w03-inner-product-spaces

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1 Inner product spaces

Let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$. Then we have the following rules apply

(i) $\langle x, x \rangle \geq 0$ only when $x = 0$
(ii) $\langle x, y \rangle = \langle y, x \rangle$
(iii) $\langle cx + dy, z \rangle = c\langle x, z \rangle + d\langle y, z \rangle$

The inner product $\sum x_i \bar{y}_i$ is not the only inner product space, there are several well-defined inner product spaces as we see below.

1.1 Regular product spaces

We define the regular inner product as

$$\langle x, y \rangle = y^* x = \bar{y}^T x \tag{1.1}$$

Furthermore, we define norm as

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x^* x} \tag{1.2}$$

Remember that if x is complex, then you have to conjugate and transpose. It's good practice to do it regardless.

For $\mathbb{P}_n([a, b])$ we have that the inner product space is

$$\langle f, g \rangle = \int_a^b f(x) \bar{g(x)} dx \tag{1.3}$$

Where $g(x)$ is conjugated.

2 Vector projections

To project a vector x onto a span Y, we have the form

$$\text{proj}_Y(x) = \frac{\langle x, y \rangle}{\langle y, y \rangle} y \tag{2.1}$$

Similarly, if $u = y/\|y\|$, we have

$$\text{proj}_Y(x) = \langle x, u \rangle u \tag{2.2}$$

Which is the projection of vector x onto span Y (which contains the vector y).

2.1 Projection matrix

From projection we can find a matrix:

$$\text{proj}_Y(x) = \langle x, u \rangle u = u \langle x, u \rangle = u(u^* x) = (uu^*)x = Px \tag{2.3}$$

Where $P = uu^*$ is our $n \times n$ projection matrix. Thus we can find the projection just using our matrix P and an x.

3 Orthonormal basis

Two vectors x, y are orthogonal if $\langle x, y \rangle = 0$. They are orthonormal if they are orthogonal, and they are normalized $\|x\| = 1$ and $\|y\| = 1$.

Extending this to a basis set of vectors, we have

$$\langle x_j, x_k \rangle = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \tag{3.1}$$

For $x = (x_1, \dots, x_j, \dots, x_n)$. Meaning all vectors are orthogonal with eachother except itself, to which it equals 1, ie. it is normalized.

All orthonormal bases are linearly independent.

4 Gram Schmidt procedure

The Gram Schmidt procedure is a method to turn a basis into being orthonormal. Meaning we have to normalize them $\|x\| = 1$, as well as make them orthogonal.

(i) Set $w_1 := v_1$, and then normalize it $u_1 = 1/\|w_1\| \cdot w_1$
(ii) Let $w_k := v_k - \sum_{j=1}^{k-1} \langle v_k, u_j \rangle u_j$
(iii) Normalize it $u_k = 1/\|w_k\| \cdot w_k$

Thus you have found your orthonormal basis $u_1, \dots, u_n$.

In step (ii), note that $k_1 + n_1 = v_k$ for some vectors k_1, n_1. Thus we take the result v_k and remove the one side n_1, such that we are left with only the orthogonal side relative to v_{k-1}.

5 Unitary matrices

An $n \times n$ matrix is called unitary if $U^* U = I$. If it's real $Q \in \mathbb{R}^{n \times n}$ then it's called real orthogonal.

The following is equivalent for $U \in \mathbb{C}^{n \times n}$

(i) U is unitary, $U^* U = I$
(ii) Columns of U are an orthonormal basis in $\mathbb{C}^n$
(iii) $UU^* = I$
(iv) U^* is unitary
(v) Rows of U are an orthonormal basis
(vi) $\forall x, y \in \mathbb{F}^n : \langle Ux, Uy \rangle = \langle x, y \rangle$