COMP 362 - Winter 2017 - Assignment 2

Due: 6pm Feb 9th

General rules: In solving these questions you may consult your book; you can discuss high level ideas with each other. But each student must find and write his/her own solution. You should drop your solutions in the assignment drop-off box located in the Trottier Building on the 2nd floor.

1. (5 Points) A farmer can choose from three feeds for his milk cows. The nutritional facts and costs of these feeds are shown in the following table. The minimum daily requirements of nutrients A, B, and C are 65, 82, 70 units, respectively. Write a linear program to determine the mixture of feeds that will supply the minimum nutritional requirement at least cost. (You do not need to solve the linear program).

Feed	A (units/lb)	B (units/lb)	C (units/lb)	Cost /lb
Feed 1	4	7	3	0.10
Feed 2	2	3	4	0.07
Feed 3	5	5	3	0.06

Solution:

Variables: x_1, x_2, x_3 for the quantity of feed $1, 2, 3$ respectively.

$$\begin{aligned} \max \quad & 0.1x_1 + 0.07x_2 + 0.06x_3 \\ \text{s.t.} \quad & 4x_1 + 2x_2 + 5x_3 \geq 65 \\ & 7x_1 + 3x_2 + 5x_3 \geq 82 \\ & 3x_1 + 4x_2 + 3x_3 \geq 70 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

2. (a) (5 Points) Draw the feasible region of the following system of

linear constraints:

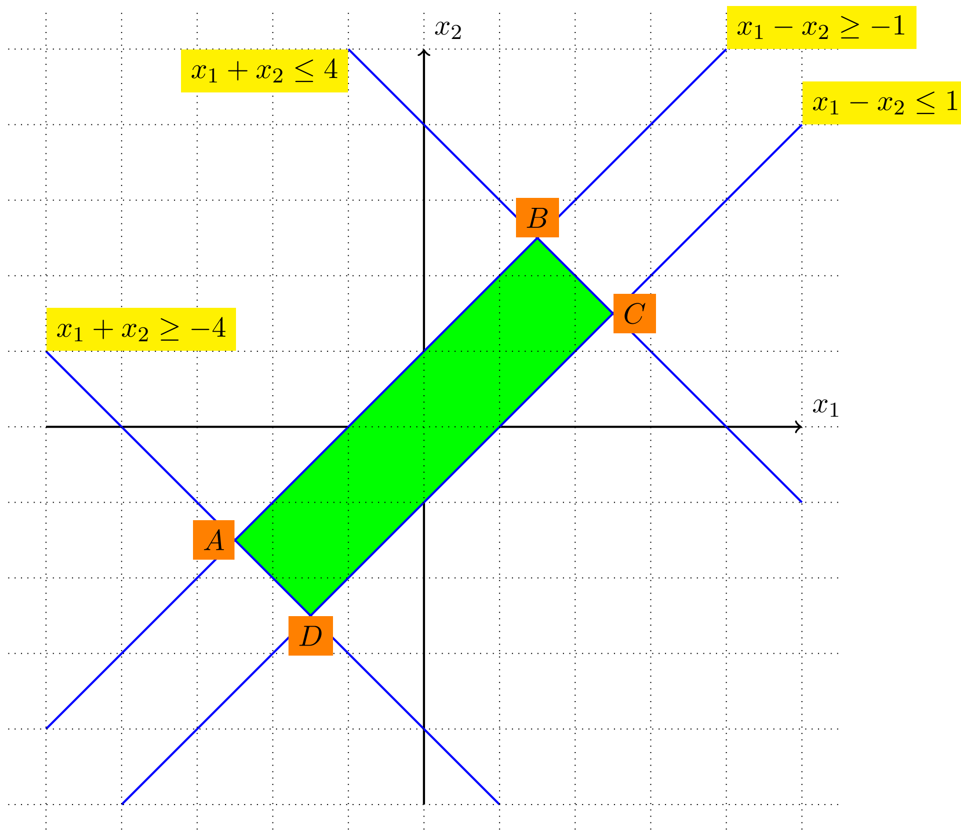
$$\begin{aligned} x_1 - x_2 &\geq -1 \\ x_1 - x_2 &\leq 1 \\ x_1 + x_2 &\leq 4 \\ x_1 + x_2 &\geq -4 \end{aligned}$$

(b) (5 Points) What are the vertices of the feasible region? For each vertex, list the two constraints that define that vertex.

