COMP 360 - Winter 2014 - Assignment 5

Due: 6pm April 11th.

General rules: In solving these questions you may consult books but you may not consult with each other. There are in total 115 points, but your grade will be considered out of 100. You should drop your solutions in the assignment drop-off box located in the Trottier Building.

1. (10 Points) Given a set P of n points on the plane, consider the problem of finding the smallest circle containing all the points in P. Show that the following is a 2-factor approximation algorithm for this problem. Pick a point x in P, and set r to be the distance of the farthest point in P from x. Output the circle centered at x with radius r.

   Solution: First we show that all the points will be in the circle outputted by algorithm: If y is the point farthest from x, then r=distance(x,y). If there's a point p outside the circle, then $distance(p, x) > r = distance(x, y)$ which can't be because y is the farthest point from x.

   Now we show that this is a 2-factor approximation: Suppose x' and r' are the center and radius of the optimal solution. Since x and y are inside the circle we have $distance(x, c) \geq r'$ and $distance(y, c) \geq r'$ so $distance(x, c) + distance(y, c) \geq 2r'$. From the triangle inequality we have $distance(x, c) + distance(y, c) \geq distance(x, y) = r$ from combining these two we have $2r' \geq r$

2. (15 Points) Problem 10 of Chapter 11: Suppose you are given an $n \times n$ grid graph G. Associated with each node v is an integer weight $w(v) \geq 0$. You may assume that all the weights are distinct. Your goal is to choose an independent set S of nodes of the grid, so that the sum of the weights of the nodes in S is as large as possible. (The sum of the weights of the nodes in S will be called its total weight.) Consider the following greedy algorithm for this problem.

   - Start with $S := \emptyset$.
   - While some node remains in G:

- Pick a node v of maximum weight.
- Add v to S.
- Delete v and its neighbors from G

• Endwhile.

Show that this algorithm returns an independent set of total weight at least $\frac{1}{4}$ times the maximum total weight of any independent set in the grid graph G.

Solution: Since for every node v picked we remove the neighbors, the algorithm will not output any connected nodes thus the algorithm gives an independent set. v has maximum weight so $weight(v) \geq weight(neighbor_i)$ for $i = 1..4$. So, $4weight(v) \geq \sum_{i=1}^{4} weight(neighbor_i)$. If the optimal algorithm didn't choose v and chose a subset(or all four) of the neighbors instead, then it could be at most 4 times better.

3. (15 Points) Consider a k-CNF (each clause contains k terms) with $m < 2^{k-1}$ clauses. Suppose further that the variables involved in any given clause are distinct. Show that the probability that the following randomized algorithm does <u>not</u> succeed in satisfying ϕ is at most 2^{-1000}.

   • For $i = 1, \ldots, 1000$:

   • Pick a random truth assignment σ.

   • If σ satisfies the CNF, then output σ and terminate.

   Solution: The probability of each of the terms in a clause evaluating to false is $\frac{1}{2}$. For a clause to be false, all the terms should be false. Since all the variables of a clause is distinct, the probability of a clause with k variables evaluating to false is $\frac{1}{2}^k$. For ϕ to be false, one of the m clauses should be false. Thus by the union bound the probability is bounded from above by $m\frac{1}{2^k} = \frac{2^{k-1}}{2^k} = \frac{1}{2}$. Therefore, the probability for none of the iterations to satisfy ϕ is at most $\frac{1}{2}^{1000}$.

4. Consider a directed bipartite graph $G = (V, E)$. We want to eliminate all the directed cycles of length 4 by removing a smallest possible set of <u>vertices</u>.

