Constrained Hamiltonian Dynamics

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`Notes (not official report)`

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1 Introduction

When you throw a rubber ball from top of a tower, it comes down in a straight line. You can write down the differential equation of motion of the ball using the Newtons's law,

$$\vec{F} = m\,\vec{a} = m\,\frac{d^2\vec{x}}{dt^2} \tag{1}$$

where the force here is the gravitational force. Solve this simple differential equation and you get to know everything that is there to know about the motion: position, velocity, acceleration at any given time.

There is one more picture that was drawn to predict the ball's motion. Instead of talking about force, we can talk in terms of energy. The ball moved from the tower top to rock bottom because it wanted to *decrease* it's potential energy. Then, the question comes, as to why didn't it choose any other path other than the straight line. Because *any* meandering path to the rocks below, would have done the job of decreasing it's potential energy.

Now, a 'weird until got used to' quantity called the Lagrangian becomes central to this picture,

$$L = T - V \tag{2}$$

which is potential energy taken away from the kinetic energy. A law, as fundamental (and equivalent to) as the Newton's force law is stated that the particle moves in *such* a path such that this quantity when calculated at each time from start to finish and summed up would be the least value compared to any other paths[1]. This picture is an equivalent picture to Newton's as one can be derived from the other. Also, we can extend this picture to other systems where working with Newton's laws is non-trivial.

The advantages of using the new law is that we can now deal with scalar quantities involving energies instead of working with more difficult vector quantities as Newton had instructed us to do. Also, this new paradigm will help a lot when the system has to satisfy some constraints. And most importantly, a very neat and elegant theory of classical mechanics was able to be built based on this principle called the, 'Stationary action principle'.

1.1 Lagrangian formalism

$L \equiv L(q, \dot{q}, t)$[2]
The action integral for a system is defined as,

$$S = \int_{t_1}^{t_2} L\,dt \tag{3}$$

When you substitute $q \equiv q(t)$ (any arbitrary function of time) and $\dot{q} = \frac{dq(t)}{dt}$ in the right hand side of the integral and evaluate the definete integral, you get S, which is a pure

[1] Read Feynman's lectures on physics, Vol 2, verbatim lecture on least action principle for a wonderful, non-technical account.

[2] L is a function of $q, \dot{q}$.

number. This number however depends on the the arbitrary function of time you chose for q, i.e. it depends on the path you have assigned to the system. S is a functional. To determine the dynamics, you use the below principle:

Hamilton's principle:

> The motion of an arbitrary mechanical system occurs in such a way that the definite integral S becomes stationary for arbitrary possible variations of the configuration of the system, provided the initial and final configurations of the system are prescribed.

We want to find a path of the system where $\delta S = 0$, i.e. if you virtually move every point on this path infinitesimally by δq, the change in action is zero upto first order in δq.

$$\delta S = \delta \int_{t_1}^{t_2} L \, dt$$

As the variation of the integral equals the integral of the variation,

$$= \int_{t_1}^{t_2} \delta L \, dt$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] dt$$

By using integration of parts on the second term and from the fact that the variation of the boundary points is zero, we get the modified form of the variation of S in terms of variation of q alone.

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q \, dt = 0 \tag{4}$$

As this should be true for *any* infinitesimal variation about the true path, we get the Euler-Lagrange equation:

$$-\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0 \tag{5}$$

For a system with n generalized co-ordinates, the EL equations are:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \qquad\qquad i = 1, 2, \cdots n \tag{6}$$

This differential equation is equivalent to stating $\delta S = 0$ for the true path. The solution $q_i \equiv q_i(t)$ to the above partial differential equation are the $q_i(t)$ s which when substituted in (3) would give the least[3] action when compared to any other $q_i(t)$ s which is arbitrarily considered.

[3] Stationary action in general.

1.2 Hamiltonian formalism

We were working with generalized co-ordinates and generalized velocities in the Lagrangian formalism. In the Hamiltonian formalism, instead of the generalized velocities, we use the conjugate canonical momentum of each co-ordinate which is defined as:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad\qquad i = 1, 2, \cdots n \tag{7}$$

where n is the number of generalized co-ordinates.

Assume Einstein convention and define a new function called the Hamiltonian, $H(q, p)$ thus:

$$H = p_i \dot{q}_i - L(q, \dot{q}) \tag{8}$$

where p_i is as defined in Eq. (7). We can see that the Hamiltonian is a function of q, p alone and not $\dot{q}_i$ by taking a variation of H and seeing what happens.

$$\delta H = p_i \delta \dot{q}_i + \delta p_i \dot{q}_i - \frac{\partial L}{\partial q_i}\delta q_i - \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i$$

From the definition of canonical momentum in 7, we get

$$\delta H = \dot{q}_i \delta p_i - \frac{\partial L}{\partial q_i}\delta q_i \tag{9}$$

The variation of H is determined only by the variations of the q and p. Thus[4] we see that the Hamiltonian is a function of q, p alone and not of the generalized-velocities $\dot{q}_i$. Such a transformation of variables from $q, \dot{q}$ to q, p is in general called a Legendre transform.

2 Constrained Hamiltonian Dynamics

Reading section (2.1), you would see that there may exist *mathematical* examples where the momenta-velocity equations cannot be inverted uniquely. However there *are* very important *physical* systems, for example, the Maxwell field theory where this does happen. This example has something called 'gauge degrees of freedom'. The Poisson brackets of such systems cannot be promoted to commutator relationships. So, the theory cannot be quantized. Dirac constructed a general procedure for such systems which we would study below. He also introduced new brackets which can be promoted to commutator relations and it is said that that will bring the theory nearer to quantization.

2.1 Constraints

Consider the Lagrangian equations (6). We can write these equations in a different form, in which we explicitly write the accelerations of the generalized co-ordinates.
The subscripts r, s runs from 1 to n.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0$$

[4]From Eqs. (6), (7), you can write Eq. (9) in a nice form, $\delta H = \dot{q}_i \delta p_i - \dot{p}_i \delta q_i$.

$$\Rightarrow \frac{\partial^2 L}{\partial q_s \partial \dot{q}_r}\dot{q}_s + \frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_r}\ddot{q}_s - \frac{\partial L}{\partial q_r} = 0$$

Defining a $(n \times n)$ matrix,

$$W_{rs} = \frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_s} = \frac{\partial p_s}{\partial \dot{q}_r} \tag{10}$$

we can rewrite the Lagrangian equation as,

$$W_{rs}\ddot{q}_s = \frac{\partial L}{\partial q_r} - \frac{\partial^2 L}{\partial \dot{q}_r \partial q_s}\dot{q}_s \tag{11}$$

This is a set of non-homogeneous simultaneous equations in the n number of accelerations. We can invert these set of n equations to get $\ddot{q}_s$ as a function of $q, \dot{q}$ only when the matrix W_{rs} is invertible.

The invertibility of one more set of equations depends on this matrix $||W||$. Consider the set of n equations (7), which form the set of equations defining the canonical momenta.

In this set of equations connecting momenta, co-ordinates and velocities, the invertibility of these equations, i.e. finding $\dot{q}_s$ as a function of q, p depends on the invertibility of the Jacobian $W_{rs} = \frac{\partial p_s}{\partial \dot{q}_r}$. The Jacobian being invertible is a necessary condition for change of variables and here we are changing the co-ordinates from $\dot{q}_s$ to p_s.

If $\det||W|| = 0$, then the rank of this matrix $R < n$. Rank of a matrix + number of null eigen vectors of the same matrix = n implies that the number of null eigen vectors of this matrix is $n - R$. It is to be remembered that the determinant of a matrix is zero if and only if not all the rows are found to be linearly independent of the other. Thus $\det||W||$=0 implies that there are linearly dependent equations in 7.

