COMP 360: Algorithm Design	Instructor: Yang Cai

Assignment 2 Solution

General Rules: In solving these questions you may consult books but you may not search on the web for solutions. You must write up your final solution yourself. You should drop your solutions in the assignment drop-off box located in the Trottier Building. No late assignments accepted.

1. (15 pts) Consider the following LP

$$\begin{aligned} \text{min } & 9y_1 + 14y_2 + 4y_3 \\ \text{s.t.} \quad & 3y_1 + y_2 \geq 2 \\ & -2y_1 + 3y_2 + y_3 \geq 1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

   (a) (5 pts) Give the dual linear program of the linear program above. [Use the usual dual format, that is, max$\{c^T x : A^T x \leq b, x \geq 0\}$.]

   (b) (5 pts) Convert the dual linear program into the standard form.

   (c) (5 pts) Run the simplex algorithm to obtain the optimal solution to the dual.

Solution:

(a) We write the dual in the suggested form; "reading off the coefficients" we get the dual:

$$\begin{aligned} \max\ & 2x_1 + x_2 \\ s.t.\ & 3x_1 - 2x_2 \leq 9 \\ & x_1 + 3x_2 \leq 14 \\ & x_2 \leq 4 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(b) Add slack variables

$$\begin{aligned} \max\ & 2x_1 + x_2 \\ s.t.\ & 3x_1 - 2x_2 + s_1 = 9 \\ & x_1 + 3x_2 + s_2 = 14 \\ & x_2 + s_3 = 4 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0. \end{aligned}$$

(c) Our LP has become: max z subject to

$$\begin{array}{ccccccccccccc} 2x_1 & + & x_2 & + & & & & & & - & z & = 0 \\ 3x_1 & - & 2x_2 & + & s_1 & & & & & & & = 9 \\ x_1 & + & 3x_2 & & & + & s_2 & & & & & = 14 \\ & & x_2 & + & & & & + & s_3 & & & = 4 \end{array}$$

which corresponds to the following tableau:

x_1	x_2	s_1	s_2	s_3	z	
2	1	0	0	0	-1	0
3	-2	1	0	0	0	9
1	3	0	1	0	0	14
0	1	0	0	1	0	4

We begin with the basis consisting the slack variables (s_1, s_2, s_3) and thus the BFS becomes

$$(x_1, x_2, s_1, s_2, s_3) = (0, 0, 9, 14, 4)$$

and $z = 0$. Observe that x_1 contributes the most to the objective function: each unit increase in x_1 increase the objective by 2. According to the min ratio rule, we choose variable x_1 on the row 2 for pivoting. The negative ratios should be ditched because they will lead to decrement in objective function when substituted into row 1. We now substitute the following into the tableau

$$x_1 = \frac{1}{3}(9 - s_1 + 2x_2)$$

and after some arithmetic, we get the 2nd tableau:

x_1	x_2	s_1	s_2	s_3	z	
0	$7/3$	$-2/3$	0	0	-1	-6
1	$-2/3$	$1/3$	0	0	0	3
0	$11/3$	$-1/3$	1	0	0	11
0	1	0	0	1	0	4

Our BFS is

$$(x_1, x_2, s_1, s_2, s_3) = (3, 0, 0, 11, 4).$$

Now, applying the same logic from above for choosing a pivot, we choose x_2 and row 3 for pivoting. We substitute the following into the 2nd tableau

$$x_2 = \frac{3}{11}(11 + \frac{1}{3}s_1 - s_2)$$

and we get the 3rd tableau.

x_1	x_2	s_1	s_2	s_3	z	
0	0	$-15/33$	$-7/11$	0	-1	-13
1	0	$7/33$	$2/11$	0	0	5
0	1	$-1/11$	$3/11$	0	0	3
0	0	$1/11$	$-3/11$	1	0	1

Our BFS is

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (5, 3, 0, 0, 1).$$

Observe that all coefficients of row 1 are now nonpositive, hence the algorithm stops in this step and the last BFS is the optimal one. The maximum value of the objective function is 13.

2. (15 pts) Let g be monotone increasing on $\mathbb{R}$. Let $p_1(\cdot)$, $p_2(\cdot)$, and $p_3(\cdot)$ be polynomials of odd degree. x, y, z satisfy the following constraints:

   $a_{11}p_1(x) + a_{12}p_2(y) \leq 1$, $a_{21}p_2(y) + a_{22}p_3(z) \leq 2$, $a_{31}p_3(z) + a_{32}p_1(x) \leq 3$, where a_{ij} is an absolute constant in $\mathbb{R}$ for all (i, j).

   Formulate an LP and show that it finds the value of $p_1(x)$, $p_2(y)$ and $p_3(z)$ that maximizes $g(p_1(x) + p_2(y) + p_3(z))$ subject to the conditions above.