   (a) (5 points) Let $\mathcal{C}_4$ denote the set of all cycles of length 4 in the graph. Show that the following integer program models the problem:

$$
\begin{array}{lll}
\min & \sum_{v \in V} x_v & \\
\text{s.t.} & \sum_{u \in C} x_u \geq 1 & \forall C \in \mathcal{C}_4 \\
& x_u \in \{0,1\} & u \in V
\end{array}
$$

Solution: For each vertex v, we have a variable x_v. These variables are 0/1 valued. The meaning of $x_v = 1$ is that we remove vertex v from the graph. The meaning of $x_v = 0$ is that we keep vertex v. Let OPT denote the optimum value for the original problem. Let OPT_{ip} denote the optimum value for the integer program. Let x^* be an optimum solution of the integer program. By the inequality constraint, the integer program will pick at least one vertex from each 4-cycle. Thus removing the vertices corresponding to $x^* = 1$ will remove all the 4-cycles. Therefore we have $\text{OPT} \leq \text{OPT}_{ip}$. On the other hand, take a minimum set of vertices whose removal kills all the 4-cycles. Setting $x_v = 1$ for these vertices clearly produces a feasible solution for the integer program. Therefore $\text{OPT}_{ip} \leq \text{OPT}$.

(b) (5 points) Why does the optimal solution to the following relaxation provides a lower bound for the optimal answer to the above integer linear program? In other words why it is not necessary to have the constraints $x_u \leq 1$ in the relaxation?

$$
\begin{array}{lll}
\min & \sum_{v \in V} x_v & \\
\text{s.t.} & \sum_{u \in C} x_u \geq 1 & \forall C \in \mathcal{C}_4 \\
& x_u \geq 0 & \forall u \in V
\end{array}
$$

Solution: We claim that in any optimum solution x^*, $x^*_u \leq 1$ for all u. Suppose there exists some u such that $x^*_u > 1$. Round down the value of this variable to 1. Note that all the inequality constraints will still be satisfied. So we still have a feasible solution. On the other hand, the optimum value will go down, which is a contradiction.

(c) (15 points) Give a simple 4-factor approximation algorithm for the problem based on rounding the solution to the above linear program.

Solution: As before, let x^* be the optimum solution. The rounding is as follows. If $x^*_u \geq 1/4$, set $x^*_u = 1$, otherwise set $x^*_u = 0$. First let's check that we get a feasible solution to our problem. In each inequality constraint, it must be the case that at least one of the variables has value $\geq 1/4$. Thus in our rounded solution, we pick at least one vertex from each 4-cycle. So we kill all the 4-cycles as required. Let OPT^* be the optimum for the linear program, let OPT be the optimum for the original problem and let A be the value obtained by rounding the optimum of the linear program. Clearly $\mathrm{OPT}^* \leq \mathrm{OPT}$. Also, by our rounding scheme, we have $A \leq 4\mathrm{OPT}^*$. Thus, $A \leq 4\mathrm{OPT}$, i.e. our solution is within a factor 4 of the optimum.

(d) (10 points) Let L and R denote the set of the vertices in the two parts of the bipartite graph. (Every edge has one endpoint in L and one endpoint in R). Let x^* denote an optimal solution to the linear program in Part (b). We round x^* in the following way:

For every $u \in V$,

- if $u \in R$ and $x^*_u \geq 1/2$, set $\widehat{x}_u = 1$.
- if $u \in L$ and $x^*_u > 0$, set $\widehat{x}_u = 1$.
- Otherwise set $\widehat{x}_u = 0$.

Show that $\widehat{x}$ is a feasible solution to the integer linear program.

Solution: Observe that each 4-cycle contains two vertices from L and two vertices from R. Consider an inequality constraint of the linear program (so we are considering a fixed 4-cycle). If $x^*_u > 0$ for one of the two vertices in L, $\widehat{x}_u$ will be set to 1 and therefore this inequality will be satisfied. On the other hand, if $x^*_u = 0$ for both vertices in L, then it must be the case that $x^*_v \geq 1/2$ for one of the vertices in R. Thus this vertex will be rounded to 1 and the inequality will be satisfied.

(e) (10 points) Consider the dual of the relaxation:

$$\begin{array}{lll} \max & \sum_{C \in \mathcal{C}_4} y_C & \\ \text{s.t.} & \sum_{C \in \mathcal{C}_4, u \in C} y_C \leq 1 & \forall u \in V \\ & y_C \geq 0 & \forall C \in \mathcal{C}_4 \end{array}$$

and let y^* be an optimal solution to the dual. Use the complementary slackness to prove the following statement: For every $C \in \mathcal{C}_4$ either we have $|\{u : \widehat{x}_u = 1\}| \leq 3$ or $y^*_C = 0$.