The $n - R$ number of *independent* null eigen vectors of the matrix can be written as,

$$\lambda^a_s(q, \dot{q})W_{rs}(q, \dot{q}) = 0 \qquad\qquad \forall\, r \tag{12}$$

This is just the summation form of the matrix multiplication (Square matrix multiplied to the Null eigen column vector gives the zero column vector). The superscript a runs from 1 to $n - R$.

For a given unique boundary conditions (satisfying the constraints (12)), you see that the accelerations are not determined uniquely because of the non-invertability of $||W||$, see (11).

2.2 Primary constraints

The $n - R$ number of constraints got as explained above are called the primary constraints[5]. They are derived from the definition of the conjugate momentum. m runs over all the number of constraints obtained from (7).

$$\phi_m(q, p) = 0 \qquad\qquad m = 1, 2, ...M = n - R \tag{13}$$

These constraints define a sub-space/shell in the phase space. The dynamical variables q, p of the Hamiltonian theory should obey these conditions during the dynamics and thus the equations of motion should evolve the initial co-ordinates in such a way such that these conditions are satisfied.

[5] Bergmann's notation.

The Hamiltonian defined here is not unique. Consider another Hamiltonian,

$$H^* = H + u_m \phi_m \tag{14}$$

where u_m can be any functions of q, p would still be as good a Hamiltonian as H. H^* is the most general function of co-ordinates and momenta, numerically equal to H when the constraints are satisfied. Thus H^* is the broadest generalization of the Hamiltonian possible. As of now there is an indeterminacy in finding the Hamiltonian and thus the equations of motion due to the presence of u_m.

Consider equation (9). When there are constraints of the form (13),

$$\delta H = \dot{q}_i \delta p_i - \frac{\partial L}{\partial q_i} \delta q_i$$

is true if the constraints are satisfied by the variations in q, p. The reason is stated below:

In the derivation of eq. (9), we had put $p_i = \frac{\partial L}{\partial \dot{q}_i}$. This is true only in the shell in the phase space where the constraints are satisfied. Otherwise, it's not.

2.3 Equations of motion

Using the least action principle for the general Hamiltonian H^*, we can find the equations of motion using calculus of variations.

$$S = \int_{t_1}^{t_2} (p_i \dot{q}_i - H^*) dt$$

To get the equations of motion, we set

$$\delta S = \delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H^*) dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} (\delta(p_i \dot{q}_i) - \delta H^*) dt = 0$$

$$= \int_{t_1}^{t_2} (\dot{q}_i p_i + \dot{q}_i \delta p_i - \delta H - \phi_m \delta u_m - u_m \delta \phi_m) dt$$

As we are interested in getting all the variations to be in terms of $\delta q, \delta p, \delta u_m$, we use integration by parts on the second term above. Also we can write δH in the form (9) because we are obeying the constraints here. Thus, we get

$$= \int_{t_1}^{t_2} (\dot{q}_i p_i - \dot{p}_i \delta q_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i + u_m \frac{\partial \phi_m}{\partial q_i} \delta q_i + u_m \frac{\partial \phi_m}{\partial p_i} \delta p_i + \phi_m \delta u_m) dt$$

The variation in each independent variable $\rightarrow$ $\delta q_i, \delta p_i, \delta u_m$ should vanish. Thus we get,

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + u_m \frac{\partial \phi_m}{\partial q_i} \tag{15}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} - u_m \frac{\partial \phi_m}{\partial p_i} \tag{16}$$

$$\phi_\rho = 0 \tag{17}$$

The first two equations give the Hamiltonian equations of motion which tell about the evolution of the co-ordinates and momentum over time. These equations still have the unknown co-efficients u_m. The last equations just gives back the constraint we had put initially ourselves.

2.4 Poisson bracket

Consider two functions of the Hamiltonian variables, $f(q,p)$ and $g(q,p)$. The Poisson bracket of them, $[f,g]$ is defined by

$$[f,g] = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \tag{18}$$

`Properties of Poisson bracket:`

1. Anti-commutation
$$[f,g] = -[g,f]$$
2. Linearity in both terms.
$$[f_1 + f_2, g] = [f_1, g] + [f_2, g]$$
$$[f, g_1 + g_2] = [f, g_1] + [f, g_2]$$
3. Product rule
$$[f_1 f_2, g] = f_1[f_2, g] + f_2[f_1, g]$$
4. Jacobi identity
$$[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$$

The time evolution of any function $g(q,p)$ over time is,

$$\dot{g} = \frac{\partial g}{\partial q_i}\dot{q}_i + \frac{\partial g}{\partial p_i}\dot{p}_i \tag{19}$$

Substituting (15), (16) above, we would re-write this as,

$$\dot{g} = [g,H] + u_m[g,\phi_m] \tag{20}$$

We can further simplify the equations of motion. Though the Poisson bracket is defined only for function of q,p, we can consider the Poisson bracket,

$$[g, H + u_m\phi_m]$$

Though u_m are not functions of q,p and are independent variables as of now, the Poisson bracket above is still defined in the subspace where the constraints (13) are obeyed. The explanation follows thus:

$$[g, H + u_m\phi_m] = [g,H] + [g, u_m\phi_m] = [g,H] + u_m[g,\phi_m] + \phi_m[g,u_m]$$

Though the Poisson bracket in the third term in RHS is not defined, we need not worry of that term if we are working in the shell where (13) is obeyed. Then that term anyway goes to zero, so we need not worry of the undefined PB.

Thus we have,

$$[g, H + u_m\phi_m] = [g, H] + [g, u_m\phi_m] \approx [g, H] + u_m[g, \phi_m] = \dot{g} \tag{21}$$

only in the shell where (13) is obeyed.

2.5 Weak and strong equations

In evaluating the Poisson bracket in (21), we should not substitute $\phi_m = 0$ before evaluating the PB. We should substitute $\phi_m = 0$ *after* evaluating the PB as we have done in the exercise in the previous section. To emphasize this point, we re-write equation (13) as

$$\phi_m \approx 0 \tag{22}$$

This is called a weak equation. The symbol $\approx$ should remind that we should not substitute $\phi_m = 0$ before evaluating the PB.

As the RHS in (21) gives $\dot{g}$ and as $[g, H + u_m\phi_m] = [g, H] + [g, u_m\phi_m] \approx \dot{g}$ ($\approx$ comes in the second equality because we first evaluated the PB and then substituted $\phi_m = 0$.) Thus,

$$\dot{g} \approx [g, H + u_m\phi_m]$$

For a weak equation, the variation of the LHS $\neq$ the variation of the RHS.

- `Example:` $\delta\phi_m \neq 0$ though $\phi_m \approx 0$. i.e. the gradient of the $\phi_m \neq 0$. In other words, the Poisson bracket of ϕ_m with any general function, $h(q, p) \neq 0$. It would have been so if the gradient of this constraint was to be zero[6].

- `Example:` $p_i \approx \frac{\partial L}{\partial \dot{q}_i}$

  In a space of $3n$ independent co-ordinates, $q, \dot{q}, p$, we see that $\delta p \neq \delta f(q, \dot{q})$ where $f(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}_i}$. This is because the variation of an independent quantity p_i is not determined by the variation of other quantities. Thus, the definition of the momentum should be rightly written as a weak equation.

Equivalent statements

Gradient of the constraint$\neq 0 \Longleftrightarrow$ PB of the constraint with an arbitrary function, in general $\neq 0$.

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Strong equations are shown by the usual = sign.

