MATHEMATICS AND MUSIC*

Two apparently distant and different topics are sometimes closely related. Such is the case with mathematics and music. In searching for the number essence of all things, the Pythagoreans discovered several simple numerical relationships for musical sounds. During the eighteenth century, mathematicians went beyond the Pythagoreans' understanding of music, completing a quite thorough mathematical description and analysis of musical sound. A knowledge of the mathematics of music is fundamental for designing telephones, radios, record players, and televisions.

One of the interesting relationships between mathematics and music shows up in the forms of musical instruments. The spacing of frets on a guitar, the curve of a grand piano, and the size of organ pipes all mirror the mathematics of sound. The lessons that follow show how "sound" mathematics is connected to the shape of pianos, guitars, and organs. Then the lessons take the mathematics further, showing how it relates not only to music but a variety of natural forms. The final lesson, *Coxeter's Golden Sequence,* brings together in one elegant example musical math, the exponential function, the golden ratio, and the logarithmic spiral.

*The ideas that follow are based on information from the following references.

"Logs, Pianos and Spirals" in *The Language of Mathematics* by Frank Land, pages 128–132 and 139–142.

Mathematics: A Human Endeavor by Harold R. Jacobs, pp. 176, 271–273, 284–285, and 290–291.

The Divine Proportion: A Study in Mathematical Beauty by H. E. Huntley, pages 100–102.

The *Teacher's Guide for Math in Nature Posters* by Alan Hoffer, page 10.

Patterns in Nature by Stevens, pages 88–89.

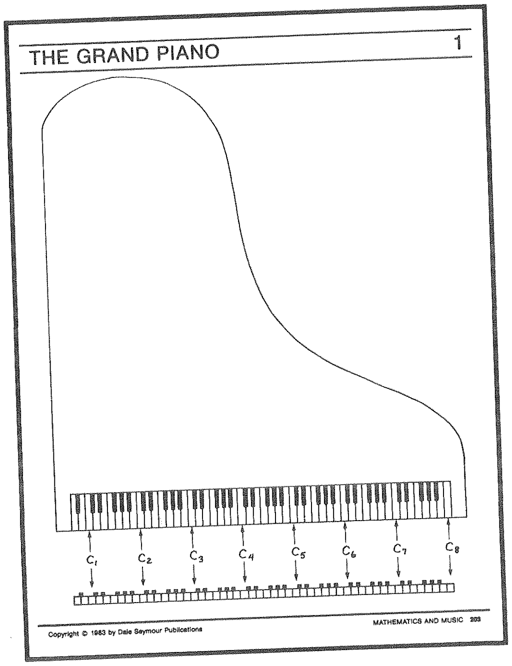
Mathematics, Its Magic and Mastery, Second Edition by Aaron Bakst, pages 295–296.

The Golden Section and Related Curiosa by Garth E. Runion, pages 54–55.

"The Diverse Pleasures of Circles that are Tangent to One Another" in the Mathematical Games section of the January 1979 issue of *Scientific American,* pages 24 and 28.

MATHEMATICS CONTENT:
Exponential Function

THE GRAND PIANO



Each of the keys of a piano is associated with a string inside the piano. Striking a piano key causes a "hammer" inside the piano to hit the string in the piano. When the hammer hits the string, the string vibrates. The vibrating string creates a musical sound of a certain pitch.

Short strings vibrate very fast, creating high pitches. Long strings vibrate at lesser frequencies, creating lower pitches. A piano is arranged so that, facing the keyboard, the keys give very low pitches at the left and gradually give higher pitches as you move to the right. The string arrangement in a grand piano corresponds to the keyboard arrangement; it has long strings at the left and short strings at the right. Page 1 shows a sketch of a grand piano and its keyboard, viewed from the top. Notice the graceful curve of the piano that accommodates the various lengths of its strings.

