up:-

$$\begin{gathered}\frac{dh_1}{dt} = A\dot{\omega_1} - (B-C)\omega_2\omega_3 = \\ \frac{A}{a}[q\ddot{x} + s\ddot{y} - n(p\dot{x} + q\dot{y}) - \dot{n}(px+qy)] - \frac{n}{a}(B-C)[p\dot{x} + q\dot{y} + n(qx+sy)] = \\ -Ma[q\ddot{x} + s\ddot{y} + n(1-2p)\dot{x} - 2nq\dot{y} + \dot{n}\{(1-p)x - qy\} \\ -n^2\{qx-(1-s)y\}] + Mga[qx-(1-s)y]\end{gathered} \tag{27}$$

Similarly for the second and the third components of equation (4),

$$\begin{gathered}\frac{dh_2}{dt} = B\dot{\omega_2} - (C-A)\omega_3\omega_1 \\ = -\frac{B}{A}\left[p\ddot{x} + q\ddot{y} + n(q\dot{x} + s\dot{y}) + \dot{n}(qx+sy) + \frac{n}{a}(C-A)(q\dot{x} + s\dot{y} - n(px+qy))\right] \\ = Ma[p\ddot{x} + q\ddot{y} + n\{2q\dot{x} - (1-s)\dot{y}\} + \dot{n}\{qx - (1-s)y\} \\ +n^2\{(1-p)x - qy\}] + Mga[x(1-p) - qy]\end{gathered} \tag{28}$$

and

$$\begin{gathered}\frac{dh_3}{dt} = C\omega_3 - (A-B)\omega_1\omega_2 = -C\dot{n}\left[1 - \frac{1}{2}\left(\frac{px+qy}{a}\right)^2 - \frac{1}{2}\left(\frac{qx+sy}{a}\right)^2\right] \\ +\frac{1}{a^2}Cn[(px+qy)(p\dot{x}+q\dot{y}) + (qx+sy)(q\dot{x}+s\dot{y})] + \frac{1}{a^2}C(ps-q^2)(\ddot{x}y - x\ddot{y}) \\ = Mx[q\ddot{x} + s\ddot{y} + \dot{n}\{(1-p)x - qy\}] - My[p\ddot{x} + q\ddot{y} + n(q\dot{x}+s\dot{y}) + \dot{n}\{qx-(1-s)y\}] \\ +Mn[x\{(1-2p)\dot{x} - 2q\dot{y}\} - y\{2q\dot{x} - (1-2s)\dot{y}\} - xn\{qx-(1-s)y\} - yn\{(1-p)x-qy\}] \\ -M\frac{g}{a}[q(x^2+y^2) + (p-s)xy]\end{gathered} \tag{29}$$

From the first equality of equation (29), $\dot{n}$ should be of second order (i.e. quadratic) because the left hand side of that equation yields only second order terms. Thus, all the terms of the right hand side should be second order terms or higher[5].

1.2.8 Simplification of equations (I)

To simplify equations,

$$\begin{gathered}A + Ma^2 = \alpha Ma^2 \\ B + Ma^2 = \beta Ma^2 \\ C = \gamma Ma^2\end{gathered} \tag{30}$$

[5]Higher order terms may be obtained on RHS because we have voluntarily ignored second order terms in forming ω_2,ω_3 etc. But lower order terms than second order is not possible on the right hand side.

such that,

$$\alpha > 1, \beta > 1, \gamma > 0, \alpha + \beta - \gamma > 2, \beta + \gamma - \alpha > 0, \gamma + \alpha - \beta > 0 \tag{31}$$

The first three inequalities of equation (31) are obvious. The last three inequalities come from the perpendicular axis theorem that states that the sum of the moment of inertia about any two axes is greater than the moment of inertia about the third axis where all the axis are perpendicular to each other.
This makes the last two inequalities obvious.
I am working out the fourth inequality below,

$$\begin{aligned} A + B + 2Ma^2 + C &= \alpha Ma^2 + \beta Ma^2 + \gamma Ma^2 \\ \Rightarrow C &= Ma^2[2 - (\alpha + \beta - \gamma)] + A + B \end{aligned} \tag{32}$$

