HA3

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Exercise A.a. Suppose that we want to prove that

$$P(n) : \frac{1}{2} \cdot \frac{3}{4} \cdot ... \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

for all positive integers n. Show that if we try to prove this inequality using mathematical induction, the basis step works, but the inductive step fails.

Basis step: For $P(1)$ we have $\frac{2 \cdot 1 - 1}{2 \cdot 1} < \frac{1}{\sqrt{3 \cdot 1}}$. Because $2^2 = 4$, the square root of 3 must be less than 2, so we see that $(1/2) < (1/\sqrt{3})$. Therefore $P(1)$ is true.

Inductive step: Assume $P(k)$, that is, $\frac{1}{2} \cdot \frac{3}{4} \cdot ... \cdot \frac{2k-1}{2k} < \frac{1}{\sqrt{3k}}$. Now we want to show that $P(k+1)$ is true, that is, $\frac{1}{2} \cdot \frac{3}{4} \cdot ... \cdot \frac{2(k+1)-1}{2(k+1)} < \frac{1}{\sqrt{3(k+1)}}$.

We can multiply $(2(k+1)-1)$ / $(2(k+1))$ to both sides of the inequality. This gives us the left hand side immediatiately. Then we can check if the right hand side is less than what we want to find, since that will hold the inequality

$$\begin{aligned}
\frac{1}{2} \cdot \frac{3}{4} \cdot ... \cdot \frac{2k-1}{2k} \cdot \frac{2(k+1)-1}{2(k+1)} &< \frac{1}{\sqrt{3k}} \cdot \frac{2(k+1)-1}{2(k+1)} \\
&< \left(\frac{1}{\sqrt{3k}} \cdot \frac{2k+1}{2k+2}\right)^2 < \left(\frac{1}{\sqrt{3(k+1)}}\right)^2 \\
&< \frac{1}{3k} \cdot \frac{(2k+1)^2}{(2k+2)^2} < \frac{1}{3(k+1)} \\
&< \frac{1}{k} \cdot \frac{(2k+1)^2}{(2k+2)^2} < \frac{1}{k+1} \\
&< \frac{(2k+1)^2}{4k(k+1)^2} < \frac{1}{k+1} \\
&< \frac{4k^2+4k+1}{4k(k+1)} < 1 \\
&< 4k^2+4k+1 < 4k(k+1) \\
&< 1 < 0
\end{aligned} \tag{1}$$

This fails.

Exercise A.b. Show that mathematical induction can be used to prove the stronger inequality

$$P(n) : \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$$

for all integers greater than 1, which, together with the verification for the case where $n = 1$, establishes the weaker inequality we originally tried to prove using mathematical induction.

Basis step: For $P(2)$ we have $\frac{1}{2} \cdot \frac{2 \cdot 2 - 1}{2 \cdot 2} < \frac{1}{\sqrt{3 \cdot 2 + 1}}$. We see that $\frac{3}{8} < \frac{1}{\sqrt{7}}$. So $P(2)$ is true.

Inductive step: Assume $P(k)$, that is, $\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2k-1}{2k} < \frac{1}{\sqrt{3k+1}}$. Now we want to show that $P(k+1)$ is true, that is, $\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2(k+1)-1}{2(k+1)} < \frac{1}{\sqrt{3(k+1)+1}}$.

We can again multiply the same term to both sides to get what we want on the LHS immediately. Now we can check the if the inequality holds for the RHS as follows

$$\begin{aligned}
\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2k-1}{2k} \cdot \frac{2(k+1)-1}{2(k+1)} &< \frac{1}{\sqrt{3k+1}} \cdot \frac{2(k+1)-1}{2(k+1)} \\
&< \left(\frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2} \right)^2 < \left(\frac{1}{\sqrt{3(k+1)+1}} \right)^2 \\
&< \frac{(2k+1)^2}{(3k+1)(2k+2)^2} < \frac{1}{3k+4} \\
&< \frac{(2k+1)^2}{4(3k+1)(k+1)^2} < \frac{1}{3k+4} \\
&< (3k+4)(2k+1)^2 < 4(3k+1)(k+1)^2 \\
&< 12k^3 + 28k^2 + 19k + 4 < 12k^3 + 28k^2 + 20k + 4 \\
&< 19k < 20k
\end{aligned} \tag{2}$$

