matrix factorizations and norm

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Summary. This is lecture notes from following the major Applied mathematics in Denmark. Generally, you might notice it set up in notes and recipes, that will help you calculate problems step-by-step.

1 Matrix factorizations

1.1 LU factorization

The naive gaussian elimination applied to a matrix A gives a factorization into a product of two simple matrices, one unit lower triangular, and the other upper triangular,

This yields $A = LU$, where L is a matrix with ones in it's diagonal, and elements below it, and U is a matrix with elements in it's diagonal and above it's diagnonal.

Note the row echelon form is the U (the matrix after forward elimination).

The L is a multiplication of the invisible matrices we multiply with A to get the row operations to be the row echelon form.

1.1.1 Solving systems using LU

We can solve $Ax = b$ by saying $LUx = b$, and then solving the systems

$$Lz = b \tag{1.1}$$

$$Ux = z \tag{1.2}$$

1.1.2 Fallbacks

The LU factorization is dependent upon not having any 0 divisors in the algorithm, thus many matrices have no LU factorization,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \tag{1.3}$$

As an example.

NOTE: Doolittle pseudocode p. 391.

1.2 LDL.T factorization

Here, L is a unit lower triangular, and D is a diagonal matrix. This can be used if A is symmetric and has an ordinary LU factorization with L as unit lower triangular.

We have

$$LU = A = A^T = (LU)^T = U^T L^T \tag{1.4}$$

Since L is invertible, we can write $U = L^{-1}U^T L^T$. Finally, we have $U = DL^T$ and thus $A = LU = LDL^T$.

1.3 Cholesky factorization

Any symmetric matrix that has an LU factorization where L is lower triangular, it also has LDL^T factorization. The Cholesky factorization $A = LL^T$ is a consequence of it for the case where A is symmetric and positive definite.

A matrix A is symmetric and postive definite if $A = A^T$ and $x^T Ax > 0$ for every nonzero vector x.

We must have that $U = L^T$.

1.4 Permutation matrix

A permutation matrix is an $n \times n$ matrix P that arises from the identity matrix by permuting it's rows.

P is a matrix corresponding to the pivoting strategy used during gaussian elimination. We have

$$PA = LU \tag{1.5}$$

Where the matrix PA is A with its rows rearranged.

2 Vector and matrix norms

I already defined vector norms as $\|x\|_2 = \sqrt{x_1^2 + \cdots + x_{n^2}}$, see [[../../../01002-mat1b/notes/w01/w01-functions-1|w01-functions-1]].

Matrix norm can be defined as

$$\|A\| = \sup\{\|Ax\| : x \in \mathbb{R}^n \text{ and } \|x\| = 1\} \tag{2.1}$$

Singular values, eigenvals

2.1 Matrix norm properties

For an $n \times n$ matrix A, it follows that

$$\|Ax\|_2 \leq \|A\|_2 \|x\|_2 \tag{2.2}$$
$$\|AB\|_2 \leq \|A\|_2 \|B\|_2 \tag{2.3}$$

These are important to have.

2.2 Error in vectors

We can calculate the absolute and relative error on vectors. Let $\delta x = \bar{x} - x$, where $\bar{x}$ is a number approximating x.

Define the absolute error as $\|\delta x\|_2 = \|\bar{x} - x\|_2$ Define the relative error as

$$\frac{\|\delta x\|_2}{\|x\|_2} \tag{2.4}$$

If $\|\delta x\|_2$ is small, then we can say that $\bar{x}$ is close to x.

2.3 Condition number

The condition number is a number telling us how precise our system is (lower = better). For $Ax = b$, the condition number is defined as

$$\kappa(A) = \|A\|_2 \|A^{-1}\|_2 = \text{cond(A)} \tag{2.5}$$

Generally, if $\kappa(A) = 10^k$, then one can expect to lose at least k digits of precision in solving the system $Ax = b$.

2.3.1 Relative error with condition number

Define the relative error with the condition number in mind as

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|} \tag{2.6}$$

Thus we have the bigger the condition number, the greater our relative error risks being.