As we want $C < A + B$, we want $\alpha + \beta - \gamma > 2$.
With these simplifications, equations (27, 28, 29) can be re-written as,

$$\alpha(q\ddot{x} + s\ddot{y}) - (\alpha + \beta - \gamma)n(p\dot{x} + q\dot{y}) - (\beta - \gamma)n^2(qx + sy) + n\dot{x} + n^2 y = \frac{g}{a}[px - (1 - s)y] \tag{33}$$

$$\beta(p\ddot{x} + q\ddot{y}) + (\alpha + \beta - \gamma)n(q\dot{x} + s\dot{y}) - (\alpha - \gamma)n^2(px + qy) - n\dot{y} + n^2 x = \frac{g}{a}[qy - (1 - p)x] \tag{34}$$

The complicate equation (29) can be simplified by taking n to have only small values,

$$a^2\gamma\dot{n} = \gamma(ps - q^2)(\ddot{x}y - \ddot{y}x) + [q(y\ddot{y} - x\ddot{x}) + py\ddot{x} - s\ddot{y}x] + \frac{g}{a}[(p - s)xy + q(x^2 + y^2)] \tag{35}$$

1.2.9 The characteristic differential equation

I will give a tutorial below on how to form the characteristic equation for coupled systems of differential equations,
Say,

$$A\ddot{x} + B\dot{y} + Cx = (AD^2 + C)x + BDy = 0 \tag{36}$$

$$G\ddot{y} + E\dot{x} + Fy = (GD^2 + F)y + EDx = 0 \tag{37}$$

where $\dot{x}$ indicates derivative with respect to t and D is the differential operator. Now multiplying equation (36) and equation (37) from the left with ED (an operator) and $(AD^2 + C)$ respectively and subtracting the two got equations (which cancels off two terms), we get the *characteristic equation*,

$$BED^2y - (AD^2 + C)(GD^2 + F)y = 0 \tag{38}$$

- *End tutorial*

Likewise we do for the equations (33, 34). Also, as we are observing an oscillation about the equilibrium, we expect the solution for x, y to be of form $Ce^{\sigma t}$ which is substituted in the characteristic equation to get the final form:-

$$\begin{gathered} (\sigma^2 + n^2)[\alpha\beta(ps - q^2)\sigma^2 + (\alpha - \beta)qn\sigma] \\ +(\sigma^2 + n^2)[n^2\{1 - p(\alpha - \gamma) - s(\beta - \gamma) + (ps - q^2)(\alpha - \gamma)(\beta - \gamma)\} + \\ \frac{g}{a}\{\alpha s + \beta p - (ps - q^2)(\alpha + \beta)\}] \\ + \frac{n^2 g}{a}[2 - (1 + \alpha + \beta - \gamma)(p + s) + 2(\alpha + \beta - \gamma)(ps - q^2)] + \left(\frac{g}{a}\right)^2[(1 - p)(1 - s) - q^2] = 0 \end{gathered} \tag{39}$$

Solving for σ in this fourth order equation will give us the equations of motion. But instead of solving, the celt behaviour can be easily observed from equation (39).

1.2.10 Inferences from the characteristic equation

If σ is the solution to the characteristic equation (39) when the spin is n, then $-\sigma$ is a solution to the same equation when the spin is $-n$. i.e. when the celt turns around and spins. This is obvious from this equation because n appears squared in every term in equation (39) except in the second term. So, when $n \to -n$, $\sigma \to -\sigma$ will keep the whole equation exactly same as before. So, $-\sigma$ is the new root for the case of $-n$.

By the way, remember that x, y changes only because of oscillation and not rotation as the co-ordinate system is attached to the body of the celt.
For example, let the celt be in the position as shown in Figure (1) and let there be no oscillation but only rotation. Then x, y does not change at all as the as the relative distances between center of mass and point of contact doesn't change at all. The rotating body is seen to be stationary in this co-ordinate system, attached to the body.

