HA4

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Exercise A. How many terms are there in the formula for the number of elements in the union of 10 sets given by the principle of inclusion–exclusion?

The inclusion-exclusion principle says that: $|A \cup B| = |A| + |B| - |A \cap B|$, which consists of 3 terms. For 3 sets, we have 7 terms: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

This means that counting the terms is the same as counting the number of possible ways to combine the amount of sets for which you want to find the number of terms, but excluding the zero sets. This means that the problem is the same as the following: a set of 10 elements can be described as

$$\left(\sum_{k=1}^{10} \binom{10}{k} \right) - 1 = 2^{10} - 1 = 1024 - 1 = 1023 \tag{1}$$

We also know that there are $2^n - 1$ terms in the formula. Therefore there are 1023 terms in the formula for the number of elements in the union of 10 sets.

Exercise B. Find the probability that when a fair coin is flipped five times tails comes up exactly three times, the first and last flips come up tails, or the second and fourth flips come up heads.

Let $P(T)$ be the probability that exactly three throws comes up tails. Let $P(F)$ be the probability that the first and last flips come up tail. Let $P(H)$ be the probability that the second and fourth flips come up heads. Then we want to find $P(T \cup F \cup H)$

We know the total possible throws are $2^5 = 32$.

$$\begin{aligned} C(5,3) = \frac{5!}{3!(5-3)!} = 10 \to P(T) &= \frac{10}{32} \\ 2^{5-2} = 8 \to P(F) &= \frac{8}{32} \\ 2^{5-2} = 8 \to P(H) &= \frac{8}{32} \end{aligned} \tag{2}$$

We can find $P(T \cap F)$ by realizing that the first and last flips are fixed to become tail. Then we have 3 flips left where we could get 1 tail. Furthermore, we find $P(T \cap H)$ by realizing that $P(H)$ takes up the second and fourth flip for heads, now we have exactly 1 combination for getting the remaining 3 tails. For $P(F \cap H)$ we see that they are independent events, so $8/32 * 8/32 = 2/32$.

For $P(T \cap F \cap H)$ we see that there is only one combination possible because the first and last flips is tail, the second and fourth is heads, so the middle value must be tails. Putting it all together:

$$
\begin{aligned}
P(T \cup F \cup H) &= P(T) + P(F) + P(H) - P(T \cap F) - P(T \cap H) - P(F \cap H) + P(T \cap F \cap H) \\
P(T \cup F \cup H) &= \frac{10}{32} + \frac{8}{32} + \frac{8}{32} - \frac{3}{32} - \frac{1}{32} - \frac{2}{32} + \frac{1}{32} \\
P(T \cup F \cup H) &= \frac{26}{32} - \frac{3}{32} - \frac{1}{32} - \frac{2}{32} + \frac{1}{32} = \frac{21}{32}
\end{aligned} \tag{3}
$$

So the probability is 21/32.

Exercise C. Show that if n is a positive integer, then

$$n! = \binom{n}{0} D_n + \binom{n}{1} D_{n-1} + ... + \binom{n}{n-1} D_1 + \binom{n}{n} D_0$$

where D_k a permutation of k-elements with no fixed points, and $C(n,i)$ is the number of ways to choose i objects from an n-set. Then $C(n,i) * D_{n-i}$ is the number of permutations of an n-set with i fixed points.

Given $\forall n((n \in \mathbb{Z}^+) \rightarrow (n! = \sum_i^n \binom{n}{i} D_{n-i})$ we will show through a combinatorial argument.

Consider the number of ways we can arrange n different candies.

LHS: When we place the first candy, we have used a spot, so there are $n-1$ spots left for the remaining candies. Then after placing another candy, its taking up yet another spot, so there are now $n-2$ spots left for the remaining candies and so on until all the candies have a placement. Therefore, there must be $n!$ ways to arrange n different candies.

RHS: We first count the number of i candies that should be in its original spot ($C(n,i)$), and then we count the $n-i$ candies remaining that should all be in different spots from its original spot (D_{n-i}). We want to count all the different original spots, so we add all the cases up to n.

Exercise D. Let R be the relation on the set K of all sets of real numbers such that SRT if and only if S and T have the same cardinality. Show that R is an equivalence relation. What are the equivalence classes of the sets $\{0,1,2\}$ and $\mathbb{Z}$?

If R is an equivalence relation, it must be reflexive, symmetric and transitive, we will check these properties.

Reflexive: For all sets A in K, (A,A) has to be in the relation. Since R contains all the pairs such that SRT iff they have the same cardinality, then each set $|A| = |A|$ will have the same cardinality as itself, so (A,A) will be in the relation for all $A \in K$.

