COMP 360 - Winter 2014 - Assignment 4

Due: 6pm March 24th.

General rules: In solving these questions can use that SAT, 3SAT, Max Independent Set, Max Clique, Min Vertex Cover, 3-Colourability, Subset Sum are NP-complete.

There are in total 110 points, but your grade will be considered out of 100. You should drop your solutions in the assignment drop-off box located in the Trottier Building.

1. (15 points) Use the complementary slackness to show that $x_1^* = x_3^* = 0.5$, $x_2^* = x_4^* = 0$, $x_5^* = 2$ is an optimal solution for the following Linear Program:

$$\begin{array}{ll}
\max & 3.1x_1 + 10x_2 + 8x_3 - 45.2x_4 + 18x_5 \\
\text{s.t.} & x_1 + x_2 + x_3 - x_4 + 2x_5 \leq 5 \\
& 2x_1 - 4x_2 + 1.2x_3 + 2x_4 + 7x_5 \leq 16 \\
& x_1 + x_2 - 3x_3 - x_4 - 10x_5 \leq -20 \\
& 3x_1 + x_2 + 3x_3 + \frac{3}{2}x_4 + \frac{7}{3}x_5 \leq 10 \\
& x_2 + x_3 + 6x_4 + 2x_5 \leq 4.5 \\
& 2x_2 - x_4 + x_5 \leq 2 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}$$

Solution: The dual is

$$\begin{array}{ll}
\min & 5y_1 + 16y_2 - 20y_3 + 10y_4 + 4.5y_5 + 2y_6 \\
\text{s.t.} & y_1 + 2y_2 + y_3 + 3y_4 \geq 3.1 \\
& y_1 - 4y_2 + y_3 + y_4 + y_5 + 2y_6 \geq 10 \\
& y_1 + 1.2y_2 - 3y_3 + 3y_4 + y_5 \geq 8 \\
& -y_1 + 2y_2 - y_3 + \frac{3}{2}y_4 + 6y_5 - y_6 \geq -45.2 \\
& 2y_1 + 7y_2 - 10y_3 + \frac{7}{3}y_4 + 2y_5 + y_6 \geq 18 \\
& y_1, y_2, y_3, y_4, y_5, y_6 \geq 0
\end{array}$$

Let $y_1^*, \ldots, y_6^*$ be an optimal solution to the dual. If $x_1^* = x_3^* = 0.5$, $x_2^* = x_4^* = 0$, $x_5^* = 2$ is an optimal solution to the primal, then since for these values, there is slack in the 2nd, 3rd, and 4th constraints of the primal, we must have $y_2^* = y_3^* = y_4^* = 0$. Furthermore since x_1^*, x_3^*, x_5^* are nonzero, we must have no slackness in the corresponding constraints of the dual. Thus we must have

$$\begin{array}{cl} y_1^* & = 3.1 \\ y_1^* + y_5^* + 2y_6^* & \geq 10 \\ y_1^* + y_5 & = 8 \\ -y_1 + 6y_5 - y_6 & \geq -45.2 \\ 2y_1 + 2y_5 + y_6 & = 18 \\ y_1, y_2, y_3, y_4, y_5, y_6 & \geq 0 \end{array}$$

Now this is easy to solve. We must have $y_1^* = 3.1$, $y_5^* = 4.9$, $y_6^* = 2$. Then we check that $y_1^* = 3.1$, $y_2^* = y^*3 = y_4^* = 0$, $y_5^* = 4.9$, $y_6^* = 2$ is a feasible solution to the dual, and has the same cost (41.55) as the optimal solution to the primal, and then the weak duality implies that the 41.55 is the optimal solution.

2. (20 points) Either prove that the following problem is NP-complete, or show that it belongs to P.

   - Input: A number k, and a formula ϕ in conjunctive normal form.
   - Output: Is there a truth assignment that satisfies ϕ and assigns False to exactly k variables?

   What happens if in the above problem we replace k with the fixed number 100?

   Solution: This is in NP as given a truth assignment as a certificate we can easily verify that it assigns False to k variables, and that it satisfies ϕ. To prove the completeness, we reduce SAT to this problem. Given an input ϕ (with n variable) to SAT, we can use an oracle to the above problem to decide whether ϕ is satisfiable efficiently using the following oracle algorithm:

   - For $k = 0, \ldots, n$ do
   - If there is a truth assignment that satisfies ϕ and assigns False to exactly k variables (ask oracle) then output YES and terminate;
   - EndFor;
   - Output NO;

If we replace k with the fixed number 100 then the problem belongs to P as there are only $\binom{n}{100} \leq n^{100}$ truth assignments that need to be verified, and since n^{100} is a polynomial, this can be done efficiently.

3. (15 points) Recall the project selection problem: There is a set of projects P, and every $i \in P$ has a revenue p_i which can be negative or positive, and certain projects are prerequisites for others. We learned how to use circulation with demands to find a subset $A \subseteq P$ such that the prerequisites of every project in A also belong to A, and its profit $\sum_{i \in A} p_i$ is maximized. Either prove that the following variant of this problem is NP-complete, or show that it belongs to P.

