COMP 360 - Winter 2014 - Assignment 3

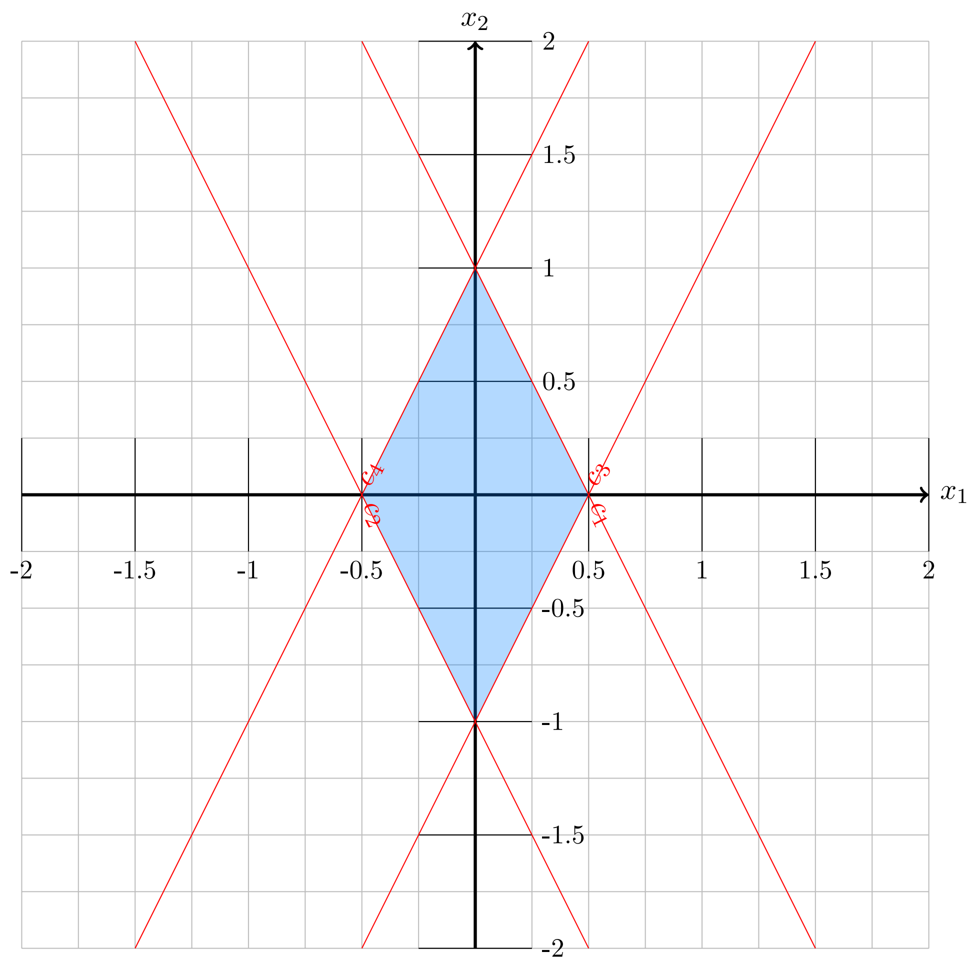
Solution

General rules: In solving these questions you may consult books but you may not consult with each other. There are in total 110 points, but your grade will be considered out of 100. You should drop your solutions in the assignment drop-off box located in the Trottier Building.

1. (15 Points) Draw the feasible region of the following system of linear constraints:

$$\begin{aligned} 2x_1 + x_2 &\leq 1 \\ 2x_1 + x_2 &\geq -1 \\ 2x_1 - x_2 &\leq 1 \\ 2x_1 - x_2 &\geq -1 \end{aligned}$$

Soloution:



2. We have a supply of m different kinds of material. Specifically, we have b_j units of raw material of kind j. They are n different kinds of products that we can make from these raw

material. Each unit of product i takes a_{i1} unit of raw material 1, a_{i2} units of raw material 2 etc. to create, and can be sold at the price c_i dollars. It is possible to sell fractional amounts of any product. Our goal is to produce the most profitable set of products with our available material.

(a) (15 Points) Formulate the above optimization problem as a linear program.

Solution: We will create a variable x_i representing the amounts of product i produced. We then have the following linear program:

$$\begin{aligned} &\max\ c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ &\text{s.t. } \sum_{i=1}^{n} a_{ik}x_i \leq b_k \ \forall\ 1 \leq k \leq m \\ &\qquad x_1, x_2, \ldots, x_n \geq 0 \end{aligned}$$

(b) (15 Points) Write the dual of this program.