- `Example:` $L = L(q, \dot{q})$. This is a strong equation because the variation of the LHS is the same as the variation of the RHS by the very definition of L itself.

[6]We know that the gradient terms occur in the PB equation.

- `Example:` Consider ϕ^2 where $\phi \approx 0$.
  Let $g(q,p)$ be any function,

$$[\phi^2, g] = 2\phi[\phi, g] \approx 0$$

  We see that the PB of ϕ with any general function g is zero. Thus $\phi^2 = 0$ (strong equality).
  You can also see that the gradient of ϕ^2 is zero too. Thus, given a weak equality to 0, you can find a strong equality to 0 by squaring the weak equality.

2.6 Secondary constraints

$$\dot{g} \approx [g, H_T] \tag{23}$$

where H_T may be called the total Hamiltonian [2].

$$H_T = H + u_m \phi_m \tag{24}$$

Now that we have the equations of motion for a general function (we still have the unknown co-efficients u_m though), we can get some consistency conditions using the constraint equations. If at $t = t_1$, a primary constraint is seen to hold, then it should hold even at $t = t_2 > t_1$. So, for the constraints to be consistent with the equations of motion, we should have

$$\dot{\phi}_n \approx 0 \tag{25}$$

Putting $g = \phi_n$ where $n = 1, 2, ...n - R$ in (20), we have

$$[\phi_n, H] + u_m[\phi_n, \phi_m] \approx 0 \tag{26}$$

We get a number of consitency conditions from these $n - R$ number of equations. These may lead to one of the four consequences.

1. We get an inconsistency in the form 1=0. This would not happen if the given Lagrangian is a physically meaningful one. We should exclude inconsistent Lagrangians from this theory[7].

2. We get equation (26) to be a linear combination of the previously known constraints. So, (26) would lead to the form 0=0.

3. We get equation (26) to be independent of the u_m terms and also the $\chi(q,p)$ term if found to be linearly independent of the previous constraints we know, then we have a new constraint of the form $\chi(q,p) \approx 0$.

4. We get equation (26) to reduce to some equation involving u_m terms equated to zero. This will impose conditions on the form of the u_m.

The new constraints obtained as in statement (3) are called secondary constraints. Primary constraints are a consequence of the definition of the set of equations defining momentum. However secondary constraints differ from the primary constraints in the sense

[7]Please refer to Appendix (5.2) for an example.

that the primary constraints and the application of Hamiltonian equations of motion was required to arrive at them.

The iterating process is done until every time derivative of all the primary constraints and all the new constraints obtained is a linear combination of the set of *all* constraints. At the end, the time derivative of any constraints either takes the form of statement (2) or (4). We can thus stop our quest of finding new constraints on the system.

Let the secondary constraints be written thus.

$$\phi_k \approx 0 \qquad\qquad k = n - R + 1, ...n - R + K \tag{27}$$

As we have to consider the secondary constraints on the same footing as the primary, it is nice to write all the constraints under one set.

$$\phi_\tau \approx 0 \qquad\qquad \tau = 1, 2, ...K + M = T \tag{28}$$

More the constraints we acquire, the sub-space in the phase space where dynamics occur keeps becoming smaller.

The updated total Hamiltonian is:

$$H_T = H + u_\tau \phi_\tau \tag{29}$$

where τ runs as defined in (28).

2.7 First and second class constraints

After the iteration program, you have the total set of constraints (28). Till now, the distinction between them we had, was whether they were primary or secondary. This is not an important distinction because if you had started with a different Lagrangian, the primary and secondary constraints would have been exchanged.

However, an important distinction among the constraints can be brought by considering the Poisson brackets of the constraints among themselves.

`Definition:` A function $R(q, p)$ is called <u>first class</u> if the Poisson-bracket of R with *all* the constraints in the set (28) is weakly zero.

$$[R, \phi_\gamma] \approx 0 \qquad\qquad \gamma = 1, 2, \cdots T$$

`Note:` If $[R, \phi_\gamma]$ is weakly zero, then $[R, \phi_\gamma]$ is strongly equal to a linear combinbation of the constraints, $[R, \phi_\gamma] = k_\tau(q, p)\ \phi_\tau$. This is true because any quantity weakly zero in this theory should be a linear combination of the constraints and cannot give a linearly independent (of the constraints) equation. Because we have already written down *all* the constraints in the system and we won't get any *new* constraint by whatever operations that are done.

`Definition:` A function $R(q, p)$ is called <u>second class</u> if the Poisson-bracket of R with *atleast* one constraint is not weakly zero.

$$[R, \phi_\gamma] \neq 0 \qquad\qquad \gamma \in \text{Set } 28$$

Putting $R = \phi_\tau$, we can classify all the constraints into two sets: First class constraints and second class constraints.

Let the first class constraints be denoted as:

$$\psi_i \approx 0 \qquad\qquad i = 1, 2, \cdots I \tag{30}$$

Second class constraints be denoted as:

$$\phi_\alpha \approx 0 \qquad\qquad \alpha = 1, 2, \cdots N \tag{31}$$

where

$$I + N = T \tag{32}$$

Both the set of first class and second class constraints are a mixture of primary and secondary constraints each.

Let's define a matrix making use of the second class constraints alone.

$$C_{\alpha\beta} = [\phi_\alpha, \phi_\beta] \tag{33}$$

Note that this is an anti-symmetric matrix.

$$C_{\beta\alpha} = -C_{\alpha\beta} \tag{34}$$

Dirac proved [2] explicitly that this matrix is invertible. Alternatively, the invertibility of matrix $||C||$ can be given by the following argument:

Let the matrix be non-invertible and it's rank be less than K. Then we would have null eigen vectors of form,

$$\lambda_k \, C_{lk} \approx 0 \qquad\qquad \forall\, l$$

Then, as $\phi_k[\lambda_k, \phi_l] \approx 0$, we can write the above as $[\lambda_k \phi_k, \phi_l] \approx 0, \ \forall\, l$. Thus we would find a linear combination of ϕ_k, the II class constraints to give a first class constraint. This is a contradiction as we have written down the K number of II class constraints such that no I class constraint can be formed by the linear combination of them (if we had got any such I class constraint, we would have promptly removed it while segregating the I and II class constraints).

Thus, the inverse of $||C||$ exists. We know that an anti-symmetric matrix is invertible if and only if it is of even order. As $||C||$ is an anti-symmetric matrix, we conclude that K is even. The II class constraints are always even in number.

Defining inverse of $||C||$:

$$C^{-1}_{\alpha\beta} C_{\beta\gamma} = \delta^\alpha_\gamma \tag{35}$$

Let's assume that all the constraints are II class. All the undetermined co-efficients corresponding to the second class constraints in (29) has the form

$$u_\beta(q, p) = -[H, \phi_\alpha] \, C^{-1}_{\alpha\beta} \tag{36}$$

These solutions can be checked for it's truth by substituting them in (26) after making sure that we now change the index m to α index because our set of constraints may have expanded after the iteration.

The Hamiltonian for a system with II class constraints alone is completely determined and has the form,

$$H' = H - [H, \phi_\alpha]\, C^{-1}_{\alpha\beta}\, \phi_\beta \tag{37}$$

The Hamiltonian above is a I class quantity whose PB with any II class constraint is 0.

Equations (26) either provided new constraints or put restrictions on the form of u_α. Eitherway, the solution (36) can be seen to satisfy every equation of the form (26) as is explicitly worked out in Appendix (5.3).

For the case of all constraints being II class, we can thus determine *all* undetermined co-efficients and thus the Hamilton becomes unique and thus too the equations of the motion.

