COMP 362 - Winter 2015 - Assignment 5

Due: 6pm April 10th.

General rules: In solving these questions you may consult books but you may not consult with each other. You should drop your solutions in the assignment drop-off box located in the Trottier Building.

1. (25 Points) In the MAX-k-COL problem we are given a graph G and we want to color the vertices of the graph with k-colors so as to maximize the number of edges whose endpoints are in different colors.

   Give an efficient randomized $\frac{k}{k-1}$-factor approximation algorithm for the MAX-k-COL problem. (That is the expected value of the output of the algorithm is at least $\frac{k-1}{k}$ times the optimal solution.)

   Solution: Consider the algorithm that assigns a color from $\{1, \ldots, k\}$ randomly and independently to every vertex. For every edge uv, o=let $X_{uv} = 1$ if u and v receive different colors and $X_{uv} = 0$ otherwise. Note that

   $$\mathbb{E}[X_{uv}] = \Pr[X_{uv} = 0] \times 0 + \Pr[X_{uv} = 1] \times 1 = \Pr[X_{uv} = 1] = \frac{k-1}{k},$$

   as obviously $\Pr[X_{uv} = 0] = 1/k$. Now $X = \sum_{uv \in E} X_{uv}$ is the number of edges whose endpoints are in different colors. Hence by linearity of expectation

   $$\mathbb{E}[X] = \sum_{uv \in E} \mathbb{E}[X_{uv}] = \frac{k-1}{k} m,$$

   where m is the total number of edges. Since the optimal answer is at most m, the output of the algorithm is at least $\frac{k-1}{k}$ times the optimal answer.

2. (25 Points) Consider a k-CNF (each clause contains k terms) with $m < 2^{k-1}$ clauses. Suppose further that the variables involved in any given clause are distinct. Show that the probability that the following randomized algorithm does <u>not</u> succeed in satisfying ϕ is at most 2^{-1000}.

- For $i = 1, \ldots, 1000$:
- Pick a random truth assignment σ.
- If σ satisfies the CNF, then output σ and terminate.

Solution: Consider one round of the algorithm. Define the random variables $X_1, \ldots, X_m$ as in the following. For every $i = 1, \ldots, m$, let $X_i = 1$ if the i-th clause is *NOT* satisfied and $X_i = 0$ otherwise. Obviously the probability that a particular clause is not satisfied is 2^{-k}. Hence

$$\mathbb{E}[X_i] = \Pr[X_i = 1] = 2^{-k}.$$

Let $X = \sum_{i=1}^{m} X_i$ be the number of unsatisfied clauses. Then by the linearity of expectation, we have

$$\mathbb{E}[X] = \sum_{i=1}^{m} \mathbb{E}[X_i] = 2^{-k} m < \frac{1}{2}.$$

This in particular shows that

$$\Pr[\text{CNF is not satisfied}] = \Pr[X \geq 1] \leq \sum_{i=i}^{m} \Pr[X = i] \times i = \mathbb{E}[X] < \frac{1}{2}.$$

Hence the probability that the algorithm fails in all the 1000 rounds is at most 2^{-1000}.

3. (20 Points) Read the excerpt from Samuel Beckett's novel, Molloy, attached to the assignment.

   (a) Consider the method that is described in the first paragraph. What is the probability that in the first $4n$ times that he is putting a pebble in his mouth, he is picking the same four stones n times?

   Solution: The probably that the first four stones are repeated n times is:

   $$\frac{1}{(4^4)^{n-1}}.$$

   (b) Consider the method described in the third paragraph (starting with *All (all!) that was necessary*). What is the probability that the first, seventh and twelfth stone of the first cycle are respectively the sixth and eleventh and sixteenth of the second cycle?

   Solution: $\frac{5! \times 1}{6!} \times (\frac{4! \times 1}{5!})^2 = \frac{1}{150}$

4. Consider the MAX-SAT problem: Given a 2-CNF ϕ (clauses of the form $(x_i \vee x_i)$ and $(x_i \vee \bar{x}_i)$ are not allowed) on n variables $x_1, \ldots, x_n$, we want to find a truth assignment that satisfies the maximum number of clauses.

   (a) (10 Points) As in the case of the MAX-CUT problem, find an "integer quadratic program" formulation of this problem in which the only constraints are $y_i \in \{-1, 1\}$. (Hint: It may help to introduce an auxiliary variable y_0 which indicates whether "True" is associated with -1 or with 1.)

