Classical Mechanics and Chaos

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Contents

Abstract

A very brief and unhelpful discussion on Classical mechanics is given. The next section reports detailed basic studies on chaos in logistic maps.

1 Classical Mechanics

1.1 Hamilton's principle

For monogenic systems, the motion of the system from time t_1 to t_2 is such that the line integral

$$I = \int_{t_1}^{t_2} L \; dt \tag{1}$$

where $L = T - V$, has a stationary value for the actual path of the motion.[1]

- L is the Lagrangian of the system.
- T is the kinetic energy.
- V is the potential energy of the system.

The theoretical framework of classical mechanics is based on taking the Hamilton's principle as the fundamental law from where we find out the equations of motion (or till the differential equations governing the motion of the system). This is in analogy to the three Newton's laws of motion from where we work out the equations of motion of a system. By the D'Alembert's principle, we arrive at the Lagrange's equations.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \tag{2}$$

Meanwhile, from the knowledge of the calculus of variations, we know that if

$$J = \int_{x_1}^{x_2} f(y, \dot{y}, x) \; dx \tag{3}$$

where

- $y = y(x)$
- $\dot{y} = \frac{dy}{dx}$

and if J has to have a stationary value then, the variation of this line integral must be zero.
i.e. δI must be zero. From this, the *Euler-Lagrange equations* are derived,

$$\frac{d}{dx}\frac{\partial f}{\partial \dot{y}_i} - \frac{\partial f}{\partial y} = 0 \tag{4}$$

But we see that we ended up with an equation of the form (2). Comparing both these equations, we see that L is the function which should have been minimized to get that form of equation as in (2). From this clue, the Principle of Least action is given as the fundamental law on which the classical mechanics is built upon.

In formulating classical mechanics, we also declare in the beginning that the space is homogeneous and isotropic and time is homogeneous[1][2][3]. Relativistic effects are not considered in the development of classical mechanics initially. [4]

1.2 Symmetries

Principle/Law	As a consequence of..
Principle of inertia (Newton's first law)	Homogeneity of space and Hamilton's principle
Newton's second law	Hamilton's principle
Newton's third law	Homogeneity of space and Hamilton's principle
Principle of conservation of energy	Homogeneity of time and Hamilton's principle
Principle of conservation of momentum	Homogeneity of space and Hamilton's principle

1.3 Noether's theorem

These are all summed up through the Noether's theorem which allows to get conserved quantities from symmetries of the laws of nature. Noether's theorem states that,

> For any continuous symmetry of the motion, there exists a corresponding constant of the motion[Noether 1918].

A dynamic observable or functional f is called a constant of motion if it's total time derivative vanishes whenever the generalized coordinates satisfy the Lagrange's equations.[6]

Every symmetry/each conserved quantity gives raise to a constant of motion and allows us to eliminate one of the degrees of freedom.

[1] No sense in talking about time being isotropic or non-isotropic as time has only one dimension unlike space

[2] The equations of motion will remain the same for moving the system as a whole and the environment influencing it to another region of space and that the equations of motion will remain the same irrespective of, at what time the experiment is started, on the condition that the environment is not dynamically evolving in time. This is the homogeneity of space and time respectively.

[3] Such a frame in which these conditions hold is called an inertial frame.

[4] The Galilean transformation holds as a consequence of the homogeneity of time which tells that time is absolute in all frames.

1.4 Central force motion

1.4.1 Theory

For the electrostatic Coulombic force, the potential is of the form $V = \frac{1}{r}$. This is the potential that can explain the dynamics of a central repulsive force between two point objects (between like charges), one of which is fixed at the origin.

$$F = -\frac{\partial V}{\partial r} = \frac{1}{r^2}\hat{r}$$

Thus the force is directed radially outward.
Let's consider constructing a Hamiltionian for this simple case.

$$d\vec{s} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\,sin\theta\,d\phi\,\hat{\phi}$$

$$\vec{v} = \frac{d\vec{s}}{dt} = \frac{dr}{dt}\,\hat{r} + r\,\frac{d\theta}{dt}\,\hat{\theta} + r\,sin\theta\,\frac{d\phi}{dt}\,\hat{\phi}^{5}$$

$$T = \frac{1}{2}\,m(\vec{v}.\vec{v})$$

$$T = \frac{1}{2}\,m\dot{r}^2 + \frac{1}{2}\,mr^2\dot{\theta}^2 + \frac{1}{2}\,mr^2sin^2\theta\dot{\phi}^2$$

$$L = \frac{1}{2}\,m\dot{r}^2 + \frac{1}{2}\,mr^2\dot{\theta}^2 + \frac{1}{2}\,mr^2sin^2\theta\dot{\phi}^2 - \frac{1}{r}$$

Finding out the conjugate momenta,

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2sin^2\theta\dot{\phi}$$

Finding out the time derivatives of r, θ, ϕ in terms of their conjugate momenta, we write the hamiltonian,

$$H = p_r\dot{r} + p_\theta\dot{\theta} - L$$

$$H = \frac{1}{2}\frac{p + r^2}{m} + \frac{1}{2}\frac{p_\theta^2}{mr^2} + \frac{1}{2}\frac{p_\phi^2}{mr^2sin^2\theta} + \frac{1}{r}$$

Using the canonical equations,

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{p_r} = -\frac{\partial H}{\partial r}$$

$$= -\frac{p_\theta^2}{mr^3} - \frac{p_\phi}{mr^3sin^2\theta} + \frac{1}{r^2}$$

[5] θ is the angle made by the radius vector with the z axis and ϕ is the angle made by the projection of the radius vector on the yx plane with the y axis.

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$
$$\dot{p_\theta} = -\frac{\partial H}{\partial \theta} = -\frac{p_\phi^2 cos\theta}{mr^2 sin^3\theta}$$
$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2 sin^2\theta}$$

For $\phi = 0$, the orbit must stay in the yz plane.[6]
For $p_\phi = 0$[7]

$$\dot{\theta} = \frac{p_\theta}{mr^2}$$
$$\dot{p_\theta} = 0$$

This implies that p_θ is just some constant.

$$p_r = m\dot{r}$$

$$\dot{p_r} = \frac{p_\theta^2}{mr^3} + \frac{1}{r^2}$$

Then,

$$m\frac{d^2r}{dt^2} = m\frac{dv}{dt} = m\frac{dv}{dr}\frac{dr}{dt} = mv\frac{dv}{dr} = \frac{p_\theta^2}{mr^3} + \frac{1}{r^2}$$

where $v = \frac{dr}{dt}$. Integrating the above equations with respect to v and r, we get

$$\frac{mv^2}{2} = -\frac{1}{2}\frac{p_\theta^2}{mr^2} - \frac{1}{r} + c$$

where c is the constant of integration. Rewriting the above equation,

$$v = \frac{dr}{dt} = \sqrt{-\frac{p_\theta^2}{m^2r^2} - \frac{2}{mr} + \frac{2c}{m}}$$

Integrating the equation with respect to r and t and a little more algebra, we get

$$r = \frac{p_\theta^2}{cos(\theta + k)\sqrt{\frac{2cp_\theta^2}{m} + 1} - 1}$$

Comparing it to the standard equation of a conic with one focus at the origin is,

$$r = \frac{1}{C[1 + e\ cos(\theta - \theta')]} \tag{5}$$

Comparing the two equations, we get eccentrity e to be,

$$e = \sqrt{\frac{2cp_\theta^2}{m} + 1} \tag{6}$$

[6]This is a system where the motion of the body about the origin is in the same plane for the whole of it's motion. So, we might as well simplify the problem by choosing an appropriate co-ordinate frame such as where we can reduce this now 3-D problem into a two dimensional problem by choosing $\phi = 0$ for the whole of the motion and taking the plane in which the moving object is revolving around the origin to be the yz plane.

[7]As ϕ is chosen to be zero for the whole motion, $\dot{\phi}$ is obviously zero making $p_\phi = 0$.

1.4.2 Plots

These are the orbits obtained for different parameters.

1. $c = 0$
   $m = 1$
   $p_\theta = 0.5$
   For which the eccentricity(e) turns out to be 1, orbit corresponds to a parabola.

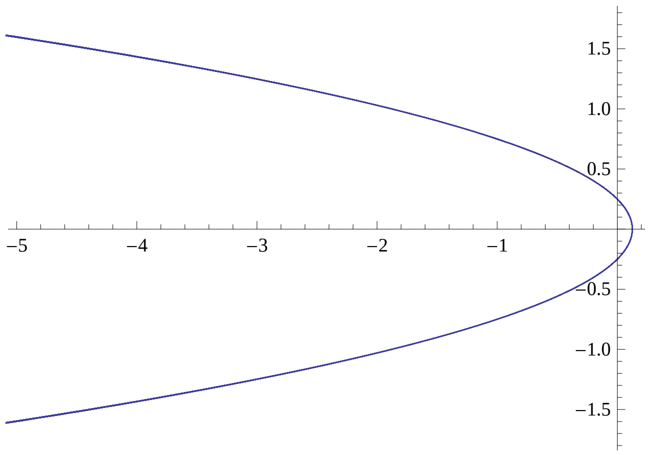


Figure 1: The orbit for $e = 1$.

