COMP 362 - Winter 2017 - Assignment 1

Due: 6pm Jan 26th

General rules: In solving these questions you may consult your book; you can discuss high level ideas with each other. But each student must find and write his/her own solution. You should drop your solutions in the assignment drop-off box located in the Trottier Building on the 2nd floor.

1. (5 points) What is the running time of the Ford-Fulkerson algorithm for finding the maximum bipartite matching?

2. (a) (10 points) Show that for every flow network, there exists an execution of the Ford-Fulkerson algorithm that never decreases the value of the flow on any of the edges (i.e. never "pushes back" the flow on any of the edges).

   (b) (5 points) On the other hand, show that if we modify the Ford-Fulkerson algorithm so that it does not decrease the flow on any of the edges (i.e. we do not add the opposite edges to the residual graph), then the algorithm might terminate without finding the maximum flow.

3. (20 Points) Consider a variation of the Ford-Fulkerson algorithm in which to find an augmenting path, we run a BFS to find the shortest path from s to t in the residual graph. Prove that throughout the algorithm, the length of the shortest path never decreases. Moreover show that after at most m augmentations the length of the shortest path increases. Here m is the number of the edges of G. What is the running time of this algorithm?

4. (a) (15 points) Show that for every flow network, there exists an execution of the Ford-Fulkerson algorithm that finds the maximum flow after at most m augmentations. Here m is the number of the edges of G.

(b) (5 points) On the other hand, construct examples of flow networks that require $\Theta(m)$ augmentations no matter how we choose the augmenting paths.

5. (20 points) Let G be a graph in which some of the edges are directed, and some of them are undirected. We want to assign an orientation to each undirected edge so that in the resulted directed graph the indegree of every vertex is equal to its outdegree. Here the indegree is the number of edges pointing towards the vertex and the outdegree is the number of the edges pointing away from the vertex. Solve this problem by reducing it to a max-flow problem.

6. (20 Points) Consider the problem of finding the largest matching in a bipartite graph H with n vertices. We saw how we can solve this problem by modeling it as a max-flow problem in a flow network G. Suppose we run the Ford-Fulkerson algorithm on G and at some point during the execution we arrive at a flow f (not necessarily maximum yet). Suppose that the length of the shortest s-t path in G_f is larger than $\sqrt{n}$. Prove that $\text{val}(f) \geq M - \sqrt{n}$ where M is max flow (which is equal to the size of the largest matching).