## soCartesian Elimination

Let $A$, $B$, and $C$ be arbitrary sets. We are now going to prove that "if $A \times B \subseteq B \times C$ then $A \subseteq C$". Now let us consider $B \neq \{\}$ and $(A \times B) \subseteq (B \times C)$. If $A = \varnothing$ then we know by definition of the empty set that $A \subseteq C$. So, we now need to prove the statement for the case when $A \neq \varnothing$. So, let us consider $A \neq \varnothing$. As, we know $A \neq \varnothing$ and $B \neq \varnothing$ we now know that $A \times B \neq \varnothing$ giving us by our assumption the fact $B \times C \neq \varnothing$ which lets us state that $C \neq \varnothing$. Now, let $a \in A$ and $b \in B$. Then $(a, b) \in (A \times B)$. By our assumption $(a, b) \in (B \times C)$. So, by definition of Cartesian Product $a \in B$ and $b \in C$. Therefore, $(a \in A) \rightarrow (a \in B)$, and $(b \in B) \rightarrow (b \in C)$. So, because $a$ was arbitrary and $(a \in A) \rightarrow (a \in B)$ we can state that $A \subseteq B$ by definition of subset. Also, because $b$ was arbitrary and $(b \in B) \rightarrow (b \in C)$ we can state that $B \subseteq C$ by definition of subset. So, because $a \in B$ and $B \subseteq C$ we can state by definition of subset that $a \in C$. Therefore $(a \in A) \rightarrow (a \in C)$. So, because $a$ was arbitrary and $(a \in A) \rightarrow (a \in C)$ we can state that $A \subseteq C$ by definition of subset. So, with the fact that we have proven the statement $((A \times B) \subseteq (B \times C)) \rightarrow (A \subseteq C)$ holds when $A = \varnothing$ or $A \neq \varnothing$, and we know that $A = \varnothing$ or $A \neq \varnothing$ is true by definition of Excluded Middle, and we can conclude with the additional fact that $A$, $B$, and $C$ were arbitrary sets that for any sets $A$, $B$, and $C$ the statement if $(A \times B) \subseteq B \times C$ then $A \subseteq C$ is a true.