2. $c = -2$
   $m = 1$
   $p_\theta = 0.5$
   $e = 0$ and orbit is a circle.

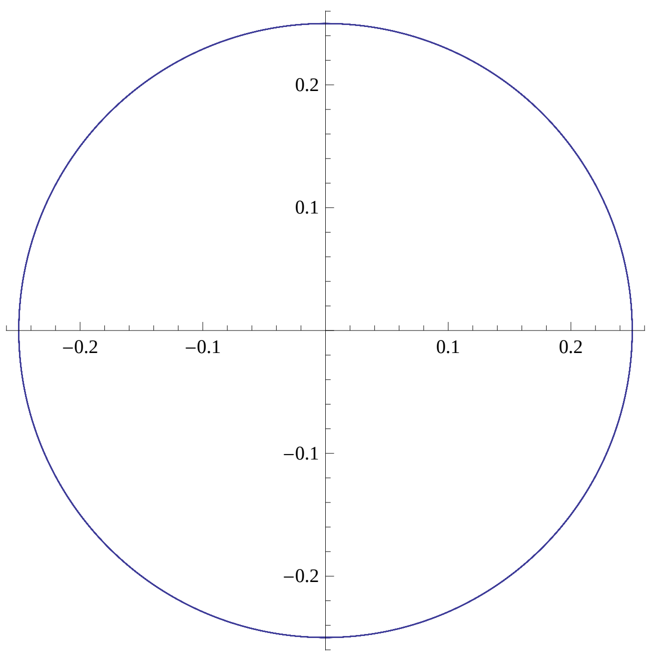


Figure 2: The orbit for $e = 0$.

3. $c = -1$
   $m = 1$
   $p_\theta = 0.5$
   $e = \sqrt{0.5}$ and the orbit is an ellipse.

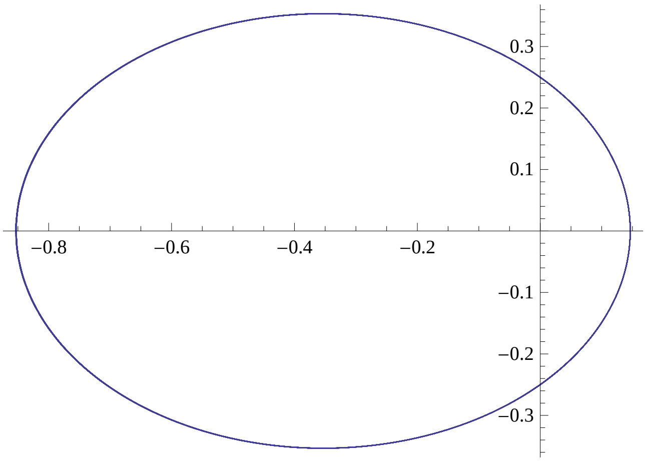


Figure 3: The orbit for $e = \sqrt{0.5}$.

4. $c = -0.5$
   $m = 1$
   $p_\theta = 0.5$
   $e = \sqrt{0.75}$ and the orbit is an ellipse.

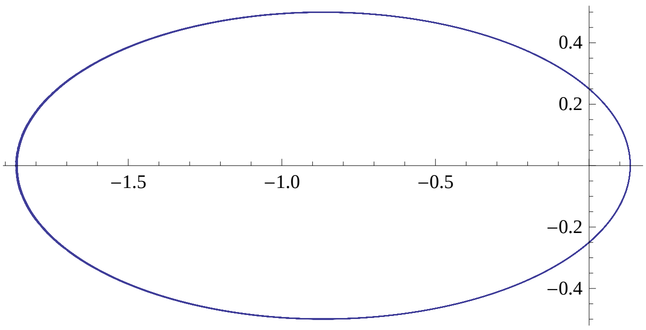


Figure 4: The orbit for $e = \sqrt{0.75}$.

5. $c = 2$
   $m = 1$
   $p = 0.5$
   $e = \sqrt{2}$ and the orbit is not closed now as it is an hyperbola.

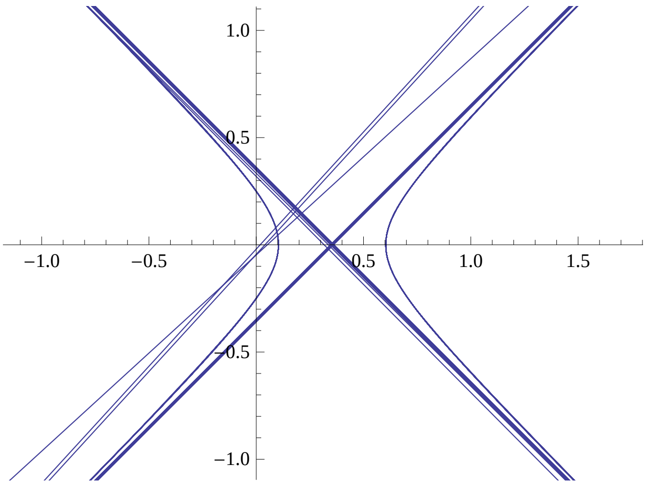


Figure 5: The orbit for $e = \sqrt{2}$.

Thus, we see that we are able to get closed orbits for the inverse square law forces. Indeed, by Bertrand's theorem,

> The only central forces that result in closed orbits for all bound particles are the inverse-square law and Hooke's law.

We see that, there are closed periodic orbits of planets around sun, satellites around planets[8], etc. So, from this theorem, we can conclude that the force acting between the bodies be either the inverse square law force of the force governed by the Hooke's law. However, as we know that the force does not tend to infinity as the separation between particles tend to infinity, we can safely conclude that the force acting between the masses has to be the inverse square law force.

[8]With minor perturbations ofcourse, due to the presence of other planets.

2 Chaos

> Predictability: Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?

-The title of the talk given by E.N.Lorenz at the December 1972 meeting of the American Association for the Advancement of Science, Washington D.C.

> Chaos is the phenomenon of occurence of bounded non-periodic evolution in completely deterministic systems with high sensitive dependence on initial conditions.[5]

The complex and seemingly irregular behaviour of a simple, well behaved system is referred to as chaos. This irregular behaviour is not a result of the external noise which is affecting the system in consideration or due to the difficulty in comprehending the motion in a large number of degrees of freedom of the system.[9] Choas is the inherent characteristic of the equation of evolution of the system itself.

> A new kind of uncertainity is brought into physics, in addition to the more familiar sources of the statistical fluctuations (noise) and the quantum fluctuations (uncertainity) relation.[3]

2.1 The logistic map

> What she's doing is, every time she works out a value for y, she using that as her next value for x. And so on. Like a feedback. She's feeding the solution back into the equation, and then solving it again. Iteration, you see.
> -Tom Stoppard, Arcadia (p.44)

$$x_n = m\ x_{n-1}\ (1 - x_{n-1}) \tag{7}$$

This equation is a non-linear function and this does not give the dependence of x_n on n, but rather the value of x_n depends upon the previous value of x, i.e. x_{n-1}. Thus, you need to run the experiment in order to find out how the evolution of the system takes place and cannot beforehand predict the value of x_n at a given n. Time, which we take to be the value of the integer n, is discontinuous. m is a parameter which we vary from 0 to 4. x_n diverges to infinity after 4 or if below 0. For the former case, the maximum value of the parabola[10] would be greater than 1 and the trajectory can no longer be bounded and diverges to $-\infty$. For the latter case, every iteration would alternate between the positive and negative values of x_n. The range of x is [0,1]. A bifurcation map is plotted. For each value of m in intevals of 0.001, 10000 iterations of (7) are performed. And the last 32 points of the iteration process for each value of m is plotted on the y axis against m.

[9] A system with a single degree of freedom, the logistic map, as we shall see, exhibits chaos
[10] explained in the discussion of the cobweb plot below.

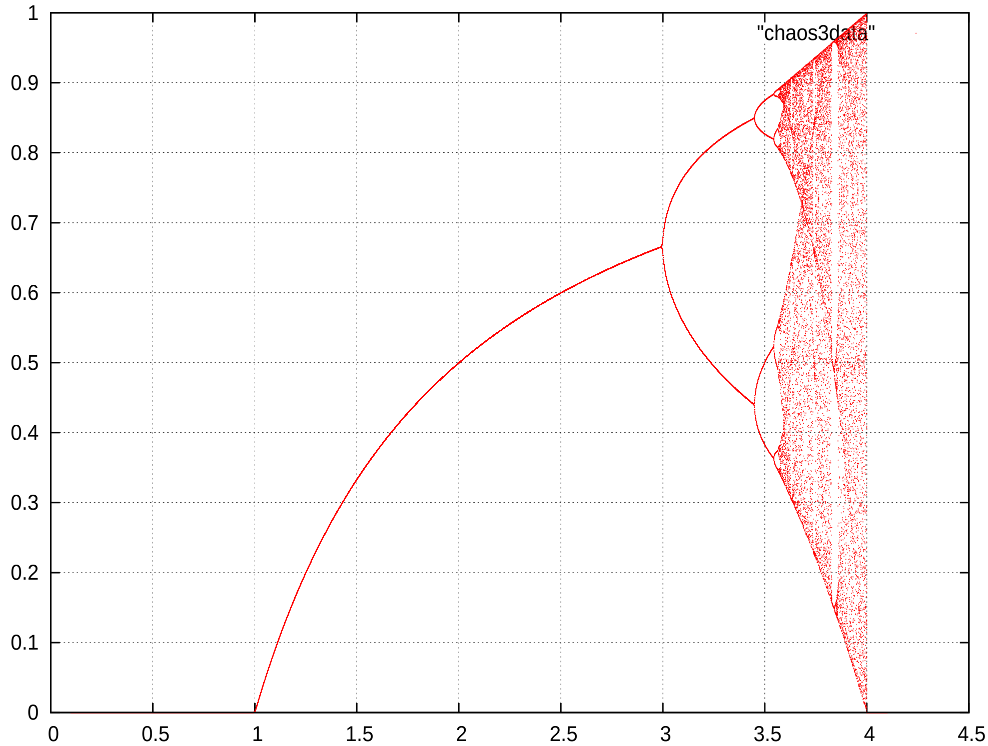


Figure 6: Bifurcation diagram - logistic map

For the range of m in $[0, 1]$, we see that the system decomposes to 0 for large values of n. For the range of m in [1,3], the system decomposes to a finite value. The first bifurcation occurs at $m = 1$, the second at $m = 3$. Bifurcation, in the sense that for a small range of m henceforth from 3, the system does not decompose to a single value of x, but to two values. That is for large n,