If you look carefully at the keyboard of a piano, you will see that the keys are arranged in a pattern. There is a group of seven white keys and five black keys that repeats seven times (with a few keys left over). Each of these groups of keys corresponds to a musical octave.

Notes in musical octaves are related in a very interesting mathematical way. The C's within each octave are marked on the piano of page 1. The *frequencies* of these C-notes double as you go from C to C. (Frequency is a measure of the vibrations of the strings.) The frequency of C_2 is twice the frequency of C_1; the frequency of C_3 is twice the frequency of C_2 and, thus, four times the frequency of C_1; and so on. In other words, the frequencies of these notes are related exponentially and their relationship can be described by the following equation.

$$C_n = C_1 2^{n-1}$$

where $n \in \{1, 2, 3, 4, 5, 6, 7, 8\}$
and C_n represents the frequency
of the nth C on the piano.

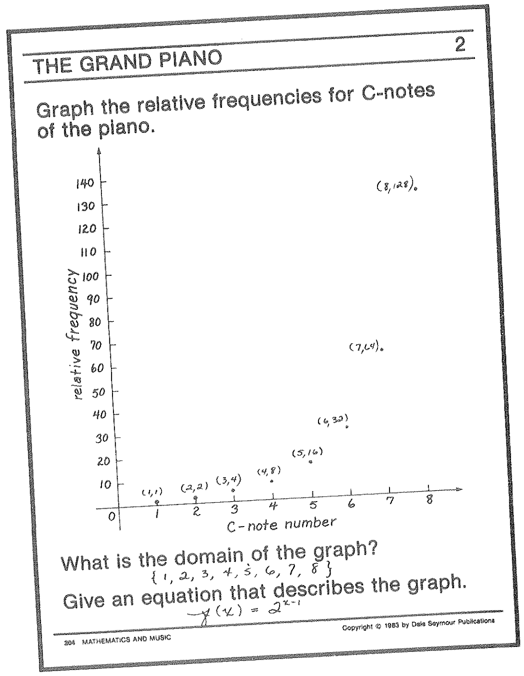
Using this equation, you can find the relationship between C_4 and C_1, for example.

$$C_4 = C_1 2^{(4-1)} = C_1 2^3 = 8C_1$$

This means that the frequency of the fourth C on the piano, C_4, is eight times the frequency of the first C on the piano, C_1.

Unless you know the actual frequency of one of the C's on the piano, you cannot find the frequencies of the other C's using the equation. You can, however, find the *relative* frequencies of the various C's. Simply evaluate 2^{n-1} for each value of n. By graphing these relative frequencies for the eight C-notes, you will obtain a visual representation of the relationship among the C's of the piano. Page 2 graphs the first two points. Complete the graph.

The completed graph has eight points. The graph represents an exponential function—an exponential function with a *restricted* domain. What is the domain of your function? (The domain is {1, 2, 3, 4, 5, 6, 7, 8}.) The function can be described by the equation $f(x) = 2^{x-1}$. If you remove the restriction on the domain and graph $f(x) = 2^{x-1}$ for all real values of x, the result will be an exponential curve. Sketch it in the first quadrant. Each point of your graph lies on this exponential curve. Do you see how the curve is related to the curve of the grand piano?



EXERCISES

1. Graph $f(x) = 2^x$ where $x \in \{1, 2, 3, 4, 5\}$.

2. Graph $f(x) = 2^x$ where x is a real number.

ANSWERS

1. The points on the graph are (1, 2), (2, 4), (3, 8), (4, 16), and (5, 32).

2. The graph intersects the vertical axis at 1, curves upward to the right and very slowly down to the left; it is asymptotic to the horizontal axis toward the left; it goes through (−1, 1/2) and (4, 16).

3. Graph $f(x) = 3^x$ where x is a real number.

3. The graph has the same basic shape as $f(x) = 2^x$ but rises faster to the right and curves downward to the left more slowly; it intersects the vertical axis at 1; it goes through $(-1, 1/3)$ and $(4, 81)$.