   - Input: A positive integer k, and as in the original problem a set of projects P, with their revenues and lists of prerequisites.
   - Output: Is there a subset $A \subseteq P$ that respects the prerequisite condition and has profit *exactly* k:
   $$\sum_{i \in A} p_i = k.$$

   Solution: We show that this problem is NP-complete. It is in NP as given a set of projects as a certificate we can easily verify that it respects the prerequisites, and that its profit is exactly k. To prove the completeness, we reduce SubsetSum to this problem. Given an input $\langle \{w_1, \ldots, w_n\}, m \rangle$ to SubsetSum, we create an instance of the above problem in the following way.

   - P consists of n projects, and the i-th project has revenue w_i.
   - The graph of prerequisites is empty. That is, no project has any prerequisites.
   - set $k = m$.

   Now note that there is a set A with profit $k = m$, if and only if the answer to the SubsetSum problem is YES. Hence we can use the oracle fo the SubsetSum problem to solve this.

4. (20 points) Either prove that the following problem is NP-complete, or show that it belongs to P.

   - Input: A positive integer k, and $(G, \{c_e\}_{e \in E}, \{b_v\}_{v \in V})$ where $G = (V, E)$ is a graph and to every edge e a positive cost $c_e > 0$ is assigned, and to every vertex v a positive benefit $b_v > 0$ is assigned.

- Output: Is there a subset S of the vertices such that the total benefit of the vertices in S minus the total cost of the edges in S is at least k? That is

$$\sum_{v \in S} b_v - \sum_{\substack{e=uv \in E \\ u,v \in S}} c_e \geq k.$$

Solution: We show that this problem is NP-complete. It is in NP as given a certificate S we can easily and efficiently verify that it satisfies the conditions. To prove the completeness, we reduce the Independent Set problem to this problem. Given an input $\langle G, m \rangle$ to the Independent Set problem (that is we want to know whether G has an Independent Set of size m), we construct an input to the above problem in the following manner.

- Consider the same graph $G = (V, E)$.
- For every $e \in E$, set $c_e = n$, where n is the number of vertices of G.
- For every $v \in V$, set $b_v = 1$.
- Set $k = m$.

Now note that if S is *not* an independent set then

$$\sum_{v \in S} b_v - \sum_{\substack{e=uv \in E \\ u,v \in S}} c_e \leq 0,$$

and otherwise it is equal to the size of S. Hence G has an independent set of size m if and only if there is a set S such that

$$\sum_{v \in S} b_v - \sum_{\substack{e=uv \in E \\ u,v \in S}} c_e \geq k = m.$$

Hence we can use an oracle for this problem to solve the independent set problem efficiently.

5. (20 points) Show that if in linear programming we allow constraints of the form $\sum_{i=1}^{n} a_i |x_i| \geq b$ for integers b and a_i, then the problem becomes NP-complete.

Solution: Note that $|x_i| \geq 1$ and $-|x_i| \leq 1$ together imply that $x_i \in \{-1, 1\}$. Then $\frac{x_i+1}{2} \in \{0, 1\}$ and we can use these to solve NP-complete problems. For example we can reduce vertex cover to this problem. For an input $\langle G = (V, E), k \rangle$ to vertex cover, we can write

$$\begin{array}{lll} \min & \sum_{u \in V} y_u & \\ \text{s.t.} & y_u + y_v \geq 1 & \forall uv \in E \\ & y_u = \frac{x_u+1}{2} & \forall u \in V \\ & |x_u| \geq 1 & \forall u \in V \\ & -|x_u| \geq 1 & \forall u \in V \end{array}$$

Hence G has a vertex cover of size at most k if and only if the solution to this program is at most k.

6. (20 points) Show that the following problem is NP-complete:

- Input: A formula ϕ in conjunctive normal form such that each clause in ϕ either involves only positive literals (i.e., variables), or it involves only negative literals (i.e., negated variables).
- Output: Is there a truth assignment that satisfies ϕ?

Solution: This is trivially in NP as it is a special case of SAT. To prove the NP-completeness we reduce SAT (in its general form) to this problem. Consider an input ψ to SAT, and let $C_j = (\vee_{i \in C_j^+} x_i \vee \vee_{i \in C_j^-} \bar{x}_i)$ be a clause in ψ with both positive and negative literals. Now to construct ϕ, we replace every such C_j with $(\vee_{i \in C_j^+} x_i \vee \vee_{i \in C_j^-} y_i)$. Here y_i are new variables. Finally for every i, we add the clauses $(x_i \vee y_i)$ and $(\bar{x}_i \vee \bar{y}_i)$ to ϕ. Note that that each clause in ϕ either involves only positive literals, or it involves only negative literals. Moreover the clauses $(x_i \vee y_i)$ and $(\bar{x}_i \vee \bar{y}_i)$ are satisfied only if $y_i = \bar{x}_i$. Hence ϕ is satisfiable if and only if ψ is satisfiable.