$x_n = x_{n+2}$

$x_{n+1} = x_{n+3}$

and $x_n \neq x_{n+1}$

We can see similar bifurcations in the graph. Few of the first values of m for which bifurcation occurs is

Table 1: Bifurcation table - Logistic map

Bifurcation	Value of m
First	1.0000
Second	3.0000
Third	3.4495
Fourth	3.5441
Fifth	3.5645

2.1.1 Fixed points

We now need to talk about the fixed points and how to find them analytically and how we can visualise them.

$$x^* = f(x^*) \tag{8}$$

where $f(x)$ is the iterative function, in this case it is $mx_{n-1}(1 - x_{n-1})$.(8) just means that once the iterative function has reached the value x^*, it will stay there as all further iterations of x^* returns to x^*. Thus, for a system starting from the initial condition where $x[0] = x^*$, then the system will remain at the starting point. For the logistic map, in the range of $[1, 3]$ for m, to find the fixed points,
$x^* = mx^*(1 - x^*)$
The two roots of this equations is $x^* = 0$, $\frac{m-1}{m}$. For, $m < 1$, 0 is a stable fixed point because trajectories which start near to 0, end up at 0 finally and $\frac{m-1}{m}$ is an unstable fixed point as the trajectories which start in the vicinity of this point, gets repelled from it and is attracted to the stable fixed point 0 where it ends up. 0 is the attractor and $\frac{m-1}{m}$ is the repellor in these conditions.

2.1.2 Cobweb plot :- Why does the orbit get attracted to a point? Graphical reasoning

Graphically the attractors can be shown in the cobweb plot. The cobweb plot is a method in which the iterations can be performed geometrically instead of using any calculation or algebra. If the initial position is x_0, the plot is made like this.

$$(x_0, 0), (x_0, f(x_0) = x_1), (x_1, x_1), (x_1, f(x_1) = x_2), (x_2, x_2), (x_2, f(x_2) = x_3), ...$$

The parabola is the iterative function for each value of x. The line is the straight line, $y = x$. Thus, we can easily follow the plotting of the above points by using these two curves, the parabola and the straight line. The second, fourth and fifth points shown above are on the parabola and the third, fourth points are on the straight line. Likewise, we keep plotting these points and join them with lines to get the cobweb plot.

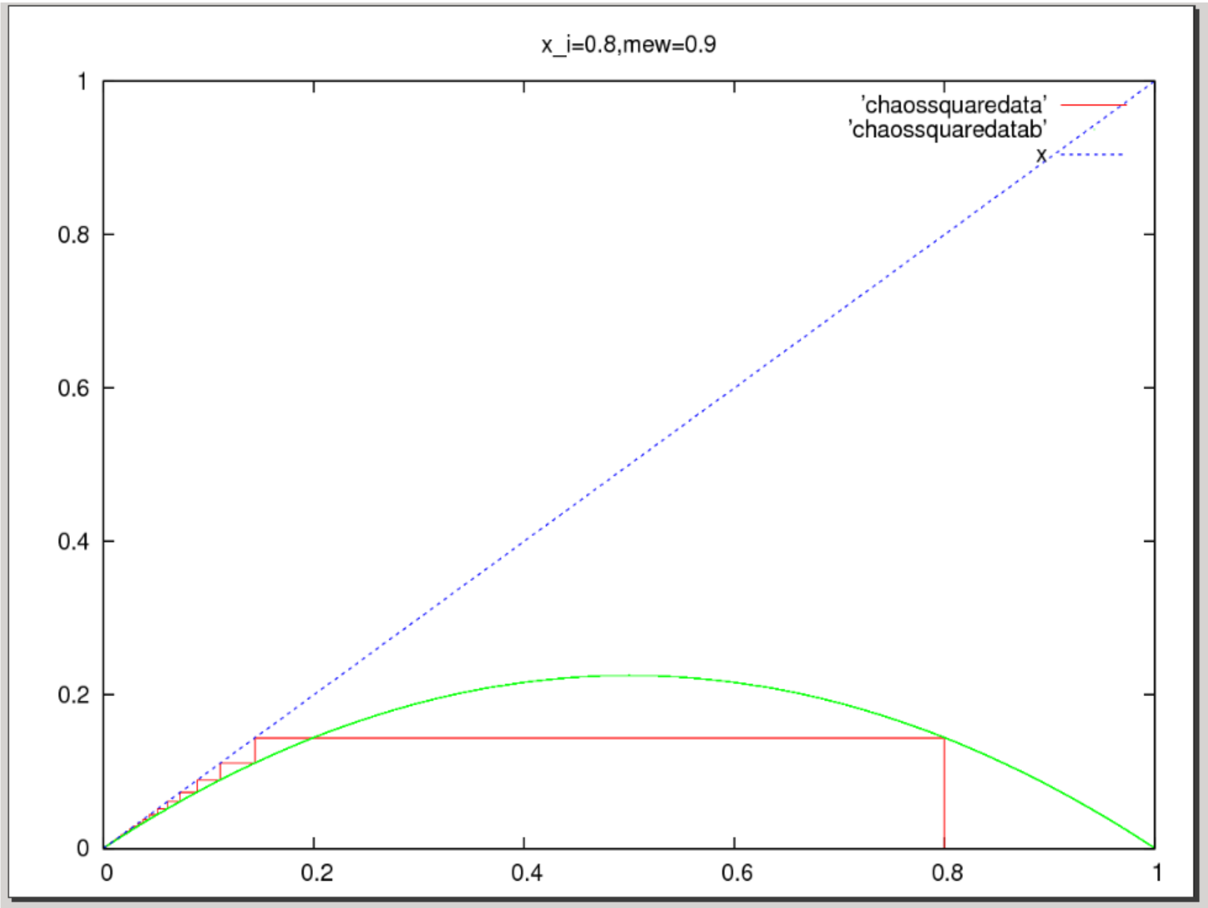


Figure 7: Logistic map - cobweb plot.**X axis**-x, **Y axis**-$f(x)$ $x[0] = 0.8, m = 0.9$

The points of intersection of the straight line with the parabola show the fixed points. Here, we see that a trajectory starting at 0.8 gets attracted to 0. We can visualise an attractor very easily through these plots.
The analytical reasoning for the same phenomenon is explained below.

2.1.3 Why does the orbit get attracted to a point? Analytical reasoning

Let us consider the case for $m < 1$. There is only one fixed point here, i.e. 0. The other root $\frac{m-1}{m}$ does not hold here as this is a negative number, outside our specified range. This number is not a part of the parabola plotted in the cobweb plot. So, considering 0, Let $x^* = 0$
and we consider a number near to it,

$$x^* + \delta$$

The initial distance between the two points is δ
Upon the first iteration,

$$f(x^* + \delta) = m(x^* + \delta)(1 - (x^* + \delta)) \approx m\delta \tag{9}$$

But $m < 1$, the above expression is lesser than δ. Thus the distance between 0 and the trajectory starting from $x^* + \delta$ decreases upon the first iteration. Upon further iterations, it will reach closer to 0 and thus 0 is an attractor.