Solution: Suppose $|\{u : \widehat{x}_u = 1\}| > 3$. Then all the variables for that cycle must be rounded to 1. For that to happen, it must be that $x^*_u \geq 1/2$ for the vertices in R and $x^*_u > 0$ for the vertices in L. Thus, we must have $\sum_{u \in C} x^*_u > 1$, i.e. the constraint is not tight. By complementary slackness, this means $y^*_C = 0$.

(f) (10 points) Use the complementary slackness and the previous parts to show that our rounding algorithm is a 3-factor approximation algorithm.

Solution: As mentioned before, we have $\sum_{u \in V} x_u^* = \text{OPT}^* \leq \text{OPT}$. Thus, we are done once we show

$$\sum_{u \in V} \widehat{x}_u \leq 3\text{OPT}^*.$$

Note that if $\widehat{x}_u = 1$, $x_u^* > 0$. Therefore, by complementary slackness, $\sum_{C \in \mathcal{C}_4, u \in C} y_C^* = 1$. The variables $\widehat{x}_u$ are 0/1 valued, so we can write

$$\sum_{u \in V} \widehat{x}_u = \sum_{u \in V} \widehat{x}_u \sum_{C \in \mathcal{C}_4, u \in C} y_C^* = \sum_{u \in V} \sum_{C \in \mathcal{C}_4, u \in C} \widehat{x}_u y_C^*.$$

We now change the order of the sums and get

$$\sum_{u \in V} \sum_{C \in \mathcal{C}_4, u \in C} \widehat{x}_u y_C^* = \sum_{C \in \mathcal{C}_4} \sum_{u \in C} \widehat{x}_u y_C^* = \sum_{C \in \mathcal{C}_4} y_C^* \sum_{u \in C} \widehat{x}_u.$$

From part (e) of the question, we know that if $y_C^* \neq 0$, then $\sum_{u \in C} \widehat{x}_u \leq 3$. Therefore the above quantity can be upper bounded by $3 \sum_{C \in \mathcal{C}_4} y_C^* = 3\text{OPT}^*$ (the equality follows from duality). Putting things together, we have shown

$$\sum_{u \in V} \widehat{x}_u \leq 3\text{OPT}^*$$

as required.

5. Read the excerpt from Samuel Beckett's novel, Molloy, attached to the assignment.

(a) (5 points) Consider the method that is described in the first paragraph. What is the probability that in the first eight iterations, the same first four stones are picked twice? Here, an iteration refers to every time that he puts a pebble in his mouth.

Solution: The probably that the first four stones are repeated twice is: $\frac{1}{(4^4)^4} = 4^{-16}$

(b) (5 points) Consider the method described in the third paragraph (starting with *All (all!) that was necessary*). What is the probability that the first, seventh and twelfth stone of the first cycle are the sixth and eleventh and sixteenth respectively of the second?

Solution: $\frac{5! \times 1}{6!} \times (\frac{4! \times 1}{5!})^2 = \frac{1}{150}$

(c) (10 points) Consider again the method that is described in the first paragraph. What is the expected number of (different) stones that are picked in the first five iterations?