Symmetric: If $(A,B) \in R$, then $(B,A) \in R$. If $(A,B) \in R$, then it's true that $|A| = |B|$. But then it's true that $|B| = |A|$, so (B,A) must also be in R.

Transitive: If $(A,B) \in R \rightarrow |A| = |B|$, and $(B,C) \in R \rightarrow |B| = |C|$, then $(A,C) \in R \rightarrow |A| = |C|$. This is true since $|A| = |B| = |C|$.

Looking at our equivalence classes:

$$\begin{aligned}
[\{0,1,2\}]_{\sim} &= \{s \mid (\{0,1,2\}, s) \in R\} \\
[\{0,1,2\}]_{\sim} &= \{..., \{-3.25, 0, 1\}, \{0,1,2\}, \{0.33, \sqrt{2}, \pi\}, ...\} \\
[\mathbb{Z}]_{\sim} &= \{s \mid (\mathbb{Z}, s) \in R\} \\
[\mathbb{Z}]_{\sim} &= \{..., \mathbb{N}, \mathbb{Q}, \mathbb{Z}, ...\}
\end{aligned} \tag{4}$$

That is, our equivalence class for $\{0,1,2\}$ is all the sets in K that has the same cardinality 3. For $\mathbb{Z}$ our equivalence class is all the sets in the set K that has the same cardinality, in this case, the cardinality is infinite.

Exercise E. Let R be the relation on the set S of ordered pairs of positive integers such that $((a,b),(c,d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

If R is an equivalence relation, it must be reflexive, symmetric and transitive, we will check these properties.

Reflexive: For all ordered pairs of positive integers $(a,b) \in S$, $((a,b),(a,b))$ has to be in the relation. Since R contains pairs $((a,b),(c,d))$ iff $ad = bc$, then each identical pair $((a,b),(a,b))$ it will be true that $ab = ba$, so $((a,b),(a,b))$ will be in the relation for all $(a,b) \in S$.

Symmetric: If $((a,b),(c,d)) \in R$, then $((c,d),(a,b)) \in R$. If $((a,b),(c,d)) \in R$, then it's true that $ad = bc$. But then it's true that $cb = da$, so $((c,d),(a,b))$ must also be in R.

Transitive: If $((a,b),(c,d)) \in R \to d/c = b/a$, and $((c,d),(e,f)) \in R \to d/c = f/e$, then $((a,b),(e,f)) \in R \to b/a = f/e$. This is true since $b/a = d/c = f/e$.

Therefore R is an equivalence relation.

Exercise F. Which of these relations on $\{0,1,2,3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

a) The relation $\{(0,0),(2,2),(3,3)\}$ is not a partial ordering since it's not reflexive as $(1,1)$ is not included. It has to be true for all pairs in the set.

b) The relation $\{(0,0),(1,1),(2,0),(2,2),(2,3),(3,3)\}$ is a partial ordering since reflexive, anti-symmetric and transitive.

c) The relation $\{(0,0),(1,1),(1,2),(2,2),(3,1),(3,3)\}$ is not a partial ordering since it is not transitive. It is the case that $(3,1)$ and $(1,2)$ then $(3,2)$ is not in R.

d) The relation $\{(0,0),(1,1),(1,2),(1,3),(2,0),(2,2),(2,3),(3,0),(3,3)\}$ is not a partial ordering since it is not transitive. That is, for $(1,2)$ and $(2,0)$ it is not the case that $(1,0)$.

e) The relation $\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,2),(3,3)\}$ is not a partial ordering since it is not anti-symmetric. That is, for $(0,1)$ and $(1,0)$ it is not the case that $0 = 1$.

The properties as mentioned are that they have to be reflexive, anti-symmetric and transitive. To expand on it, it has to include all (a, a) in the set. Furthermore, the relations that are partial orderings does not have a lot of pairs that arent the reflexive pairs. For example, in b), outside of the reflexive pairs, it has $(2, 0), (2, 3)$ both of which do not have a similar pair that reflects it like $(0, 2)$ which would have greater scrutiny in the anti-symmetry property, or $(0, 1)$ which might violate the transitive properties.

Exercise G. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree.

Given $G(n, e)$ where $n \geq 2$. We will show by contradiction. Assume all vertices $\{v_1, ..., v_n\}$ must have distinct degrees $\{0, 1, ..., n - 1\}$.

Then, there is a vertex with degree $n - 1$ and another with degree 0. This is a contradiction, since the vertex (call it v_k) with degree $n - 1$ is connected by an edge to every other vertex besides itself (as there are n vertices and $n - 1$ vertices excluding itself), then there cannot be a vertex with a degree of 0.

Therefore, there must be two vertices with the same degree in a simple graph with at least two vertices. $\square$