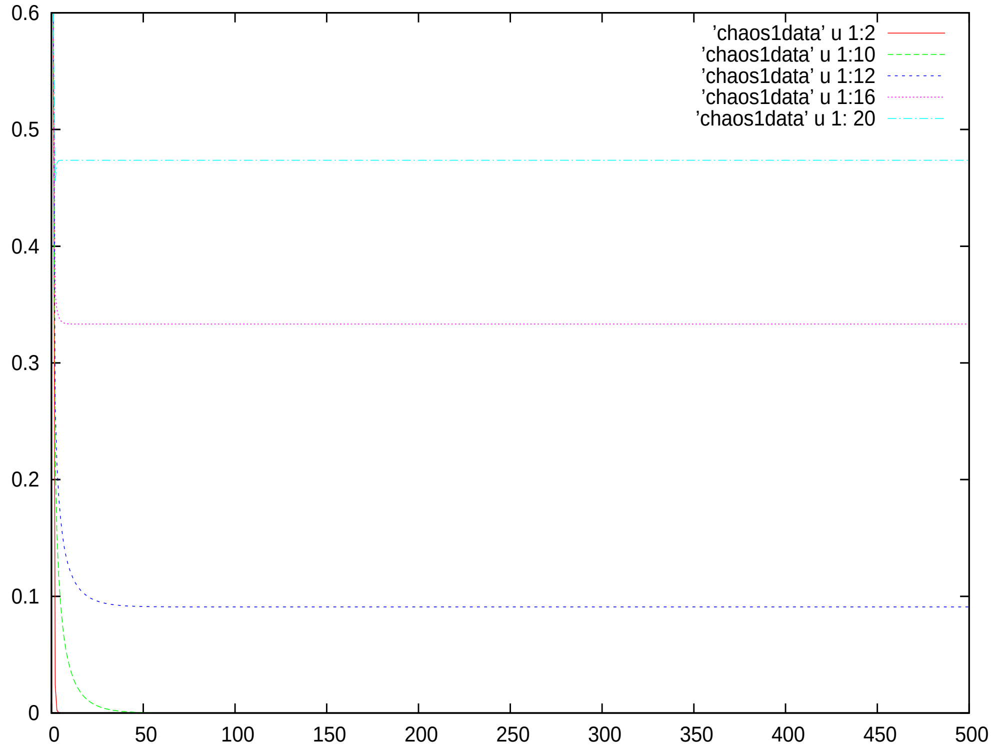


Figure 8: The orbit reaching the fixed point for various values of m. X-axis-iterations, Y axis - x_n. Red, green, dark blue, pink, light blue representing m=0.1, 0.9, 1.1, 1.5, 1.9 respectively. All the orbits are starting from the initial condition 0.6

2.1.4 More cobweb, more analysis

The other cobweb plots for varying m is shown below.

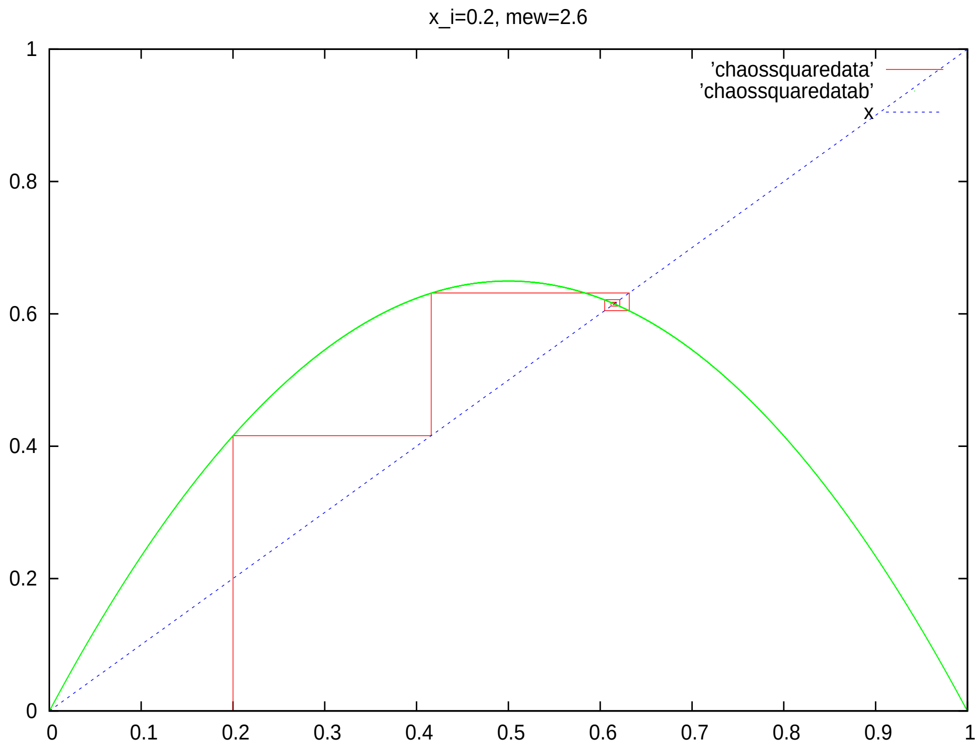


Figure 9: Cob web plot for $x_0 = 0.2, m = 2.6$

The slope of the parabola at $x = 0$ was less than one before the first bifurcation at $m = 1$. After this bifurcation, the slope at 0 is greater than one. Thus the trajectory can no more converge to that point.

We can now see the attractor to be at the point $\frac{m-1}{m}$.

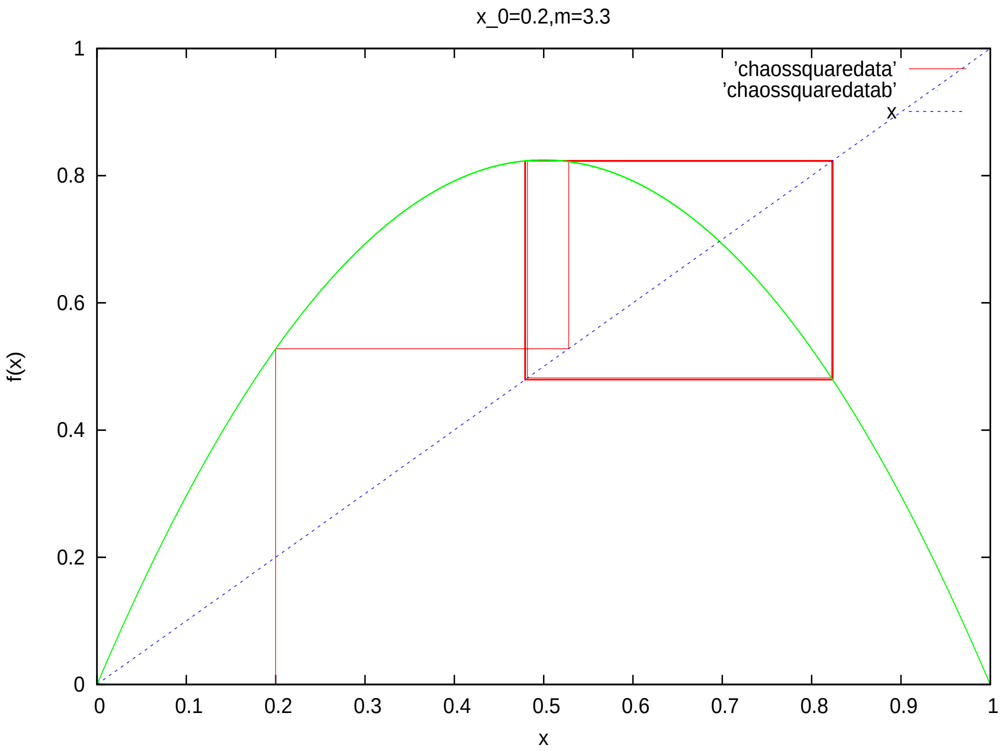


Figure 10: Cob web plot for $x_0 = 0.2, m = 3.3$

We see that here, the trajectory does not decompose into a single point but into two points as indicated by the two thick lines on the cobweb plot. The trajectory keeps oscillating between these two values after a large number of iterations. The points between which they oscillate are not the stable points of the logistic map but they are the stable points of the square map. i.e. the map in which

$$F(x_n) = f^2(x_{n-1}) \tag{10}$$

where f is the old logistic map function.

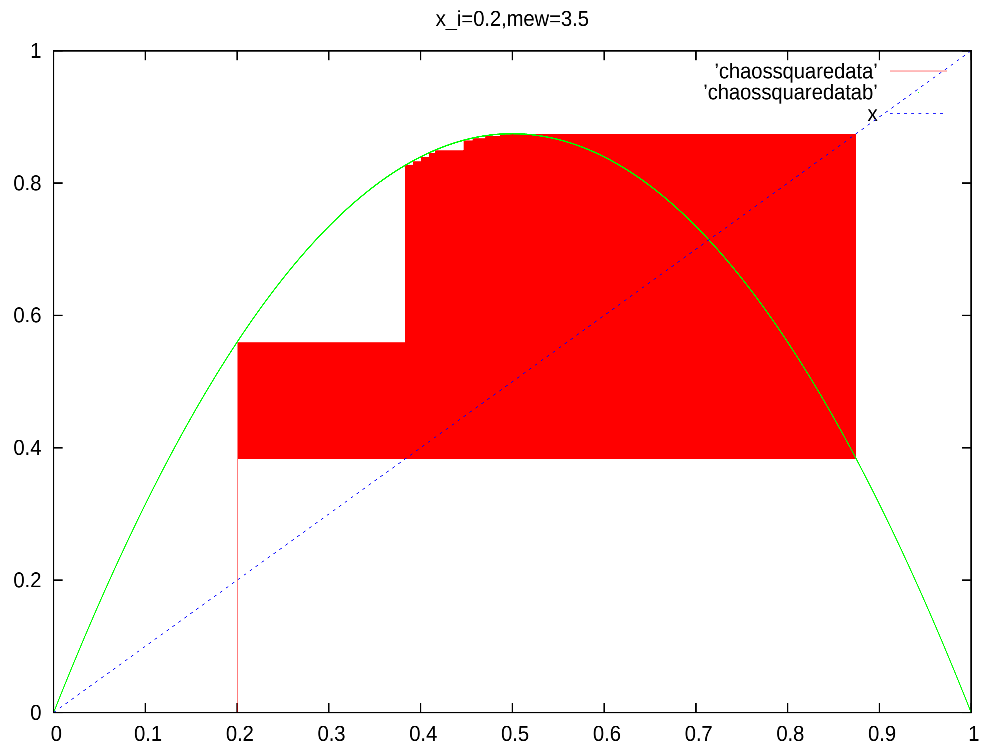


Figure 11: The cobweb plot for $x_0 = 0.2, m = 3.5$

For $m \in (1,3)$, the equation (9) is greater than δ, thus 0 is a repellor in this scenario. Considering the other root,
Let $x^* = 1 - \frac{1}{m}$
Considering a point near to this root, $1 - \frac{1}{m} + \delta$, the initial distance between the two points is δ. Upon the first iteration we get,