4. Graph $f(x) = 3^{-x}$ where x is a real number.

4. The graph is the reflection of $f(x) = 3^x$ over the vertical axis.

5. Graph $f(x) = 4(2^x)$ where x is a real number.

5. The graph is steeper than the graph of $f(x) = 2^x$; it intersects the vertical axis at 4; it goes through $(1, 8)$ and $(-1, 2)$.

6. Graph $f(x) = 4(2^{-x})$ where x is a real number.

6. The graph is the reflection of the graph in Exercise 5 over the vertical axis.

7. Graph $f(x) = 5(2^x)$ where x is a real number.

7. The graph is steeper than both $f(x) = 2^x$ and $f(x) = 4(2^x)$; it intersects the vertical axis at 5; it goes through $(1, 10)$ and $(-1, 2\text{-}1/2)$.

8. Graph $f(x) = 5(2^{-x})$ where x is a real number.

8. The graph is the reflection of the graph in Exercise 7 over the vertical axis.

MATHEMATICS CONTENT:
Geometric Sequence,
Logarithms,
Exponential Function

MORE PIANO RELATIONSHIPS

Every octave of the piano contains thirteen notes from C to C. There are eight white keys and five black keys—two and three. (Have you seen the numbers 2, 3, 5, 8 and 13 before?) The figure on page 1 shows a close-up of the middle octave, or scale, on a piano keyboard.

Today pianos are usually tuned so that the ratios of frequencies of consecutive notes are constant. For example, the ratio of frequencies from middle C to C# is the same as the ratio of frequencies from C# to D and from D to D# and so on. (The notes in order are C, C#, D, D#, E, F, F#, G, G#, A, A#, B, and C.) Suppose r stands for that ratio and f_n stands for the frequency of the nth note of the scale starting from middle C. Can you write an equation describing how f_n is related to f_1, the frequency of middle C? Recall that the numbers in a geometric sequence are related by a common ratio.

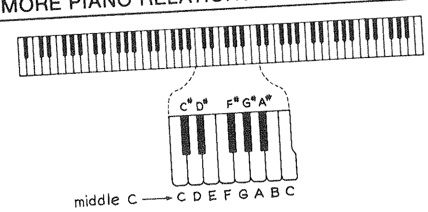
The equation relating the frequencies is like the equation describing the nth term of a geometric sequence. The equation you're looking for is

$$f_n = f_1 r^{n-1}$$

The frequency of middle C is about 261.6 cycles per second. Because the frequencies of C-notes double from octave to octave, the frequency of the C above middle C is about 523.2 cycles per second. Using these two values you can find a value for the common ratio r between notes of the scale. If you don't have a calculator, you can use logarithms to solve the problem. (The value of r is $\sqrt[12]{2}$. You will get $\log r = (1/12)\log 2 \approx 0.0251$, so $r \approx 1.0595$. Using a TI-35 calculator, you will find r is approximately 1.0594631.)

Once you've found a value for r, you can find approximate frequencies for all the notes of the middle octave. Then you can graph the frequencies with respect to the numbers you've assigned to the notes. What is the domain of your graph?

MORE PIANO RELATIONSHIPS 1



Frequencies of consecutive notes are related by a common ratio, r.
Let f_n = frequency of nth note.
Let f_1 = frequency of middle C.
Write an equation relating f_n, f_1, and r.

$f_n = f_1 r^{n-1}$

Frequency of middle C is 261.6 cycles per second.
Frequency of next higher C is 523.2 cycles per second.
Find r.