Since $19k < 20k$, we can replace it and keep the inequality as such

$$\begin{aligned}
\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2(k+1)-1}{2(k+1)} &< \frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3(k+1)+1}} < \frac{1}{\sqrt{3(k+1)}} \\
\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2(k+1)-1}{2(k+1)} &< \frac{1}{\sqrt{3(k+1)}}
\end{aligned} \tag{3}$$

That proves the stronger inequality for $n \geq 2$. $\square$

Furthermore, this shows that our weaker inequality in Exercise A.a. holds for all positive integers $n \geq 2$. We can verify for the case where $n = 1$ to show that the weaker inequality also holds for all positive integers. For $P(1)$ we have $(2 \cdot 1 - 1)/(2 \cdot 1) < (1)/(\sqrt{3 \cdot 1})$. We see that $(1/2) < (1/\sqrt{3})$. Therefore $P(1)$ is true. Hence we have established that the weaker inequality $P(n)$ in Exercise A.a. holds for all positive integers n.

Exercise B. Use strong induction to show that every positive integer can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1, 2^1 = 2, 2^2 = 4$, and so on.

Let $P(n)$ be the statement "n can be written as a sum of distinct powers of two."

Basis step: For $P(1)$ we have $2^0 = 1$. Therefore $P(1)$ is true.

Inductive step: Assume $P(k)$ true for $1, ..., k$, that is, assume $1, ..., k$ can be written as a sum of distinct powers of two. $2^i + ... + 2^n = k$. We then want to show that $k+1$ can be written as a sum of distinct powers of two $2^i + ... + 2^n = k+1$.

Consider the case when $k+1$ is even:

By the inductive hypothesis, $k = 2^i + ... + 2^n$. Then $(k+1)/2 = m$ can also be written as $(k+1)/2 = m = 2^i + ... + 2^n$ since we assumed it is true for all up to k. Then 2 is divible by k+1 s.t.

$$\begin{aligned} 2 \mid k+1 \rightarrow k+1 &= 2m \\ k+1 &= 2 \cdot (2^i + ... + 2^n) \\ k+1 &= 2^{i+1} + .. + 2^{n+1} \end{aligned} \tag{4}$$

Since $k = 2^i + .. + 2^n$ can be written as a sum of distinct powers of two, then $k+1 = 2^{i+1} + ... + 2^{n+1}$ will also be distinct, that is, all even positive integers.

Consider the case when $k+1$ is odd:

Note that, if $k+1$ is odd, then k is even and can be written as the sum of distinct powers of two by the inductive hypothesis, so

$$\begin{aligned} k &= 2^i + ... + 2^n \\ k+1 &= 2^i + ... + 2^n + 1 \\ k+1 &= 2^i + ... + 2^n + 2^0 \end{aligned} \tag{5}$$

Then it can be written as above, where $2^0 = 1$ can be added onto the term. Since k is even, we know it will be some $2c$, $c \in \mathbb{Z}$, and therefore adding 2^0 will still be distinct. We have now shown that for both even and odd $k+1$, they can be written as the sum of distinct powers of two, and hence it will be true for all positive integers. $\square$

Exercise C. Use structural induction to show that $l(T)$, the number of leaves of a full binary tree T, is 1 more than $i(T)$, the number of internal vertices of T.

Basis step: For one root r we have $l(T) = 1$ and $0 = i(T)$. We see that $l(T)$ is 1 more than $i(T)$. Therefore the basis step is true.

Recursive step: Now we can split the tree up into T_1 and T_2. Assume that for some $k, n \in \mathbb{Z}^+$, then $l(T_1) = k$ and $i(T_1) = k-1$ holds true as well as $l(T_2) = n$ and $i(T_2) = n-1$. Now we want to show that it holds for the full tree, that is, $l(T) = i(T) + 1$.

We know that $l(T) = l(T_1) + l(T_2)$ and $i(T) = r + i(T_1) + i(T_2)$ from the recursive definition. So

$$
\begin{aligned}
l(T) &= k + n \\
i(T) &= 1 + (k-1) + (n-1) = k + n - 1 = l(T) - 1 \\
i(T) + 1 &= l(T)
\end{aligned} \tag{6}
$$

Therefore $l(T)$ is 1 more than $i(T)$ in our full binary tree. $\square$

Exercise D. How many bit strings of length seven either begin with two 0s or end with three 1s?