$$f(x^* + \delta) = m(1 - \frac{1}{m} + \delta)(1 - (1 - \frac{1}{m} + \delta))$$

$$= (1 - \frac{1}{m}) + \delta(2 - m)$$

For the given range of m, the distance between this trajectory and the fixed point is lesser than δ and upon further iterations comes closer to the fixed point. Thus, the root $1 - \frac{1}{m}$ is an attractor in this scenario. All the above derivations are for two cases of m, i.e. for the ranges $[0,1)$ and $(1,3)$. What about the behaviour at $m = 1$? For this we need to take the second degree term of δ which we had ignored for the above derivations. Then we can prove that 0 is the attractor for m=1.[11] The distance between the trajectory and the fixed point 0 after the first iteration would then be
$\delta - (m\delta - m\delta^2) = \delta^2$
Thus the distance decreases and the trajectory reaches 0.
Now that we have some knowledge of the fixed points, we can as well define the bifurcation map to be the map of the fixed points of the map for each value of m. For m, when it is greater than 3.57, the trajectory does not seem to repeat itself, we can see this to be as a case where there are infinite number of fixed points and the behaviour to be chaotic.

2.1.5 Feigenbaum's constant

From the bifurcation diagram, we see that the logistic map reaches chaos through period doubling. To predict the next period doubling from the knowledge of the previous values of m at the period doubling occurs, can be approximately calculated by the **Feigenbaum's constant**.

$$F = \lim_{j \to \infty} \frac{m_{j+1} - m_j}{m_{j+2} - m_{j+1}} \tag{11}$$

where
m_j is the value of m at where the period of the logistic map was 2^j (the j^{th} bifurcation)and likewise for the other terms involved in the equation. The Feigenbaum's constant is 4.669201. This is a universal constant and holds for all logistic maps which has a quadratic maximum. We see that this constant doesn't hold for the tent map as it doesn't have a quadratic maximum and is non-differntiable at the maximum. This constant holds for both the sine map and the map that was done using the Gaussian function. From the Feigenbaum's constant, we find that the period doubling ends at $m = 3.57$. What happens hereafter on moving to the right of this value of m is *chaos*. So, how do we check that this region is chaotic? We can find out by plotting two trajectories whose initial positions are very near to each other and if there is chaos in the system, then the

[11]For m=1, both the fixed points are the same point 0 as $\frac{m-1}{m}$ for $m = 1$ is 0.

distance between the trajectories increases exponentially till it is as big as the domain from where it can no longer increase further ofcourse due to the restrictions we have placed on the domain. This is the fundamental difference between chaotic behaviour and the behaviour of the system due to external noise. Similar phenomenon is not possible due to the influence of noise alone. The same phenomenon does not happen for $m < 3.57$ and the distance between such trajectories reach zero and don't diverge exponentially.

2.1.6 Lyapunov exponent

It is time to define the Lyapunov exponent which quantitatively tells whether the system is chaotic or not.

$$\lambda = \frac{1}{n}\, ln \left[\frac{f^n(x_o + \epsilon) - f^n(x_o)}{\epsilon} \right] \tag{12}$$

where ϵ is a very small number and $x_o, x_o + \epsilon$ are the initial points of the two trajectories with which we are calculating the Lyapunov exponent with.

> We can legitimately conclude that the behaviour of the system is chaootic only if the average Lyapunov exponent is positive[4].

Thus,

$$s(n) = s_o\, e^{\lambda n} \tag{13}$$

where $s(n)$ is the separation between the two trajectories after n number of iterations and s_o is the initial separation between the two trajectories. If the exponent is positive, it indicates that the separation exponentially increases with iterations[12] indicating a chaotic region. In the non-chaotic region, the exponent is negative, showing that the trajectories exponentially converge upon iterations indicating a non-chaotic region.

[12]Ofcourse, as the range is bounded, the magnitude of separation cannot increase forever.

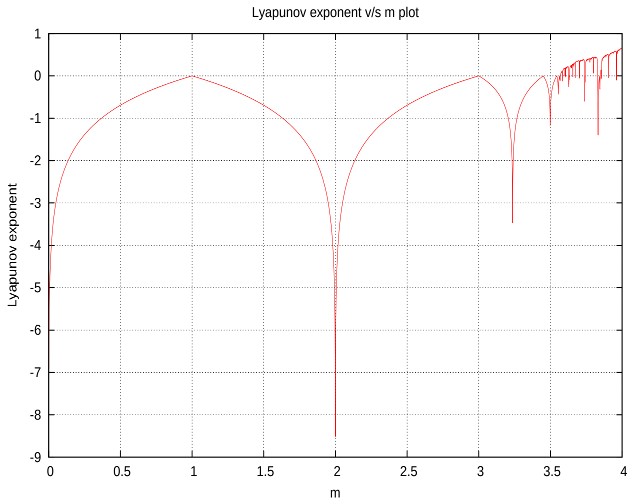


Figure 12: Lyapunov exponent values versus m.

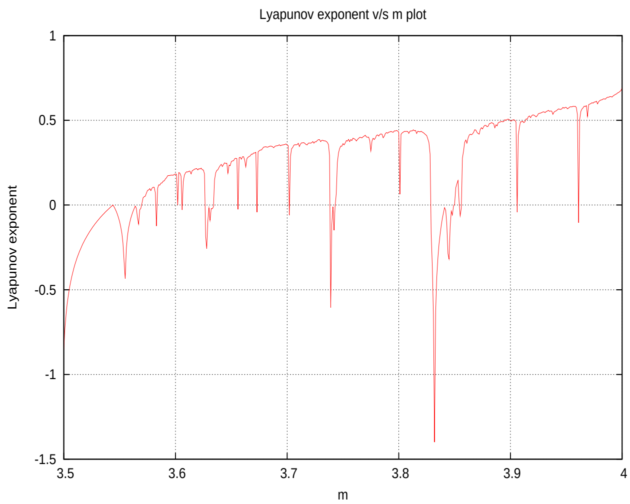


Figure 13: Lyapunov exponent values versus m zoomed in on m = [3.5,4]

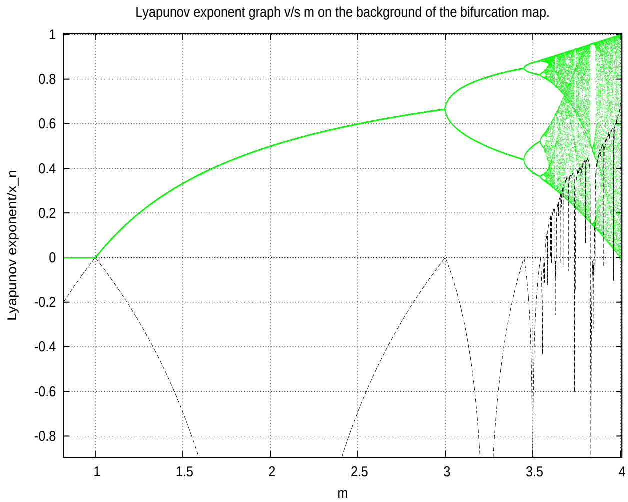


Figure 14: Lyapunov exponent values versus m, the bifurcation map on top.

We see that the Lyapunov exponent values are negative for all the values of m till the chaotic region, $m = 3.57$ is reached. After this the Lyapunov exponent is positive indicating the start of the chaotic region. Again, we should not fail to observe that even after $m = 3.57$, we see the exponents to be negative at the periods of stability in the chaotic region as these period 3 window of stability($m = 3.83$) are non-chaotic regions. The Lyapunov exponent values differ for each set of initial starting points of the two trajectories. The Lyapunov exponent graph plotted below was the average of 50 such pair of initial points where the initial separation between the trajectories was 0.001. Higher the Lyapunov exponent, more is the chaotic behaviour of the trajectory. We see that the highest Lyapunov exponent was obtained at m=4 where the value was found out to be very near to the natural logarithm of two.

Table 2: Few of the values obtained from the C++ program to calculate Lyapunov exponent.

m	Lyapunov exponent
0.001	-6.90786
0.002	-6.21471
0.003	-5.80925
0.004	-5.52156
0.781	-0.247289
0.782	-0.246009
2	-8.5168
2.001	-6.72506
2.999	-0.000733705
3	0.000188261
3.569	-0.0192453
3.57	0.0107575
3.582	0.071169
3.583	-0.123647
3.584	0.0984561
3.655	0.273284
3.656	-0.0254116
3.657	0.278803
4	0.69317

2.1.7 Behavioural dependence of the logistic map on the parameter m

- $m \in [0, 1]$ For all the initial values, the trajectory reaches the attractor at 0 over succesive iterations.