Solution: $4 \times \frac{1}{4^4} + 5 \times (1 - \frac{1}{4^4}) \approx 4.99$

1 An excerpt from Molloy by Samuel Beckett

I took advantage of being at the seaside to lay in a store of sucking-stones. They were pebbles but I call them stones. Yes, on this occasion I laid in a considerable store. I distributed them equally between my four pockets, and sucked them turn and turn about. This raised a problem which I first solved in the following way. I had say sixteen stones, four in each of my four pockets these being the two pockets of my trousers and the two pockets of my greatcoat. Taking a stone from the right pocket of my greatcoat, and putting it in my mouth, I replaced it in the right pocket of my greatcoat by a stone from the right pocket of my trousers, which I replaced by a stone from the left pocket of my trousers, which I replaced by a stone from the left pocket of my greatcoat, which I replaced by the stone which was in my mouth, as soon as I had finished sucking it. Thus there were still four stones in each of my four pockets, but not quite the same stones. And when the desire to suck took hold of me again, I drew again on the right pocket of my greatcoat, certain of not taking the same stone as the last time. And while I sucked it I rearranged the other stones in the way I have just described. And so on. But this solution did not satisfy me fully. For it did not escape me that, by an extraordinary hazard, the four stones circulating thus might always be the same four. In which case, far from sucking the sixteen stones turn and turn about, I was really only sucking four, always the same, turn and turn about. But I shuffled them well in my pockets, before I began to suck, and again, while I sucked, before transferring them, in the hope of obtaining a more general circulation of the stones from pocket to pocket. But this was only a makeshift that could not long content a man like me.

So I began to look for something else and the first thing that I hit upon was that I might do better to transfer the stones four by four, instead of one by one, that is to say, during the sucking, to take the three stones remaining in the right pocket of my greatcoat and replace them by the four in the right pocket of my trousers, and these by the four in the left pocket of my trousers, and these by the four in the left pocket of my greatcoat, and finally these by the three from the right pocket of my greatcoat, plus the one, as soon as I had finished sucking it, which was in my mouth. Yes, it seemed to me at first that by so doing I would arrive at a better result. But on further reflection I had to change my mind and confess that the circulation of the stones four by four came to exactly the same thing as their circulation one by one. For if I was certain of finding each time, in the right pocket of my greatcoat, four stones totally different from their immediate predecessors, the possibility nevertheless remained of my always chancing on the same stone, within each group of four, and consequently of my sucking, not the sixteen turn and turn about as I wished, but in fact four only, always the same, turn and turn about. So I had to seek elsewhere than in the mode of circulation. For no matter how I caused the stones to circulate, I always ran the same risk. It was obvious that by increasing the number of my pockets I was bound to increase my chances of enjoying my stones in the way I planned, that is to say one after the other until their number was exhausted. Had I had eight pockets, for example, instead of the four I did have, then even the most diabolical hazard could not have prevented me from sucking at least eight of my sixteen stones, turn and turn about. The truth is I should have needed sixteen pockets in order to be quite easy in my mind. And for a long time I could see no other conclusion than this, that short of having sixteen pockets, each with its stone, I could never reach the goal I had set myself, short of an extraordinary hazard. And if at a pinch I could double the number of my pockets, were it only by dividing each pocket in two, with the help of a few safety-pins let us say, to quadruple them seemed to be more than I could manage. And I did not feel inclined to take all that trouble for a half-measure. For I was beginning to lose all sense of measure, after all this wrestling and wrangling, and to say, All or nothing. And if I was tempted for an instant to establish a more equitable proportion between my stones and my pockets, by reducing the former to the number of the latter, it was only for an instant. For it would have been an admission of defeat. And sitting on the shore, before the sea, the sixteen stones spread out before my eyes, I gazed at them in anger and perplexity. For just as I had difficulty in sitting on a chair, or in an armchair, because of my stiff leg you understand, so I had none in sitting on the ground, because of my stiff leg and my stiffening leg, for it was about this time that my good leg, good in the sense that it was not stiff began to stiffen. I needed a prop under the ham you understand, end even under the whole length of the leg, the prop of the earth. And while I gazed thus at my stones, revolving interminable martingales all equally defective, and crushing handfuls of sands, so that sand ran through my fingers and fell back on the sand, yes, while thus I lulled my mind and part of my body, one day suddenly it dawned on the former, dimly, that I might perhaps achieve my purpose without increasing the number of my pockets or reducing the number of my stones, but simply by sacrificing the principle of trim. The meaning of this illumination, which suddenly began to sing within me, like a verse of Isaiah, or of Jeremiah, I did not penetrate at once, and notably the word trim, which I had never met with, in this sense, long remained obscure. Finally I seemed to grasp that this word trim could not here mean anything else, anything better, than the distribution of the sixteen stones in four groups of four, one group in each pocket, and that it was my refusal to consider any distribution other than this had vitiated my calculations until then and rendered the problem literally insoluble. And it was on the basis of this interpretation, whether right or wrong, that I finally reached a solution inelegant assuredly, but sound, sound. Now I am willing to believe, indeed I firmly believe, that other solutions to this problem might have been found, and indeed nay still be found, no less sound, but much more elegant, than the one I shall ow describe, if I can. And I believe too that had I been a little more insistent, a little more resistant, I could have found them myself. But I was tired, but I was tired, and I contended myself ingloriously with the first solution that was a solution, to this problem. But not go over the heartbreaking stages through which I passed before I came to it, here it is, in all its hideousness.