   Solution: Introduce variables $y_i \in \{\pm 1\}$ for each boolean variable x_i and also also introduce an extra variable $y_0 \in \{\pm 1\}$ interpreted in the following way. We will set $x_i = T$ if $y_i = y_0$ and $x_i = F$ otherwise. For every clause involving two variables x_i and x_j we will consider a formula

$$1 - \left(\frac{1 \pm y_i y_0}{2}\right)\left(\frac{1 \pm y_j y_0}{2}\right) = \left(\frac{1}{4} \pm \frac{y_i y_0}{4}\right) + \left(\frac{1}{4} \pm \frac{y_j y_0}{4}\right) + \left(\frac{1}{4} \pm \frac{y_i y_j}{4}\right), \tag{1}$$

   where the $\pm$ signs on the left hand side are picked according to whether the variable is appear as itself or negated in the clause ($-y_i y_0$ if it appears as itself and $-y_i y_0$ if it is negated). For example the clause $(x_i \vee \overline{x_j})$ corresponds to

$$1 - \left(\frac{1 - y_i y_0}{2}\right)\left(\frac{1 + y_j y_0}{2}\right) = \left(\frac{1}{4} - \frac{y_i y_0}{4}\right) + \left(\frac{1}{4} + \frac{y_j y_0}{4}\right) + \left(\frac{1}{4} - \frac{y_i y_j}{4}\right).$$

   Note that this expression is equal to 1 if the clause is satisfied and it is equal to 0 otherwise. This is because $\frac{1 - y_i y_0}{2} = 0$ if x_i is True (i.e. $y_i = y_0$) and it is 1 otherwise, and similarly $\frac{1 + y_i y_0}{2} = 0$ if x_i is False and it is 1 otherwise.

   Now if we sum the expression in the right hand side of (1) over all clauses we obtain an expression $\Phi = \frac{1}{4} \sum_{i<j} \alpha_{ij}(1 + y_i y_j)$ where the coefficients $\alpha_{ij} \in \mathbb{Z}_{\geq 0}$. Note that Φ counts the number of satisfied clauses. Hence the corresponding quadratic program will maximize the number of satisfied clauses.

$$\begin{array}{lll} \max & \frac{1}{4} \sum_{i<j} \alpha_{ij}(1 - y_i y_j) + \sum_{i<j} \beta_{ij}(1 + y_i y_j)) & \\ \text{s.t.} & y_i^2 = 1 & \forall i = 0, \ldots, n \end{array}$$

(b) (20 Points) Formulate the SDP relaxation of your integer quadratic program, and use it to find a randomized $\frac{1}{0.878}$-factor approximation algorithm for the MAX-2-SAT problem.

Solution: The SDP relaxation is

$$\begin{array}{ll} \max & \frac{1}{4}\left(\sum_{i<j} \alpha_{ij}(1-\langle v_i, v_j\rangle) + \sum_{i<j} \beta_{ij}(1+\langle v_i, v_j\rangle)\right) \\ \text{s.t.} & \langle v_i, v_i\rangle = 1 \qquad \forall i = 0, \ldots, n \end{array}$$

Now to round the solution of the SDP to a solution for the quadratic program we pick a random unit vector $w \in \mathbb{R}^n$ and for $i = 0, \ldots, n$ we set $y_i = 1$ if $\langle v_i, w\rangle \geq 0$ and $y_i = -1$ if $\langle v_i, w\rangle < 0$. Now from the analysis of the SDP rounding for MAX-CUT we know that

$$\frac{\mathbb{E}[1 - y_i y_j]}{1 - \langle v_i, v_j\rangle} \geq 0.878,$$

and using a similar analysis we can see that

$$\frac{\mathbb{E}[1 + y_i y_j]}{1 + \langle v_i, v_j\rangle} \geq 0.878.$$

This as in in the analysis of MAX-CUT implies

$$\frac{\mathbb{E}[\sum_{i<j} \alpha_{ij}(1 - y_i y_j) + \sum_{i<j} \beta_{ij}(1 + y_i y_j)]}{\sum_{i<j} \alpha_{ij}(1 - \langle v_i, v_j\rangle) + \sum_{i<j} \beta_{ij}(1 + \langle v_i, v_j\rangle)} \geq 0.878.$$

Since the optimal answer to the SDP is at least as large as the optimal solution to 2-SAT we conclude that the expected value of the rounded solution is at least 0.878 times the optimal solution.