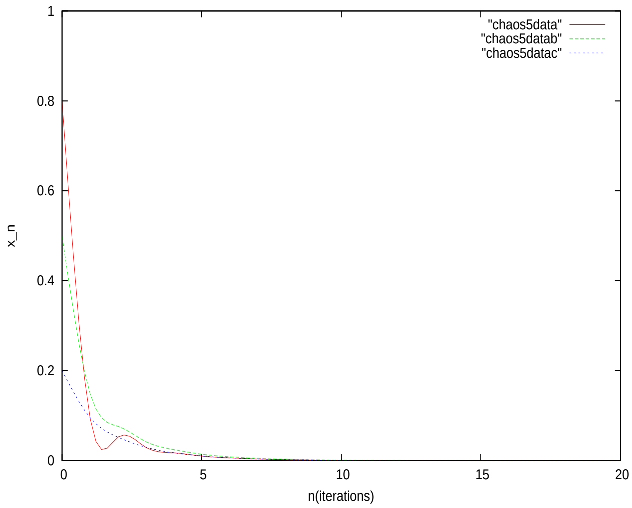
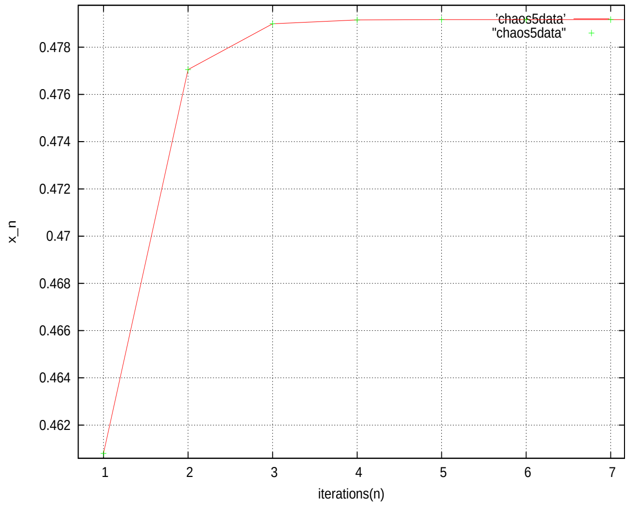


Figure 15: Trajectories reaching the attractor 0

- $m \in (1, 2]$
  The trajectory decomposes to a stable non zero finite value $(\frac{m-1}{m})$. The trajectory is seen to move only upwards towards the positive Y axis till the stable value is reached. No other value greater than the stable value is attained in the whole trajectory. This behaviour holds for this range, including $m = 2$.



.

Figure 16: x_n v/s n(iteration) graph for $x_0 = 0.6, m = 1.92$

- $m \in (2, 3]$
  The trajectory decomposes to a stable non zero finite value ($\frac{m-1}{m}$). The trajectory reaches values greater than the stable value in this range. Once the trajectory reaches a higher value than the value of the stable point, the next iteration, it attains a value lesser than the stable value and the immediate next iteration, it again reaches a higher value than the stable value but this value will be lesser than the previous higher value attained two iterations back. It is exactly like the damping of an oscillator. The trajectory oscillates about the stable value till it reaches the stable fixed point.

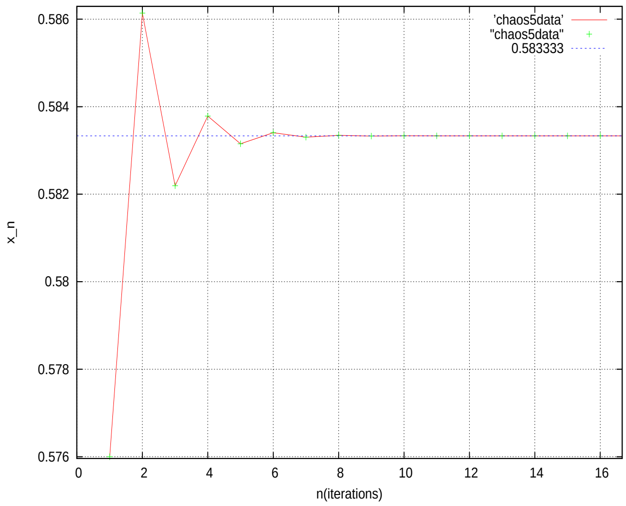


Figure 17: x_n v/s n(iteration) graph for $x_=0.6, m = 2.4$.

For every progressive plot, we see that the fixed points increase to the power of 2, starting downwards from 4 till 16 have been plotted and shown.

- $m \in (3, 3.4495)$

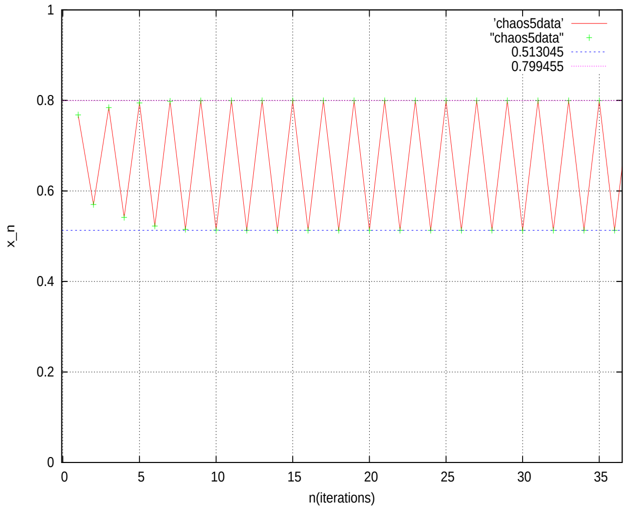


Figure 18: x_n v/s n(iteration) graph for $x_0 = 0.6, m = 3.2$.

- $m \in (3.4495, 3.5411)$

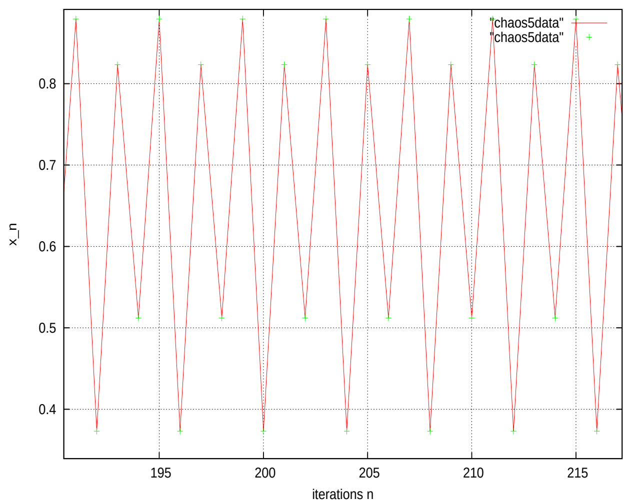


Figure 19: x_n v/s n(iteration) graph for $x_0 = 0.6, m = 3.52$.

- $m \in (3.5411, 3.5645)$

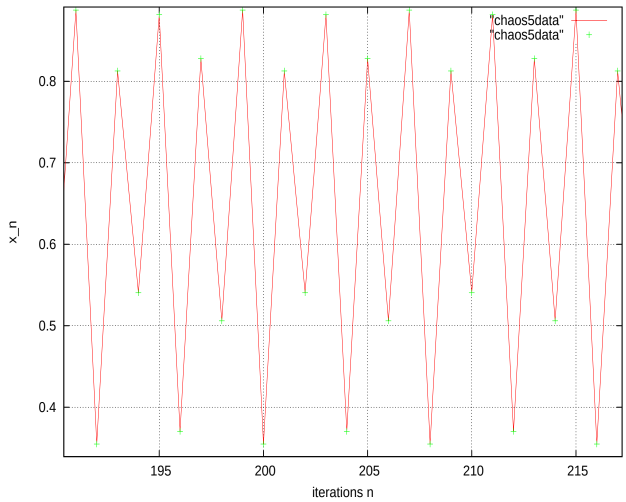


Figure 20: x_n v/s n(iteration) graph for $x_0 = 0.6, m = 3.55$.

- $m \in (3.5645, 3.5688)$

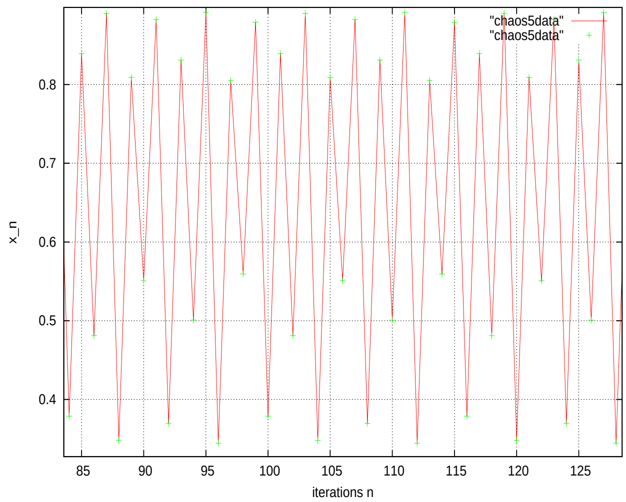


Figure 21: x_n v/s n(iteration) graph for $x_0 = 0.6, m = 3.567$.

