1. (5 Points) Show that the following language is NP-complete:

   - Input: A CNF ϕ.
   - Question: Is G satisfiable with at least 2 different assignments?

   Solution: First note that the problem is in NP. Indeed if $\langle \phi \rangle \in X$, then the two different assignments satisfying ϕ are certificates, and their validity can be verified in polytime.

   To prove the completeness we reduce SAT to this. Let $\langle \psi \rangle$ be an instance of the SAT problem. Let y be a new variable that does not appear in ψ and set $\phi := \psi \wedge (y \vee \overline{y})$. Note that any satisfying assignment to ψ can be extended to a satisfying assignment to ϕ in two different ways (setting $y = true$ or $y = false$). Thus

   $$(\psi \text{ is satisfiable}) \iff (\phi \text{ has at least two different solutions}).$$

   So an oracle for this problem can be used to decide whether ψ is satisfiable efficiently.

2. (10 Points) A path of length k contains $k+1$ vertices. Show that the following problem is NP-complete:

   - Input: A graph G and a number k.
   - Question: Is is possible to remove k vertices from G to eliminate all paths of length 2?

   Solution: First note that the problem is in NP. Indeed if $\langle G, k \rangle$ is a YES instance, then the set of k vertices whose removal eliminates all paths of length 2 is a certificate, and its validity can be verified in polytime: We remove those vertices and verify that no vertex has more than one neighbour.

   To prove the completeness we reduce VertexCover to this. Let $\langle G, k \rangle$ be an instance of the vertex cover. Construct the graph H from G by attaching a pendant edge to every vertex in G. First note that if S is a vertex cover in G, then removing the vertices of S from H leaves no path of length 2 in H.

   On the other hand let T be a minimum set of vertices whose removal eliminates all paths of length 2 in H. Without loss of generality we can assume that all vertices in T are in the original graph G (why?). Then T must be a vertex cover for G as otherwise there will be an edge in G whose endpoints are not removed by T and that leaves a path of length 2 in H (using the pendant edges).

   We conclude that the size of the minimum vertex cover in G is equal to the minimum number of vertices required to be removed from H to eliminate all paths of length 2. So $\langle G, k \rangle$ is a yes input to vertex cover if and only if $\langle H, k \rangle$ is a yes input to the above problem, and thus an oracle for the above problem can be used to solve the vertex cover problem efficiently.

3. (15 points) The complete bipartite graph $K_{m,n}$ is the bipartite graph with parts of sizes m and n, respectively, such that every vertex from the first part is adjacent to every vertex in the second part.

   Either prove that the following problem is NP-complete or show that it belongs to P by giving a polynomial time algorithm:

- Input: A bipartite graph G, and a positive integer m.
- Question: Does G contain a copy of $K_{m,m}$ as a subgraph?

Solution: First note that the problem is in NP. Indeed if $\langle G, m\rangle$ is a YES instance, then the set of $2m$ that form a copy of $K_{m,m}$ is a certificate, and its validity can be verified in polytime.

To prove completeness we reduce the clique problem to this. Consider an instance $\langle H, m\rangle$ for the clique problem. Construct the bipartite graph G as following. For every vertex v of H add put two vertices v_1 and v_2, together with the edges v_1v_2 in G. Nnext for every edge $uv \in E(H)$, add the two edges u_1v_2 and v_1u_2 to G. Note that a clique of size m in H corresponds to a copy of $K_{m,m}$ in G and vice versa. Hence if we have an oracle that solves the above problem, we can use it to solve the clique problem.

4. (10 Points) Let $G = (V, E)$ be a graph. Recall that $S \subseteq V$ is a vertex cover if and only if $V - S$ is an independent set. Also recall that the following is a 2-factor approximation algorithm for vertex cover: Pick any maximal matching M in G and let S be the set of all vertices involved in M. Output S.

   Is it true that the following is a 2-factor approximation algorithm for the maximum independent set problem? Pick any maximal matching M in G and let S be the set of all vertices involved in M. Output $V - S$.

   Solution: No it is not true. For example if G is a single edge, then the algorithm outputs 0 while the size of the largest independent set is 1 (and $0 \leq \frac{1}{2}$).