However, when there are I class constraints too in the system, then our final Hamiltonian is still not unique and can be written as

$$\mathbf{H} = H' + v_i \psi_i \tag{38}$$

Here v_α are arbitrary functions of time. The system then is said to possess I degrees of gauge freedom. Because of this arbitrariness, given an initial point, the evolution of the system can happen to different points in the phase space for different arbitrary functions v_α chosen. However, physically we know that at a give time, the system has a definite phase. In such cases, though two different points in phase space are not equal, the phase/state of the system corresponding to both points would still be the same.

The equations of motion in general is:

$$\begin{aligned} \dot{q}_i &= [q_i, H'] + v_j [q_i, \psi_j] \\ \dot{p}_i &= [p_i, H'] + v_j [p_i, \psi_j] \end{aligned} \tag{39}$$

Physical quantities are gauge independent quantities.

2.8 Example I

If we get a Lagrangian which is linear in velocities, then the momentum definition (7) would have a RHS independent of velocities while the LHS would be having momenta. Thus, you cannot find $\dot{q}$ in terms of momentum.

Such an example occurs in classical mechanics when a massive charged particle is in the presence of a homogeneous magnetic field perpendicular to the plane in which the particle is confined to move.

Let the charge of particle be q and mass be m which is confined to move in $x - y$ plane and there is a magnetic field pointing in the $\hat{z}$ direction. The Lagrangian for the particle is

$$L = \frac{1}{2} m v^2 + \frac{q}{c} \vec{A}.\vec{v} - V(\vec{r}) \tag{40}$$

V is an arbitrary potential. $\vec{B} = \nabla \times \vec{A}$.

$$\vec{A} = \frac{B}{2}(x\hat{y} - y\hat{x}) \tag{41}$$

$\Rightarrow$

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{qB}{2c}(x\dot{y} - y\dot{x}) - V(x, y) \tag{42}$$

The Euler-Lagrange equations (6) lead to the following equations of motion for the particle.

$$\begin{aligned} m\ddot{x} &= -\frac{\partial V}{\partial x} + \frac{qB}{c}\dot{y} \\ m\ddot{y} &= -\frac{\partial V}{\partial y} - \frac{qB}{c}\dot{x} \end{aligned} \tag{43}$$

When a very large magnetic field is applied or when the charge is high or when the mass of particle is very less, then $\frac{qB}{mc} >> 1$. For such case, we may approximate the Lagrangian to get a simpler Lagrangian which is linear in velocities and approximate solutions are obtained.

$$L' = \frac{qB}{2c}(x\dot{y} - y\dot{x}) - V \tag{44}$$

The EL equations now give,

$$\begin{aligned} \dot{x} &= -\frac{c}{qB}\frac{\partial V}{\partial y} \\ \dot{y} &= \frac{c}{qB}\frac{\partial V}{\partial x} \end{aligned} \tag{45}$$

We see that the equation of motion using an approximate Lagrangian is an approximation of the equation of motion of the exact Lagrangian. This however does not happen if we use the Hamiltonian.

Let's define momenta using L' as the Lagrangian.

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = -\frac{qB}{2c}y \\ p_y &= \frac{\partial L}{\partial \dot{y}} = \frac{qB}{2c}x \end{aligned} \tag{46}$$

As can be seen, these cannot be inverted to find $\dot{q}$ as a function of q, p.

$$H = \dot{x}p_x + \dot{y}p_y - L' = V(x, y) \tag{47}$$

This Hamiltonian would give $\dot{x} = \dot{y} = 0$ which are obviously wrong.

Dirac Algorithm Let's solve the same problem for L' using Dirac's algorithm of writing the Hamiltonian.

There are two constraints as seen in eq. (46).

$$\begin{aligned} \phi_1 &= p_x + \frac{qB}{2c}y \approx 0 \\ \phi_2 &- p_y - \frac{qB}{2c}x \approx 0 \end{aligned} \tag{48}$$

$$H^* = H + u_1\phi_1 + u_2\phi_2 = V + u_1\phi_1 + u_2\phi_2 \tag{49}$$

Now, we should check for consistency conditions.

$$\begin{aligned} \dot{\phi}_1 &\approx [\phi_1, H] + u_1[\phi_1, \phi_1] + u_2[\phi_1, \phi_2] = -\frac{\partial V}{\partial x} + u_2\frac{qB}{c} \approx 0 \\ \dot{\phi}_2 &\approx [\phi_2, H] + u_1[\phi_2, \phi_1] + u_2[\phi_2, \phi_2] = -\frac{\partial V}{\partial y} - u_1\frac{qB}{c} \approx 0 \end{aligned} \tag{50}$$

Both the consistency conditions in eqs (50) put a restriction on the form of u and no further (secondary) constraints are got.

The undetermined co-efficients u are found out to be

$$\begin{aligned} u_1 &= -\frac{c}{qB}\frac{\partial V}{\partial y} \\ u_2 &= \frac{c}{qB}\frac{\partial V}{\partial x} \end{aligned} \tag{51}$$

The equation of motion for a function $g(q,p)$ here is

$$\dot{g} = [g,V] + u_1[g,\phi_1] + u_2[g,\phi_2]$$

So, the Hamiltonian equations of motion are

$$\begin{aligned} \dot{x} &= -\frac{c}{qB}\frac{\partial V}{\partial y} \\ \dot{y} &= \frac{c}{qB}\frac{\partial V}{\partial x} \\ \dot{p_x} &= -\frac{1}{2}\frac{\partial V}{\partial x} \\ \dot{p_y} &= -\frac{1}{2}\frac{\partial V}{\partial y} \end{aligned} \tag{52}$$

As we see these equations are consistent with the EL solutions (45) obtained using the approximate Lagrangian L'.

2.9 Dirac bracket

If our intention was just to find the equations of motion, then we could have stopped at this point because we did find them.

However, the most important reason for developing constrained Hamiltonian dynamics was to help quantize the system.

To go to quantum mechanics, the Poisson bracket $[-,-]$ is promoted to the commutatator of operators $i\hbar[-,-]$. When there are constraints in the system, problems arise as is illustrated below. The example is taken from [2].

Let there be only II class constraints in the system and they be:

$$\begin{aligned} \phi_1 &= q_1 \approx 0 \\ \phi_2 &= p_1 \approx 0 \end{aligned} \tag{53}$$

In QM, we would then get $q_1\psi \approx 0$, $p_1\psi \approx 0$ which would imply that $(q_1p_1 - p_1q_1)\psi \approx 0$. However, the PB $[q_1, p_1] = 1$ when given the commutatator status goes to $[q_1, p_1] = i\hbar$ and this would imply $[q_1, p_1]\psi = i\hbar\psi$. Thus we get a contradiction.

In this case, as the constraints are simple, the simple thing to avoid such contradictions would be to change the PB being evaluated with respect to $q_1, ...q_n; p_1, ...p_n$ to the PB being evaluated with respect to $q_2, ...q_n; p_2, ...p_n$. We are removing the co-ordinates q_1, p_1 from the theory because they are not of any physical importance.

However, when the constraints are in general of form $\phi(q,p) \approx 0$, Dirac has given a general procedure to go from the old PB to a new PB which has only those co-ordinates which are of physical importance.

Dirac in his book [2] writes:

> That is the idea which one uses for quantizing a theory which involves II class constraints. The existence of II class constraints means that there are some degrees of freedom which are not physically important. We have to pick out these degrees of freedom and set up new Poisson brackets referring only to the other degrees of freedom which *are* of physical importance. Then in terms of those new Poisson brackets we can pass over to the quantum theory. I would like to discuss a general procedure for carrying that out.

