COMP 360 - Winter 2016 - Assignment 3

Due: 6pm Feb 29th.

General rules: In solving these questions you may consult books but you may not consult with each other. You should drop your solutions in the assignment drop-off box located in the Trottier Building.

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1. (10 Points) A farmer can choose from three feeds for his milk cows. The nutritional facts and costs of these feeds are shown in the following table. The minimum daily requirements of nutrients A, B, and C are 65, 82, 70 units, respectively. Write a linear program to determine the mixture of feeds that will supply the minimum nutritional requirement at least cost.

Feed	A (units/lb)	B (units/lb)	C (units/lb)	Cost /lb
Feed 1	4	7	3	0.10
Feed 2	2	3	4	0.07
Feed 3	5	5	3	0.06

Solution:

Variables: x_1, x_2, x_3 for the quantity of feed $1, 2, 3$ respectively.

$$\begin{aligned} \max \quad & 0.1x_1 + 0.07x_2 + 0.06x_3 \\ \text{s.t.} \quad & 4x_1 + 2x_2 + 5x_3 \geq 65 \\ & 7x_1 + 3x_2 + 5x_3 \geq 82 \\ & 3x_1 + 4x_2 + 3x_3 \geq 70 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

2. (a) (5 Points) Draw the feasible region of the following system of linear constraints:

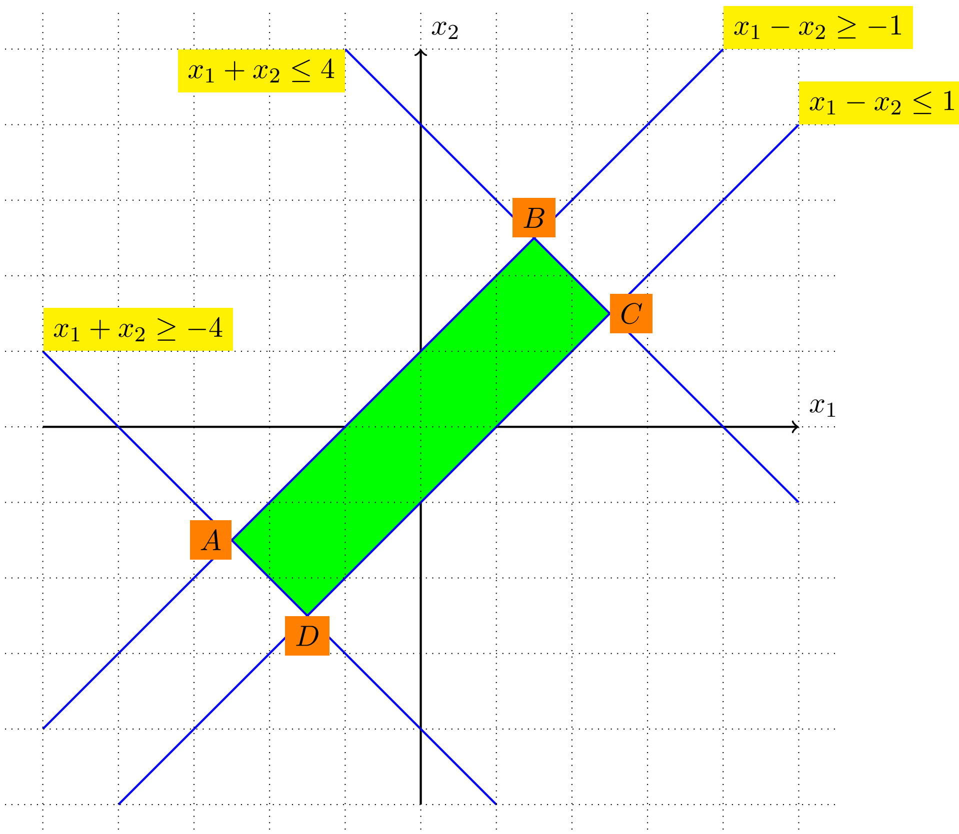
$$\begin{aligned} x_1 - x_2 &\geq -1 \\ x_1 - x_2 &\leq 1 \\ x_1 + x_2 &\leq 4 \\ x_1 + x_2 &\geq -4 \end{aligned}$$

(b) (10 Points) What are the vertices of the feasible region? For each vertex, list the two constraints that define that vertex.

(c) (10 Points) Which vertex maximizes the value of the function $5x_1 + x_2$? Now suppose that we wanted to find this vertex using the simplex algorithm starting from the vertex with the smallest x_1. In other words we start at that vertex and each time we move to a neighbouring vertex that increases the value of $5x_1 + x_2$. List the two possible paths that the algorithm can take starting from that vertex and finishing at the optimal vertex. For each path, explain how the linear constraints that define the vertices on the path change as we move from one vertex to the next vertex.