(c) (5 Points) Which vertex maximizes the value of the function $5x_1 + x_2$? Now suppose that we wanted to find this vertex using the simplex algorithm starting from the vertex with the smallest x_1. In other words we start at that vertex and each time we move to a neighbouring vertex that increases the value of $5x_1 + x_2$. List the two possible paths that the algorithm can take starting from that vertex and finishing at the optimal vertex. For each path, explain how the linear constraints that define the vertices on the path change as we move from one vertex to the next vertex.

Solution:

(a) The feasible region is the green part in the following picture:



(b) The vertices of the feasible regions and corresponding constraints:

A	$(-2.5, -1.5)$	$x_1 + x_2 \geq -4$	$x_1 - x_2 \geq -1$
B	$(1.5, 2.5)$	$x_1 + x_2 \leq 4$	$x_1 - x_2 \geq -1$
C	$(2.5, 1.5)$	$x_1 + x_2 \leq 4$	$x_1 - x_2 \leq 1$
D	$(-1.5, -2.5)$	$x_1 + x_2 \geq -4$	$x_1 - x_2 \leq 1$

(c) Vertex C maximizes $5x_1 + x_2$. Vertex A has the smallest value of x_1.

Starting from A, the algorithm has two paths to reach C:

(i) $A \to B \to C$. First replace $x_1 + x_2 \geq -4$ with $x_1 + x_2 \leq 4$. Then replace $x_1 - x_2 \geq -1$ with $x_1 - x_2 \leq 1$.

(ii) $A \to D \to C$. First replace $x_1 - x_2 \geq -1$ with $x_1 - x_2 \leq 1$. Then replace $x_1 + x_2 \geq -4$ with $x_1 + x_2 \leq 4$.

3. (10 Points) Formulate the following problem as a linear program: Given $a_1, \ldots, a_n \geq 0$, and an $n \times n$ matrix A, we want to find $x_1, \ldots, x_n \geq 0$ (and not all of them zero) such that

$$\frac{a_1x_1 + a_2x_2 + \ldots + a_nx_n}{x_1 + x_2 + \ldots + x_n}$$

is maximized and

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \leq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Solution:

$$\begin{aligned} \max \quad & a_1x_1 + a_2x_2 + \ldots + a_nx_n \\ \text{s.t.} \quad & x_1 + x_2 + \ldots + x_n = 1 \\ & A\vec{x} \leq \vec{0} \\ & x_1, \ldots, x_n \geq 0 \end{aligned}$$

4. Consider the following variant of the Rock-paper-scissors game. Alice and Bob each can form one of the n shapes with an outstretched hand. There is an $n \times n$ matrix $A = [a_{ij}]_{1 \leq i,j \leq n}$ that determines who wins the game. If Alice forms the i-th shape and Bob forms the j-th shape, then the value of $a_{ij} \in \{0, 1\}$ tells us who wins (there are no ties): Here $a_{ij} = 1$ means Alice wins, and $a_{ij} = 0$ means Bob wins.

   (a) (10 points) Suppose that Alice has a strategy: She forms the 1-st shape with probability p_1, the 2-nd shape with probability p_2, etc. Obviously $p_1, \ldots, p_n \in [0, 1]$ and they add up to one. Bob also has a strategy according to which he forms the 1-st shape with probability q_1, the 2-nd shape with probability q_2, etc. If they play according to these strategies, what is the formula for the probability that Alice wins?

   Solution:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} p_i q_j A_{ij}.$$

(b) (20 points) Alice wants to maximize the probability that she wins, and Bob wants to minimize the probability that Alice wins (thus maximize the probability that he wins). Consider now two scenarios: In the first scenario Alice chooses her strategy (i.e. p_i's) first, and then announces it to Bob, and then Bob chooses his best strategy knowing Alice's strategy. In the second scenario first Bob chooses his strategy (i.e. q_i's) first and announces it to Alice, and then Alice chooses her best strategy knowing Bob's strategy. Use strong duality theorem to prove that in both scenarios Alice ends up with the same probability of winning (and consequently Bob too). In other words, prove that

$$\max_p \min_q \Pr[\text{Alice wins}] = \min_q \max_p \Pr[\text{Alice wins}].$$

Solution: Note that for a fixed p,

$$\min_q \Pr[\text{Alice wins}] = \min_j \sum_{i=1}^{n} p_i A_{ij},$$

and thus the left hand side corresponds to the following linear program:

$$\begin{array}{lll} \max & \alpha & \\ \text{s.t.} & p_1 + p_2 + \ldots + p_n = 1 & \\ & \displaystyle\sum_{i=1}^{n} A_{ij} p_i - \alpha \geq 0 & \forall j \\ & p_1, \ldots, p_n \geq 0 & \end{array}$$