2.9.1 Definition

In this section, we will consider those systems which have II class constraints alone. For general II class constraints, the new Poisson bracket called the Dirac bracket is defined as,

$$[f,g]_{DB} = [f,g] - [f,\phi_\alpha]C^{-1}_{\alpha\beta}[\phi_\beta, g] \tag{54}$$

2.9.2 Equation of motion

The equation of motion for a system with second class constraints can be written as

$$\dot{g} \approx [g,H]_{DB} \tag{55}$$

i.e. the Dirac bracket of g with the canonical Hamiltonian is the same as the Poisson bracket of the final Hamiltonian $\mathbf{H}$ $(= H')$ [8] with g.

`Proof:`

$$[g,H]_{DB} = [g,H] - [g,\phi_\alpha]C^{-1}_{\alpha\beta}[\phi_\beta, H]$$

By the anti-symmetricity of the PB and the matrix $||C^{-1}||$, we have

$$= [g,H] - [H,\phi_\beta]C^{-1}_{\beta\alpha}[g,\phi_\alpha]$$

From (36), we get

$$= [g,H] + u_\alpha[g,\phi_\alpha] = \dot{g}$$

Thus, we prove (55).

2.9.3 Setting II class constraints strongly to 0

Let's consider the Dirac bracket of any second class constraint ϕ_γ with *any* function $f(q,p)$.

$$[f,\phi_\gamma]_{DB} = [f,\phi_\gamma] - [f,\phi_\alpha]C^{-1}_{\alpha\beta}[\phi_\beta,\phi_\gamma]$$
$$= [f,\phi_\gamma] - [f,\phi_\alpha]C^{-1}_{\alpha\beta}C_{\beta\gamma}$$

[8] Because there are no first class constraints in the system.

$$= [f, \phi_\gamma] - [f, \phi_\alpha]\delta^\alpha_\gamma$$
$$= [f, \phi_\gamma] - [f, \phi_\gamma] = 0$$

As the Dirac bracket of any quantity with a II class constraint is weakly zero, we can set all II class constraints to be strongly zero if we agree to use Dirac brackets instead of Poisson brackets in the theory. So, now in a Dirac bracket, because of the strong equality of the II class constraint, you can set the constraint to zero *before* you evaluate the brackets. A freedom we didn't have before.

2.9.4 Getting a first class quantity

Given a function $A(q,p)$, we can define a new function $A'(q,p)$ such that A' has vanishing PBs with all the second class constraints.

$$A' = A - [A, \phi_\alpha]C^{-1}_{\alpha\beta}\phi_\beta \tag{56}$$

The claim can be seen to be true as shown below:
Let ϕ_γ be any II class constraint.

$$[A', \phi_\gamma] = [A - [A, \phi_\alpha]C^{-1}_{\alpha\beta}\phi_\beta, \phi_\gamma]$$
$$= [A, \phi_\gamma] - [[A, \phi_\alpha]C^{-1}_{\alpha\beta}\phi_\beta, \phi_\gamma]$$
$$= [A, \phi_\gamma] - [A, \phi_\alpha]C^{-1}_{\alpha\beta}[\phi_\beta, \phi_\gamma] - \phi_\beta[[A, \phi_\alpha]C^{-1}_{\alpha\beta}, \phi_\gamma]$$
$$\approx [A, \phi_\gamma] - [A, \phi_\alpha]C^{-1}_{\alpha\beta}C_{\beta\gamma} = [A, \phi_\gamma] - [A, \phi_\alpha]\delta^\alpha_\gamma$$
$$= [A, \phi_\gamma] - [A, \phi_\gamma] = 0$$

As ϕ_γ was any arbitrary II class constraint, it is seen that A' is a first class quantity[9]. Thus,

$$[A', \phi_\gamma] \approx 0 \tag{57}$$

for any ϕ_γ $\in$ constraints set.

The following result can be verified.

$$[A, B]_{DB} \approx [A', B'] \approx [A', B] \approx [A, B'] \tag{58}$$

It is easy to see that H' has vanishing brackets with all II class constraints and is a first class quantity if the system has II class constraints alone.

2.9.5 Respecting constraints

Let ϕ_γ be a constraint and f be any function. We know that $\phi_\gamma f\psi \approx 0\psi$, $f\phi_\gamma\psi \approx 0$. But we see that the PB in general will be of form $[\phi_\gamma, f] \approx g(q,p)$. When the PB is promoted to a commutator, we would then have according to the commutation relation that $\phi_\gamma f\psi - f\phi_\gamma\psi \approx i\hbar g$ which is a contradiction.

However, we see that the Dirac bracket respects the constraints. If we instead promote the Dirac bracket as a commutator in QM, as $[\phi_\gamma, f]_{DB} \approx 0$, we see that we have no constradiction here.

`NOTE:` Please refer to appendix (5.5) where the Dirac bracket for the trivial case $q_1 \approx p_1 \approx 0$ has been worked out.

[9]Remember that we are considering systems whose all constraints are II class.

3 O(3) model - Rigid Rotator

Constraint:

$$x^2 + y^2 + z^2 - r^2 \approx 0 \tag{59}$$

The Lagrangian for the particle is

$$L' = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{60}$$

Notation:

$$q_1 = x \qquad q_2 = y \qquad q_3 = z \qquad q_4 = F \tag{61}$$

The Lagrangian written now with the Lagrange multiplier F is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - F(x^2 + y^2 + z^2 - r^2) \tag{62}$$

The matrix $||W||$ is

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{63}$$

The rank of the matrix is $3 < 4$. So, we can expect 1 primary constraint. The determinant of $||W||$ is 0.
By the canonical momentum definition, we have

$$p_x \approx m\dot{x} \qquad p_y \approx m\dot{y} \qquad p_z \approx m\dot{z} \qquad p_F \approx 0 \tag{64}$$

Primary constraint:

$$p_F \approx 0 \tag{65}$$

This is a primary constraint because it was born when we were defining the momenta-velocity relationship.
The canonical-Hamiltonian is

$$\begin{aligned} H &= \dot{q}_i p_i - L \\ &= \dot{x}p_x + \dot{y}p_y + \dot{z}p_z + \dot{F}p_F - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + F(x^2 + y^2 + z^2 - r^2) \end{aligned}$$

Thus,

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + F(x^2 + y^2 + z^2 - r^2) \tag{66}$$

The effective Hamiltonian is

$$H^* = H + \lambda p_F \tag{67}$$

where λ is the undetermined multiplier.

Consistency conditions:

1. $[p_F, H^*] \approx 0$

$$\Rightarrow -1 \times \frac{\partial H^*}{\partial F} = -(x^2 + y^2 + z^2 - r^2) \approx 0$$

Thus, we get a secondary constraint (which was our hand-put constraint in the Lagrangian L) when we impose a consistency condition on the primary constraint.

