Homework no. 1

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Problem 1. Show that the polynomials $B_0^2(x) = (1-x)^2, B_1^2(x) = 2(1-x)x$, and $B_2^2(x) = x^2$ form a basis for $\mathbb{P}_2$. Let $\mathcal{B}_2 = \left\{B_0^2, B_1^2, B_2^2\right\}$.

Proof

Let us convert the polynomials into vectors so we can do calculations on them. We have the coordinates with respect to their standard basis as such

$$[B_0^2]_{\mathcal{S}_2} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad [B_1^2]_{\mathcal{S}_2} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}, \quad [B_2^2]_{\mathcal{S}_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now that we have transformed our vectors, we can put them in a matrix and do gaussian elimination to see if they are linearly independent like so

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the polynomials form linearly independent vectors spanning $\dim(\mathbb{P}_2) = 3$, they are a basis for $\mathbb{P}_2$. $\square$

Problem 2. Determine the "change of coordinates matrices" $[I]_{\mathcal{S}_2,\mathcal{B}_2}$ and $[I]_{\mathcal{B}_2,\mathcal{S}_2}$.

From problem 1, we have already found the change of coordinate matrix by switching from basis $\mathcal{B}_2$ to $\mathcal{S}_2$ as such

$$[I]_{\mathcal{S}_2\mathcal{B}_2} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

Furthermore, we can find the other change of coordinate matrix by taking the inverse of the one we have already as such

$$[I]_{\mathcal{B}_2\mathcal{S}_2} = ([I]_{\mathcal{S}_2\mathcal{B}_2})^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

Therefore we have that our other basis change matrix is

$$[I]_{\mathcal{B}_2\mathcal{S}_2} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1/2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Thus the change of coordinate matrices are given above.

Problem 3. Show that the polynomials $B_0^3(x) = (1-x)^3, B_1^3(x) = 3(1-x)^2x$, $B_2^3(x) = 3(1-x)x^2$, and $B_3^3(x) = x^3$ form a basis for $\mathbb{P}_3$. Let $\mathcal{B}_3 = \{B_0^3, B_1^3, B_2^3, B_3^3\}$.

Proof

Let us similarly convert the polynomials into vectors so we can do calculations on them. By expanding the polynomials, we have

$$[B_0^3]_{\mathcal{S}_3} = \begin{pmatrix} 1 \\ -3 \\ 3 \\ -1 \end{pmatrix}, \quad [B_1^3]_{\mathcal{S}_3} = \begin{pmatrix} 0 \\ 3 \\ -6 \\ 3 \end{pmatrix}, \quad [B_2^3]_{\mathcal{S}_3} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix}, \quad [B_3^3]_{\mathcal{S}_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Then we can again perform gaussian elimination on it's matrix to see if they form linearly independent vectors as such

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -6 & 3 & 0 \\ 0 & 3 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the polynomials form linearly independent vectors spanning $\dim \mathbb{P}_3 = 4$, they are a basis for $\mathbb{P}_3$. $\square$

Problem 4. Determine the "change of coordinates matrices" $[I]_{\mathcal{S}_3,\mathcal{B}_3}$ and $[I]_{\mathcal{B}_3,\mathcal{S}_3}$.

We have found the change of coordinate matrix

$$[I]_{\mathcal{S}_3\mathcal{B}_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

We can find the other by taking the inverse as such

$$[I]_{\mathcal{B}_3\mathcal{S}_3} = ([I]_{\mathcal{S}_3\mathcal{B}_3})^{-1} = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & -6 & 3 & 0 & 0 & 0 & 1 & 0 \\ -1 & 3 & -3 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & -6 & 3 & 0 & -3 & 0 & 1 & 0 \\ 0 & 3 & -3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & -3 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

Therefore we have that the other change of coordinate matrix is

$$[I]_{\mathcal{B}_3\mathcal{S}_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 0 & 0 \\ 1 & 2/3 & 1/3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Thus the change of coordinate matrices are given above.

Problem 5. Consider the map $i : \mathbb{P}_2 \to \mathbb{P}_3$ given by $i(p) = p$. Determine the matrix for this map both with respect to the standard bases $\mathcal{S}_2, \mathcal{S}_3$ and with respect to the bases $\mathcal{B}_2, \mathcal{B}_3$.

We have a transformation that takes a polynomial p and turns it into itself p, but in $\mathbb{P}_3$. Thus we can form the matrix

$$[i]_{\mathcal{S}_3\mathcal{S}_2} = \begin{bmatrix}[i(1)]_{\mathcal{S}_3} & [i(x)]_{\mathcal{S}_3} & [i(x^2)]_{\mathcal{S}_3}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

To find the matrix for the map with respect to the bases $\mathcal{B}_2, \mathcal{B}_3$, we can use the rule that $[i]_{\mathcal{B}_3\mathcal{B}_2} = [I]_{\mathcal{B}_3\mathcal{S}_3} \cdot [i]_{\mathcal{S}_3\mathcal{S}_2} \cdot [I]_{\mathcal{S}_2\mathcal{B}_2}$. We have

$$[i]_{\mathcal{B}_3\mathcal{B}_2} = \begin{bmatrix}1 & 0 & 0 & 0\\ 1 & 1/3 & 0 & 0\\ 1 & 2/3 & 1/3 & 0\\ 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ -2 & 2 & 0\\ 1 & -2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 1/3 & 2/3 & 0\\ 0 & 2/3 & 1/3\\ 0 & 0 & 1\end{bmatrix}$$

Thus we have found the matrix for the map with respect to standard bases and $\mathcal{B}_2, \mathcal{B}_3$ as seen above.