2.2 Similar maps

$$x_n = m\,sin(\pi x_{n-1}) \tag{14}$$

$$x_{n+1} = e^{-\lambda(x-\frac{1}{2})^2} - e^{-\frac{1}{4}} \tag{15}$$

We need to notice how the maps are being designed. We see that all the maps have been written in such a manner that they have a range in $[0,1]$ and that the values of the function is zero at the end points. This makes the boundary condition for the map and all the maps should now have to stay within this 'parabola' of the function. The range of the function is immaterial except that the function be zero at the end points of the range. One more necessity is that the maximum of the function occurs exactly half way at the range. This holds naturally for the parabolic, sine or the gaussian functions ofcourse.

We see that the sine map and the Gaussian function's map, both have the same kind of behaviour as a quadratic function does when the functions reach their maximum. Thus, the Feigenbaum's constants should hold here. These maps too follow the period doubling route to chaos. The bifurcation maps for these functions are given below:

2.2.1 Plots

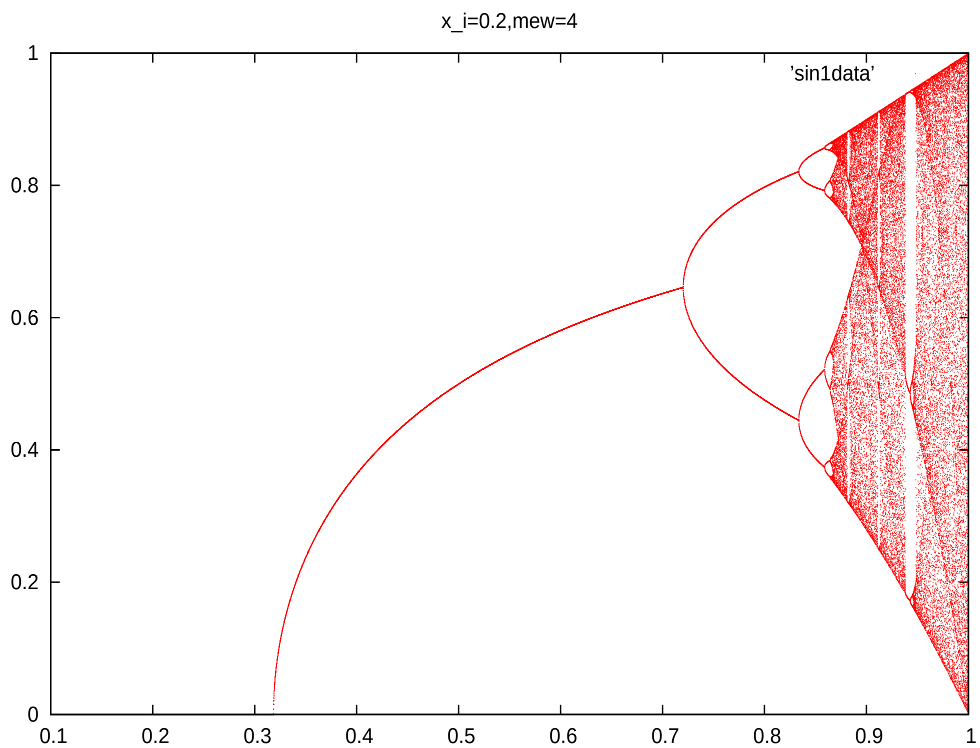


Figure 22: Bifurcation diagram for Sine map.

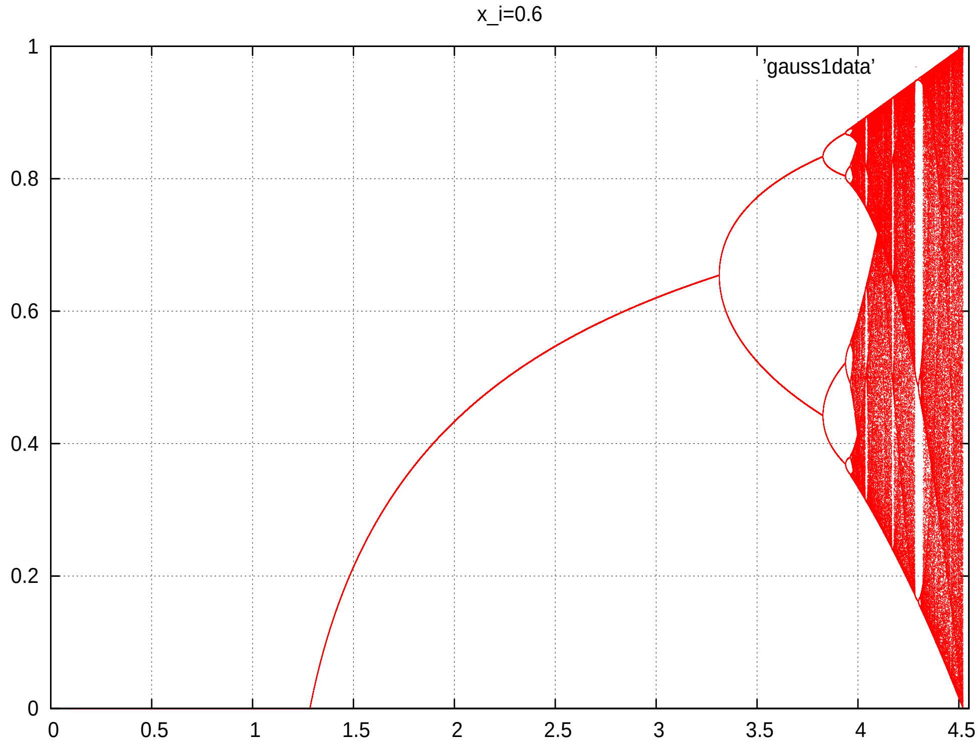


Figure 23: Bifurcation diagram for gaussian map.

2.2.2 Numbers

Similar bifurcation values are found for these two maps as done for logistic map and the Feigenbaum constants too are verified.

Table 3: Bifurcation table for Sine and Gaussian map

Bifurcation	Value of m for Sine map	Value of m for gaussian map
First	0.7204	3.3128
Second	0.8337	3.8269
Third	0.8591	3.9389
Fourth	0.8646	3.9631

Table 4: Feigenbaum constants for Sine and gaussian map.

Feigenbaum constant	Sine map	gaussian map
First	4.4606	4.59018
Second	4.6182	4.62809

We see that even here the Feigenbaum constant is approaching the value 4.669201. Thus, we verify that the Feigenbaum constants hold for these maps too.
The Lyapunov exponents are calculated for the Gaussian map too by taking the average value of the Lyapunov exponent for 50 pairs of initial set of conditions. The plots of them are attached below.

2.2.3 Lyapunov exponent plots

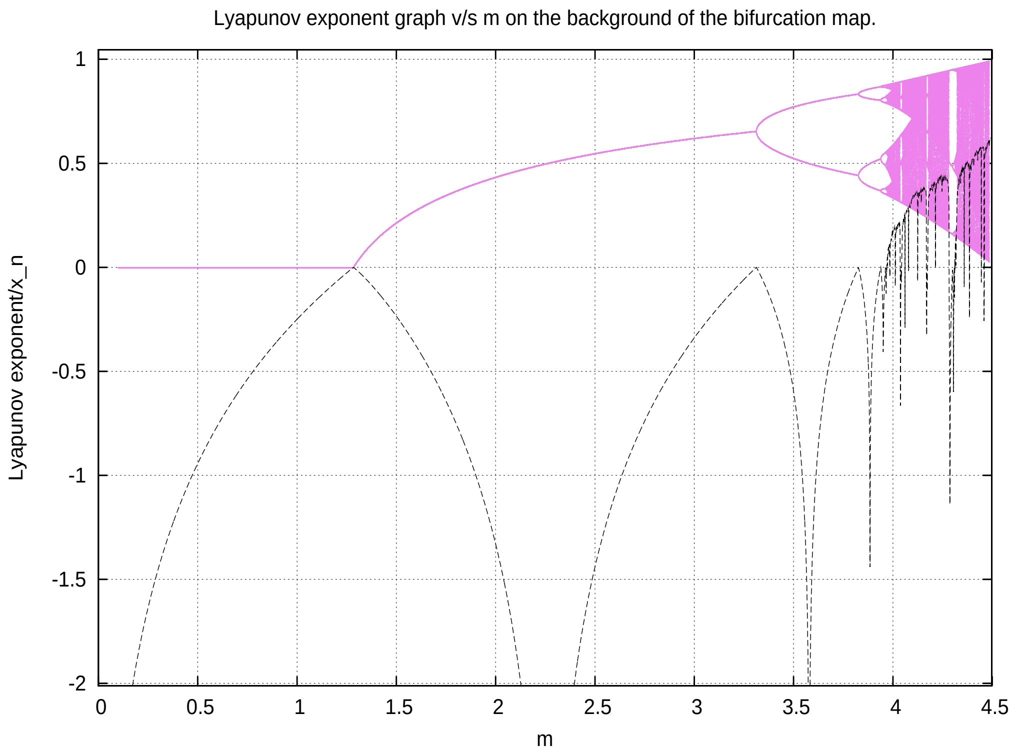


Figure 24: Lyapunov exponent v/s m plot for the Gaussian map.