Secondary constraint:

$$x^2 + y^2 + z^2 - r^2 \approx 0 \tag{68}$$

2. $[x^2 + y^2 + z^2 - r^2, H^*] \approx 0$ This is our next consistency imposition.

$$\begin{aligned}
&\Rightarrow 2x\frac{\partial H^*}{\partial p_x} + 2y\frac{\partial H^*}{\partial p_y} + 2z\frac{\partial H^*}{\partial p_z} \approx 0 \\
&= x\frac{p_x}{m} + y\frac{p_y}{m} + z\frac{p_z}{m} \approx 0
\end{aligned}$$

Thus we get one more constraint in phase-space.

$$xp_x + yp_y + zp_z \approx 0 \tag{69}$$

3. $[xp_x + yp_y + zp_z, H^*] \approx 0$

$$\Rightarrow p_x\frac{\partial H^*}{\partial p_x} - x\frac{\partial H^*}{\partial x} + p_y\frac{\partial H^*}{\partial p_y} - y\frac{\partial H^*}{\partial y} + p_z\frac{\partial H^*}{\partial p_z} - z\frac{\partial H^*}{\partial z} \approx 0$$

Thus,

$$(p_x^2 + p_y^2 + p_z^2) - 2mF(x^2 + y^2 + z^2) \approx 0 \tag{70}$$

4. $[(p_x^2 + p_y^2 + p_z^2) - 2mF(x^2 + y^2 + z^2), H^*] \approx 0$

$$\begin{aligned}
&\Rightarrow -4mFx\frac{\partial H^*}{\partial p_x} - 2p_x\frac{\partial H^*}{\partial x} - 4mFy\frac{\partial H^*}{\partial p_y} - 2p_y\frac{\partial H^*}{\partial y} - 4mFz\frac{\partial H^*}{\partial p_z} - 2p_z\frac{\partial H^*}{\partial z} \\
&\quad - 2m(x^2 + y^2 + z^2) \approx 0 \\
&\Rightarrow -4mFx\frac{p_x}{m} - 2p_x(2Fx) - 4mFy\frac{p_y}{m} - 2p_y(2Fx) - 4mFz\frac{p_z}{m} - 2p_z(2Fz) \\
&\quad - 2m(x^2 + y^2 + z^2) \approx 0
\end{aligned}$$

Thus, we have the final consistency condition,

$$4F(xp_x + yp_y + zp_z) + mr^2\lambda \approx 0 \tag{71}$$

This is a consistency equation which puts a condition on the form of the undetermined multiplier λ. Substituting the secondary constraint (69) in (71), we get

$$\lambda \approx 0 \tag{72}$$

Also using (70), we can find out the Lagrange multiplier F in terms of the dynamical co-ordinates. So,

$$F \approx \frac{p_x^2 + p_y^2 + p_z^2}{2mr^2} \tag{73}$$

Note that (71) is our final consistency condition because on further requiring this condition's PB with H^* to be weakly zero, we would obtain the result to be a linear combination of the previously obtained consistency conditions and thus no new linearly independent consistency conditions are found.
Let's rename the complete set of constraints we have.

$$\begin{aligned} \chi_1 &: \ x^2 + y^2 + z^2 - r^2 \approx 0 \\ \chi_2 &: \ xp_x + yp_y + zp_z \approx 0 \end{aligned} \tag{74}$$

Let us take the PB of these two constraints.

$$\begin{aligned} [x^2 + y^2 + z^2 - r^2, xp_x + yp_y + zp_z] &= 2x^2 - 0 + 2y^2 - 0 + 2z^2 - 0 \\ &= 2(x^2 + y^2 + z^2) \approx 2(r^2) \neq 0 \end{aligned}$$

Thus the two constraints on the phase space are II class.
Towards defining the Dirac brackets, lets find out the matrix $||C||$ where $C_{\alpha\beta} = [\chi_\alpha, \chi_\beta]$.

$$\begin{bmatrix} 0 & 2r^2 \\ -2r^2 & 0 \end{bmatrix} \tag{75}$$

The inverse of $||C||$ is

$$\begin{bmatrix} 0 & -\frac{1}{2r^2} \\ \frac{1}{2r^2} & 0 \end{bmatrix} \tag{76}$$

By the definition of Dirac bracket,

$$[F, G]_{DB} = [F, G] - [F, \chi_1]C^{-1}_{12}[\phi_2, G] - [F, \phi_2]C^{-1}_{21}[\phi_1, G] \tag{77}$$

For example, lets work out the DB of x and p_x.

$$\begin{aligned} [x, p_x]_{DB} &= [x, p_x] + \frac{1}{2r^2}[x, x^2 + y^2 + z^2 - r^2][xp_x + yp_y + zp_z, p_x] \\ &- \frac{1}{2r^2}[x, xp_x + yp_y + zp_z][x^2 + y^2 + z^2 - r^2, p_x] \\ &= 1 + 0 - \frac{1}{2r^2}(x)(2x) \\ &= 1 - \frac{x^2}{r^2} \end{aligned}$$

With similar calculations, we get the following Dirac brackets:

$$[q_i, p_j]_{DB} = \delta_{ij} - \frac{q_i q_j}{r^2}$$

$$[q_i, q_j]_{DB} = 0$$

$$[p_i, p_j]_{DB} = \frac{1}{r^2}(-q_i p_j + p_i q_j)$$

Finding F using equation (71), we get

$$F \approx \frac{p_x^2 + p_y^2 + p_z^2}{2mr^2} \tag{78}$$

All the constraints may be set strongly equal to zero now as our new brackets of the theory is not Poisson brackets but Dirac brackets.

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) \tag{79}$$

The final Hamiltonian **H** is[10],

$$\begin{aligned}
\mathbf{H} &= H - [H, \chi_\alpha] C^{-1}_{\alpha\beta} \chi_\beta \\
&= H - [H, \chi_1] C^{-1}_{12} \chi_2 - [H, \chi_2] C^{-1}_{21} \chi_1 \\
[H, \chi_1] &= \frac{1}{2m} [p_x^2 + p_y^2 + p_z^2, \chi_1] \\
&= \frac{1}{2m} ([p_x^2, x^2] + [p_y^2, y^2] + [p_z^2, z^2]) \\
&- \frac{2}{m}(x_i p_i) \approx 0 \qquad \text{(From (74).)} \\
[H, \chi_2] &= \frac{1}{2m} [p_x^2 + p_y^2 + p_z^2, x p_x + y p_y + z p_z] \\
&= \frac{1}{m}(-p_i p_i)
\end{aligned}$$

Thus the final Hamiltonian may be written as (The indices i, j are on 1,2,3)

$$\mathbf{H} = H + \frac{p_i p_i}{2mr^2}(x_j x_j - r^2) \tag{80}$$

The equation of motion of a dynamical variable $f(x, p)$ is

$$\dot{f} = [f, \mathbf{H}] \approx [f, H]_{DB} \tag{81}$$

4 Maxwell Field Theory

Consider the 4-dimensional curl of the 4-component potential tensor.

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{82}$$

$F^{\mu\nu}$ is called the field strength tensor.

The scalar got by contracting the field strength tensor gives the Lagrangian density of the theory.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{83}$$

[10] The final Hamiltonian is a first class quantity because we have only II class constraints in the theory.

The canonical momenta fields are derived as shown:

$$
\begin{aligned}
\pi^i &\approx \frac{\partial \mathcal{L}}{\partial \dot{A}_i} \\
&= -\frac{1}{4}\frac{\partial}{\partial \dot{A}_i}(F_{\mu\nu}F^{\mu\nu}) \\
&= -\frac{g^{\mu\alpha}g^{\nu\beta}}{4}\frac{\partial}{\partial \dot{A}_i}(F_{\mu\nu}F_{\alpha\beta}) \\
&= -\frac{g^{\mu\alpha}g^{\nu\beta}}{4}[F_{\mu\nu}(\delta_0^\alpha\delta_i^\beta - \delta_i^\alpha\delta_0^\beta) + F_{\alpha\beta}(\delta_o^\mu\delta_i^\nu - \delta_i^\mu\delta_0^\nu)] \\
&= -\frac{g^{\mu 0}g^{\nu i}}{4}F_{\mu\nu} + \frac{g^{\mu i}g^{\nu 0}}{4}F_{\mu\nu} - \frac{g^{0\alpha}g^{i\beta}}{4}F_{\alpha\beta} + \frac{g^{i\alpha}g^{0\beta}}{4}F_{\alpha\beta} \\
&= -\frac{g^{00}g^{ii}}{2}F_{0i} + \frac{g^{ii}g^{00}}{2}F_{i0} \\
&= -g^{ii}g^{00}F_{0i}
\end{aligned}
$$