1 An excerpt from Molloy by Samuel Beckett

I took advantage of being at the seaside to lay in a store of sucking-stones. They were pebbles but I call them stones. Yes, on this occasion I laid in a considerable store. I distributed them equally between my four pockets, and sucked them turn and turn about. This raised a problem which I first solved in the following way. I had say sixteen stones, four in each of my four pockets these being the two pockets of my trousers and the two pockets of my greatcoat. Taking a stone from the right pocket of my greatcoat, and putting it in my mouth, I replaced it in the right pocket of my greatcoat by a stone from the right pocket of my trousers, which I replaced by a stone from the left pocket of my trousers, which I replaced by a stone from the left pocket of my greatcoat, which I replaced by the stone which was in my mouth, as soon as I had finished sucking it. Thus there were still four stones in each of my four pockets, but not quite the same stones. And when the desire to suck took hold of me again, I drew again on the right pocket of my greatcoat, certain of not taking the same stone as the last time. And while I sucked it I rearranged the other stones in the way I have just described. And so on. But this solution did not satisfy me fully. For it did not escape me that, by an extraordinary hazard, the four stones circulating thus might always be the same four. In which case, far from sucking the sixteen stones turn and turn about, I was really only sucking four, always the same, turn and turn about. But I shuffled them well in my pockets, before I began to suck, and again, while I sucked, before transferring them, in the hope of obtaining a more general circulation of the stones from pocket to pocket. But this was only a makeshift that could not long content a man like me.

So I began to look for something else and the first thing that I hit upon was that I might do better to transfer the stones four by four, instead of one by one, that is to say, during the sucking, to take the three stones remaining in the right pocket of my greatcoat and replace them by the four in the right pocket of my trousers, and these by the four in the left pocket of my trousers, and these by the four in the left pocket of my greatcoat, and finally these by the three from the right pocket of my greatcoat, plus the one, as soon as I had finished sucking it, which was in my mouth. Yes, it seemed to me at first that by so doing I would arrive at a better result. But on further reflection I had to change my mind and confess that the circulation of the stones four by four came to exactly the same thing as their circulation one by one. For if I was certain of finding each time, in the right pocket of my greatcoat, four stones totally different from their immediate predecessors, the possibility nevertheless remained of my always chancing on the same stone, within each group of four, and consequently of my sucking, not the sixteen turn and turn about as I wished, but in fact four only, always the same, turn and turn about. So I had to seek elsewhere than in the mode of circulation. For no matter how I caused the stones to circulate, I always ran the same risk. It was obvious that by increasing the number of my pockets I was bound to increase my chances of enjoying my stones in the way I planned, that is to say one after the other until their number was exhausted. Had I had eight pockets, for example, instead of the four I did have, then even the most diabolical hazard could not have prevented me from sucking at least eight of my sixteen stones, turn and turn about. The truth is I should have needed sixteen pockets in order to be quite easy in my mind. And for a long time I could see no other conclusion than this, that short of having sixteen pockets, each with its stone, I could never reach the goal I had set myself, short of an extraordinary hazard. And if at a pinch I could double the number of my pockets, were it only by dividing each pocket in two, with the help of a few safety-pins let us say, to quadruple them seemed to be more than I could manage. And I did not feel inclined to take all that trouble for a half-measure. For I was beginning to lose all sense of measure, after all this wrestling and wrangling, and to say, All or nothing. And if I was tempted for an instant to establish a more equitable proportion between my stones and my pockets, by reducing the former to the number of the latter, it was only for an instant. For it would have been an admission of defeat. And sitting on the shore, before the sea, the sixteen stones spread out before my eyes, I gazed at them in anger and perplexity. For just as I had difficulty in sitting on a chair, or in an armchair, because of my stiff leg you understand, so I had none in sitting on the ground, because of my stiff leg and my stiffening leg, for it was about this time that my good leg, good in the sense that it was not stiff began to stiffen. I needed a prop under the ham you understand, end even under the whole length of the leg, the prop of the earth. And while I gazed thus at my stones, revolving interminable martingales all equally defective, and crushing handfuls of sands, so that sand ran through my fingers and fell back on the sand, yes, while thus I lulled my mind and part of my body, one day suddenly it dawned on the former, dimly, that I might perhaps achieve my purpose without increasing the number of my pockets or reducing the number of my stones, but simply by sacrificing the principle of trim. The meaning of this illumination, which suddenly began to sing within me, like a verse of Isaiah, or of Jeremiah, I did not penetrate at once, and notably the word trim, which I had never met with, in this sense, long remained obscure. Finally I seemed to grasp that this word trim could not here mean anything else, anything better, than the distribution of the sixteen stones in four groups of four, one group in each pocket, and that it was my refusal to consider any distribution other than this had vitiated my calculations until then and rendered the problem literally insoluble. And it was on the basis of this interpretation, whether right or wrong, that I finally reached a solution inelegant assuredly, but sound, sound. Now I am willing to believe, indeed I firmly believe, that other solutions to this problem might have been found, and indeed nay still be found, no less sound, but much more elegant, than the one I shall ow describe, if I can. And I believe too that had I been a little more insistent, a little more resistant, I could have found them myself. But I was tired, but I was tired, and I contended myself ingloriously with the first solution that was a solution, to this problem. But not go over the heartbreaking stages through which I passed before I came to it, here it is, in all its hideousness.

