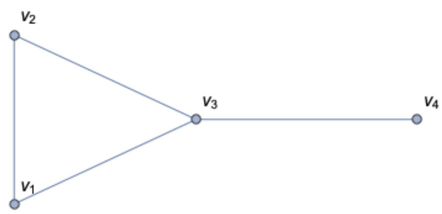
This project is fundamentally about studying graphs, and the key object we use to study those graphs is the graph Laplacian. In this section we will define the graph Laplacian, and explore a few of its properties and how it connects to every other area of the project.

The Graph Laplacian

The graph Laplacian is a matrix, an array of numbers. It is derived from a graph, a set of vertices connected with edges. One can assign an orientation to edges, turning them from lines into arrows. While we will do that at some points in the project, it is not fundamentally needed to define the graph Laplacian. To define the Laplacian, we first need to define a couple other graph matrices. Say we have a graph such as this one, call it Γ:



The *degree* of a vertex v_i is the number of edges incident to that vertex, and denoted $\deg(v_i)$. For example, vertex v_2 has degree 2. We denote the number of vertices in a graph as $|V|$. The *valency matrix* val_Γ of a graph is the $|V| \times |V|$ matrix whose ijth entry (that is, its ith row and jth column) is given by

$$\mathrm{val}_\Gamma(i,j) = \begin{cases} \deg(v_i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

In other words, this is a matrix which is zero every except the diagonal, and the diagonal will contain the degree of each vertex. The valency matrix for the graph above is

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The other matrix one needs to define the graph Laplacian is the *adjacency matrix* A_Γ. This is also a $|V| \times |V|$ square matrix, and its ijth entry is given by

$$A_\Gamma(i,j) = \begin{cases} 1 & \text{if an edge connects } v_i \text{ and } v_j \\ 0 & \text{otherwise} \end{cases}.$$

For our graph, the adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now that we have the valency and adjacency matrices, defining the *graph Laplacian* Δ_Γ is simple:

$$\Delta_\Gamma = \mathrm{val}_\Gamma - A_\Gamma.$$

Thus the graph Laplacian of this graph is

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Why the graph Laplacian?

Now, you may be wondering why the graph Laplacian is defined this way, especially if you're familiar with the Laplacian differential operator from multivariable calculus denoted by the same symbol. Imagine the following: take a graph in the shape of a grid, and successively add more and more boxes to grid. At the same time, take the size of the boxes in the grid and shrink them. As this process repeats indefinitely, the graph will get closer and closer to a model of continuous space. As it turns out, if one looks at how the Laplacian acts in this limit, it will turn into the calculus Laplacian! So in a way, this really is just the discrete version of the calculus Laplacian.

What we really care about , though, is what are called the *eigenvalues* of the Laplacian.

Eigenvalues and Eigenvectors

To understand eigenvalues (and their associated eigenvectors) of a matrix, we must change the way we think about matrices. We can multiply matrices times vectors. Recall that a vector can be thought of as an arrow pointing in space (though for the Laplacian given above, you will need to multiply it by a 4-dimensional vector, so don't try to picture it too hard). For square matrices, this will take a matrix and a vector and give back a vector of the same size. In this way, a square matrix A acts as a function that takes in a vector $\vec{x}$ and outputs another vector $\vec{b}$:

$$A\vec{x} = \vec{b}.$$

The field of linear algebra studies these types matrix functions. One question one can ask about such functions is the following: when does the matrix preserve the direction of a vector, when thought of as an arrow in space? More specifically, is there a vector, call it $\vec{x}$, such that

$$A\vec{x} = \lambda\vec{x},$$

for some number λ? Such a vector and associated value are not guaranteed to exist. If they do, the vector $\vec{x}$ is called an *eigenvector* and λ is called the *eigenvalue* of that eigenvector.

Eigenvalues are one of the most important topics in linear algebra, the field which studies vectors, matrices, and their properties. Eigenvalues are used ubiquitously in computer science, engineering, and physics. For more information about why they are so useful, see the section on Spectral Graph Theory. We will limit our discussion, however, to the use of the eigenvalues of the graph Laplacian.