Taking our metric convention to be

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{84}
$$

We get,

$$
\pi^0(x) \approx 0 \qquad \pi^1(x) \approx E^1(x) \qquad \pi^2(x) \approx E^2(x) \qquad \pi^3(x) \approx E^3(x) \tag{85}
$$

Thus we get the electric fields to be the space component of the momentum vector. However, we can easily see that $\pi^0(x) \approx 0$ is a constraint on phase space. It is a primary constraint as it was born out of the momenta-velocity relation.

$$
\phi_1(x) = \pi^0(x) \approx 0 \tag{86}
$$

We get the canonical Hamiltonian density to be

$$
\mathcal{H} = \frac{1}{2}(\pi^2(x) + B^2(x)) - A_0(\nabla.\vec{\pi}(x)) \tag{87}
$$

The total Hamiltonian density is

$$
\mathcal{H}' = \frac{1}{2}(\pi^2(x) + B^2(x)) - A_0(\nabla.\vec{\pi}(x)) + \lambda\pi^0(x) \tag{88}
$$

Let's get the secondary constraint.

$$
\begin{aligned}
\dot{\pi^0(x)} &= [\pi^0(x), H_p] \\
&= \int d^3z \left(\frac{\delta\pi_0(x)}{\delta A_\mu(z)}\frac{\delta H_p}{\delta A_\mu(z)} - \frac{\delta\pi_o(x)}{\delta\pi_\mu(z)}\frac{\delta H_p}{\delta A_\mu(z)}\right) \\
&= \int d^3z \left(-\delta_\mu^0\delta(x-z)\frac{\delta H_p}{\delta A_\mu(z)}\right) \\
&= -\frac{\delta H_p}{\delta A_o(x)} = -\frac{\partial \mathcal{H}_p}{\partial A_0(x)} = \nabla.\vec{\pi}(x)
\end{aligned}
$$

Setting this weakly to zero gives us the secondary constraint:

$$\phi_2(x) = \nabla.\vec{\pi}(x) \approx 0 \tag{89}$$

It can be shown that the time derivative of this constraint is weakly zero and thus our chain of constraints end here. The two constraints are easily seen to be I class as the PB between them is zero.

To apply Dirac constraint analysis, we choose two gauge conditions such that all the four constraints then together can become II class. One of the gauge choice is taking $A_o(x) \approx 0$ and $\nabla.\vec{A}(x) \approx 0$.

5 Appendix

5.1 Lagrange Multipliers

In this section, the physical meaning of Lagrange multipliers will be discussed for holonomic constraints by considering an example of a constrained system.
System: is a particle in a two dimensional space constrained to move on a plank (a straight line).

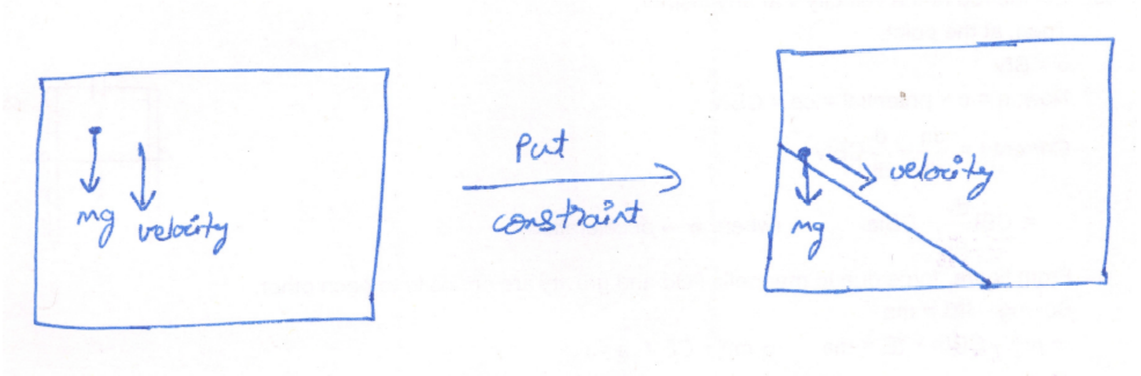

Figure 1

One can easily see that the smart way of going through this problem is choosing a generalized co-ordinate, say s which would be the distance the particle would have moved along the planck. However, lets choose 2 co-ordinates x, y to describe the motion.

$$L = T - V = m\left[\frac{\dot{x}^2 + \dot{y}^2}{2}\right] - mgy \tag{90}$$

and the corresponding constraint is

$$f(x, y) = x + y - 1 = 0 \tag{91}$$

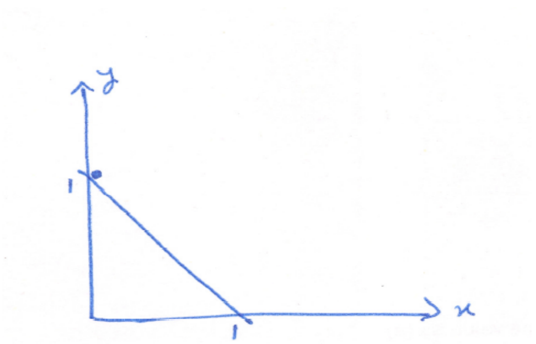

Figure 2: The constraint line

We write now a new Lagrangian L' which is,

$$L' = T - V' = m\left[\frac{\dot{x}^2 + \dot{y}^2}{2}\right] - mgy - \lambda(x + y - 1) \tag{92}$$

where

$$V' = mgy + \lambda(x + y - 1) \tag{93}$$

In this new Lagrangian, we treat x, y as independent. Why?

Because after all what does a constraint (plank) do? It provides a force such that the particle moves only on the constrained path. It does nothing else. Providing this force is the only job of the plank. Lagrange multiplier method uses this information and replaces the constraint surface by adding an external force to the system such that this external force is equal to the force which the constraint would have given to the particle.

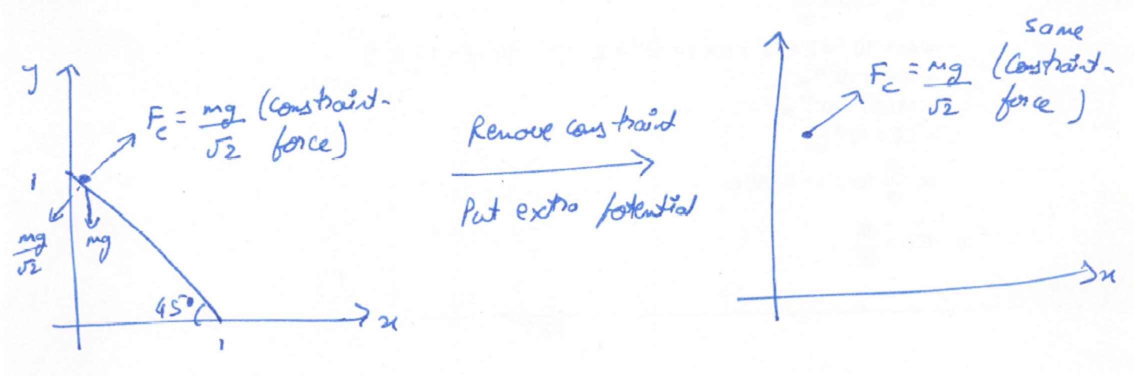

Figure 3: Replacement of the plank by an extra-potential

For a monogenic force, we can replace this external force by a potential. So, we thus modify the old Lagrangian, L with dependent co-ordinates (dependent because the planck was there) with a new Lagrangian, L' (new V' that is) with all x, y as independent (because no planck is there now).