If we have a string of length 7 we can use the product rule and the subtraction rule. We use the product rule to determine the amount of ways to get a bit string that either begin with two 0s or end with three 1s. We then use the subtraction rule that subtracts the number of ways by the number of ways that are common to both ways. Observe our visualisation of ways to pick:

$$
\begin{aligned}
\underline{1}\ \underline{1}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2} &= 2^5 = 32 \\
\underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{1}\ \underline{1}\ \underline{1} &= 2^4 = 16 \\
\underline{1}\ \underline{1}\ \underline{2}\ \underline{2}\ \underline{1}\ \underline{1}\ \underline{1} &= 2^2 = 4
\end{aligned} \tag{7}
$$

A bit string consists of a "list" of states, either 0 or 1. Meaning 2 states. The full number of combinations of a bit string of length seven is: $2^7 = 128$. Let us consider the different scenarios. Two 0's at the beginning of the bit string: We know that the first two bits are fixed, but there are five bits of which value we do not know:

$$2^{7-2} = 2^5 = 32 \tag{8}$$

Another scenario is three 1's at the end of the bit string:

$$2^{7-3} = 2^4 = 16 \tag{9}$$

But the sum of these scenarios is not the number of possible combinations of a bit string of length seven with either two 0's at the beginning or three 1's at the end of the bit string. We need to subtract the number of combinations in which both are true:

$$
\begin{gathered}
2^{7-2-3} = 2^2 = 4 \\
32 + 16 - 4 = 44
\end{gathered} \tag{10}
$$

Therefore, 44 bit strings of length seven either begin with two 0s or end with three 1s.

Exercise E. Let $n_1, n_2, ..., n_t$ be positive integers. Show that if $n_1 + n_2 + ... + n_t - t + 1$ objects are placed into t boxes, then for some i, $i = 1, 2, ..., t$, the ith box contains at least n_i objects.

Let $P \rightarrow Q$ be the statement "if $n_1 + ... + n_t - t + 1$ objects are placed into t boxes, then for some $i = 1, 2, ..., t$, the ith box contains at least n_i objects".

We will prove by contraposition ($\neg Q \rightarrow \neg P$). Assume that $\exists i$, $i = 1, 2, ..., t$ s.t. the i.th box contains at most $n_i - 1$ objects. Then there are $\sum_i^t (n_i - 1) = n_1 + n_2 + ... + n_t - t$ objects, which contradicts that $n_1 + n_2 + ... + n_t - t + 1$ objects are placed into t boxes. $\square$

Exercise F. How many subsets with an odd number of elements does a set with 10 elements have?

A set of 10 elements has $2^{10} = 1024$ subsets per the definition of the power set. There are half amount of odd numbers in that list, so

$$2^{10-1} = 2^9 = 512 \tag{11}$$

So a set of 10 elements has 512 subsets with an odd number of elements.

Exercise G. Show that if n is a positive integer, then $\binom{2n}{2} = 2 \cdot \binom{n}{2} + n^2$ by

a) Using a combinatorial argument

We are given

$$\binom{2n}{2} = 2 \cdot \binom{n}{2} + n^2 \tag{12}$$

Consider the number of ways to select 2 people from a group of $2n$ people to receive candy.

LHS. We can choose 2 people directly, in $\binom{2n}{2}$ ways, by counting the no. of ways to select 2 people from a group twice as large as n to receive candy.

RHS. Alternatively, we can count the number of ways by first counting twice the no. of ways to pick 2 people from an n group, that is, two groups of n counts the no. of ways to pick 2 people, and then we account for the missing counts of the $2n$ group by adding n^2.

b) By algebraic manipulation

By the definition of the combinatorial formula, we can manipulate it as such

$$\begin{aligned}
\binom{2n}{2} &= \frac{(2n)!}{2!(2n-2)!} = \frac{2n(2n-1)(2n-2)...(2)}{2 \cdot (2n-2)(2n-3)...(2)} \\
&= \frac{2n(2n-1)}{2} \\
&= n(2n-1) \\
&= 2n^2 - n \\
&= n(n-1) + n^2 \\
&= \frac{n(n-1)(n-2)...(2)}{(n-2)(n-3)...(2)} + n^2 \\
&= \frac{n!}{(n-2)!} + n^2 \\
&= 2 \cdot \frac{n!}{2!(n-2)!} + n^2 \\
\binom{2n}{2} &= 2 \cdot \binom{n}{2} + n^2
\end{aligned} \tag{13}$$

Which is exactly what we wanted to show. $\square$