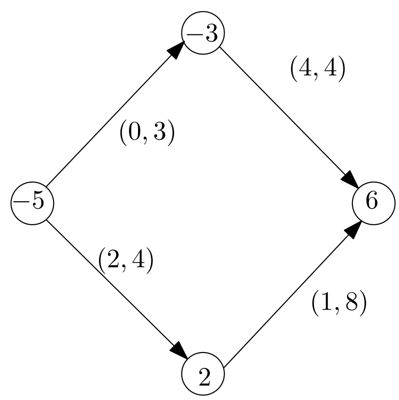
COMP 360 - Winter 2014 - Assignment 2

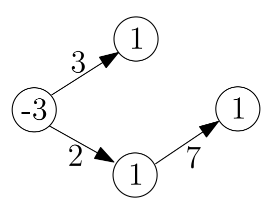
Due: 6pm Feb 17th.

General rules: In solving these questions you may consult books but you may not consult with each other. There are in total 115 points, but your grade will be considered out of 100. You should drop your solutions in the assignment drop-off box located in the Trottier Building.

1. (20 Points) Consider the following circulation with demands and lower-bounds problem. The demands are written on the vertices, and the lower-bounds and capacities are written as pairs next to the edges.

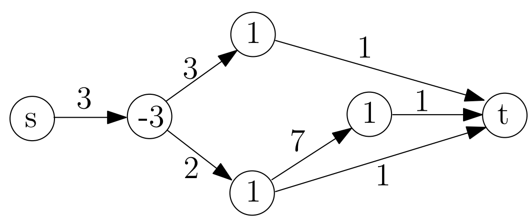


   (a) Using the approach discussed in the lecture, convert this problem to a circulation with demands problem (no lower-bounds).
   Solution:



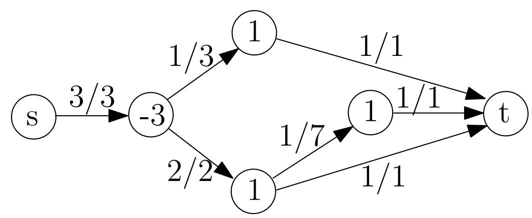
(b) Convert the circulation with demands that you obtained in the previous part to a max-flow problem using the approach discussed in the lecture.

Solution:

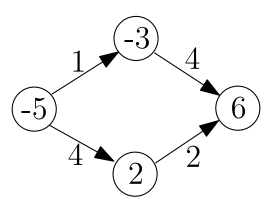


(c) Give a maximum flow for this problem, and convert it back to a solution to the original circulation with demands and lower-bounds problem.

Solution: flow is 3.

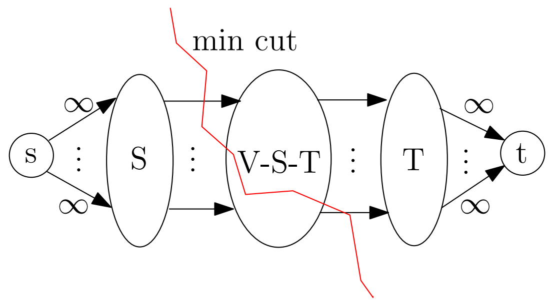


After converting back, we have:



2. (15 Points) Consider a flow network $(G, s, t, \{c_e\})$ and two disjoint subsets of the internal vertices $S, T \subseteq V - \{s, t\}$. Design an efficient algorithm that finds the smallest cut (A, B) with the extra condition that $S \subseteq A$ and $T \subseteq B$. That is all the vertices in S are in A and all the vertices in T are in B.

Solution: add edges (s,v) $\forall v \in S$ and (v,t) $\forall v \in T$ with capacities of ∞. The minimum cut of the new graph will be the desired smallest cut. The reason behind this is that edges with capacity of ∞ will never be a bottleneck and thus the new minimum cut will completely separate S and T.