5. (10 Points) Let G be a 4-regular graph on n vertices (4-regular means that every vertex is adjacent to 4 edges). We want to color the *edges* of G with two colors Red and Blue such that the number of vertices that are adjacent to exactly two Red and two Blue edges is maximized. If we color the edges at random, then what is the expected number of vertices that satisfy the above condition?

   Solution: For a every vertex $v \in G$, define the random variable

$$X_v = \begin{cases} 1 & \text{two red and two blue edges are incident to } v \\ 0 & \text{otherwise} \end{cases}$$

Note that $\sum_{v \in V} X_v$ is the number of vertices that are adjacent to exactly two Red and two Blue edges. Thus the expected number of vertices that satisfy this condition is $\mathbb{E}[\sum_{v \in V} X_v] = \sum_{v \in V} \mathbb{E}[X_v]$. Now

$$\mathbb{E}[X_v] = \Pr[X_v = 1] \times 1 + \Pr[X_v = 0] \times 0 = \Pr[X_v = 1] = \binom{4}{2}\left(\frac{1}{2}\right)^4,$$

where in the last equality we used the fact that there are $\binom{4}{2}$ ways to color the 4 edges incident to v so that exactly two of them are Red and two Blue, and that each such coloring happens with probability $\left(\frac{1}{2}\right)^4$. We conclude that the expected number of vertices that satisfy the desired condition is $n\binom{4}{2}\left(\frac{1}{2}\right)^4 = \frac{3n}{8}$.

6. (10 Points) Consider the following optimization problem: Given a graph G on $\underline{2n}$ vertices we want to eliminate the maximum possible number of edges from G by deleting exactly n vertices. Show that the following is a $\frac{1}{2}$-factor approximation algorithm: Let $v_1, \ldots, v_{2n}$ be all the vertices. Try deleting each one of the two sets $\{v_1, \ldots, v_n\}$ and $\{v_{n+1}, \ldots, v_{2n}\}$ separately and output the one that removes more edges.

   Solution: Let E_1 be the set of the edges removed by deleting $\{v_1, \ldots, v_n\}$ and E_2 be the set of the edges removed after deleting $\{v_{n+1}, \ldots, v_{2n}\}$. Obviously $E_1 \cup E_2 = E$ and hence either $|E_1| \geq |E|/2$ or $|E_2| \geq |E|/2$. Thus the output of the algorithm is at least $|E|/2$ which is at least half of the optimal solution.

7. Consider a graph $G = (V, E)$. The chromatic number of G is the minimum number of colors required to color the vertices of G properly. Let $\mathcal{I}$ be the set of all independent sets in G (Note that every element in $\mathcal{I}$ is a set).

   (a) (10 Points) Prove that the solution to the following linear program provides a lower-bound for the chromatic number of G.

$$\begin{array}{lll} \min & \sum_{I \in \mathcal{I}} x_I & \\ \text{s.t.} & \sum_{I : v \in I} x_I \geq 1 & \forall v \in V \\ & x_I \geq 0 & \forall I \in \mathcal{I} \end{array}$$

   Solution: Let $x_I = 1$ if I is one of the color classes and $x_I = 0$ otherwise. Since there are $\chi(G)$ color classes the cost of this solution is $\chi(G)$, and since every vertex belongs to some color class, this is a feasible solution.

   (b) (10 Points) Write the dual of the above linear program.

$$\begin{array}{lll} \max & \sum_{v \in V(G)} y_v & \\ \text{s.t.} & \sum_{v \in I} y_v \leq 1 & \forall I \in \mathcal{I} \\ & y_v \geq 0 & \forall v \in V(G) \end{array}$$

   (c) (5 Points) Prove that every clique in G provides a solution to the dual linear program.

   Solution: Let $y_v = 1$ if v belongs to the clique, and $y_v = 0$. No independent can have more than 1 vertex from the clique. Hence this is a feasible solution.

8. (15 Points) Consider the SDP relaxation for chromatic number:

$$\begin{array}{lll} \min & t & \\ \text{s.t.} & \langle v_i, v_j \rangle \leq t & \forall\ ij \in E \\ & \langle v_i, v_i \rangle = 1 & \end{array}$$

   What is the relation of the optimial solution to this SDP and the chromatic number?

   Solution: See the notes for Lecture 24.