Coming back to analyzing equation (39), in general,

$$x = e^{\sigma t} = e^{(X+iY)t} = e^{Xt}e^{iYt} = e^{Xt}\,(cos(Yt) + i\; sin(Yt)) \tag{40}$$

The position co-ordinates (x,y) is not to be confused with X and Y.
When you rotate the celt with your finger in the direction it does not prefer, let the spin be n and $X = k > 0$ and the equation of motion of the system is given by equation (40) where X is replaced with k. So, with increasing time, the amplitude of oscillation keeps increasing. The increase in energy required for oscillation is fed by the decrease in the kinetic energy of the spin of the celt as the spin slows down with time. When there is no more energy left for the spin motion to give to the oscillatory motion, the celt stops spinning. As the celt stopped due to the instability of it rotating in that direction, it is unlikely that it will again move in that direction. It will turn around and now rotate in it's stable direction.
The spin is now $-n$ and the root of equation (39) is $-\sigma$. Now the equation of motion is equation (40) where $X = -k < 0$. So, amplitude of oscillation keeps decreasing with time. And this energy is fed into the spin of the celt and the celt starts rotating faster as the oscillation keeps damping with time.

It is to be noticed that the second term of equation (39) is critical in making possible the characteristic rattleback motion. It is the only term where n and σ occur in first order in the equation. Everywhere else they occur in second order and no where else are they multiplied to each other.
The coefficient of this important term is zero, when:-

1. $\alpha = \beta$
   $\Rightarrow A = B$.
   That's why it is necessary that the two moments of inertia in the horizontal axes of the celt should not be equal. If equal, the rattleback behaviour cannot be observed in the object.

2. $q = 0$
   q represents the non-overlap of the moment of inertia axis and the symmetry axis.

Also $q = 0$ when the smallest and the largest radius of curvature of the body be the same. These can be more evidently seen upon further analysis by Bondi.[6]

1.2.11 Simplification of equations (II)

Bondi defined the parameters as follows:-

$$\begin{aligned} \theta + \phi &= p + s \\ \theta\phi &= ps - q^2 \\ \psi(\theta - \phi) &= p - s \end{aligned} \tag{41}$$

Choosing $q > 0$ and inverting these equations,

$$\begin{aligned} p &= \frac{1}{2}(\theta + \phi) + \frac{1}{2}\psi(\theta - \phi) \\ q &= \frac{1}{2}(\theta - \phi)(1 - \psi^2)^{\frac{1}{2}} \\ s &= \frac{1}{2}(\theta + \phi) - \frac{1}{2}\psi(\theta - \phi) \end{aligned} \tag{42}$$

By defining two dimensionless parameters,

$$\begin{aligned} \rho &= \frac{\sigma}{n} \\ \Omega &= \frac{g}{an^2} \end{aligned} \tag{43}$$

where ρ characterizes the roots of the equation and Ω characterizes the spin of the object, the characteristic differential equation (39) of the system can now be written as,

$$(\rho^2 + 1)(\rho^2 + \chi\rho + \kappa + \Omega\lambda) + \Omega\mu + \Omega^2\nu = 0 \tag{44}$$

where the auxiliary parameters are defined as follows:-

$$\begin{aligned} \alpha\beta\theta\phi\chi &= \frac{1}{2}(\alpha - \beta)(\theta - \phi)(1 - \psi^2)^{\frac{1}{2}} \\ \alpha\beta\theta\phi\kappa &= 1 - \frac{1}{2}(\alpha + \beta - 2\gamma)(\theta + \psi) + (\alpha - \gamma)(\beta - \gamma)\theta\phi - \frac{1}{2}(\alpha - \beta)(\theta - \phi)\psi \\ \alpha\beta\theta\phi\lambda &= \frac{1}{2}(\alpha + \beta)(\theta + \phi - 2\theta\phi) - \frac{1}{2}(\alpha - \beta)(\theta - \phi)\psi \\ \alpha\beta\theta\phi\mu &= 2 - (\theta + \phi) - (\alpha + \beta - \gamma)(\theta + \phi - 2\theta\phi) \\ \alpha\beta\theta\phi\nu &= (1 - \theta)(1 - \phi) \end{aligned} \tag{45}$$

In the new characteristic equation (44), it can be seen that the new critical term is the second term of the second bracket.
Garcia and Hubbard [3] defined the parameters (41) as curvature parameters,

- $\theta =$ the ratio of a to the smallest radius of curvature.

