5. (1) Let $K > \max(T, \sum_{i}^{n} \text{abs}(a_i))$

($\Rightarrow$)

it's trivial the if there $\exists S_1 \subseteq S$

such that $\text{sum}(S_1) = T$

Let $S_2 = \{a + k \mid a \in S_1\}$, $\text{sum}(S_2) = T + |S_2| \cdot K$

and we can get $n - |S_2|$ Ks

to reach $T + n \cdot K$

($\Leftarrow$)

since k is larger than T and sum

of the S if there exist $S_a \subseteq S$, $S_K \subseteq (k, \dots, k)$

① $|S_a| + |S_K| \geq n$ because it impossible

to get k from the a_i s.

② $|S_a| + |S_K| \leq n$ because $T < K$, so

$T + nk < (n+1)k < |S_a| + |S_K|$

if $|S_a| + |S_K| > n$

$\Rightarrow |S_a| + |S_K| = n$ so it's clear the

$S_1 = \{a - k \mid a \in S_a\}$ has sum T #

$(T + nk - (|S_a| + |S_K|) k)$

(3) DDBP:

Given a collection of n balls B with weights a1, · · · , an kilograms respectively, with constraint that ai ∈ [0,1], i.e., a single ball's weight is at most 1 kilogram and at least 0 kilogram. The Dragon Ball Problem find if it's possible to have a partition balls that number of the bin is less than K and all the bins weigh at most 1 kilogram

$QQP \leq_p DDBP$:

$S = \{a_1 .. a_n)$ $\quad V = \sum_{i=0}^{n} a_i$

Let $B = \{ \frac{2b}{V} \mid b \in S \}$ $\quad K = 2$

($\Rightarrow$) if exist $S_1 \subseteq S$ s.t. $sum(S_1) = \frac{V}{2}$

Let $S_2 = S - S_1$ and $sum(S_2) = V - \frac{V}{2} = \frac{V}{2}$

we can get $B_1 = \{ b_i \mid a_i \in S_1 \}$

$B_2 = \{ b_i \mid a_i \in S_2 \}$

and $Sum(B_1) = Sum(B_2) = \frac{V}{2} \times \frac{2}{V} = 1$

we have a partition with 2 bins

($\Leftarrow$) if we have a partition with 2 bins

since $sum(B) = \frac{2}{V} \times \left(\sum_{i=0}^{n} a_i \right) = \frac{2}{V} \times V = 2$

we must partition into 2 bins with 1kg

$\Rightarrow$ exist $B_1 \subseteq B$ $sum(B_1) = 1$

$\Rightarrow$ exist $S_1 = \{ a_i \mid b_i \in B_1 \}$ s.t. $sum(S_1) = Sum(B_1) \times \frac{V}{2} = \frac{V}{2}$

#

(4) we can check QQP by sum the output and check if it $= \frac{U}{2}$

check DDBP by checking each bin have at most 1 kg and has less than k bin

DBP and QQP are both NP

by (1) (2) (3)

$JJBAP \leq_p JJBAP_t \leq_p QQP$

$\leq_p DDBP$

Since JJBAP is NPC

QQP and DDBP are NP-Hard and NP

$\Rightarrow$ NPC #

(5) by (2) is we have a

poly $\left(\frac{3}{2}-\epsilon\right)$- approximation algo

since $\lfloor 2 \times (\frac{3}{2}-\epsilon) \rfloor = \lfloor 3-2\epsilon \rfloor = 2$

we can still solve QQP

in poly. $(\rightarrow\leftarrow)$

(6)

$S = \{a_1 \dots a_n\}$ be the set or m uniqe possible value

$$1 + \sum_{i=1}^{\lfloor \frac{1}{c} \rfloor} \frac{m^i}{i} \leq \sum_{i=0}^{\lfloor \frac{1}{c} \rfloor} m^i = \frac{1 - m^{\lfloor \frac{1}{c} \rfloor}}{1-m}$$

(7) Let x_i be the nuber of ith set appear in k bin for $i \in 1 \sim N$

$$x_1 + \dots + x_M = k$$

the way have $H^M_k = C^{M+k-1}_{k} = C^{M+k-1}_{M-1}$

$$= \frac{(M+k-1) \times \dots (k)}{(M-1) \times \dots 1} \leq \frac{(M+k)^M}{M!} \leq \frac{(M \cdot k)^M}{M!} = \frac{M^M}{M!} \cdot k^M$$

Let $C_M = \frac{M^M}{M!}$

(8) the algorithm is the Let $k = 1 \sim n$ (atmost n bin needed)

find $1^M + 2^M \dots + n^M$ time needed

Let $c = 1$, $n_0 = 1$, $\forall n \geq n_0$

$$1 \cdot (1^M + 2^M + \dots n^M) \leq n \cdot n^M = n^{M+1}$$

#

7. (1) every degree are even

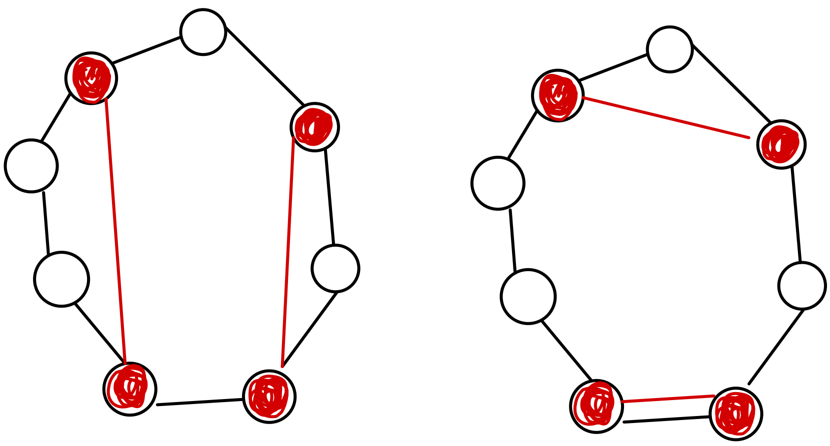
(2) since one Edge provide 2 unique degrees

sum of the degree must be $2|E|$ is even

if $|V'|$ is odd, sum of the degree

$= \underbrace{\sum_{v_i \in V'} \deg(v_i)}_{\text{odd}} + \underbrace{\sum_{v_i \notin V'} \deg(v_i)}_{\text{even}} \Rightarrow \text{odd} \; (\Longleftrightarrow)$

(3) Let the OPT output be the cycle below



We have two way to match by the same order through the cycle

and $cost(M_1) \leq cost(\text{black edge outside})$ by the triangle inequality

$cost(M_2)$

$cost(M_1) + cost(M_2) \leq OPT$

for min match M

$cost(M) \leq \min(cost(M_1) + cost(M_2)) \leq \frac{OPT}{2}$

(4)

find Min Spanning Tree T (poly)

find out the odd degree vertice (poly)

find min match M (Oracle poly)

get $T \cup M$

run the Eular cycle and cut the overlap vertice (poly)

$$cost(T) \leq OPT \text{ and } cost(M) \leq \frac{OPT}{2}$$

$$cost(T \cup M) \leq \frac{3OPT}{2}$$

Last step only cut some edge (by triangle inequality)

it only decrease