COMP 362 - Winter 2017 - Assignment 5

Due: 6pm April 6th

General rules: In solving these questions you may consult your book; you can discuss high level ideas with each other. But each student must find and write his/her own solution. You should drop your solutions in the assignment drop-off box located in the Trottier Building on the 2nd floor.

1. Consider the integer program for vertex cover problem that was discussed in class.
   (a) (5 Points) Suppose G has a triangle (three pairwise adjacent vertices x_1, x_2, x_3). Then we can add to the integer program the constraint $x_1 + x_2 + x_3 \geq 2$. Why is this constraint valid?
   (b) (5 Points) Generalize the above to an odd cycle C. In other words, write a valid inequality associated with C that can be added to the integer program that still describes the original problem, but which will allow for a tighter relaxation when the integer program is relaxed to a linear program.
   (c) (10 Points) Suppose we add all constraints for all odd cycles. Let H be a graph on n vertices, that has no odd cycle of size at most $\log(n)/10$ and whose minimum vertex cover is of size $n - o(n)$. The existence of such graphs can be shown using probabilistic methods. Consider the integer program for H that is obtained by adding <u>all</u> odd-cycle constraints to the original integer program for the vertex cover. Show that even this integer program has integrality gap[1] of $2 - o(1)$.
2. (20 Points) Given a set P of n points on the plane, consider the problem of finding the smallest r such that there exist 10 circles of radius r such that together they contain all the points in P. Design a PTAS algorithm for this problem. In other words, given any fixed $\epsilon > 0$,

[1] integrality gap is the maximum ratio between the optimal solutions to the integer program, and its linear program relaxation.

design an algorithm whose running time is polynomial in n, and its output is at most $1+\epsilon$ times the optimal output.

3. Consider the MAX-SAT problem: Given a 2-CNF ϕ (clauses of the form $(x_i \vee x_i)$ and $(x_i \vee \bar{x}_i)$ are not allowed) on n variables $x_1, \ldots, x_n$, we want to find a truth assignment that satisfies the maximum number of clauses.

   (a) (10 Points) As in the case of the MAX-CUT problem, find an "integer quadratic program" formulation of this problem in which the only constraints are $y_i \in \{-1, 1\}$. (Hint: It may help to introduce an auxiliary variable y_0 which indicates whether "True" is associated with -1 or with 1.)

   (b) (10 Points) Formulate the SDP relaxation of your integer quadratic program, and use it to find a randomized $\frac{1}{0.878}$-factor approximation algorithm for the MAX-2-SAT problem.

4. (20 Points) Let $v_1, \ldots, v_n$ be *unit* vectors in $\mathbb{R}^n$. Prove that there exists $\epsilon_1, \ldots, \epsilon_n = \pm 1$ such that

$$|\epsilon_1 v_1 + \ldots + \epsilon_n v_n| \leq \sqrt{n}.$$

   Hint:

$$|\epsilon_1 v_1 + \ldots + \epsilon_n v_n|^2 = \langle \epsilon_1 v_1 + \ldots + \epsilon_n v_n, \epsilon_1 v_1 + \ldots + \epsilon_n v_n \rangle.$$

5. (20 Points) Give a polynomial algorithm for the following problem: Let G be a graph with a given proper k-vertex coloring of G. Find a vertex cover whose size is at most $(2 - \frac{2}{k})$OPT, where OPT is the size of the smallest vertex cover for G.