COMP 362

1. (5 Points) Show that the following language is NP-complete:

   - Input: A CNF ϕ.
   - Question: Is G satisfiable with at least 2 different assignments?

2. (10 Points) A path of length k contains $k+1$ vertices. Show that the following problem is NP-complete:

   - Input: A graph G and a number k.
   - Question: Is is possible to remove k vertices from G to eliminate all paths of length 2?

3. (15 points) The complete bipartite graph $K_{m,n}$ is the bipartite graph with parts of sizes m and n, respectively, such that every vertex from the first part is adjacent to every vertex in the second part.

   Either prove that the following problem is NP-complete or show that it belongs to P by giving a polynomial time algorithm:

   - Input: A bipartite graph G, and a positive integer m.
   - Question: Does G contain a copy of $K_{m,m}$ as a subgraph?

4. (10 Points) Let $G = (V, E)$ be a graph. Recall that $S \subseteq V$ is a vertex cover if and only if $V - S$ is an independent set. Also recall that the following is a 2-factor approximation algorithm for vertex cover: Pick any maximal matching M in G and let S be the set of all vertices involved in M. Output S.

   Is it true that the following is a 2-factor approximation algorithm for the maximum independent set problem? Pick any maximal matching M in G and let S be the set of all vertices involved in M. Output $V - S$.

5. (10 Points) Let G be a 4-regular graph on n vertices (4-regular means that every vertex is adjacent to 4 edges). We want to color the *edges* of G with two colors Red and Blue such that the number of vertices that are adjacent to exactly two Red and two Blue edges is maximized. If we color the edges at random, then what is the expected number of vertices that satisfy the above condition?

6. (10 Points) Consider the following optimization problem: Given a graph G on $\underline{2n}$ vertices we want to eliminate the maximum possible number of edges from G by deleting exactly n vertices. Show that the following is a $\frac{1}{2}$-factor approximation algorithm: Let $v_1, \ldots, v_{2n}$ be all the vertices. Try deleting each one of the two sets $\{v_1, \ldots, v_n\}$ and $\{v_{n+1}, \ldots, v_{2n}\}$ separately and output the one that removes more edges.

7. Consider a graph $G = (V, E)$. The chromatic number of G is the minimum number of colors required to color the vertices of G properly. Let $\mathcal{I}$ be the set of all independent sets in G (Note that every element in $\mathcal{I}$ is a set).

   (a) (10 Points) Prove that the solution to the following linear program provides a lower-bound for the chromatic number of G.

$$
\begin{array}{lll}
\min & \sum_{I\in\mathcal{I}} x_I & \\
\text{s.t.} & \sum_{I:v\in I} x_I \geq 1 & \forall v \in V \\
& x_I \geq 0 & \forall I \in \mathcal{I}
\end{array}
$$

(b) (10 Points) Write the dual of the above linear program.

(c) (5 Points) Prove that every clique in G provides a solution to the dual linear program.

8. (15 Points) Consider the SDP relaxation for chromatic number:

$$
\begin{array}{lll}
\min & t & \\
\text{s.t.} & \langle v_i, v_j\rangle \leq t & \forall\ ij \in E \\
& \langle v_i, v_i\rangle = 1 &
\end{array}
$$

What is the relation of the optimial solution to this SDP and the chromatic number? Show that for the cycle of length 5, the optimal solution is at most $\frac{1}{1-\sqrt{5}}$.