COMP 362 - Winter 2015 - Assignment 4

Due: 6pm March 27th.

General rules: In solving these questions you may consult books but you may not consult with each other. You should drop your solutions in the assignment drop-off box located in the Trottier Building.

1. (10 Points) Show that the following problem is in PSPACE:

   - Input: An undirected graph G and a positive integer m.
   - Question: Is the number of proper m-vertex colorings of G divisible by 7?

   (Note that the size of the input is $O(n^2 + \log m)$ where n is the number of vertices of G.)

2. (15 Points) Given a set P of n points on the plane, consider the problem of finding the smallest circle containing all the points in P. Show that the following is a 2-factor approximation algorithm for this problem. Pick a point x in P, and set r to be the distance of the farthest point in P from x. Output the circle centered at x with radius r.

3. (15 Points) Problem 10 of Chapter 11: Suppose you are given an $n \times n$ grid graph G. Associated with each node v is an integer weight $w(v) \geq 0$. You may assume that all the weights are distinct. Your goal is to choose an independent set S of nodes of the grid, so that the sum of the weights of the nodes in S is as large as possible. (The sum of the weights of the nodes in S will be called its total weight.) Consider the following greedy algorithm for this problem.

   - Start with $S := \emptyset$.
   - While some node remains in G:
     - Pick a node v of maximum weight.
     - Add v to S.

– Delete v and its neighbors from G

• Endwhile.

Show that this algorithm returns an independent set of total weight at least $\frac{1}{4}$ times the maximum total weight of any independent set in the grid graph G.

4. Consider a directed bipartite graph $G = (V, E)$. We want to eliminate all the directed cycles of length 4 by removing a smallest possible set of <u>vertices</u>.

(a) (5 points) Let $\mathcal{C}_4$ denote the set of all cycles of length 4 in the graph. Show that the following integer program models the problem:

$$
\begin{aligned}
\min \quad & \sum_{v \in V} x_v \\
\text{s.t.} \quad & \sum_{u \in C} x_u \geq 1 && \forall C \in \mathcal{C}_4 \\
& x_u \in \{0, 1\} && u \in V
\end{aligned}
$$

(b) (5 points) Why does the optimal solution to the following relaxation provides a lower bound for the optimal answer to the above integer linear program? In other words why it is not necessary to have the constraints $x_u \leq 1$ in the relaxation?

$$
\begin{aligned}
\min \quad & \sum_{v \in V} x_v \\
\text{s.t.} \quad & \sum_{u \in C} x_u \geq 1 && \forall C \in \mathcal{C}_4 \\
& x_u \geq 0 && \forall u \in V
\end{aligned}
$$

(c) (15 points) Give a simple 4-factor approximation algorithm for the problem based on rounding the solution to the above linear program.

(d) (15 points) Let L and R denote the set of the vertices in the two parts of the bipartite graph. (Every edge has one endpoint in L and one endpoint in R). Let x^* denote an optimal solution to the linear program in Part (b). We round x^* in the following way:

For every $u \in V$,

• if $u \in R$ and $x_u^* \geq 1/2$, set $\widehat{x}_u = 1$.
• if $u \in L$ and $x_u^* > 0$, set $\widehat{x}_u = 1$.
• Otherwise set $\widehat{x}_u = 0$.

Show that $\widehat{x}$ is a feasible solution to the integer linear program.

(e) (10 points) Consider the dual of the relaxation:

$$\begin{array}{ll} \max & \sum_{C \in \mathcal{C}_4} y_C \\ \text{s.t.} & \sum_{C \in \mathcal{C}_4, u \in C} y_C \leq 1 \qquad \forall u \in V \\ & y_C \geq 0 \qquad\qquad\qquad \forall C \in \mathcal{C}_4 \end{array}$$

and let y^* be an optimal solution to the dual. Use the complementary slackness to prove the following statement: For every $C \in \mathcal{C}_4$ either we have $|\{u : \widehat{x}_u = 1\}| \leq 3$ or $y_C^* = 0$.

(f) (10 points) Use the complementary slackness and the previous parts to show that our rounding algorithm is a 3-factor approximation algorithm.