COMP 362 - Winter 2015 - Midterm

Name:

1. Consider a flow network G. Prove or Disprove each one of the following statements.

   - (10 points) If e has the lowest capacity among all edges of G, then e is in a minimum cut.

     Solution: Consider the network with the vertices s, v_1, v_2, v_3, a, t, and the edges $sv_1, sv_2, sv_3, v_1a, v_2a, v_3a$ of capacity 1, and the edge at of capacity 2. Then the cut (A, B) with $B = \{t\}$ is the only minimum cut which contains only the edge at.

   - (10 points) If the capacity of e is strictly greater than the capacity of every other edge of G, then e cannot be in a minimum cut.

     Solution: Same as the previous part.

   - (10 points) If G has at least two different maximum flows f and f', then it has at least two different minimum cuts (A, B) and (A', B').

     Solution: Same as the previous part. One can push the 2-unit of flow from s to t through different paths, yet the network has a unique minimum cut.

2. (20 points) Find an efficient algorithm for the following problem: Given a flow network G and an edge e, we want to find out whether e is in every minimum cut.

   Solution: Increase the capacity of e say by 1, and compare the maximum flow in this network with the original network. If they are different it means that the increase in the capacity of e has increased the capacity of every minimum cut, and thus e had to be in every minimum cut. The other direction is also obvious. If e is in every minimum cut, then increasing its capacity will increase the value of every minimum cut, and thus the new value of minimum cut will be higher.

3. (20 Points) Consider a graph where some of the edges have "directions", but some are undirected edges. We want to assign direction to all the undirected edges such that in the resulting directed graph, the incoming degree of every vertex is equal to its outgoing degree. Note that we are not allowed to change the direction of the edges that have directions in the beginning. Show that this problem can be solved using the max-flow problem.

   Solution: The idea is that every undirected edge ab can contribute 1 to the indegree of a or b, and we want to decide these contributions so that every vertex a will finally have indegree $\deg(a)/2$, where $\deg(a)$ is the number of edges (directed and undirected) incident to a.

   Construct a network flow as in the following. For every vertex a in the graph put a vertex v_a in the network, and for every edge ab in the graph put a vertex v_{ab} in the network. If ab has no direction, then add the edges $v_{ab}v_a$ and $v_{ab}v_b$, each with capacity 1 to the network. If ab has already a direction, say from a to b, then only add the edge $v_{ab}v_b$, and if the direction is from b to a then add $v_{ab}v_a$ instead.

   Add a source s and connect it to each vertex v_{ab} with an edge of capacity 1. Add a sink t, and for every vertex v_a, add the edge v_at with capacity $\deg(a)/2$.

4. (10 points) Draw the feasible region to the following linear program.

$$\begin{array}{ll} \max & x_1 + x_2 \\ \text{s.t.} & x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution: The area below the line $x_1 + x_2 = 4$ and $x_1 + 2x_2 = 6$ and above $x_2 = 0$, and to the right of $x_1 = 0$.

5. (20 points) Formulate the following problem as a linear program. We want to assign non-negative numbers to the edges of a graph such that their sum is maximized, and furthermore the sum of the numbers on the edges incident to every vertex is at most 1.

Solution:

$$\begin{array}{lll} \max & \sum_{uv \in E} x_{uv} & \\ \text{s.t.} & \sum_{uv \in E} x_{uv} \leq 1 & \forall v \in V \\ & x_{uv} \geq 0 & \forall uv \in E \end{array}$$