[6] It may be noted that it was the co-efficient $q \neq 0$ which ensured the chirality of the object in equation (7).

- $\phi =$ the ratio of a to the biggest radius of curvature.
- $\psi = cos^2(\xi) - sin^2(\xi)$.

ξ is the small angle between the direction of the smaller radius of curvature and the axis of inertia deviating from it.
Franti [1] brought about the equivalence of the two ways of defining the parameters by Bondi [2] and Garcia and Hubbard [3].

It can be seen that $q = 0$ when,

- $\theta = \phi$. i.e. the smallest radius of curvature = biggest radius of curvature. So, basically the radius of curvature is the same in all directions.
  Thus it is necessary that the radii of curvature be different to exhibit the rattleback behaviour.
- $\psi = 1$. This will happen when $\xi = 0,\ \pi$. i.e. when the moment of inertia axis overlaps with the symmetry axis, the rattleback behaviour can not be observed.

1.2.12 Explaining equation (8)

As we want the oscillations to be stable, we want all the radii of curvatures of the body to be greater than a.
Thus, we would need,

$$\theta < 1,\ \phi < 1 \tag{46}$$

From the given conditions in equation (8) and from equation (41), we get

$$\theta + \phi < 2 \tag{47}$$

and also,

$$(1-p)(1-s) > q^2$$
$$\Rightarrow 1 - (p - s) > q^2 - ps$$

Thus,

$$\theta + \phi < 1 + \theta\phi \tag{48}$$

Equations (47) and (48) ensure the compliance of condition (46).
Thus equation (8) ensures that condition (46) is followed and the oscillation be stable.

1.2.13 Summary of this theory

We now know by this model:-

- The necessity of chirality of celt - we discussed the importance of the term q, in regards to bringing chirality
- The necessity of the non-overlap of principal moment of inertial axes and the symmetry axes - this condition is necessary so that $q \neq 0$. This co-efficient is crucial in bringing about the behaviour of the celt.

- How the amplitude of oscillation builds up when the celt is spinned in the non-preferred direction was seen. It stops when all the spin energy is transferred to the vertical oscillations. Now, as $-\sigma$ is a solution to the characteristic equation for spin $-n$ if σ is for spin n. The opposite directed spin is allowed too and that solution is chosen and here the celt reverses it's spin and the amplitude of the vibrations is seen to decrease and all this energy is taken up by the spin as it spins faster.

2 'A chiral granular gas' [4]

2.1 Introduction

Motivated by the property of spin preference of a chiral object (rattleback toys), Tsai et.al in Physical Review Letters (2005) [4] verified whether other simple 3D chiral objects too can show spin preference.

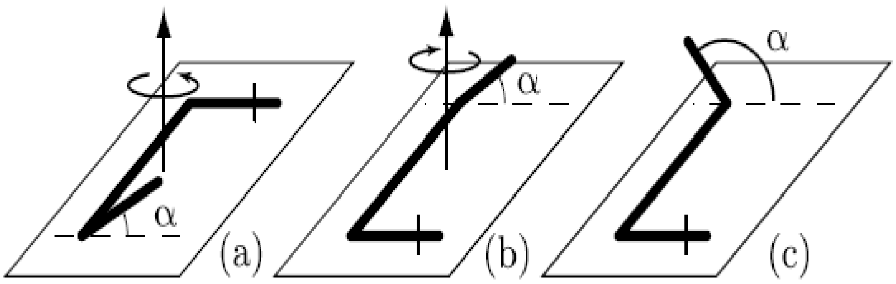

Figure 2: (a) Chiral wire (b) Mirror image of the chiral wire (c) The wire used in the experiments by the author, $|\alpha| = \frac{3\pi}{4}$ [4]

The objects in Figure (2) are made out of wires and are chiral for $\alpha \neq 0,\ \pi$.

One single wire was put into a cyclindrical container which has an oscillating flat lower platform and smooth vertical walls. Upon oscillation of the platform, the wire was found to show an average spin preference. This is analogous to pressing the tip of the rattleback and watching it convert the energy given to make it oscillate to spin motion.