Solution: We introduce variables $y_1, \ldots, y_m$ for each b_i. We then have the following linear program:

$$\begin{aligned} &\min\ b_1y_1 + b_2y_2 + \ldots + b_my_m \\ &\text{s.t. } \sum_{i=1}^{m} a_{ki}y_i \geq c_k \ \forall\ 1 \leq k \leq n \\ &\qquad y_1, y_2, \ldots, y_m \geq 0 \end{aligned}$$

3. Consider an undirected graph $G = (V, E)$, and the following linear program with variables x_{uv} for edges $uv \in E$.

$$\begin{array}{lll} \max & \sum_{uv \in E} x_{uv} & \\ \text{s.t.} & \sum_{v:uv \in E} x_{uv} & \leq 1 \quad \forall u \in V \\ & x_{uv} & \geq 0 \quad uv \in E \end{array}$$

(a) (10 Points) Show that the optimum solution to this linear program is at least the size of the largest matching in G.

Solution: Given a largest matching M I will show that we can build a feasible (although not necessarily optimal) solution to the problem. For every edge e in the matching, set $x_e = 1$. For every edge e not in the matching, set $x_e = 0$. Clearly every variable has a value ≥ 0. For every vertex $v \in V$, $\sum_{v:uv \in E} x_{uv} \leq 1$ as the vertex can be incident to either 0 or 1 edge in the matching by definition. Thus this solution is feasible, and $Opt(P) \geq |M|$.

(b) (15 Points) Write the dual of this linear program.

Solution: For the dual, we introduce a variable y_v $\forall v \in V$. We then have the following linear program:

$$\begin{aligned} &\min \sum_{v \in V} y_v \\ &\text{s.t. } y_u + y_v \geq 1 \ \forall\ uv \in E \\ &\qquad y_1, y_2, \ldots, y_m \geq 0 \end{aligned}$$

4. Consider a directed graph $G = (V, E)$, and let s and t be two distinct vertices in G. Consider the following linear program with variables x_u for vertices $u \in V$:

$$\begin{array}{lll} \max & x_s - x_t & \\ \text{s.t.} & x_u - x_v & = 0 \quad \forall uv \in E \end{array}$$

(a) (10 Points) What does the solution to this linear program tell us about s and t?

Solution: If there is a path (ignoring the directions) in G from s to t, then the answer to the maximization problem is 0. This implies that if the answer to the maximization problem is not 0, then s and t are in different connected components of G.

(b) (15 Points) Write the dual of the program.

Solution: Assume the variables x_u are free since there are no restrictions on them. The variables are y_{uv} where $uv \in E$.

$$\begin{aligned} \min\ & 0 \\ \text{s.t. } & \textstyle\sum_{u:su\in E} y_{su} - \sum_{u:us\in E} y_{us} = 1 \\ & \textstyle\sum_{u:wu\in E} y_{wu} - \sum_{u:wu\in E} y_{wu} = 0, \text{ for } w \in V \setminus \{s,t\} \\ & \textstyle\sum_{u:tu\in E} y_{tu} - \sum_{u:ut\in E} y_{ut} = -1 \\ & y_{uv} \text{ free for } uv \in E \end{aligned}$$

5. (15 Points) Formulate the following problem as a Linear Program:

- Input: An undirected graph $G = (V, E)$.
- Question: What is the maximum of

$$\frac{\sum_{uv\in E} x_{uv}}{\sum_{u,v\in V} x_{uv}},$$

where x_{uv} are non-negative numbers (and not all zeros) and satisfy the triangle inequality $x_{uv} + x_{vw} \geq x_{uw}$ for all $u, v, w \in V$?

Solution: Let $s = \frac{1}{\sum\limits_{u,v\in V} x_{uv}}$. Then we can write the problem as:

$$\begin{aligned} \max\ & \sum_{uv\in E} x_{uv}s \\ \text{s.t. } & x_{uv} + x_{vw} \geq x_{uw}, \text{ for } u, v, w \in V \\ & \sum_{u,v\in V} x_{uv}s = 1 \end{aligned}$$

and then as

$$\begin{aligned} \max\ & \sum_{uv\in E} x_{uv}s \\ \text{s.t. } & x_{uv}s + x_{vw}s \geq x_{uw}s, \text{ for } u, v, w \in V \\ & \sum_{u,v\in V} x_{uv}s = 1 \end{aligned}$$

Let $y_{uv} = x_{uv}s$. Then we have

$$\begin{aligned} \max\ & \sum_{uv\in E} y_{uv} \\ \text{s.t. } & y_{uv} + y_{vw} \geq y_{uw}, \text{ for } u, v, w \in V \\ & \sum_{u,v\in V} y_{uv} = 1 \\ & y_{uv} \geq 0 \end{aligned}$$

Optimize this problem to get optimal y vector.