All (all!) that was necessary was to put for example to begin with six stones in the right pocket of my greatcoat, or supply-pocket, five in the right pocket of my trousers, and five in the left pocket of my trousers, that makes the lot, twice five ten plus six, sixteen, and none, for none remained in the left pocket of my greatcoat, which for the time being remained empty, empty of stones that is, for its usual contents remained, as well as occasional objects. For do you think I hid my vegetable knife, my silver, my horn and other things that I have not yet named, perhaps shall never name. Good. Now I can begin to suck. Watch me closely. I take a stone from the right pocket of my greatcoat, suck it, stop sucking it, put it in the left pocket of my greatcoat, the one empty (of stones). I take a second stone from the right pocket of my greatcoat, suck it put it in the left pocket of my greatcoat. And so on until the right pocket of my greatcoat is empty (apart from its usual and casual contents) and the six stones I have just sucked, one after the other, are all in the left pocket of my greatcoat. Pausing then, and concentrating, so as not to make a balls of it, I transfer to the right pocket of my greatcoat, in which there are no stones left, the five stones in the right pocket of my trousers, which I replace by the five stones in the left pocket of my trousers, which I replace by the six stones in the left pocket of my greatcoat. At this stage then the left pocket of my greatcoat is again empty of stones, while the right pocket of my greatcoat is again supplied, and in the right way, that is to say with other stones than those I have just sucked. These other stones I then begin to suck, one after the other, and to transfer as I go along to the left pocket of my greatcoat, being absolutely certain, as far as one can be in an affair of this kind, that I am not sucking the same stones as a moment before, but others. And when the right pocket of my greatcoat is again empty (of stones), and the five I have just sucked are all without exception in the left pocket of my greatcoat, then I proceed to the same redistribution as a moment before, or a similar redistribution, that is to say I transfer to the right pocket of my greatcoat, now again available, the five stones in the right pocket of my trousers, which I replace by the six stones in the left pocket of my trousers, which I replace by the five stones in the left pocket of my greatcoat. And there I am ready to begin again. Do I have to go on? No, for it is clear that after the next series, of sucks and transfers, I shall be back where I started, that is to say with the six stones back in the supply-pocket, the next five in the right pocket of my stinking old trousers and finally the last five in left pocket of same, and my sixteen stones will have been sucked once at least in impeccable succession, not one sucked twice, not one left unsucked. It is true that the next time I could scarcely hope to suck my stones in the same order as the first time and that the first, seventh and twelfth for example of the first cycle might very well be the sixth and eleventh and sixteenth respectively of the second, if worst came to the worst. But that was a drawback I could not avoid. . .

There was something more than a principle I abandoned, when I abandoned the equal distribution, it was a bodily need. But to suck the stones in the way I have described, not haphazard, but with method, was also I think a bodily need. Here then were two incompatible bodily needs, at loggerheads. Such things happen. But deep down I didn't give a tinker's curse about being off my balance, dragged to the right hand and the left, backwards and forwards. And deep down it was all the same to me whether I sucked a different stone each time or always the same stone, until the end of time. For they all tasted exactly the same. And if I had collected sixteen, it was not in order to ballast myself in such and such a way, or to suck them turn about, but simply to have a little store, so as never to be without. But deep down I didn't give a fiddler's curse about being without, when they were all gone they would be all gone, I wouldn't be any the worse off, or hardly any. And the solution to which I rallied in the end was to throw away all the stones but one, which I kept now in one pocket, now in another, and which of course I soon lost, or threw away, or gave away, or swallowed. . .