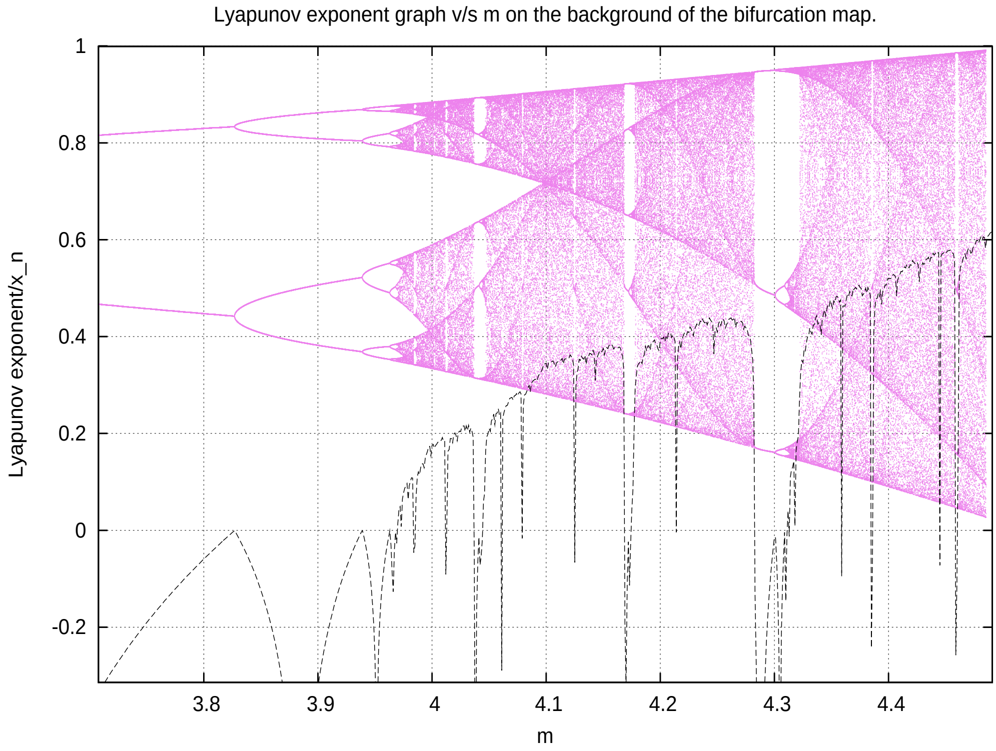


Figure 25: Lyapunov exponent v/s m plot for the Gaussian map.

2.2.4 Self similarity of the logistic maps

The two Feigenbaum numbers suggest that the bifurcation diagram are just smaller replicas of other parts of the diagram.[4] By contracting the first bifurcation fork's vertical axis by a the value of the square of the second Feigenbaum constant, α^2 and contracting the horizontal axis by the value of the first Feigenbaum constant, we get the same sized fork as the second bifurcation fork and the modified first fork and the second fork match exactly in all characteristics. A geometric structure that has this replicating behaviour under appropriate magnifications is said to be self similar[4]. If the period 3 or the period 5 window is zoomed, we get the same bifurcation diagram in the blown up picture. These self similar objects are called fractals as their geometric dimension is a fraction, not an integer.

3 Conclusions, remarks and questions

Plotting the trajectories in Mathematica is a good method of learning how the orbit differs on changing the constants of the equations of the motion.
The other problems which I worked on were,

- The system consisting of a massless rod connected to two equal masses, one at each end of it where the center of mass of the system is moving at a constant angular velocity and describing a circle was considered. Here the rod had the freedom to rotate about the center of mass in the plane of the motion of the center of mass and the system is kept vertical in a gravitational field.The Lagrangian equations were derived (exact solution is possible) and the trajectories of the masses for different angular velocities of the center of mass about the center of the circle and initial angular velocity of the masses about the center of the mass was studied.

- The above problem was tried, with different masses at each end. The exact solution of the differential equations is not possible and on numerical integration and plotting on Mathematica, it was found that there is an onset of chaos (the trajectory of the system diverges from the real path) in the plot generated due to the limited amount of precision with which the computer can numerically integrate the differential equation.

- The phase space diagram of a one-dimensional pendulum was generated for different energies and superposed on the same graph, where the pendulum's trajectory closes in on itself in an ellipse for $E < mgl$ and for higher energies, follows a cycloid path.

- For a system consisting of two oscillating springs in perpendicular direction (x and y), it was shown that the trajectory (plot of x v/s y) generated in the computer closes in on itself as expected when the ratio of the angular frequencies of both the oscillators is a rational number. On the special case, when the ratio is 1, we get the plot to be an ellipse. If it is an irrational ratio, the path never closes in on itself.

- Solving the three coupled differential equations of the Lorenz model [Classical chaos - Exercise - 6][1] and plotting the trajectory in Mathematica to observe the Lorenz attractor.

- The Duffing oscillator's differential equation was solved [Classical Chaos - Exercise - 8][1] and it was shown by plotting the numerically obtained solution on Mathematica, that the amplitude of the steady state oscillation shows hysteresis.

In chaos, I have just covered about some basics of the logistic maps. I still have not figured out a lot of things in chaos. One of the questions that came up in my mind, I have mentioned below:

- Is there any relation we can find out as to how many iterations a computer would take for a particular value of m to reach the stable orbit? I had plotted the number of iterations the computer takes to reach stable orbit v/s m and got an interesting plot. The plot however, to a certain extend depends upon the initial conditions too. However, the large scale structure of the plot for all the initial values is almost the very same.

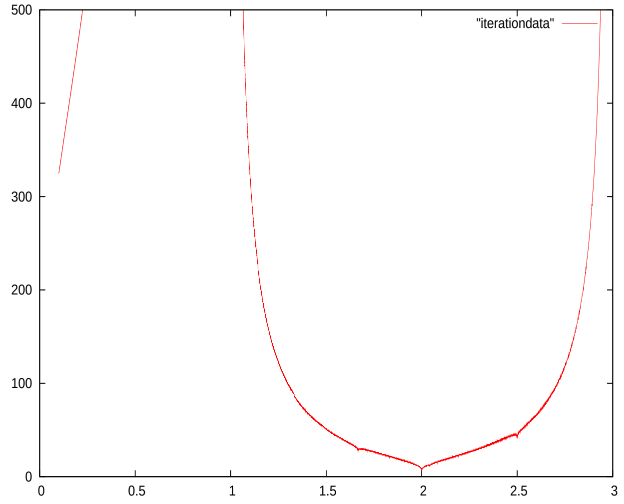


Figure 26: Y axis - number of iterations. X axis - m.

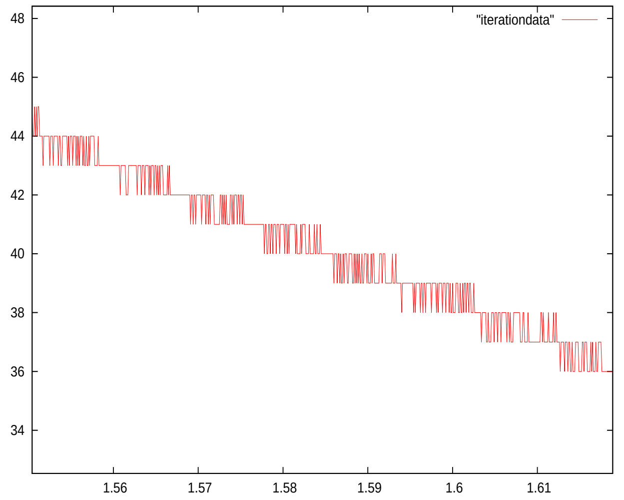


Figure 27: Y axis - number of iterations. X axis - m.

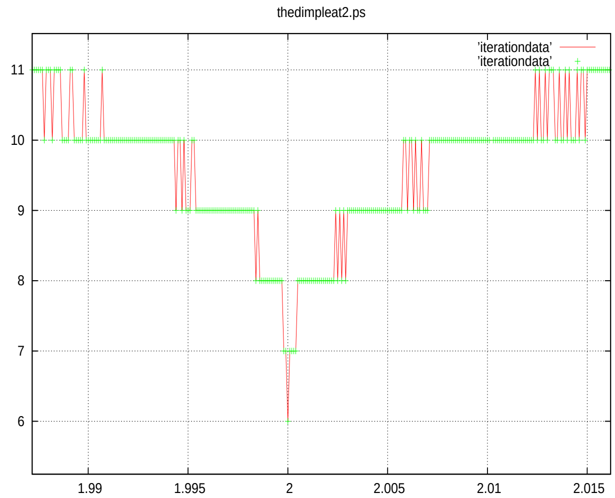


Figure 28: Y axis - number of iterations. X axis - m.

4 Acknowledgment

I would like to thank Prof. Phatak for giving me a chance to do my reading project under him in the summer of 2011 at NISER. The discussions we had in LH 107 along with Sourabh, Maneesha and Arka were very helpful.

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