Problem 6. Consider the map $T : \mathbb{P}_3 \to \mathbb{P}_2$ given by $T(p) = p'$ (differentiation). Determine the matrix for this map with respect to the bases $\mathcal{B}_3, \mathcal{B}_2$.

We have a map that transforms a polynomial into it's differentiated. Thus we have the map with respect to the standard bases

$$[T]_{\mathcal{S}_2\mathcal{S}_3} = \begin{bmatrix}[T(1)]_{\mathcal{S}_2} & \dots & [T(x^3)]_{\mathcal{S}_2}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$$

We can again use the rule where $[T]_{\mathcal{B}_2\mathcal{B}_3} = [I]_{\mathcal{B}_2\mathcal{S}_2} \cdot [T]_{\mathcal{S}_2\mathcal{S}_3} \cdot [I]_{\mathcal{S}_3\mathcal{B}_3}$. Now insert the matrices and calculate

$$[T]_{\mathcal{B}_2\mathcal{B}_3} = \begin{bmatrix}1 & 0 & 0\\ 1 & 1/2 & 0\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ -3 & 3 & 0 & 0\\ 3 & -6 & 3 & 0\\ -1 & 3 & -3 & 1\end{bmatrix} = \begin{bmatrix}-3 & 3 & 0 & 0\\ 0 & -3 & 3 & 0\\ 0 & 0 & -3 & 3\end{bmatrix}$$

Thus we have found the matrix for this map with respect to the bases $\mathcal{B}_2, \mathcal{B}_3$ as given.

Problem 7. We can also regard T as a map $T : \mathbb{P}_3 \to \mathbb{P}_3$. Determine the matrix for this map with respect to the bases $\mathcal{B}_3, \mathcal{B}_3$.

We can apply the rule $[T]_{\mathcal{B}_3\mathcal{B}_3} = [I]_{\mathcal{B}_3\mathcal{S}_3} \cdot [T]_{\mathcal{S}_3\mathcal{S}_3} \cdot [I]_{\mathcal{S}_3\mathcal{B}_3}$. Thus we must find the matrix with respect to it's standard basis

$$[T]_{\mathcal{S}_3\mathcal{S}_3} = \begin{bmatrix}[T(1)]_{\mathcal{S}_3} & \dots & [T(x^3)]_{\mathcal{S}_3}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Now we can insert and calculate as such

$$[T]_{\mathcal{B}_3\mathcal{B}_3} = \begin{bmatrix}1 & 0 & 0 & 0\\ 1 & 1/3 & 0 & 0\\ 1 & 2/3 & 1/3 & 0\\ 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ -3 & 3 & 0 & 0\\ 3 & -6 & 3 & 0\\ -1 & 3 & -3 & 1\end{bmatrix} = \begin{bmatrix}-3 & 3 & 0 & 0\\ -1 & -1 & 2 & 0\\ 0 & -2 & 1 & 1\\ 0 & 0 & -3 & 3\end{bmatrix}$$

Thus we have found the matrix for this map with respect to the bases $\mathcal{B}_2, \mathcal{B}_3$ as given.

Problem 8. Solve the equation $Tp = B_1^2$ and write the general solution in the $\mathcal{B}_3$ basis.

We are given that $T(p) = 2(1-x)x$ but are missing the unknown initial polynomial p that makes it happen. Therefore we can use the formula

$$[T]_{\mathcal{B}_3\mathcal{B}_3} \cdot [p]_{\mathcal{B}_3} = [B_1^2]_{\mathcal{B}_3}$$

To find the polynomial p with respect to the $\mathcal{B}_3$ basis, since we already have the matrix for the map T. Let us first find $[B_1^2]_{\mathcal{B}_3}$ by doing gaussian elimination with our change of basis coordinate matrix $[I]_{\mathcal{S}_3\mathcal{B}_3}$ with B_1^2. We get that

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & 2 \\ 3 & -6 & 3 & 0 & -2 \\ -1 & 3 & -3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 \\ 0 & -6 & 3 & 0 & -2 \\ 0 & 3 & -3 & 1 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & -3 & 1 & -2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2/3 \\ 0 & 0 & 1 & 0 & 2/3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

So we have

$$[B_1^2]_{\mathcal{B}_3} = \begin{pmatrix} 0 & 2/3 & 2/3 & 0 \end{pmatrix}^T$$

Now we can apply the formula by doing gaussian elimination with the changed coordinate $[T]_{\mathcal{B}_3\mathcal{B}_3}$ as such

$$\left[\begin{array}{cccc|c} -3 & 3 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 2/3 \\ 0 & -2 & 1 & 1 & 2/3 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 2/3 \\ 0 & -2 & 1 & 1 & 2/3 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1/3 \\ 0 & 1 & -1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -1/3 \\ 0 & 1 & 0 & -1 & -1/3 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus we have the full solution

$$[p]_{\mathcal{B}_3} = \begin{pmatrix} -1/3 \\ -1/3 \\ 0 \\ 0 \end{pmatrix} + k \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad k \in \mathbb{R}$$

Converted back into the polynomial, we have

$$p(x) = (-1/3 + k)B_0^3 + (-1/3 + k)B_1^3 + kB_2^3 + kB_3^3$$

Which is the general solution of the equation.