Proofs (Limits)

Problem 7. Suppose that a_n is a sequence of real numbers that satisfies $a_n < 0$, and $\lim_{n\to\infty} a_n = L$. Prove that $L \leq 0$.

Assume that $L > 0$. By the definition of a limit, we know that $|a_n - L| < \epsilon$ for some ϵ. Choose $\epsilon = L/2$. Then since $a_n < 0$:

$$\begin{aligned} -a_n + L &< \frac{L}{2} \\ \frac{L}{2} &< a_n \end{aligned} \tag{1}$$

Since $L > 0$, then $L/2 > 0$, but then $a_n > L/2$ which is a contradiction, since we assumed that $a_n < 0$. $\square$

Problem 8. Let a_n and b_n be two convergent sequences of real numbers. Using the definition of limit, prove or disprove:

if $a_n < b_n$ for all n, and $\lim_{n\to\infty} b_n = L$, then $\lim_{n\to\infty} a_n < L$.

We will show with a counter-example. Let $a_n = 2$ and $b_n = 2 + 1/n$. We know that $a_n < b_n$. Then $\lim_{n\to\infty} b_n = 2$, and $\lim_{n\to\infty} a_n = 2$. But $2 \not< 2$. $\square$

if $a_n < b_n$ for all n, and $\lim_{n\to\infty} b_n = L_b$, then $\lim_{n\to\infty} a_n \leq L_a$.

By theorem 2.3, we know that $\lim_{n\to\infty}(a_n + b_n) = L_a + L_b$. Then $\lim_{n\to\infty}(a_n - b_n) = L_a - L_b \leq 0$, because $a_n < b_n$. Thus $L_a \leq L_b$. $\square$

Problem 9. Define $\kappa : \mathbb{R} \to \mathbb{R}$ by

$$\kappa(x) = \begin{cases} 1, & \text{if x is rational} \\ 0, & \text{if x is irrational.} \end{cases}$$

For each of the following, either find the limit or prove that it does not exist.

(1) We will prove that $\lim_{x\to 0} \kappa(x)$ does not exist by contradiction. Suppose there exists $\lim_{x\to 0} \kappa(x) = L$. We will split this proof into three cases.

Case 1: $L > 0$ Choose $\epsilon = L/2$, then by our definition of our limit,

$$\begin{aligned} |\kappa(x) - L| &< \epsilon \\ -\kappa(x) + L &< \frac{L}{2} \\ L - \frac{L}{2} &< \kappa(x) \\ \frac{L}{2} &< \kappa(x) \end{aligned} \tag{2}$$

This does not hold, since, for any $\delta > 0$, there is an irrational number x within the interval. For this x we have that $\kappa(x) = 0$. If we substitute that into our equation we have $L/2 < 0$, which is a contradiction since our assumption was $L > 0$.

Case 2: $L = 0$ Choose $\epsilon = 1/2$, then by our definition of our limit,

$$\begin{aligned} |\kappa(x) - 0| &< \epsilon \\ |\kappa(x)| &< 1/2 \\ \kappa(x) &< 1/2 \end{aligned} \tag{3}$$

This does not hold, since, for any $\delta > 0$, there is a rational number x within the interval. For this x we have that $\kappa(x) = 1$. If we substitute that into our equation we have $1 < 1/2$, which is a contradiction.

Case 3: $L < 0$ Choose $\epsilon = -L/2$, then by our definition of our limit,

$$\begin{aligned} |\kappa(x) - L| &< \epsilon \\ \kappa(x) - L &< -\frac{L}{2} \\ \kappa(x) &< \frac{L}{2} \end{aligned} \tag{4}$$

This case does not hold, since, for any $\delta > 0$, there is an irrational number x within the interval. For this x we have that $\kappa(x) = 0$. If we substitute that into our equation we have $0 < L/2$, which is a contradiction. Thus we have proven that the limit does not exist. $\square$

(2) We will now prove that $\lim_{x\to 0} x^2\kappa(x)$ exists by the squeeze theorem. Let $g(x) = x^2, \quad h(x) = 0$ and suppose that $\lim_{x\to 0} g(x) = \lim_{x\to 0} h(x) = 0$. We know that $0 \leq x^2\kappa(x) \leq x^2 \Rightarrow h(x) \leq x^2\kappa(x) \leq g(x)$. Then

$$\begin{aligned} \lim_{x\to 0} 0 = 0 \leq \lim_{x\to 0} x^2\kappa(x) \leq \lim_{x\to 0} x^2 = 0 \\ 0 \leq \lim_{x\to 0} x^2\kappa(x) \leq 0 \end{aligned} \tag{5}$$

It follows that $\lim_{x\to 0} x^2\kappa(x) = 0$, which is what we wanted. $\square$

(3) We will now prove that $\lim_{x\to 0}(x^2 + \kappa(x))$ does not exist by contradiction. Suppose there exists $\lim_{x\to 0}(x^2 + \kappa(x)) = L$.

Case 1: $L > 0$ Choose $\epsilon = L/2$, then by our definition of our limit,

$$\begin{aligned} |(x^2 + \kappa(x)) - L| &< \epsilon \\ -x^2 - \kappa(x) + L &< \frac{L}{2} \\ -x^2 + L/2 &< \kappa(x) \end{aligned} \tag{6}$$

This does not hold, since, for any $\delta > 0$, there is an irrational number x within the interval. For this x we have that $\kappa(x) = 0$. If we substitute that into our equation we have $0 > L/2 - x^2$, which is a contradiction.

Case 2: $L = 0$ Choose $\epsilon = 1/2$, then by our definition of our limit,

$$\begin{aligned} |(x^2 + \kappa(x)) - 0| &< \epsilon \\ x^2 + \kappa(x) - 0 &< \frac{1}{2} \\ \kappa(x) &< \frac{1}{2} - x^2 \end{aligned} \tag{7}$$

This does not hold, since, for any $\delta > 0$, there is a rational number x within the interval. For this x we have that $\kappa(x) = 1$. If we substitute that into our equation we have $1 < 1/2 - x^2$, which is a contradiction.

Case 3: $L < 0$ Choose $\epsilon = -L/2$, then by our definition of our limit,

$$\begin{aligned} |(x^2 + \kappa(x)) - L| &< \epsilon \\ x^2 + \kappa(x) - L &< -\frac{L}{2} \\ \kappa(x) &< \frac{L}{2} - x^2 \end{aligned} \tag{8}$$

This case does not hold, since, for any $\delta > 0$, there is a rational number x within the interval. For this x we have that $\kappa(x) = 1$. If we substitute that into our equation we have $1 < L/2 - x^2$, which is a contradiction. Thus we have proven that the limit does not exist. $\square$