Solution:

(a) The feasible region is the green part in the following picture:



(b) The vertices of the feasible regions and corresponding constraints:

A	$(-2.5, -1.5)$	$x_1 + x_2 \geq -4$	$x_1 - x_2 \geq -1$
B	$(1.5, 2.5)$	$x_1 + x_2 \leq 4$	$x_1 - x_2 \geq -1$
C	$(2.5, 1.5)$	$x_1 + x_2 \leq 4$	$x_1 - x_2 \leq 1$
D	$(-1.5, -2.5)$	$x_1 + x_2 \geq -4$	$x_1 - x_2 \leq 1$

(c) Vertex C maximizes $5x_1 + x_2$. Vertex A has the smallest value of x_1.

Starting from A, the algorithm has two paths to reach C:

(i) $A \to B \to C$. First replace $x_1 + x_2 \geq -4$ with $x_1 + x_2 \leq 4$. Then replace $x_1 - x_2 \geq -1$ with $x_1 - x_2 \leq 1$.

(ii) $A \to D \to C$. First replace $x_1 - x_2 \geq -1$ with $x_1 - x_2 \leq 1$. Then replace $x_1 + x_2 \geq -4$ with $x_1 + x_2 \leq 4$.

3. (a) (15 Points) Write a linear program to solve the following problem: Given an undirected graph G as an input, we want to assign nonnegative numbers to the *edges* of the graph so that the following two conditions hold simultaneously:

- For every vertex, the sum of the numbers on the edges incident to it is at most 1.
- The total sum of the numbers on all the edges is maximized.

Solution:
Let E be the set of edges in G, let V be the set of vertices in G. For $(u, v) \in E$, let x_{uv} be the number assigned to the edge uv.

$$\begin{array}{lll} \max & \sum_{(u,v)\in E} x_{uv} & \\ \text{s.t.} & \sum_{(u,v)\in E} x_{uv} & \leq 1 \quad \forall u \in V \\ & x_{uv} & \geq 0 \quad \forall (u,v) \in E \end{array}$$

Remark: Always define notations like V, E **before** using them.

(b) (5 Points) If we also require that the numbers assigned to edges are integers, then what does the above problem correspond to?
Solution:
Then it would find the size of the maximal matching in G.

4. (15 Points) Convert the following linear program to canonical form.

$$\begin{array}{lll} \max & x_1 + 2x_2 - x_3 & \\ \text{s.t.} & 2x_1 - 5x_2 + x_3 & = 0 \\ & -x_1 - x_2 & = 2 \\ & x_1 - x_2 + 3x_3 & \geq 5 \\ & 10x_1 + x_2 - x_3 & \leq 5 \\ & x_1 & \geq 0 \\ & x_2 & \leq 0 \end{array}$$

Solution:

$$\begin{array}{lll} \max & x_1 - 2\bar{x}_2 - x_3' + x_3'' & \\ \text{s.t.} & 2x_1 + 5\bar{x}_2 + x_3' - x_3'' & \leq 0 \\ & -2x_1 - 5\bar{x}_2 - x_3' + x_3'' & \leq 0 \\ & -x_1 + \bar{x}_2 & \leq 2 \\ & x_1 - \bar{x}_2 & \leq -2 \\ & -x_1 - \bar{x}_2 - 3x_3' + 3x_3'' & \leq -5 \\ & 10x_1 - \bar{x}_2 - x_3' + x_3'' & \leq 5 \\ & x_1 & \geq 0 \\ & \bar{x}_2 & \geq 0 \\ & x_3' & \geq 0 \\ & x_3'' & \geq 0 \end{array}$$

5. (15 Points) Write the dual of the following linear program:

$$\begin{array}{lll} \max & x_1 + 2x_2 - x_3 & \\ \text{s.t.} & 2x_1 - 5x_2 + x_3 & \leq 0 \\ & -x_1 - x_2 & \leq 2 \\ & x_1 + 3x_3 & \leq 5 \\ & 10x_1 + x_2 - x_3 & \leq 5 \end{array}$$

Solution:

$$\begin{array}{ll} \min & 2y_2 + 5y_3 + 5y_4 \\ \text{s.t.} & 2y_1 - y_2 + y_3 + 10y_4 = 1 \\ & -5y_1 - y_2 + y_4 = 2 \\ & y_1 + 3y_3 - y_4 = -1 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

6. (15 Points) Formulate the following problem as a linear program: Given $a_1, \ldots, a_n \geq 0$, and an $n \times n$ matrix A, we want to find $x_1, \ldots, x_n \geq 0$ (and not all of them zero) such that

$$\frac{a_1 x_1 + a_2 x_2 + \ldots + a_n x_n}{x_1 + x_2 + \ldots + x_n}$$

is maximized and

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \leq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Solution: Note that if $(x_1, \ldots, x_n)$ is an optimal solution, then so is $(x_1/r, \ldots, x_n/r)$ where $r = x_1 + \ldots + x_n$. Hence we can use the following linear program.

$$\begin{array}{ll} \max & a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \\ \text{s.t.} & x_1 + x_2 + \ldots + x_n = 1 \\ & Ax \leq \vec{0} \end{array}$$