# COMP 362 - Winter 2017 - Assignment 4

Due: 6pm Mar 23th

**General rules:** In solving these questions you may consult your book; you can discuss high level ideas with each other. But each student must find and write his/her own solution. You should drop your solutions in the assignment drop-off box located in the Trottier Building on the 2nd floor.

1. (10 Points) Show that the following problem is in PSPACE:

   - Input: An undirected graph $G$ and a positive integer $m$.
   - Question: Is the number of proper $m$-vertex colorings of $G$ divisible by $(m+1)$?

2. (15 Points) Problem 10 of Chapter 11: Suppose you are given an $n \times n$ grid graph $G$. Associated with each node $v$ is an integer weight $w(v) \geq 0$. You may assume that all the weights are distinct. Your goal is to choose an independent set $S$ of nodes of the grid, so that the sum of the weights of the nodes in $S$ is as large as possible. (The sum of the weights of the nodes in $S$ will be called its total weight.) Consider the following greedy algorithm for this problem.

   - Start with $S := \emptyset$.
   - While some node remains in $G$:
     - Pick a node $v$ of maximum weight.
     - Add $v$ to $S$.
     - Delete $v$ and its neighbors from $G$
   - Endwhile.

   Show that this algorithm returns an independent set of total weight at least $\frac{1}{4}$ times the maximum total weight of any independent set in the grid graph $G$.

3. (15 Points) Given a set $P$ of $n$ points on the plane, consider the problem of finding the smallest circle containing all the points in $P$. Show that the following is a 2-factor approximation algorithm for this problem. Pick a point $x$ in $P$, and set $r$ to be the distance of the farthest point in $P$ from $x$. Output the circle centered at $x$ with radius $r$.

4. Consider a directed bipartite graph $G = (V, E)$. We want to eliminate all the directed cycles of length 4 by removing a smallest possible set of vertices.

   (a) (5 points) Let $\mathcal{C}_4$ denote the set of all cycles of length 4 in the graph. Show that the following integer program models the problem:

$$\begin{array}{lll} \min & \sum_{v\in V} x_v & \\ \text{s.t.} & \sum_{u\in C} x_u \geq 1 & \forall C \in \mathcal{C}_4 \\ & x_u \in \{0,1\} & u \in V \end{array}$$

   (b) (5 points) Why does the optimal solution to the following relaxation provides a lower bound for the optimal answer to the above integer linear program? In other words why it is not necessary to have the constraints $x_u \leq 1$ in the relaxation?

$$\begin{array}{lll} \min & \sum_{v\in V} x_v & \\ \text{s.t.} & \sum_{u\in C} x_u \geq 1 & \forall C \in \mathcal{C}_4 \\ & x_u \geq 0 & \forall u \in V \end{array}$$

   (c) (15 points) Give a simple 4-factor approximation algorithm for the problem based on rounding the solution to the above linear program.

   (d) (15 points) Let $L$ and $R$ denote the set of the vertices in the two parts of the bipartite graph. (Every edge has one endpoint in $L$ and one endpoint in $R$). Let $x^*$ denote an optimal solution to the linear program in Part (b). We round $x^*$ in the following way:

   For every $u \in V$,

   - if $u \in R$ and $x^*_u \geq 1/2$, set $\widehat{x}_u = 1$.
   - if $u \in L$ and $x^*_u > 0$, set $\widehat{x}_u = 1$.
   - Otherwise set $\widehat{x}_u = 0$.

   Show that $\widehat{x}$ is a feasible solution to the integer linear program.

(e) (10 points) Consider the dual of the relaxation:

$$\begin{array}{lll} \max & \sum_{C\in\mathcal{C}_4} y_C & \\ \text{s.t.} & \sum_{C\in\mathcal{C}_4, u\in C} y_C \leq 1 & \forall u \in V \\ & y_C \geq 0 & \forall C \in \mathcal{C}_4 \end{array}$$

and let $y^*$ be an optimal solution to the dual. Use the complementary slackness to prove the following statement: For every $C \in \mathcal{C}_4$ either we have $|\{u : \widehat{x}_u = 1\}| \leq 3$ or $y^*_C = 0$.

(f) (10 points) Use the complementary slackness and the previous parts to show that our rounding algorithm is a 3-factor approximation algorithm.