# COMP 362 - Winter 2017 - Midterm

Name:

1. (20 points) Consider the variant of the maximum flow problem where every node $v$ also has an integer capacity $c_v \geq 0$. We are interested in finding the maximum flow as before, but now with the extra restriction that $f^{\text{in}}(v) \leq c_v$ for every node $v$. Solve this problem using the original maximum flow problem.

   **Solution (sketch):** Split every vertex $v$ into two vertices $v^{in}$ and $v^{out}$ with an edge of capacity $c_v$ directed from $v^{in}$ to $v^{out}$. Now all the incoming edges to $v$ will be joined to $v^{in}$, and all the outgoing edges from $v$ will leave $v^{out}$.

2. Recall that in Assignment 1, we considered a variation of the Ford-Fulkerson algorithm in which to find an augmenting path, we run the Breadth First Search algorithm to find the shortest path from the source to the sink in the residual graph. We showed that throughout the algorithm, the length of the shortest path never decreases. Thus we can write the algorithm in the following form:

   - Set $f \equiv 0$, and construct the residual graph $G_f$.
   - For $k = 0, \ldots, |V(G)|$ do
   -     While there is an $s - t$-paths of length $k$ in $G_f$, augment the path and update $G_f$.
   - EndFor

   (a) (20 points) Let $G$ be a flow network constructed to find the largest matching in a bipartite graph $H$ with $n$ vertices. Let us call the line "while there is" phase $k$; In this phase we are augmenting $s - t$-paths of length exactly $k$. Show that for $G$, the phase $k$ can be implemented to have total running time $O(m)$. *Hint: Run the BFS only at the beginning of this phase; Keep the needed information, and update it as you augment more paths in this phase.*

   **Solution:** We run the BFS and partition the vertices according to their distances from the source $s$. In $t$ is not in distance $k$, then we finish the phase, otherwise we start from the source and try to trace a path to $t$ by moving from each level (here level corresponds to the distance from $s$) to the next level. If we are at a vertex at level $i < k$ that has no edges to the next level, then we delete that vertex and backtrack in our path. Similarly if we are at a vertex at level $k - 1$ that has no edge going to $t$, then delete that vertex and backtrack. Otherwise we pick an edge to the next level and follow the edge (if we are at level $k - 1$, the edge must end in $t$). Once we find an $s - t$-path, we augment it, and delete all the edges in that path. Repeat.

   Since we never reuse an edge in a phase, the total running time in that phase is $O(m)$.

   (b) (20 Points) We showed in Assignment 1 that if at some point the length of the shortest $s - t$-path in $G_f$ is larger than $\sqrt{n}$, then $\text{val}(f) \geq M - \sqrt{n}$, where $M$ is the size of the maximum matching. Use this with the previous part to design an algorithm with running time $O(m\sqrt{n})$ for finding the maximum matching in a bipartite graph with $n$ vertices and $m$ edges.

   **Solution:** We run the previous part until the length of the shortest $s - t$-path becomes equal to $\sqrt{n}$. This requires $O(\sqrt{n}m)$ running time. Then we run the usual Ford-Fulkerson algorithm for another $\sqrt{n}$ steps to find the max-flow. This also requires only $O(\sqrt{n}m)$ time. Thus the total running time is $O(\sqrt{n}m)$.

3. (20 points) Formulate the following problem as a linear program: Consider a graph $G$ in which to every edge $e$ a number $\alpha_e$ is assigned. We want to assign non-negative numbers to the vertices of $G$ such that the total sum of these numbers is minimized and furthermore the sum of the two numbers on the endpoints of every edge $e$ is at least $\alpha_e$.

   **Solution:**

$$\begin{array}{lll l} \min & \sum_{u \in V} x_u & & \\ \text{s.t.} & x_u + x_v & \geq \alpha_e & \forall e = uv \in E \\ & x_u & \geq 0 & \forall u \in V \end{array}$$

4. (20 points) Write the dual of your linear program from the previous question.

   **Solution:**

$$\begin{array}{lll l} \min & \sum_{e \in E} \alpha_e y_e & & \\ \text{s.t.} & \sum_{e = uv \in E} y_e & \leq 1 & \forall u \in V \\ & y_e & \geq 0 & \forall e \in E \end{array}$$