

7.4 Linear Programming: The Simplex Method

For linear programming problems with more than two variables, the graphical method is usually impossible, so the **simplex method** is used. Because the simplex method is used for problems with many variables, it is usually not practical to use letters such as x , y , z , or w as variables, instead the symbols x_1 , x_2 , x_3 , and x_4 are used. In the simplex method, all constraints must be expressed in the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots \leq b$$

The first step is to convert each constraint (a linear inequality) into a linear equation. This is done by adding a nonnegative variable, called a **slack variable** to each constraint.

1. For the linear programming problem, use slack variables to convert each constraint into a linear equation.

$$\begin{aligned} \text{Maximize: } z &= 8x_1 + 3x_2 + x_3 \\ \text{Subject to: } 3x_1 - x_2 + 4x_3 &\leq 95 \\ 7x_1 + 6x_2 + 8x_3 &\leq 118 \\ 4x_1 + 5x_2 + 10x_3 &\leq 220 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

Solution: The Slack Variables are x_4 , x_5 , and x_6

$$\begin{aligned} 3x_1 - x_2 + 4x_3 + x_4 &= 95 \\ 7x_1 + 6x_2 + 8x_3 + x_5 &= 118 \\ 4x_1 + 5x_2 + 10x_3 + x_6 &= 220 \end{aligned}$$

A linear programming problem can be written in matrix form called the **simplex tableau**. The last row of the simplex tableau corresponds to the objective function with all variables on the left-hand side.

2. Introduce slack variables as necessary and then write the initial simplex tableau for the linear programming problem.

$$\begin{aligned} \text{Maximize: } z &= 5x_1 + 3x_2 + 7x_3 \\ \text{Subject to: } 4x_1 + 3x_2 + 2x_3 &\leq 60 \\ 3x_1 + 4x_2 + x_3 &\leq 24 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

Solution: The Slack Variables are x_4 and x_5

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 + x_4 &= 60 \\ 3x_1 + 4x_2 + x_3 + x_5 &= 24 \end{aligned}$$

The objective function $z = 5x_1 + 3x_2 + 7x_3$ needs to be rewritten so that all the variables are on the left-hand side: $-5x_1 - 3x_2 - 7x_3 + z = 0$.

The initial simplex tableau is

x_1	x_2	x_3	x_4	x_5	z	
4	3	2	1	0	0	60
3	4	1	0	1	0	24
-5	-3	-7	0	0	1	0

Except for the entries 1 and 0 on the right end-the numbers in the bottom row of a simplex tableau are called **indicators**. The indicators in this tableau are -5 -3 -7 0 and 0. The column containing the most negative indicator is called the **pivot column**. In this example, the third column is the pivot column, since -7 is the most negative indicator. Now for each positive entry in the pivot column, divide the number in the far right column of the same row by the positive number in the pivot column. The row with the smallest quotient is called the **pivot row**. The entry in the pivot row and pivot column is the **pivot**. The pivot is the number in the matrix chosen to change to a 1 and all other entries in that column changed to zeros.

Quotients : $\frac{60}{2} = 30$ and $\frac{24}{1} = 24$. The smallest quotient is 24, so the pivot is 4. Pivoting is complete once all the indicators are nonnegative.

Examples: Use the simplex method to solve.

3. Maximize $z = x_1 + 3x_2$

Subject to: $x_1 + x_2 \leq 10$

$5x_1 + 2x_2 \leq 20$

$x_1 + 2x_2 \leq 36$

$x_1 \geq 0, x_2 \geq 0$

The Slack Variables are x_3, x_4 , and x_5

$$x_1 + x_2 + x_3 = 10$$

$$5x_1 + 2x_2 + x_4 = 20$$

$$x_1 + 2x_2 + x_5 = 36$$

The objective function $z = x_1 + 3x_2$ needs to be rewritten so that all the variables are on the left-hand side: $-x_1 - 3x_2 + z = 0$.

The initial simplex tableau is

x_1	x_2	x_3	x_4	x_5	z	
1	1	1	0	0	0	10
5	2	0	1	0	0	20
1	2	0	0	1	0	36
-1	-3	0	0	0	1	0

The second column is the pivot column, since -3 is the most negative indicator.