They also put a lot of wires, about 200 of them in the container and oscillated the platform again. Each wire would be wanting to spin in it's preferred direction and the observed collective motion of all the wires together was that wires in the center of the container were stationary on an average and the center of mass of the wires performed a rotation about the center of the container everywhere else.

These videos can all be seen at,

`http://www.haverford.edu/physics-astro/Gollub/chiral`

To explain this behaviour, they developed a 2D continuum hydrodynamic theory considering the particles to form a chiral granular gas.

2.2 Theory

2.2.1 The general continuity equation

$$\frac{\partial \phi}{\partial t} + \vec{\nabla}.\vec{f} = \sigma \tag{49}$$

Here ϕ is a quantity which is being transported by the flow of the fluid. $\vec{f} = \phi\,\vec{v}$ is the flux of the quantity where $\vec{v}$ is the velocity field of the fluid.
If $\sigma > 0$, then it acts as a source of the quantity ϕ.
If $\sigma < 0$, then it acts a sink.
If $\sigma = 0$, then ϕ is conserved and equation (49) would be the strong local form of the conservation law for ϕ.

2.2.2 Equations of the system

- The angular momentum equation[7]

$$\frac{\partial l}{\partial t} = -\frac{\partial(lv_j)}{\partial j} - \Gamma^{\Omega}\Omega - \Gamma(\Omega - \omega) + D_{\Omega}\nabla^2\Omega + \tau \tag{50}$$

  - $l(\vec{x}, t) = I\Omega$, l is the spin angular momentum density due to the spin of wires about their center of mass. Ω is defined by this equation. As our gas is 2D, the angular momentum referred here is the angular momentum in the direction perpendicular to the gas. I is the moment of inertia density. $\vec{x}$ is the two component vector.
  - The first term and fourth terms are the advection and diffusion terms respectively.
  - Γ^{Ω} is the co-efficient of friction with the substrate (the flat bottom).
  - The third term forces Ω to equal ω which will happen 'when the whole sample rotates rigidly [4]'.
  - $\omega = \frac{(\vec{\nabla}\times\vec{v}_z)}{2}$ is the vorticity or the coarse-grained angular velocity.
  - τ is the source for angular momentum which is provided by the vertical oscillations of the substrate and is a function of frequency, amplitude of oscillation and shape of particles.
  - These terms are phenomenologically added to the continuity equation (49). All the extra terms can be considered to make up σ.

- The linear momentum equation

$$\begin{aligned}\frac{\partial g_i}{\partial t} &= -\frac{\partial(g_i v_j)}{\partial j} - \frac{\partial p}{\partial i} + \eta\nabla^2 v_i - \Gamma^{\nu}v_i + \frac{1}{2}\epsilon_{ij}\frac{\partial[\Gamma(\Omega-\omega)]}{\partial j} \\ &= \frac{\partial\sigma_{ij}}{\partial j} - \Gamma^{\nu}v_i\end{aligned} \tag{51}$$

  - g_i is the linear momentum in the i^{th} direction.
  - η is the viscosity.
  - p is pressure.
  - $\epsilon_{ij} = -\epsilon_{ji}$ is the anti-symmetric symbol.
  - The first three terms of the equation are from the Navier-Stokes equation.
  - σ_{ij} is the stress tensor.

[7]Einstein-summation-convention followed.

3 The mixture of chiral objects and their mirror image objects

3.1 Introduction

We tried to write down the equations of the system for the same situation as in [4] but which contains a *mixture* of chiral objects and their mirror image objects.

3.2 Theory

The below equations were written in collaboration with Ananyo Maitra[8] and Sriram Ramaswamy.
Let n_1 and n_2 be the number density of the right and left handed particles respectively. Let I_i, Ω_i, v_i, and l_i be the moment of inertia density, angular velocity about centre of mass, velocity, and angular momentum density of the ith species ($i = 1, 2$). Let

$$\omega_i = \frac{(\nabla \times v_i)_z}{2} \tag{52}$$

be the vorticity.