3. (20 Points) Consider a variant of the max-flow problem where to every internal vertex $v \in V - \{s, t\}$ a cost $\mathrm{ct(v)} \geq 0$ is assigned. The cost of a flow f at a vertex v is $\mathrm{ct(v)} \times f^{\mathrm{out}}(v)$, and the total cost of f is the total sum of these costs for all the internal vertices. Given a number $K \geq 0$ we want to find the minimum cost flow with value at least K. Formulate this problem as a linear program.

   Solution:

$$\min \quad \sum_{u \in V - \{s,t\}} (ct(u). \sum_{(u,v) \in E} f(u,v))$$

$$\text{s.t.} \quad 0 \leq f(u,v) \quad \forall (u,v) \in E$$

$$f(u,v) \leq capacity(u,v) \quad \forall (u,v) \in E$$

$$\sum_{(u,v) \in E} f(u,v) - \sum_{(v,w) \in E} f(v,w) = 0 \quad \forall v \in V - \{s,t\}$$

$$\sum_{(s,u) \in E} f(s,u)) \geq K$$

4. (15 points) A factory produces two products (A, B) using two machines (M, N). Product A requires processes on both machines, but Product B can be produced on either machine M or N. The following table shows the processing times (in minutes) on each machine

Product	Machine N	Machine M
A	15	18
B	20	25

Each machine works for 8 hours every day. Due to marketing limitations, the number of Product A sold must be at least the number of Product B sold. When sold, each unit of Product A and B contributes to profit \$16 and \$20, respectively. Write a linear program to maximize the daily profit. What is the optimal solution to this linear program?

Solution:
x : number of A products
y : number of B products produced with N
z : number of B products produced with M

$$\begin{aligned} \max \quad & 16x + 20y + 20z \\ s.t. \quad & 15x + 20y \leq 480 \\ & 18x + 25z \leq 480 \\ & x - y - z \geq 0 \\ & x, y, z \geq 0 \end{aligned}$$

The optimal point for integer values are x=17, y=11, z=6 so the profit will be 612.

5. (10 Points) Write the dual of the following linear program without converting it to canonical form:

$$\begin{aligned} \max \quad & x_1 + x_2 + 4x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \leq 8 \\ & x_1 + 2x_2 = 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution:

$$\begin{aligned} \min \quad & 8y_1 + 4y_2 \\ \text{s.t.} \quad & y_1 + y_2 \geq 1 \\ & y_1 + 2y_2 \geq 1 \\ & y_1 \geq 4 \\ & y_1 \geq 0 \\ & y_2 \ is\ free \end{aligned}$$

6. (a) (10 Points) Express the following problem as a linear program: Given numbers $a_1, \ldots, a_n$, find numbers $0 \le x_1 \le \ldots \le x_n \le 1$ such that $a_1 x_1 + \ldots + a_n x_n$ is maximized.
   Solution:

$$
\begin{array}{ll}
\max & a_1 x_1 + \cdots + a_n x_n \\
\text{s.t.} & x_n \le 1 \\
 & x_{n-1} - x_n \le 0 \\
 & \vdots \\
 & x_1 - x_2 \le 0 \\
 & x_1 \ge 0
\end{array}
$$

   (b) (10 Points) Write the dual of your linear program of the previous part.

   Solution:

$$
\begin{array}{ll}
\min & y_n \\
\text{s.t.} & y_1 \ge a_1 \\
 & y_2 - y_1 \ge a_2 \\
 & \vdots \\
 & y_n - y_{n-1} \ge a_n \\
 & y_i \ge 0 \quad 1 \le i \le n
\end{array}
$$

7. (Bonus: 15 Points) Write a linear program for solving the following problem: Given an $n \times n$ matrix A and an n-dimensional vector b, we want to find a vector

$$
x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}
$$

   such that $Ax = b$ and that our solution is as "flat" as possible. More precisely we want to <u>minimize</u> the largest difference $x_i - x_j$:

$$
\max_{i=1}^{n} \max_{j=1}^{n} x_i - x_j .
$$

   Solution:
   We solve this by introducing a new variable y

$$
\begin{array}{ll}
\min & y \\
\text{s.t.} & Ax = b \\
 & x_i - x_j \le y \quad \forall i, j
\end{array}
$$