Why did we choose this extra potential to be of the form $\lambda(x + y - 1)$?

Again consider the plank. The constraint force (i.e. the normal force) given by the plank to the particle such that the particle stays on the plank is perpendicular to the plank. i.e. constraint force $\propto \vec{\nabla}$(surface).

$\Rightarrow$ constraint force $= \lambda\vec{\nabla}$(surface) where λ is an undetermined multiplier, a function of (x, y, t) yet to be found out. Notice that,

$$-\vec{\nabla}(\text{extra-potential}) = -\vec{\nabla}(\lambda f) = -\lambda\vec{\nabla}f - f\vec{\nabla}\lambda$$

The last term is zero because we are evaluating for the constraint force on the surface $f = 0$.

Thus, it is appropriate to add $\lambda(x + y - 1)$ to the potential as the gradient of this potential is the constraint-force.

As co-ordinates are now independent, using Euler-Lagrange equations and the constraint equation, we have respectively,

$$\begin{aligned} m\ddot{x} &= -\lambda \\ m\ddot{y} &= -mg - \lambda \\ \ddot{x} &= -\ddot{y} \end{aligned} \tag{94}$$

$$\Rightarrow -\lambda = mg + \lambda$$

Thus,

$$\lambda = -\frac{mg}{2} \tag{95}$$

After substituting for λ, we can solve for x, y. For the particle being left from the top of the plank with zero velocity and the solutions are,

$$\begin{aligned} x(t) &= \frac{gt^2}{4} \\ y(t) &= 1 - \frac{gt^2}{4} \end{aligned} \tag{96}$$

Thus, we were able to solve the equations of motion. We can also know the constraint force using λ which is an advantage of using this method. As told earlier,

$$\text{Constraint-force} = F_c = -\lambda \vec{\nabla}(x + y - 1) = \frac{mg}{\sqrt{2}} \left(\frac{\hat{x} + \hat{y}}{\sqrt{2}} \right) = \frac{mg}{\sqrt{2}}\, \hat{u}$$

where $\hat{u}$ is a unit vector in the direction perpendicular to the planck. We see that the Constraint-force is correct.

The particle will move such that the action integral $(\int (T - (V + \lambda f))dt)$ is minimized. Putting a plank is equivalent to the situation of changing the potential of a particle in independent x, y co-ordinates. Because in both situations, the same constraint-force is being provided.

5.2 Inconsistent Lagrangian

Here an example of an 'inconsistent' Lagrangian is shown as given in [5].

Let the Lagrangian be taken to be

$$L(q, \dot{q}) = \dot{q} - q \tag{97}$$

Then,

$$p \approx \frac{\partial L}{\partial \dot{q}} = 1 \tag{98}$$

This gives a constraint.

$$\phi(q, p) = p - 1 \approx 0 \tag{99}$$

Defining the canonical Hamiltonian, H

$$H = \dot{q}p - L \tag{100}$$

$$H = \dot{q} - (\dot{q} - q) = q$$

Thus,

$$H(q, p) = q \tag{101}$$

For consistency of the system, we need $\dot{\phi} \approx 0$. From the relation (26), we get

$$\begin{aligned} \dot{\phi} &= [\phi, q] + u[\phi, \phi] \approx 0 \\ \Rightarrow\ & [p - 1, q] + 0 \approx 0 \\ \Rightarrow\ & -1 \approx 0 \ \text{ or } 1 \approx 0 \end{aligned}$$

This is an inconsistency.

We exclude all such Lagrangians which give such non-sensical [5] results and assume that no such inconsistencies arise in the Lagrangian we are working with.

5.3 Undetermined co-efficients for II class constraints

Derivation

$$\dot{\phi}_\gamma = [\phi_\gamma, H] + u_\beta[\psi_\gamma, \psi_\beta] \approx 0$$
$$\Rightarrow u_\beta C_{\gamma\beta} = -[\phi_\gamma, H]$$
$$\Rightarrow u_\beta C_{\beta\gamma} = [\phi_\gamma, H]$$
$$\Rightarrow u_\beta C_{\beta\gamma} C^{-1}_{\gamma\alpha} = [\phi_\alpha, H] C^{-1}_{\gamma\alpha}$$
$$\Rightarrow u_\beta \delta^\alpha_\beta = u_\alpha = -[H, \phi_\alpha] C^{-1}_{\gamma\alpha}$$

Verification

$$\dot{\phi}_\gamma = [\phi_\gamma, H] + u_\beta[\psi_\gamma, \psi_\beta] \approx 0$$

Substituting the solution (36)

$$= [\phi_\gamma, H] - [H, \phi_\alpha] C^{-1}_{\alpha\beta}[\psi_\gamma, \psi_\beta]$$

As all constraints are II class, we have

$$= [\phi_\gamma, H] - [H, \phi_\alpha] C^{-1}_{\alpha\beta} C_{\gamma\beta}$$

By the anti-symmetricity of matrix $||C||$,

$$= [\phi_\gamma, H] + [H, \phi_\alpha] C^{-1}_{\alpha\beta} C_{\beta\gamma}$$

$$= [\phi_\gamma, H] + [H, \phi_\alpha] \delta^\alpha_\gamma$$
$$\Rightarrow [\phi_\gamma, H] + [H, \phi_\gamma] \approx 0$$

by the anti-symmetricity of the PB.

Thus we see that the solutions we had written for the undetermined co-efficients are right.

5.4 Poisson bracket of first-class quantities

`Theorem:` The Poisson bracket of any two first class quantities is a first class quantity too.
`Proof:` Let R, S be two first class constraints, ϕ_τ be any constraint, then

$$[[R, S], \phi_\tau] + [[S, \phi_\tau], R] + [[\phi_\tau, R], S] = 0$$

$$\approx [[R, S], \phi_\tau] + [0, R] + [0, S] = 0$$
$$\Rightarrow [[R, S], \phi_\tau] \approx 0$$

This is true for any ϕ_τ. Thus, we see that the Poisson bracket of any two first class quantities is a first class quantity.

5.5 Trivial example for Dirac bracket

I will here work out the DB for the simple constraints we considered in (2.9). $\phi_1 = q_1 \approx 0$, $\phi_2 = p_1 \approx 0$. We expect the DB to reduce to the PB which is evaluated with respect to $q_2, ...q_n;\ p_2, ...p_n$.

Let ξ and η be any two function of $q_1, ...q_n;\ p_1, ...p_n$

The matrix $C_{\alpha\beta}$ is a 2×2 matrix.

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{102}$$

The inverse $C^{-1}_{\alpha\beta}$ matrix is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{103}$$

The Dirac bracket (54) is

$$[\xi, \eta]_{DB} = [\xi, \eta] - [\xi, q_1](-1)[p_1, \eta] - [\xi, p_1](1)[q_1, \eta] \tag{104}$$

A little simplification and then,

$$[\xi, \eta]_{DB} = [\xi, \eta] - \left(\frac{\partial \xi}{\partial q_1}\frac{\partial \eta}{\partial p_1} - \frac{\partial \xi}{\partial p_1}\frac{\partial \eta}{\partial q_1} \right)$$

We see that we get the RHS to be a simple Poisson bracket now which is evaluated just with respect to $q_2, ...q_n;\ p_2, ...p_n$. As expected. The Dirac bracket (new Poisson bracket) removed the unphysical degrees of freedom to give an appropriate Poisson bracket for the system.

6 References

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