Solutions for assignment 3 of COMP 360

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Exercise 1 (10 points)

For example, both the problem

$$\begin{aligned} \max \quad & 2x_1 - x_2 \\ \text{s.t} \quad & x_1 - x_2 \le 1 \\ & -x_1 + x_2 \le -2 \\ & x_1, x_2 \ge 0, \end{aligned}$$

and its dual

$$\begin{aligned} \min \quad & y_1 - 2y_2 \\ \text{s.t} \quad & y_1 - y_2 \ge 2 \\ & -y_1 + y_2 \ge -1 \\ & y_1, y_2 \ge 0, \end{aligned}$$

are infeasible.

Exercise 2 (40)

(a)

$$\begin{aligned} \min \quad & \sum_{u \in V} y_u \\ \text{s.t} \quad & y_u + y_v \ge c_{uv} \quad \forall (u,v) \in E \\ & y_u \ge 0 \quad \forall u \in V. \end{aligned}$$

(b)

Let $\mathcal{C}$ be the set of cycles in G.

$$\begin{aligned} \max \quad & \sum_{C \in \mathcal{C}} y_C \\ \text{s.t} \quad & \sum_{\substack{C \in \mathcal{C}: \\ (u,v) \in C}} y_C \le c_{uv} \quad \forall (u,v) \in E \\ & y_C \ge 0 \quad \forall C \in \mathcal{C}. \end{aligned}$$

(c)

$$\begin{aligned}
\min \quad & 100y' + \sum_{u \in V} y_u \\
\text{s.t} \quad & y' + y_u \geq \sum_{(u,v) \in E} c_{uv} && \forall u \in V \\
& y' \geq 0 \\
& y_u \geq 0 && \forall u \in V.
\end{aligned}$$

(d)

$$\begin{aligned}
\max \quad & \sum_{u \in V} y_u \\
\text{s.t} \quad & \sum_{u \in S} y_u \leq 1 && \forall S \in \mathcal{I} \\
& y_u \geq 0 && \forall u \in V.
\end{aligned}$$

Exercise 3 (30 points)

Note: The minimum number needed to color a graph is often referred to as the **chromatic number**.

Assume that we have a valid coloring of G with n colors. Let S_i be the set of vertices that are colored with color i. Since all vertices are colored, we have

$$\bigcup_{j=1}^{n} S_j = V.$$

Since no adjacent vertices can have the same color, $S_1, \ldots, S_n$ are all independent sets. Let

$$x_S = \begin{cases} 1 & \text{if } S \in \{S_1, \ldots, S_n\} \\ 0 & \text{otherwise} \end{cases} \quad (S \in \mathcal{I}).$$

We claim that this is a solution for the primal problem in $2(d)$. As $\cup_{j=1}^{n} S_j = V$, given an vertex $u \in V$, we must be able to find a S_i such that $x \in S_i$. In other words, we can find $x_S = 1$ for some independent set S with $u \in S$. Put differently, we have

$$\sum_{S \in \mathcal{I}: u \in S} x_S \geq 1.$$

Therefore, $(x_S, S \in \mathcal{I})$ is indeed a solution of $2(d)$.

In other words, a coloring with n colors is equivalent to a valid solution of the linear programming problem with n as the value of objective function. The optimal solution can not be bigger than the chromatic number. Put differently, the optimal solution is a lower bound of the chromatic number.

Note: Many of you based your argument on the claim that the optimal solution actually gives the chromatic number. This is **NOT** true. The optimal solution might not be an integers, and thus can not be used to derive a coloring.

Exercise 4 (20 points)

Let $y = \sum_{i=1}^{n} x_i$. Let $x'_i = x_i/y$. The original problem is equivalent to

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} a_i x'_i \\
\text{s.t} \quad & A \begin{bmatrix} x'_1 \\ \dots \\ x'_n \end{bmatrix} \leq \begin{bmatrix} 0 \\ \dots \\ 0 \end{bmatrix} \\
& \sum_{i=1}^{n} x'_i = 1 \\
& x'_i \geq 0 \qquad\qquad 1 \leq i \leq n.
\end{aligned}$$