COMP 360 - Winter 2016 - Sample Final Exam

1. (10 points) Prove that the following problem belongs to P: Given a graph G, we want to know whether G has an independent set of size 100.

2. (10 Points) Prove that the following problem belongs to PSPACE: Given a graph G and an integer k, we want to know whether the number of independent sets in G is equal to k.

3. (10 Points) Show that the following problem is NP-complete:

   - Input: An undirected graph G and an edge e.
   - Question: Does G have a Hamiltonian cycle that passes through the edge e.

4. (10 points) In the MAX-CUT problem, given an undirected graph G we want to partition the vertices of G into two parts (A, B) such that the number of edges between A and B is maximized. Prove that the following is a $\frac{1}{2}$-factor approximation algorithm for this problem:

   - Let $v_1, \ldots, v_n$ be all the vertices of G.
   - Initially set $A = B = \emptyset$.
   - For $i = 1, \ldots, n$ do
   -     IF v_i has more neighbours in A than in B THEN
   -         add v_i to B
   -     Else
   -         add v_i to A
   - EndFor

5. (10 points) Prove that the following algorithm is a 2-factor approximation algorithm for the minimum vertex cover problem:

   - While there is still an edge e left in G:
   -     Delete all the two endpoints of e from G
   - EndWhile
   - Output the set of the deleted vertices

6. (10 points) Prove that the following algorithm is a $\frac{1}{2}$-factor approximation algorithm for the MAX-SAT problem: Given a CNF ϕ on n variables $x_1, \ldots, x_n$:

   - For $i = 1, \ldots, n$ do
   -     IF x_i appears in more clauses than $\overline{x_i}$ THEN
   -         Set $x_i = T$

- Else
- Set $x_i = F$
- Remove all True clauses from ϕ and remove x_i and $\overline{x_i}$ from all the other clauses
- EndFor

7. (10 points) A *kite* is a graph on an even number of vertices, say $2k$, in which k of the vertices form a clique and the remaining k vertices are connected in a tail that consists of a path joined to one of the vertices of the clique. Prove that KITE problem defined as in the following is NP-complete.

   - Input: An undirected graph G, and a positive integer k.
   - Question: Does G contain a kite on $2k$ vertices as a subgraph?

8. Consider a graph $G = (V, E)$. The chromatic number of G is the minimum number of colors required to color the vertices of G properly. Let $\mathcal{I}$ be the set of all independent sets in G (Note that every element in $\mathcal{I}$ is a set).

   (a) (10 Points) Prove that the solution to the following linear program provides a lower-bound for the chromatic number of G.

$$\begin{array}{lll} \min & \sum_{I \in \mathcal{I}} x_I & \\ \text{s.t.} & \sum_{I : v \in I} x_I \geq 1 & \forall v \in V \\ & x_I \geq 0 & \forall I \in \mathcal{I} \end{array}$$

   (b) (10 Points) Write the dual of the above linear program.

   (c) (10 Points) Prove that every clique in G provides a solution to the dual linear program.