3.2.1 Definitions

$$n_1 + n_2 = n \tag{53}$$

$$n_1\mathbf{v}_1 + n_2\mathbf{v}_2 = n\mathbf{v} \tag{54}$$

$$n_1 - n_2 = m \tag{55}$$

$$n_1\mathbf{v}_1 - n_2\mathbf{v}_2 = m\mathbf{v}^{ch} \tag{56}$$

$$I_1 + I_2 = I \tag{57}$$

$$I_1\Omega_1 + I_2\Omega_2 = I\Omega \tag{58}$$

$$I_1\omega_1 + I_2\omega_2 = I\omega \tag{59}$$

$$I_1 - I_2 = \tilde{I} \tag{60}$$

$$I_1\Omega_1 - I_2\Omega_2 = \tilde{I}\tilde{\Omega} \tag{61}$$

$$I_1\omega_1 - I_2\omega_2 = \tilde{I}\tilde{\omega} \tag{62}$$

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$$l_1 + l_2 = l \tag{63}$$

$$l_1 - l_2 = \tilde{l} \tag{64}$$

3.2.2 Relations

$$\mathbf{v}_1 = \frac{n\mathbf{v} + m\mathbf{v}^{ch}}{n+m} \tag{65}$$

$$\mathbf{v}_2 = \frac{n\mathbf{v} - m\mathbf{v}^{ch}}{n-m} \tag{66}$$

$$\Omega_1 = \frac{I\Omega + \tilde{I}\tilde{\Omega}}{I + \tilde{I}} \tag{67}$$

$$\Omega_2 = \frac{I\Omega - \tilde{I}\tilde{\Omega}}{I - \tilde{I}} \tag{68}$$

$$\mathbf{v}_1 + \mathbf{v}_2 = 2\frac{n^2\mathbf{v} - m^2\mathbf{v}^{ch}}{(m+n)(m-n)} \tag{69}$$

$$\mathbf{v}_1 - \mathbf{v}_2 = 2mn\frac{\mathbf{v}^{ch} - \mathbf{v}}{(m+n)(m-n)} \tag{70}$$

$$\Omega_1 + \Omega_2 = 2\frac{I^2\Omega - \tilde{I}^2\tilde{\Omega}}{(I+\tilde{I})(I-\tilde{I})} \tag{71}$$

$$\Omega_1 - \Omega_2 = 2I\tilde{I}\frac{\tilde{\Omega} - \Omega}{(I+\tilde{I})(I-\tilde{I})} \tag{72}$$

3.2.3 Consider these equations, they are needed further down the theory

We need the four equations below later to derive equation (83), (85), (89) and (91).

$$n_1\mathbf{v}_1\mathbf{v}_1 + n_2\mathbf{v}_2\mathbf{v}_2 = \frac{n}{(n+m)(n-m)}[n^2\mathbf{v}\mathbf{v} + m^2\mathbf{v}^{ch}[\mathbf{v}^{ch} - 2\mathbf{v}]] \tag{73}$$

Here products like $\mathbf{v}_1\mathbf{v}_1$ indicate tensorial products. Here it is a 3×3 matrix.

$$n_1\mathbf{v}_1\mathbf{v}_1 - n_2\mathbf{v}_2\mathbf{v}_2 = \frac{-m}{(n+m)(n-m)}[m^2\mathbf{v}^{ch}\mathbf{v}^{ch} + n^2\mathbf{v}[\mathbf{v} - 2\mathbf{v}^{ch}]] \tag{74}$$

$$l_1\mathbf{v}_1 + l_2\mathbf{v}_2 = l\frac{n^2\mathbf{v} - m^2\mathbf{v}^{ch}}{(n+m)(n-m)} + \tilde{l}\frac{mn(\mathbf{v}^{ch} - \mathbf{v})}{(n+m)(n-m)} \tag{75}$$

$$l_1\mathbf{v}_1 - l_2\mathbf{v}_2 = \tilde{l}\frac{n^2\mathbf{v} - m^2\mathbf{v}^{ch}}{(n+m)(n-m)} + l\frac{mn(\mathbf{v}^{ch} - \mathbf{v})}{(n+m)(n-m)} \tag{76}$$