All (all!) that was necessary was to put for example to begin with six stones in the right pocket of my greatcoat, or supply-pocket, five in the right pocket of my trousers, and five in the left pocket of my trousers, that makes the lot, twice five ten plus six, sixteen, and none, for none remained in the left pocket of my greatcoat, which for the time being remained empty, empty of stones that is, for its usual contents remained, as well as occasional objects. For do you think I hid my vegetable knife, my silver, my horn and other things that I have not yet named, perhaps shall never name. Good. Now I can begin to suck. Watch me closely. I take a stone from the right pocket of my greatcoat, suck it, stop sucking it, put it in the left pocket of my greatcoat, the one empty (of stones). I take a second stone from the right pocket of my greatcoat, suck it put it in the left pocket of my greatcoat. And so on until the right pocket of my greatcoat is empty (apart from its usual and casual contents) and the six stones I have just sucked, one after the other, are all in the left pocket of my greatcoat. Pausing then, and concentrating, so as not to make a balls of it, I transfer to the right pocket of my greatcoat, in which there are no stones left, the five stones in the right pocket of my trousers, which I replace by the five stones in the left pocket of my trousers, which I replace by the six stones in the left pocket of my greatcoat. At this stage then the left pocket of my greatcoat is again empty of stones, while the right pocket of my greatcoat is again supplied, and in the right way, that is to say with other stones than those I have just sucked. These other stones I then begin to suck, one after the other, and to transfer as I go along to the left pocket of my greatcoat, being absolutely certain, as far as one can be in an affair of this kind, that I am not sucking the same stones as a moment before, but others. And when the right pocket of my greatcoat is again empty (of stones), and the five I have just sucked are all without exception in the left pocket of my greatcoat, then I proceed to the same redistribution as a moment before, or a similar redistribution, that is to say I transfer to the right pocket of my greatcoat, now again available, the five stones in the right pocket of my trousers, which I replace by the six stones in the left pocket of my trousers, which I replace by the five stones in the left pocket of my greatcoat. And there I am ready to begin again. Do I have to go on? No, for it is clear that after the next series, of sucks and transfers, I shall be back where I started, that is to say with the six stones back in the supply-pocket, the next five in the right pocket of my stinking old trousers and finally the last five in left pocket of same, and my sixteen stones will have been sucked once at least in impeccable succession, not one sucked twice, not one left unsucked. It is true that the next time I could scarcely hope to suck my stones in the same order as the first time and that the first, seventh and twelfth for example of the first cycle might very well be the sixth and eleventh and sixteenth respectively of the second, if worst came to the worst. But that was a drawback I could not avoid...

There was something more than a principle I abandoned, when I abandoned the equal distribution, it was a bodily need. But to suck the stones in the way I have described, not haphazard, but with method, was also I think a bodily need. Here then were two incompatible bodily needs, at loggerheads. Such things happen. But deep down I didn't give a tinker's curse about being off my balance, dragged to the right hand and the left, backwards and forwards. And deep down it was all the same to me whether I sucked a different stone each time or always the same stone, until the end of time. For they all tasted exactly the same. And if I had collected sixteen, it was not in order to ballast myself in such and such a way, or to suck them turn about, but simply to have a little store, so as never to be without. But deep down I didn't give a fiddler's curse about being without, when they were all gone they would be all gone, I wouldn't be any the worse off, or hardly any. And the solution to which I rallied in the end was to throw away all the stones but one, which I kept now in one pocket, now in another, and which of course I soon lost, or threw away, or gave away, or swallowed...