Quotients; $10/1 = 10$, $20/2 = 10$, $36/2 = 18$. Since there are two equally small quotients, we can choose either entry in column 2 to be the pivot. Lets choose the number 1 in column 2 to be the pivot.

$$\begin{array}{c}
 \begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ 5 & 2 & 0 & 1 & 0 & 0 & 20 \\ 1 & 2 & 0 & 0 & 1 & 0 & 36 \\ -1 & -3 & 0 & 0 & 0 & 1 & 0 \end{array} \\
 r_2 \rightarrow -2r_1 + r_2
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ 3 & 0 & -2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 & 36 \\ -1 & -3 & 0 & 0 & 0 & 1 & 0 \end{array} \\
 r_3 \rightarrow -2r_1 + r_3
 \end{array}$$

$$\begin{array}{r}
 -2 \quad -2 \quad -2 \quad 0 \quad 0 \quad 0 \quad -20 \\
 + \quad 5 \quad 2 \quad 0 \quad 1 \quad 0 \quad 0 \quad 20 \\
 \hline
 3 \quad 0 \quad -2 \quad 1 \quad 0 \quad 0 \quad 0
 \end{array}
 \quad
 \begin{array}{r}
 -2 \quad -2 \quad -2 \quad 0 \quad 0 \quad 0 \quad -20 \\
 + \quad 1 \quad 2 \quad 0 \quad 0 \quad 1 \quad 0 \quad 36 \\
 \hline
 -1 \quad 0 \quad -2 \quad 0 \quad 1 \quad 0 \quad 16
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ 3 & 0 & -2 & 1 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 & 16 \\ -1 & -3 & 0 & 0 & 0 & 1 & 0 \end{array} \\
 r_4 \rightarrow 3r_1 + r_4
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ 3 & 0 & -2 & 1 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 & 16 \\ 2 & 0 & 3 & 0 & 0 & 1 & 30 \end{array}
 \end{array}$$

$$\begin{array}{r}
 3 \quad 3 \quad 3 \quad 0 \quad 0 \quad 0 \quad 30 \\
 + \quad -1 \quad -3 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \\
 \hline
 2 \quad 0 \quad 3 \quad 0 \quad 0 \quad 1 \quad 3
 \end{array}$$

Since there are no negatives in the last row, we are done pivoting. The corresponding equation is

$2x_1 + 3x_3 + z = 30$ or $z = 30 - 2x_1 - 3x_3$. The basic variables are x_2 , x_4 , x_5 , and z the nonbasic variables are x_1 and x_3 . The solution is found by setting the nonbasic variables equal to zero. So $x_1 = 0$ and $x_3 = 0$, then from the last Tableau we have the following equations:

$$\begin{array}{rcl}
 x_1 + x_2 + x_3 & = & 10 \\
 3x_1 & - & x_3 + x_4 = 0 \\
 -x_1 & - & 2x_3 + x_5 = 16
 \end{array}$$

Substituting $x_1 = 0$ and $x_3 = 0$ gives us $x_2 = 10$, $x_4 = 0$, and $x_5 = 16$ and the maximum value for the variable z is $z = 30$.

Simplex Method:

1. Convert each constraint to an equation by adding slack variables
2. Set up the initial tableau
3. Locate the most negative indicator. This indicator determines the pivot column
4. Use the positive entries in the pivot column to form the necessary quotients for determining the pivot. If there are no positive entries, no maximum solution exists. If two quotients are equally the smallest, let either determine the pivot.
5. Divide the pivot row by the pivot to change the pivot to 1. Then use row operations to change all other entries in the pivot column to 0.
6. If the indicators are all positive or 0, you have found the final tableau
7. Set each nonbasic variable equal to 0 and solve the system for the basic variables. The maximum value is the number in the lower right hand corner of the tableau.

4. Maximize $z = 3x_1 + 2x_2 + x_3$
 Subject to $\begin{cases} 2x_1 + 2x_2 + x_3 \leq 10 \\ x_1 + 2x_2 + 3x_3 \leq 15 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{cases}$

Introduce slack variables and rewriting the objective function as $-3x_1 - 2x_2 - x_3 + z = 0$ gives us the initial tableau

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \\ \hline 2 & 2 & 1 & 1 & 0 & 0 & 10 \\ 1 & 2 & 3 & 0 & 1 & 0 & 15 \\ -3 & -2 & -1 & 0 & 0 & 1 & 0 \end{array} \quad \text{The pivot column is column 1. Quotients } 10/2 = 5 \text{ and } 15/1 = 15.$$

The smallest quotient is 5, so the 2 in row 1 column 1 is the pivot. Divide row 1 by 2 to get