3.2.4 Conservation laws for each type of particle

The two continuity equations for the conservation of n_1 particles and n_2 particles are:

$$\partial_t n_1 = -\nabla \cdot (n_1 \mathbf{v}_1) \tag{77}$$

$$\partial_t n_2 = -\nabla \cdot (n_2 \mathbf{v}_2) \tag{78}$$

∂_t is $\frac{\partial}{\partial t}$

3.2.5 The linear momentum equations

$$\partial_t(n_1\mathbf{v}_1) = -\lambda\nabla\cdot(n_1\mathbf{v}_1\mathbf{v}_1) - n_1\nabla\frac{\delta F}{\delta n_1} + \eta\nabla^2\mathbf{v}_1 + \frac{1}{2}\epsilon\cdot\nabla\Gamma I_1(\Omega_1-\omega_1) - \Gamma^v n_1\mathbf{v}_1 - \tilde{\Gamma}(\mathbf{v}_2-\mathbf{v}_1) \tag{79}$$

$$\partial_t(n_2\mathbf{v}_2) = -\lambda\nabla\cdot(n_2\mathbf{v}_2\mathbf{v}_2) - n_2\nabla\frac{\delta F}{\delta n_2} + \eta\nabla^2\mathbf{v}_2 + \frac{1}{2}\epsilon\cdot\nabla\Gamma I_2(\Omega_2-\omega_2) - \Gamma^v n_2\mathbf{v}_2 - \tilde{\Gamma}(\mathbf{v}_1-\mathbf{v}_2) \tag{80}$$

where $n_i\nabla\frac{\delta F}{\delta n_i} = \nabla n_i$, $i = 1, 2$. This is equivalent to the pressure p term in [4] as pressure $\propto \vec{\nabla} n_i$. $\epsilon_{kl} = -\epsilon_{lk}$.

These equations are similar to the linear momentum equations of [4] other than the co-efficient λ and one extra term, $\tilde{\Gamma}(\mathbf{v_1} - \mathbf{v_2})$ for equation (79) and the negative of this term for equation (80). This is the friction term due to the relative velocities between the n_1 and n_2 particles. So, by Newton's III law, this term should bear the opposite sign in the above two equations.
The values of the co-efficients in equations for the two particles is taken to be the same for simplicity.
Now we use the earlier definitions and relations to transform these equations.

$$\partial_t n = -\nabla \cdot (n\mathbf{v}) \tag{81}$$

$$\partial_t m = -\nabla \cdot (m\mathbf{v}^{ch}) \tag{82}$$

These equations come due to the conservation of n and m.
By equations (79), (80), (73) and (54),

$$\begin{aligned}\partial_t(n\mathbf{v}) = -\lambda\nabla\cdot[\frac{n}{(n+m)(n-m)}(n^2\mathbf{v}\mathbf{v} + m^2\mathbf{v}^{ch}(\mathbf{v}^{ch} - 2\mathbf{v}))] - \\ \nabla n + 2\eta\nabla^2[\frac{n^2\mathbf{v} - m^2\mathbf{v}^{ch}}{(n+m)(n-m)}] + \frac{1}{2}\Gamma\epsilon\cdot\nabla I(\Omega-\omega) - \Gamma^v n\mathbf{v}\end{aligned} \tag{83}$$

Similarly, by equations (79), (80), (74) and (56)

$$\partial_t(m\mathbf{v}^{ch}) = -\lambda\nabla\cdot[\frac{n}{(n+m)(n-m)}(m^2\mathbf{v}^{ch}\mathbf{v}^{ch} + n^2\mathbf{v}(\mathbf{v}-2\mathbf{v}^{ch}))] - \nabla m + 2\eta\nabla^2[\frac{nm(\mathbf{v}^{ch}-\mathbf{v}}{(n+m)(n-m)}] + \frac{1}{2}\Gamma\epsilon\cdot\nabla\tilde{I}(\tilde{\Omega}-\tilde{\omega}) - \Gamma^v m\mathbf{v}^{ch} + 4\tilde{\Gamma}\frac{mn}{(n+m)(n-m)}(\mathbf{v}^{ch}-\mathbf{v}) \quad (84)$$