$f_n = f_1 r^{n-1}$
$523.2 = 261.6 r^{13-1}$
$2 = r^{12}$
$r = \sqrt[12]{2}$
≈ 1.05946309

or $\log r = \frac{1}{12}\log 2$
$= \frac{1}{12}(0.3010)$
$= 0.0251$
$r \approx 1.0595$

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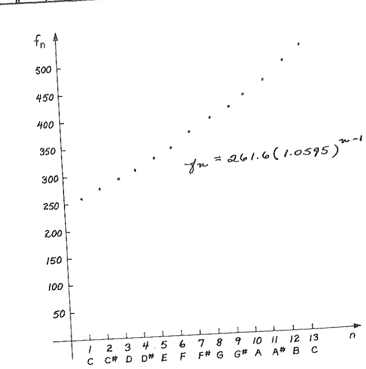
MORE PIANO RELATIONSHIPS 2

Find the frequencies for all notes of the middle octave. Then complete the graph.

name	C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C
n	1	2	3	4	5	6	7	8	9	10	11	12	13
f_n	261.6	277.2	293.6	311.1	329.6	349.2	370.0	392.0	415.3	440.0	466.1	493.8	523.2



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(The domain is 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13.) What is the range? (The range is 261.6, 277.2, 293.6, 311.1, 329.6, 349.2, 370.0, 392.0, 415.3, 440.0, 466.1, 493.8, 523.2. These values are approximate and your results may differ by a few tenths.)

EXERCISES

1. Use the equation $f_n = f_1 r^{n-1}$ to verify that f_{13}, the frequency of C above middle C, is 523.2 cycles per second given that f_1 is 261.6 and $r \approx 1.05946$.

2. Describe how to extend the graph of this lesson to show the frequencies of notes in the next octave above the middle octave of a piano.

3. What is the frequency of F in the next octave of a piano?

4. What is the frequency of the upper C in the next octave above the middle octave of a piano? Describe *two* different methods for finding the frequency.

5. How is the frequency of F in the middle octave of a piano related to the frequency of F in the next octave above?

ANSWERS

1. $f_{13} \approx 261.6 \times (1.05946)^{13-1} \approx 261.6(1.99993) \approx 523.2.$

2. Continue numbering the notes of the next octave consecutively. The first C is the last C of the middle octave and is numbered 13. C# is numbered 14, D is numbered 15, and so on. Then use the equation $f_n = f_1 r^{n-1}$ to find the values of the range. The extended domain is $\{1, 2, \ldots, 25\}$.

3. Approximately 698.4 cycles per second.

4. You can multiply the frequency of middle C by 2^2 or you can find $261.6r^{24}$.

5. The frequencies are related by a factor of r^{12}; that is, by 2.

GUITAR FRETS

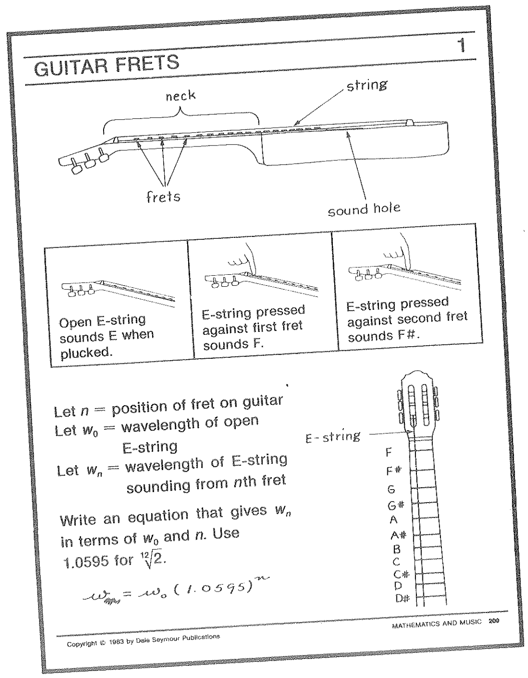
MATHEMATICS CONTENT:
Exponential Function