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \\ \hline 1 & 1 & 1/2 & 1/2 & 0 & 0 & 5 \\ 1 & 2 & 3 & 0 & 1 & 0 & 15 \\ -3 & -2 & -1 & 0 & 0 & 1 & 0 \end{array} \quad r_1 \rightarrow -r_1 + r_2 \quad \begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \\ \hline 1 & 1 & 1/2 & 1/2 & 0 & 0 & 5 \\ 0 & 1 & 5/2 & -1/2 & 1 & 0 & 10 \\ -3 & -2 & -1 & 0 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{cccccc|c} -1 & -1 & -1/2 & -1/2 & 0 & 0 & -5 \\ + & 1 & 2 & 3 & 0 & 1 & 0 & 15 \\ \hline 0 & 1 & 5/2 & -1/2 & 1 & 0 & 10 \end{array}$$

$$r_3 \rightarrow 3r_1 + r_3$$

$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \\ \hline 1 & 1 & 1/2 & 1/2 & 0 & 0 & 5 \\ 0 & 1 & 5/2 & -1/2 & 1 & 0 & 10 \\ 0 & 1 & 1/2 & 3/2 & 0 & 1 & 15 \end{array}$$

$$\begin{array}{cccccc|c} 3 & 3 & 3/2 & 3/2 & 0 & 0 & 15 \\ -3 & -2 & -1 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 1/2 & 3/2 & 0 & 1 & 15 \end{array}$$

The nonbasic variables are $x_2, x_3, \text{ and } x_4$ set these equal to 0. The corresponding system is

$$\begin{cases} x_1 + x_2 + 1/2 x_3 + 1/2 x_4 = 5 \\ x_2 + 5/2 x_3 - 1/2 x_4 + x_5 = 10 \end{cases}$$

This gives $x_1 = 5, x_5 = 10$ and the maximum value of z is 15.

5. Maximize: $z = 4x_1 - 3x_2 + 2x_3$

Subject to:
$$\begin{cases} 2x_1 - x_2 + 8x_3 \leq 40 \\ 4x_1 - 5x_2 + 6x_3 \leq 60 \\ 2x_1 - 2x_2 + 6x_3 \leq 24 \end{cases}$$

$$\left[\begin{array}{ccccccc|c} 2 & -1 & 8 & 1 & 0 & 0 & 0 & 40 \\ 4 & -5 & 6 & 0 & 1 & 0 & 0 & 60 \\ 2 & -2 & 6 & 0 & 0 & 1 & 0 & 24 \\ \hline -4 & 3 & -2 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

the pivot column is column 1.

Quotients $40/2 = 20$, $60/4 = 15$, and $24/2 = 12$. The smallest quotient is 12, the pivot is the 2 in column 1 row 3. Divide row 3 by 2, the pivot.

$$\left[\begin{array}{ccccccc|c} 2 & -1 & 8 & 1 & 0 & 0 & 0 & 40 \\ 4 & -5 & 6 & 0 & 1 & 0 & 0 & 60 \\ 1 & -1 & 3 & 0 & 0 & 1/2 & 0 & 12 \\ \hline -4 & 3 & -2 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$r_1 \rightarrow -2r_3 + r_1 \quad \left[\begin{array}{ccccccc|c} 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \\ 4 & -5 & 6 & 0 & 1 & 0 & 0 & 60 \\ 1 & -1 & 3 & 0 & 0 & 1/2 & 0 & 12 \\ \hline -4 & 3 & -2 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$\begin{array}{ccccccc} -2 & 2 & -6 & 0 & 0 & -1 & 0 & -24 \\ 2 & -1 & 8 & 1 & 0 & 0 & 0 & 40 \\ \hline 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \end{array}$$

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \\ 4 & -5 & 6 & 0 & 1 & 0 & 0 & 60 \\ 1 & -1 & 3 & 0 & 0 & 1/2 & 0 & 12 \\ \hline -4 & 3 & -2 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$r_2 \rightarrow -4r_3 + r_2 \quad \left[\begin{array}{ccccccc|c} 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \\ 0 & -1 & -6 & 0 & 1 & -2 & 0 & 12 \\ 1 & -1 & 3 & 0 & 0 & 1/2 & 0 & 12 \\ \hline -4 & 3 & -2 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$\begin{array}{ccccccc} -4 & 4 & -12 & 0 & 0 & -2 & 0 & -48 \\ 4 & -5 & 6 & 0 & 1 & 0 & 0 & 60 \\ \hline 0 & -1 & -6 & 0 & 1 & -2 & 0 & 12 \end{array}$$