Defining $\Gamma^f = 4\tilde{\Gamma}\frac{mn}{(n+m)(n-m)}$, we have

$$\partial_t(m\mathbf{v}^{ch}) = -\lambda\nabla\cdot[\frac{n}{(n+m)(n-m)}(m^2\mathbf{v}^{ch}\mathbf{v}^{ch} + n^2\mathbf{v}(\mathbf{v}-2\mathbf{v}^{ch}))] - \nabla m + 2\eta\nabla^2[\frac{nm(\mathbf{v}^{ch}-\mathbf{v}}{(n+m)(n-m)}] + \frac{1}{2}\Gamma\epsilon\cdot\nabla\tilde{I}(\tilde{\Omega}-\tilde{\omega}) - \Gamma^v m\mathbf{v}^{ch} + \Gamma^f(\mathbf{v}^{ch}-\mathbf{v}) \quad (85)$$

Equations (83) and (85) are the two linear momentum equations of the systems we got.

3.2.6 The angular momentum equations

$$\partial_t l_1 = -\zeta\nabla\cdot(l_1\mathbf{v}_1) - \Gamma^\Omega I_1\Omega_1 - \Gamma^\omega I_1(\Omega_1-\omega_1) + D\nabla^2\Omega_1 + \tau_1 \quad (86)$$

$$\partial_t l_2 = -\zeta\nabla\cdot(l_2\mathbf{v}_2) - \Gamma^\Omega I_2\Omega_2 - \Gamma^\omega I_2(\Omega_2-\omega_2) + D\nabla^2\Omega_2 + \tau_2 \quad (87)$$

These equations too are the same as in [4], except the ζ co-efficient.
By equations (86), (87), (63), (75) and (76),

$$\partial_t l = -\zeta\nabla\cdot(l\frac{n^2\mathbf{v}-m^2\mathbf{v}^{ch}}{(m+n)(n-m)} + \tilde{l}\frac{mn(\mathbf{v}^{ch}-\mathbf{v})}{(m+n)(n-m)}) - \Gamma^\Omega I\Omega - \Gamma^\omega I(\Omega-\omega) + (\tau_1+\tau_2) \quad (88)$$

which with redefinition of constants can be written as,

$$\partial_t l = -\zeta\nabla\cdot(l\frac{n^2\mathbf{v}-m^2\mathbf{v}^{ch}}{(m+n)(n-m)} + \tilde{l}\frac{mn(\mathbf{v}^{ch}-\mathbf{v})}{(m+n)(n-m)}) - \Gamma'\Omega - \Gamma^R(\Omega-\omega) + \tau \quad (89)$$

Similarly,

$$\partial_t \tilde{l} = -\zeta\nabla\cdot(\tilde{l}\frac{n^2\mathbf{v}-m^2\mathbf{v}^{ch}}{(m+n)(n-m)} + l\frac{mn(\mathbf{v}^{ch}-\mathbf{v})}{(m+n)(n-m)}) - \Gamma^\Omega\tilde{I}\tilde{\Omega} - \Gamma^\omega\tilde{I}(\tilde{\Omega}-\tilde{\omega}) + (\tau_1-\tau_2) \quad (90)$$

which with redefinition of constants can be written as,

$$\partial_t \tilde{l} = -\zeta\nabla\cdot(\tilde{l}\frac{n^2\mathbf{v}-m^2\mathbf{v}^{ch}}{(m+n)(n-m)} + l\frac{mn(\mathbf{v}^{ch}-\mathbf{v})}{(m+n)(n-m)}) - \tilde{\Gamma}'\tilde{\Omega} - \tilde{\Gamma}^R(\tilde{\Omega}-\tilde{\omega}) + \tilde{\tau} \quad (91)$$

Equations (89) and (91) are the two angular momentum equations we have now for our system.
The Tsai et.al.[4] limit is obtained with carefully taking the $n = m$ limit.

3.3 Conclusions

1. In this section, we were able to derive the continuum hydrodynamic equations for a mixture of chiral particles and their mirror image particles.

2. The steady states of the equations obtained can be studied further as in [4]. But the differential equations obtained in section (3) are tough to handle.

3. It should be noted that the knowledge of the dynamics of a single rattleback was used nowhere to work out the theories in sections (2) and (3). All the interactions between the wires will be accounted for in macroscopic co-efficients like η.

4 Acknowledgements

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