Similarly for a fixed q,

$$\max_p \Pr[\text{Alice wins}] = \max_i \sum_{j=1}^{n} q_j A_{ij},$$

and thus the right hand side corresponds to the following linear program:

$$\begin{aligned}
\min \quad & \beta \\
\text{s.t.} \quad & q_1 + q_2 + \ldots + q_n = 1 \\
& \sum_{j=1}^{n} A_{ij} q_i - \beta \leq 0 && \forall i \\
& q_1, \ldots, q_n \geq 0
\end{aligned}$$

It is straightforward to see that these two linear programs are dual, and thus by strong duality have the same answer.

5. Consider a finite set $\Phi = \{\vec{v}_1, \ldots, \vec{v}_m\}$ consisting of n-dimensional vectors, and let $\vec{w}$ be another n-dimensional vector. We want to approximate $\vec{w}$ by a vector $\vec{v} \in \text{span}(\Phi)$ so as to minimize the largest difference in the entries (i.e. we want to minimize $\max_{j=1,\ldots,n} |\vec{w}(j) - \vec{v}(j)|$ over all $\vec{v} \in \text{span}(\Phi)$).

   (a) (20 Points) Formulate this as a linear program.

   Solution: We want to solve the following problem with variables α and $x_1, \ldots, x_n$:

$$\begin{aligned}
\min \quad & \alpha \\
\text{s.t.} \quad & \left|\sum_{i=1}^{m} \vec{v}_i(j) x_i - \vec{w}(j)\right| \leq \alpha && \forall j \in \{1, \ldots, n\}
\end{aligned}$$

   This can be formulated as a linear program:

$$\begin{aligned}
\min \quad & \alpha \\
\text{s.t.} \quad & \sum_{i=1}^{m} \vec{v}_i(j) x_i + \alpha \geq \vec{w}(j) && \forall j \in \{1, \ldots, n\} \\
& \sum_{i=1}^{m} -\vec{v}_i(j) x_i + \alpha \geq -\vec{w}(j) && \forall j \in \{1, \ldots, n\}
\end{aligned}$$

(b) (20 Points) Use the strong duality theorem to show that

$$\min_{\vec{v}\in \text{span}(\Phi)} \max_{j=1,\ldots,n} |\vec{w}(j) - \vec{v}(j)|$$

is equal to

$$\max_{\vec{u}} \langle \vec{w}, \vec{u} \rangle$$

where the maximum is over all n-dimensional vectors $\vec{u}$ that satisfy $\sum_{j=1}^{n} |\vec{u}(j)| \leq 1$, and are orthogonal to all the vectors in Φ (i.e. $\langle \vec{u}, \vec{v}_i \rangle = 0$ for $i = 1, \ldots, m$).

Solution: Taking the dual of the previous program we get the following linear program

$$\begin{array}{lll}
\max & \displaystyle\sum_{j=1}^{n} \vec{w}(j)(y_j - z_j) & \\
\text{s.t.} & \displaystyle\sum_{j=1}^{n} y_j + z_j = 1 & \\
 & \displaystyle\sum_{i=1}^{m} \vec{v}_i(j)(y_j - z_j) = 0 & \forall i \in \{1, \ldots, m\} \\
 & y_j, z_j \geq 0 & \forall i \in \{1, \ldots, n\}
\end{array}$$

Consider the change of variables: $u_j = y_j - z_j$ and $a_i = y_j + z_j$. Then the conditions $y_j, z_j \geq 0$ translate to $a_j + u_j \geq 0$ and $a_j - u_j \geq 0$, or equivalently $a_j \geq |u_j|$. Thus we can rewrite the above as the following optimization problem (no longer in LP form):

$$\begin{array}{lll}
\max & \displaystyle\sum_{j=1}^{n} \vec{w}(j)u_j & \\
\text{s.t.} & \displaystyle\sum_{j=1}^{n} a_j = 1 & \\
 & \displaystyle\sum_{i=1}^{m} \vec{v}_i(j)u_j = 0 & \forall i \in \{1, \ldots, m\} \\
 & a_j \geq |u_j| & \forall i \in \{1, \ldots, n\}
\end{array}$$

One can easily get rid of the a_j variables and simplify this to the following as desired.

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} \vec{w}(j) u_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} |u_j| \leq 1 \\
& \sum_{i=1}^{m} \vec{v}_i(j) u_j = 0 \qquad\qquad \forall i \in \{1, \ldots, m\}
\end{aligned}
$$