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \\ 0 & -1 & -6 & 0 & 1 & -2 & 0 & 12 \\ 1 & -1 & 3 & 0 & 0 & 1/2 & 0 & 12 \\ \hline -4 & 3 & -2 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$r_4 \rightarrow 4r_3 + r_4 \quad \left[\begin{array}{ccccccc|c} 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \\ 0 & -1 & -6 & 0 & 1 & -2 & 0 & 12 \\ 1 & -1 & 3 & 0 & 0 & 1/2 & 0 & 12 \\ \hline 0 & -1 & 8 & 0 & 0 & 2 & 1 & 48 \end{array} \right]$$

$$\begin{array}{ccccccc} 4 & -4 & 12 & 0 & 0 & 2 & 0 & 48 \\ -4 & 3 & -2 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 8 & 0 & 0 & 2 & 1 & 48 \end{array}$$

The new pilot column is column 2 and the first row is the only possible pivot row.

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \\ 0 & -1 & -6 & 0 & 1 & -2 & 0 & 12 \\ 1 & -1 & 3 & 0 & 0 & 1/2 & 0 & 12 \\ \hline 0 & -1 & 8 & 0 & 0 & 2 & 1 & 48 \end{array} \right]$$

$$r_2 \rightarrow r_1 + r_2 \quad \left[\begin{array}{ccccccc|c} 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \\ 0 & 0 & -4 & 1 & 1 & -3 & 0 & 28 \\ 1 & -1 & 3 & 0 & 0 & 1/2 & 0 & 12 \\ \hline 0 & -1 & 8 & 0 & 0 & 2 & 1 & 48 \end{array} \right]$$

$$\begin{array}{ccccccc} 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \\ 0 & -1 & -6 & 0 & 1 & -2 & 0 & 12 \\ \hline 0 & 0 & -4 & 1 & 1 & -3 & 0 & 28 \end{array}$$

$$\begin{array}{c}
 \left[\begin{array}{cccccc|c} 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \\ 0 & 0 & -4 & 1 & 1 & -3 & 0 & 28 \\ 1 & -1 & 3 & 0 & 0 & 1/2 & 0 & 12 \\ \hline 0 & -1 & 8 & 0 & 0 & 2 & 1 & 48 \end{array} \right] \\
 r_3 \rightarrow r_1 + r_3
 \end{array}
 \quad
 \begin{array}{c}
 \left[\begin{array}{cccccc|c} 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \\ 0 & 0 & -4 & 1 & 1 & -3 & 0 & 28 \\ 1 & 0 & 5 & 1 & 0 & -1/2 & 0 & 28 \\ \hline 0 & -1 & 8 & 0 & 0 & 2 & 1 & 48 \end{array} \right] \\
 \begin{array}{cccccc|c} 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \\ 1 & -1 & 3 & 0 & 0 & 1/2 & 0 & 12 \\ \hline 1 & 0 & 5 & 1 & 0 & -1/2 & 0 & 28 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \left[\begin{array}{cccccc|c} 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \\ 0 & 0 & -4 & 1 & 1 & -3 & 0 & 28 \\ 1 & 0 & 5 & 1 & 0 & -1/2 & 0 & 28 \\ \hline 0 & -1 & 8 & 0 & 0 & 2 & 1 & 48 \end{array} \right] \\
 r_4 \rightarrow r_1 + r_4
 \end{array}
 \quad
 \begin{array}{c}
 \left[\begin{array}{cccccc|c} 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \\ 0 & 0 & -4 & 1 & 1 & -3 & 0 & 28 \\ 1 & 0 & 5 & 1 & 0 & -1/2 & 0 & 28 \\ \hline 0 & 0 & 10 & 1 & 0 & 1 & 1 & 64 \end{array} \right] \\
 \begin{array}{cccccc|c} 0 & 1 & 2 & 1 & 0 & -1 & 0 & 16 \\ 0 & -1 & 8 & 0 & 0 & 2 & 1 & 48 \\ \hline 0 & 0 & 10 & 1 & 0 & 1 & 1 & 64 \end{array}
 \end{array}$$

Setting the nonbasic valuables equal to zero and solving the corresponding system yields

$$\begin{cases} x_2 + 2x_3 + x_4 - x_6 = 16 \\ -4x_3 + x_4 + x_5 - 3x_6 = 28 \\ x_1 + 5x_3 + x_4 - 1/2 x_6 = 28 \end{cases}$$

$$x_3 = 0, x_4 = 0, x_6 = 0$$

Gives $x_2 = 16$, $x_5 = 28$, and $x_1 = 28$. The maximum value of z is 64