   Solution Consider the following LP, A, and a problem similar to the original, B:

$$\begin{aligned} & \max\ s + t + u \\ \text{s.t.} \quad & a_{11}s + a_{12}t \leq 1 \\ & a_{21}t + a_{22}u \leq 2 \\ & a_{31}u + a_{32}s \leq 3 \end{aligned}$$

$$
\begin{aligned}
&\max p_1(x) + p_2(y) + p_3(z) \\
\text{s.t.} \quad & a_{11}p_1(x) + a_{12}p_2(y) \le 1 \\
& a_{21}p_2(y) + a_{22}p_3(z) \le 2 \\
& a_{31}p_3(z) + a_{32}p_1(x) \le 3
\end{aligned}
$$

Let $x = x^*, y = y^*, z = z^*$ be a feasible solution for B. Then $s = p_1(x^*), t = p_2(y^*), u = p_3(z^*)$ is a feasible solution for A and the objective functions are equal.
Conversely, let $s = s^*, t = t^*, u = u^*$ a feasible solution for A. Since $p_i(\mathbb{R}) = \mathbb{R}$ for $i \in \{1, 2, 3\}$, we may choose x', y', z' such that $x' \in p_1^{-1}(s^*), y' \in p_2^{-1}(t^*), z' \in p_3^{-1}(u^*)$. Then $x = x', y = y', z = z'$ is a feasible solution for B and the objective functions are equal.
Finally, since g is monotone increasing, argmax$_{x,y,z} g(f(x, y, z))$ = argmax$_{x,y,z} f(x, y, z)$ for any $f : \mathbb{R}^3 \to \mathbb{R}$. Thus a solution (x^*, y^*, z^*) to B is a solution to the original problem.

3. (15 pts) Given a flow network $G(V, E)$, for every edge (u, v),

   - $l(u, v)$ is the lower bound on flow from node u to node v;
   - $c(u, v)$ is the upper bound on flow from node u to node v;
   - $e(u, v)$ is the expense for sending a unit of flow on (u, v).

   Formulate an LP and show that if it has an optimal solution, the optimal solution corresponds to a flow that satisfies lower bounds and upper bounds on all edges and minimizes the total expense. The total expense is the sum of expenses on all edges.

   Solutions:

   For each edge $(u, v) \in E$, we use variable $f(u, v)$ to represent the amount of flow though edge (u, v).

$$
\begin{aligned}
\min \quad & \sum_{(u,v)\in E} e(u, v) \cdot f(u, v) \\
\text{s.t.} \quad & f(u, v) \le c(u, v), \quad \forall (u, v) \in E \\
& f(u, v) \ge l(u, v), \quad \forall (u, v) \in E \\
& \sum_{v:(v,u)\in E} f(v, u) = \sum_{v:(u,v)\in E} f(u, v), \quad \forall u \in V \\
& f(u, v) \ge 0, \quad \forall (u, v) \in E
\end{aligned}
$$

4. (15 pts) Given a graph $G(V, E)$, the Coverage problem asks you to find a minimum subset of vertices $S \subseteq V$ such that for each edge, at least one of its endpoints is in S. Let the minimum coverage set be S^*.

   (a) (10 pts) Formulate the LP for the minimum coverage set problem and write its dual.

   (b) (5 pts) Let M be a matching of graph G, and $|M|$ be the number of edges in M. Use weak LP duality to prove that $|M| \le |S^*|$.

Solution: (a)

$$
\begin{aligned}
\min \quad & \sum_{i \in V} x_i \\
\text{s.t.} \quad & x_i + x_j \geq 1, \quad \forall e = (i, j) \in E \\
& x_i \geq 0, \quad \forall i \in V
\end{aligned}
$$

The dual of this problem is:

$$
\begin{aligned}
\max \quad & \sum_{e \in E} y_e \\
\text{s.t.} \quad & \sum_{e:\text{one of e's endpoint is i}} y_e \leq 1, \quad \forall i \in V \\
& y_e \geq 0, \quad \forall e \in E
\end{aligned}
$$

(b) For the optimal coverage set S^*, define variable vector x' for the primal as

$$
x'_i = \begin{cases} 1 & , \quad \text{if } i \in S^* \\ 0 & , \quad \text{if } i \notin S^* \end{cases}
$$

Then x' is a feasible solution for the primal. That's because for each edge $(i, j) \in E$, since S^* is a coverage set, either i or j is in S^*, which means either x_i or x_j is 1. We have $x_i + x_j \geq 1$. The objective of x' is $\sum_{i \in V} x'_i = |S^*|$.

Samely, for the matching M, define variable vector y' for the dual as

$$
y'_e = \begin{cases} 1 & , \quad \text{if } e \in M \\ 0 & , \quad \text{if } e \notin M \end{cases}
$$

Then y' is a feasible solution for the dual. That's because for each node i, since M is a matching, among those edges that have endpoint i, at most one of them is in M (and thus have $y'_e = 1$).

The objective of y' in the dual is $\sum_{e \in E} y'_e = |M|$.

Thus by weak duality, $|S^*| \geq |M|$.