COMP 360 - Winter 2016 - Assignment 1
Solutions

January 27, 2016

1. (a) Take $c = 2$ and $n_0 = 0$. We have $\sqrt{n} + n\sqrt{n} = n^{0.5} + n^{1.5} \le n^2 + n^2 = 2n^2$. So $\sqrt{n} + n\sqrt{n} \le cn^2$ for $n > 0$.

   (b) We need to show $n^5 = O((n + \log_2 n)^5)$ and $(n + \log_2 n)^5 = O(n^5)$. For the first equality, note that $n \le n + \log_2 n$ and therefore $n^5 \le (n + \log_2 n)^5$. So $c = n_0 = 1$ satisfies the definition of big-O. To show $(n + \log_2 n)^5 = O(n^5)$, note that $(n + \log_2 n)^5 \le (n + n)^5 = (2n)^5 = 32n^5$. So in this case $c = 32$ and $n_0 = 1$ works.

   (c) Observe that $\frac{n!}{n^n} = \prod_{i=1}^{n} \frac{i}{n} \le \frac{1}{n}$ since each term in the product is at most 1. So

   $$\lim_{n\to\infty} \frac{n!}{n^n} \le \lim_{n\to\infty} \frac{1}{n} = 0,$$

   and we are done by the definition of small-o.

   (d) We can use L'Hôpital's rule. So

   $$\lim_{n\to\infty} \frac{\log_2 n}{n^{1/100}} = \lim_{n\to\infty} \frac{100n^{99/100}}{n \ln 2} = \lim_{n\to\infty} \frac{100}{n^{1/100} \ln 2} = 0.$$

   (e) From part (d) we know that $\log_2 n = o(n^{1/100})$. In fact the same proof shows that $\log_2 n = o(n^{1/2})$. This implies that there is some n_0 such that $\log n < \frac{1}{2}\sqrt{n}$ for all $n > n_0$. Using this, we have

   $$\frac{n^{\sqrt{n}}}{2^n} = \frac{2^{\sqrt{n}\log_2 n}}{2^n} \le \frac{2^{\sqrt{n}\sqrt{n}/2}}{2^n} = \frac{2^{n/2}}{2^n} = \frac{1}{2^{n/2}},$$

   for $n > n_0$. Therefore $\lim_{n\to\infty} \frac{n^{\sqrt{n}}}{2^n} = 0$ and hence $n^{\sqrt{n}} = o(2^n)$.

2. (a) False. It can be shown that $2^{2^n} = o(2^{2^{n+1}})$ since

   $$\frac{2^{2^n}}{2^{2^{n+1}}} = \frac{2^{2^n}}{2^{2\cdot 2^n}} = \frac{2^{2^n}}{4^{2^n}} = \frac{1}{2^{2^n}},$$

   and hence

   $$\lim_{n\to\infty} \frac{2^{2^n}}{2^{2^{n+1}}} = 0.$$

   Because $2^{2^n} = o(2^{2^{n+1}})$, it cannot be the case that $2^{2^{n+1}} = O(2^{2^n})$.

(b) False. It can be shown that $\log_2 n^5 = o((\log n)^5)$ since

$$\frac{\log_2 n^5}{(\log n)^5} = \frac{5 \log_2 n}{(\log n)^5} \leq \frac{5}{(\log n)^4},$$

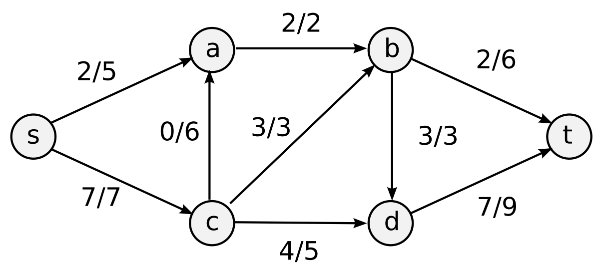
and hence

$$\lim_{n \to \infty} \frac{\log_2 n^5}{(\log n)^5} = 0.$$

(c) True. To show $n^{1/n} = O(1)$, we need to show there are c and n_0 such that $n^{1/n} \leq c$ for $n > n_0$. Pick $c = 2$ and $n_0 = 1$. Observe that $n \leq 2^n$ for $n \geq 1$ and this implies $n^{1/n} \leq 2$. To show $n^{1/n} = \Omega(1)$, we need to show there are c and n_0 such that $1 \leq cn^{1/n}$ for $n > n_0$. Let $c = n_0 = 1$ and observe that $1^n \leq n$ implies $1 \leq n^{1/n}$.

3. Two out of many possible solutions are given below.

(1) $scdbt + 3$, $scabt + 2$, $scdt + 2$, $sacdt + 2$, maximum flow is 9.



(2) $sabt + 2$, $scbt + 3$, $scdt + 4$, maximum flow is 9.