COMP 362 - Winter 2015 - Assignment 2

Due: 6pm Feb 17th.

General rules: In solving these questions you may consult books but you may not consult with each other. You should drop your solutions in the assignment drop-off box located in the Trottier Building.

1. (10 Points) Formulate the following problem as a linear program. We want to assign non-negative numbers to the edges of a graph such that they add up to 1, and furthermore the maximum load on a vertex is minimized. Here the load of a vertex is the sum of the numbers on the edges incident to that vertex.

   Solution:

$$\begin{array}{lll} \min & z & \\ \text{s.t.} & z - \sum_{v:uv\in E} x_{uv} & \geq 0 \quad \forall u \in V \\ & \sum_{uv\in E} x_{uv} & = 1 \\ & x_{uv} & \geq 0 \quad uv \in E \end{array}$$

2. (15 Points) Write a linear program for solving the following problem: Given an $n \times n$ matrix A and an n-dimensional vector b, we want to find a vector

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

   such that $Ax = b$ and that our solution is as "flat" as possible. More precisely we want to <u>minimize</u> the largest difference $x_i - x_j$:

$$\max_{i=1}^{n} \max_{j=1}^{n} x_i - x_j.$$

   Solution:

$$\begin{array}{lll} \min & z & \\ \text{s.t.} & z - x_i + x_j & \leq 0 \quad \forall i, j \\ & Ax & = b \end{array}$$

3. (a) (10 Points) Prove that if $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$ are feasible solutions to a linear program, then so is every point on the line segment between these two points. (Note that the points on the line segment are $(\alpha x_1 + (1-\alpha)x'_1, \ldots, \alpha x_n + (1-\alpha)x'_n)$ where $0 \leq \alpha \leq 1$. Also for simplicity you may assume that all the constraints are of the form $a_1x_1 + \ldots + a_nx_n \leq b$ where $a_1, \ldots, a_n, b \in \mathbb{R}$.)

   Solution: Obviously

$$\begin{aligned}
&a_1(\alpha x_1 + (1-\alpha)x'_1) + \ldots + a_n(\alpha x_1 + (1-\alpha)x'_1) \\
&= \alpha(a_1x_1 + \ldots + a_nx_n) + (1-\alpha)(a_1x_1 + \ldots + a_nx_n) \\
&\leq \alpha b + (1-\alpha)b \leq b.
\end{aligned}$$

   (b) (5 Points) Is there a linear program whose feasible region consists of the sides of a triangle?

   Solution: No that's not a convex set.

   (c) (5 Points) Is there a linear program whose feasible region consists of the surface of a 3-dimensional cube?

   Solution: No that's not a convex set.

   (d) (5 Points) Is there a linear program with 3 variables whose feasible region is a 2-dimensional plane?

   Solution: Yes, for example if we have three variables x_1, x_2, x_3 and the only constraint is $x_3 = 0$.

4. (10 Points) Write the dual of the following linear program without converting it to canonical form:

$$\begin{array}{ll}
\max & x_1 + x_2 + 4x_3 \\
\text{s.t.} & x_1 + x_2 + x_3 \leq 8 \\
& x_1 + 2x_2 = 4 \\
& x_1, x_2 \geq 0
\end{array}$$

Solution:

$$\begin{array}{ll}
\min & 8y_1 + 4y_2 \\
\text{s.t.} & y_1 + y_2 \geq 1 \\
& y_1 + 2y_2 \geq 1 \\
& y_1 = 4 \\
& y_1 \geq 0
\end{array}$$

5. (20 points) Let $G = (V, E, \{c_e\})$ be an undirected graph where every edge e has a cost c_e. Write the duals of the following Linear Programs:

   (a)

$$\begin{array}{lll} \min & \sum_{uv \in E} c_{uv} x_{uv} & \\ \text{s.t.} & \sum_{uv \in C} x_{uv} & \geq 1 \quad \text{for every cycle } C \text{ in } G \\ & x_{uv} & \geq 0 \quad uv \in E \end{array}$$

   Solution: variables are y_C for every cycle C in G.

$$\begin{array}{lll} \max & \sum_C y_C & \\ \text{s.t.} & \sum_{C \ni e} Y_c & \leq c_e \quad \text{for every edge } e \text{ in } G \\ & y_C & \geq 0 \quad \text{for every cycle } C \text{ in } G \end{array}$$

   (b)

$$\begin{array}{llll} \min & \sum_{uv \in E} c_{uv}(x_u + x_v) & & \\ \text{s.t.} & \sum_{u \in V} x_u & \leq 100 & \\ & x_u & \leq 1 & u \in V \\ & x_u & \geq 0 & u \in V \end{array}$$

   Solution: variables are z, and y_u for every vertex $u \in V$.

$$\begin{array}{llll} \max & 100z + \sum_{u \in V} y_u & & \\ \text{s.t.} & z + y_u \geq \sum_{uv \in E} c_{uv} & \forall u \in V & \\ & y_u & \geq 0 & \forall u \in V \\ & z & \geq 0 & \end{array}$$

6. (20 Points) Formulate the following problem as a linear program: Given $a_1, \ldots, a_n \geq 0$, and an $n \times n$ matrix A, we want to find $x_1, \ldots, x_n \geq 0$ (and not all of them zero) such that

$$\frac{a_1 x_1 + a_2 x_2 + \ldots + a_n x_n}{x_1 + x_2 + \ldots + x_n}$$

   is maximized and

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \leq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

   Solution: Note that if $(x_1, \ldots, x_n)$ is an optimal solution, then so is $(x_1/r, \ldots, x_n/r)$ where $r = x_1 + \ldots + x_n$. Hence we can use the following linear program.

$$
\begin{array}{ll}
\max & a_1x_1 + a_2x_2 + \ldots + a_nx_n \\
\text{s.t.} & x_1 + x_2 + \ldots + x_n = 1 \\
& Ax \leq \vec{0}
\end{array}
$$