w04-spectral-theory

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1 Definitions

Positive definite if $A^* = A$ and $\langle Ax, x\rangle > 0$

Positive semi-definite if $A^* = A$ and $\langle Ax, x\rangle \geq 0$

Negative definite if $-A$ is pos.def

Negative semi-definite if $-A$ is pos.semi-def.

2 Hermitian definition

Let $A \in \mathbb{F}^{n\times n}$ be hermitian, then the following is equivalent

(i) A is positive definite
(ii) A's eigenvals are positive
(iii) $\exists c \geq 0 : \langle Ax, x\rangle \geq c\langle x, x\rangle, \quad \forall x \in \mathbb{F}^n$.

3 Spectral theory (real case)

Let $A \in \mathbb{R}^{n\times n}$, the following are equivalent

(i) A is real symmetric ($A^T = A$)
(ii) $\mathbb{R}^n$ has an orthonormal basis of eigenvectors of A
(iii) Spectral decomposition ($\Lambda = Q^T AQ$) exists where Λ is a diagonal matrix and Q is a matrix of orthonormal eigenvectors

4 Spectral theory (complex case)

Let $A \in \mathbb{C}^{n\times n}$, the following are equivalent

(i) A is normal ($AA^* = A^*A$)
(ii) $\mathbb{C}^n$ has an orthonormal basis of eigenvectors of A
(iii) Spectral decomposition ($\Lambda = U^*AU$) exists where Λ is a diagonal matrix and U is a unitary matrix of orthonormal eigenvectors

Notice $A^T = A \implies A^* = A \implies AA^* = A^*A$

In the hierarchy, this means that if a matrix is real symmetric, it is also normal and adjoint.

5 How to construct Q matrix from spectral decomposition

Given a matrix A,

(i) Find the eigenvalues and eigenspace of A
(ii) Immediately read off Λ and start of Q
(iii) Notice the eigenspaces are lin.ind. if the eigenvalues are different. In this case they are orthogonal in Q and you just need to normalize them.
(iv) If the eigenvalues are not different, you need to use Gram Schmidt to make them orthonormal basis.

Once this is done for all eigenspaces, set them into a matrix and you have your Q matrix.

6 Reduction of polynomials

The goal is to find Q in the spectral decomposition and turn the polynomial $q(x)$ into a diagonal matrix s.t.

$$q(y) = y^T Q^T A Q y + Q^T y^T b + c$$

Where $x = Qy$ for Q in the decomp.

Note: If A is not symmetric, create it by $B = \frac{1}{2}(A + A^T)$. We get the sam $q(x)$.

Now you have your Q from the procedure above, and you ideally want $\det(Q) = 1$. Then you can use that to reduce polynomials using the $q(y)$ formula above given a set of equations. You can split them up into andengrad, førstegrad and c.

Andengrad:

$$y^T \Lambda y$$

Førstegrad:

$$Q^T y^T b$$

Where b vector can be read off of the equation. c: c

Thus we dont need to look at the bindeled where $(xy, xz, \ldots)$, as we can use the diagonalization instead, since

$$Q = [M]_{e\beta}, \quad Q^T = [M]_{\beta e}$$

Where e is the standard basis and M is the change of basis matrix.

Thus if we have b, we can say

$$Qb = [M]_{e\beta} \cdot [b]_\beta$$

To get our result in the standard basis, or vice versa.

The lesson to learn here is that diagonalization is essentially just a change of basis procedure where

$$[A]_\beta = Q^T A Q = [M]_{\beta e}[\Lambda]_{ee}[M]_{e\beta}$$