Spectral Graph Theory

The field of *spectral graph theory* studies the eigenvalues of the graph Laplacian (and other matrices associated with graphs) and connects them with properties of the graph. (The name comes from the fact that the set of eigenvalues of a matrix is often called its *spectrum*.)

To start, we have to know some things about the Laplacian: does it always have eigenvalues? How many does it have? Do they always fall within some range?

The first thing to notice about the example of the Laplacian given above is that it is *symmetric*. This means if you flip it over its diagonal, also known as taking its *transpose*, you get the same thing. In other words, the ijth entry is equal to the jith entry:

$$\Delta_\Gamma(i, j) = \Delta_\Gamma(j, i).$$

This is not a coincidence: for any graph, the Laplacian will be symmetric (think about why that is). It's also very lucky for us, since symmetric matrices have some really nice properties. There's a well-known theorem in linear algebra called the *spectral theorem* that says if an $n \times n$ matrix is symmetric, then it has n eigenvalues, up to multiplicity. Thus the Laplacian has $|V|$ eigenvalues, up to multiplicity.

That last phrase, "up to multiplicity," is a little tricky. Here's what it means: say you have an eigenvector $\vec{x}$ with eigenvalue λ. Now say there's another vector $\vec{y}$ that also has eigenvector λ, but is pointing in a different direction than $\vec{x}$. We say in that case that the eigenvalue λ has *multiplicity 2*. If there were another vector $\vec{z}$ in a different direction than $\vec{x}$ or $\vec{y}$, then λ would be said to have multiplicity 3, and so on.

So the spectral theorem is saying that if you count an eigenvalue with multiplicity 2 as being 2 separate eigenvalues, then a symmetric matrix such as the Laplacian will have as many eigenvalues as there are vertices. That gives us a lot of eigenvalues to work with, especially for big graphs.

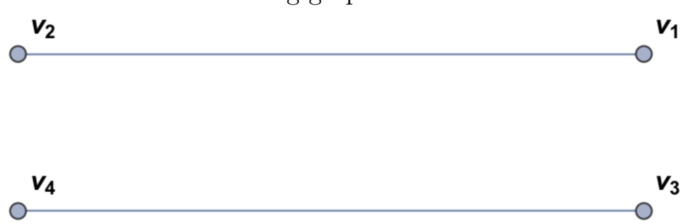
One can also prove that the Laplacian is what's called *positive semidefinite*. This means that all of the eigenvalues are nonnegative (the "semi" part means that some can be 0. In fact, the Laplacian always has at least one 0 eigenvalue). Additionally, one can prove that all the eigenvalues are going to be less than $|V|$.

The next article will give more specific theorems about what the eigenvalues of the Laplacian can tell us about a graph. We state one result here, however:

Theorem 1. *The multiplicity of the eigenvalue 0 of the graph Laplacian is equal to the number of connected components of the graph.*

For example, the example we've been using has eigenvalues $\{4, 3, 1, 0\}$. We can see its eigenvalues are in the range 0 to 4, and it has one 0 eigenvalue, corresponding to the single connected component of the graph.

Here's another example. Let Γ be the following graph:



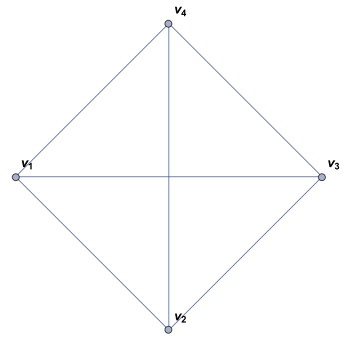
Then the Laplacian will be given by

$$\Delta_\Gamma = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

with eigenvalues $\{2, 2, 0, 0\}$, with two zeroes corresponding to the two connected components of the graph.

Recap

Given a graph Γ:



we can derive a matrix:

$$\Delta_\Gamma = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

which is going to have eigenvalues. Here, the eigenvalues are 4, 4, 4, and 0.

We know there will always be $|V|$ eigenvalues, and they will always lie between 0 and $|V|$. The number of 0 eigenvalues corresponds to the number of connected components of the graph.

To see more about what these eigenvalues can tell us about a graph, see the section on Spectral Graph Theory.