COMP 362 - Winter 2017 - Assignment 3

Due: 6pm March 7th

General rules: In solving these questions you may consult your book; you can discuss high level ideas with each other. But each student must find and write his/her own solution. You should drop your solutions in the assignment drop-off box located in the Trottier Building on the 2nd floor.

1. (15 points) Use the complementary slackness to show that $x_1^* = x_3^* = 0.5$, $x_2^* = x_4^* = 0$, $x_5^* = 2$ is an optimal solution for the following Linear Program:

$$\begin{array}{rl} \max & 3.1x_1 + 10x_2 + 8x_3 - 45.2x_4 + 18x_5 \\ \text{s.t.} & x_1 + x_2 + x_3 - x_4 + 2x_5 \leq 5 \\ & 2x_1 - 4x_2 + 1.2x_3 + 2x_4 + 7x_5 \leq 16 \\ & x_1 + x_2 - 3x_3 - x_4 - 10x_5 \leq -20 \\ & 3x_1 + x_2 + 3x_3 + \frac{3}{2}x_4 + \frac{7}{3}x_5 \leq 10 \\ & x_2 + x_3 + 6x_4 + 2x_5 \leq 4.5 \\ & 2x_2 - x_4 + x_5 \leq 2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

Solution: The dual is

$$\begin{array}{rl} \min & 5y_1 + 16y_2 - 20y_3 + 10y_4 + 4.5y_5 + 2y_6 \\ \text{s.t.} & y_1 + 2y_2 + y_3 + 3y_4 \geq 3.1 \\ & y_1 - 4y_2 + y_3 + y_4 + y_5 + 2y_6 \geq 10 \\ & y_1 + 1.2y_2 - 3y_3 + 3y_4 + y_5 \geq 8 \\ & -y_1 + 2y_2 - y_3 + \frac{3}{2}y_4 + 6y_5 - y_6 \geq -45.2 \\ & 2y_1 + 7y_2 - 10y_3 + \frac{7}{3}y_4 + 2y_5 + y_6 \geq 18 \\ & y_1, y_2, y_3, y_4, y_5, y_6 \geq 0 \end{array}$$

Let $y_1^*, \ldots, y_6^*$ be an optimal solution to the dual. If $x_1^* = x_3^* = 0.5$, $x_2^* = x_4^* = 0$, $x_5^* = 2$ is an optimal solution to the primal, then since for these values, there is slack in the 2nd, 3rd, and 4th constraints of the primal, we must have $y_2^* = y_3^* = y_4^* = 0$. Furthermore since x_1^*, x_3^*, x_5^* are nonzero, we must have no slackness in the corresponding constraints of the dual. Thus we must have

$$\begin{array}{cl} y_1^* & = 3.1 \\ y_1^* + y_5^* + 2y_6^* & \geq 10 \\ y_1^* + y_5 & = 8 \\ -y_1 + 6y_5 - y_6 & \geq -45.2 \\ 2y_1 + 2y_5 + y_6 & = 18 \\ y_1, y_2, y_3, y_4, y_5, y_6 & \geq 0 \end{array}$$

Now this is easy to solve. We must have $y_1^* = 3.1$, $y_5^* = 4.9$, $y_6^* = 2$. Then we check that $y_1^* = 3.1$, $y_2^* = y^*3 = y_4^* = 0$, $y_5^* = 4.9$, $y_6^* = 2$ is a feasible solution to the dual, and has the same cost (41.55) as the optimal solution to the primal, and then the weak duality implies that the 41.55 is the optimal solution.

2. (10 Points) Show that if P = NP, then P = NP = CoNP.

   Solution: Suppose P = NP and consider a problem $X \in$ CoNP. Then $\overline{X} \in$ NP where $\overline{X}$ denotes the problem obtained from X by changing YES inputs to NO and vice versa. Now since $\overline{X} \in$ NP, if P = NP then there is a polynomial-time algorithm that solves $\overline{X}$. To solve X, we can run that algorithm and flip the value of its output (YES to NO and NO to YES). That would show $X \in$ P, and thus CoNP = P.

3. (20 points) Prove that the following problem is NP-complete:

   - Input: A number k, and a formula ϕ in conjunctive normal form.
   - Output: Is there a truth assignment that satisfies ϕ and assigns False to exactly k variables?

   What happens if in the above problem we replace k with the fixed number 100?

   Solution: This is trivially in NP as the truth assignment with k False variables that satisfies ϕ would be a certificate and can be verified efficiently. To prove the completeness, we reduce SAT to this problem. Given an oracle for this problem, we can run it n times for $k = 1, \ldots, n$ and if any of them outputs YES then the input is satisfiable.

   If we replace k with 100, then the problem belongs to P. Indeed there are at most n^{100} (which is a polynomial) truth assignments with 100 False variables. We can check all of them in polytime.

4. (15 points) Prove that the following problem is NP-complete.

   - Input: A graph G and a vertex v of G.
   - Output: Does G have a Hamiltonian path that starts from the vertex v?

   Solution: A Hamiltonian path that starts from the node v is a certificate for a YES input and can be verified efficiently. To prove the completeness we reduce the Hamiltonian path problem to this. To solve the Hamiltonian path using the above problem, we run a for loop over all the possible choices of v, and if the answer to anyone is YES, we output YES.

5. (20 points) Show that if in the decision version of linear programming we allow constraints of the form $|\sum_{i=1}^{n} a_i x_i| \geq b$ for integers b and a_i, then the problem becomes NP-complete.

   Solution: Note that $|x_i| \geq 1$, $x_i \geq -1$ and $x_i \leq 1$ together imply that $x_i \in \{-1, 1\}$. Then $\frac{x_i+1}{2} \in \{0, 1\}$ and we can use these to solve NP-complete problems. For example we can reduce vertex cover to this problem. For an input $\langle G = (V, E), k\rangle$ to vertex cover, we can write

$$\begin{array}{lll} \min & \sum_{u \in V} y_u & \\ \text{s.t.} & y_u + y_v \geq 1 & \forall uv \in E \\ & y_u = \frac{x_u+1}{2} & \forall u \in V \\ & |x_u| \geq 1 & \forall u \in V \\ & x_u \geq -1 & \forall u \in V \\ & x_u \leq 1 & \forall u \in V \end{array}$$

   Hence G has a vertex cover of size at most k if and only if the solution to this program is at most k.

6. (20 points) Show that the following problem is NP-complete:

   - Input: A formula ϕ in conjunctive normal form such that each clause in ϕ either involves only positive terms (i.e., variables), or it involves only negative terms (i.e., negated variables).
   - Output: Is there a truth assignment that satisfies ϕ?

   Solution: This is trivially in NP as it is a special case of SAT. To prove the NP-completeness we reduce SAT (in its general form) to this problem. Consider an input ψ to SAT, and let $C_j = (\vee_{i \in C_j^+} x_i \vee \vee_{i \in C_j^-} \bar{x}_i)$ be a clause in ψ with both positive and negative literals. Now to construct ϕ, we replace every such C_j with $(\vee_{i \in C_j^+} x_i \vee \vee_{i \in C_j^-} y_i)$. Here y_i are new variables. Finally for every i, we add the clauses $(x_i \vee y_i)$ and $(\bar{x}_i \vee \bar{y}_i)$ to ϕ. Note that that each clause in ϕ either involves only positive literals, or it involves only negative literals. Moreover the clauses $(x_i \vee y_i)$ and $(\bar{x}_i \vee \bar{y}_i)$ are satisfied only if $y_i = \bar{x}_i$. Hence ϕ is satisfiable if and only if ψ is satisfiable.