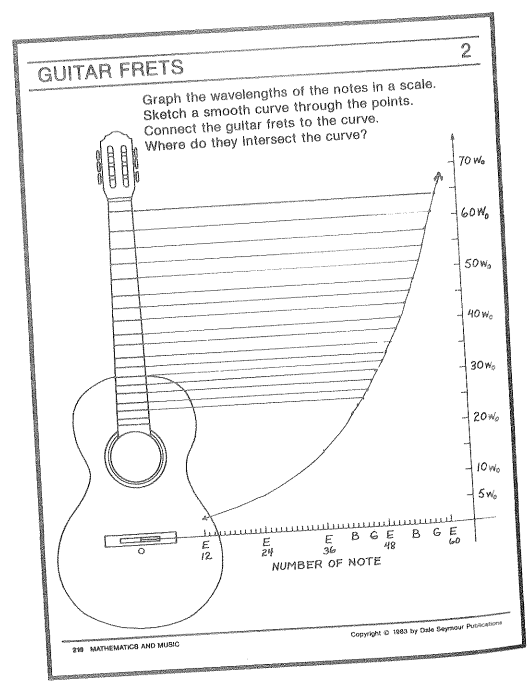
Just as the frequencies of notes in a musical scale are related, the wavelengths of those notes are related. In fact, the relationships are very similar. The wavelength of a musical note is exactly twice the wavelength of the note one octave *higher*. (Compare this to the frequency relationship of the two notes.) Also, the wavelengths of consecutive notes within the octave differ by $\sqrt[12]{2}$ or about 1.0595.

The mathematics of sound is reflected in the shape of a grand piano. The construction of a guitar also shows the mathematics of sound. The frets of a guitar serve to change the lengths of its strings. When a musician presses a finger against a string, the string is pressed against a fret near the finger. This action effectively cuts off the string at the fret, allowing only the string below the fret to sound. The musician makes different notes from the guitar strings by using different frets.

The top string of a guitar is the E-string. When it is open (not pressed by a finger), it sounds an E. As you run a finger down the neck of the guitar on the E-string, you will get successively higher notes of the scale at each fret—F, F#, G, G#, and so on. In other words, the frets on a guitar are arranged so that the wavelength of sound produced by a string pressed against one fret is about 1.0595 times the wavelength of sound produced at the very next fret (moving toward the sound hole). Can you write an equation that describes this situation? Use the diagram on page 1 and let n stand for the number assigned to the frets, w_0 represent the wavelength of an open E, and w_n stand for the wavelength of a string sounding from the nth fret. (The equation is $w_n = w_0(1.0595)^n$.)

Use your equation to find relative wavelengths for E-notes from $n = 0$ to $n = 60$. Graph those points on page 2. Also graph the B-notes and G-notes indicated. Sketch a smooth curve through the points you've graphed. What kind of curve do you get? (It should be an exponential curve.)





You may have noticed that the frets on a guitar are not spaced evenly. The spacing is related to the graph of $w_n = w_0(1.0595)^n$. Connect the guitar frets in the diagram on page 2 to the graph you drew. Explain the results. (The frets connect to the corresponding wavelengths of the notes they make.)

EXERCISES

1. The equation $f_n = (261.6)(1.0595)^{n-1}$ where n is an integer from 1 to 13 approximates the frequencies of notes in the middle octave of a piano. Approximate the frequencies to the nearest tenth (or get them from your work in the lesson on Piano Relationships). Graph the frequencies on a line.

2. The equation $w_n = w_0(1.0595)^n$ approximates the wavelengths of notes in the musical scale where w_0 stands for the wavelength of a high open E on the guitar and n stands for the number assigned to the frets of a guitar. Determine the relative wavelengths (values of $(1.0595)^n$) for the first 13 frets of a guitar. Graph the wavelengths on a line.

3. Compare the graph from Exercise 1 to the graph from Exercise 2.

4. In music, frequencies of notes and wavelengths of notes double over a full twelve-note octave. Within the octave there is an increase of about 5.95% for each successive note. In business, an investment will approximately double over a period of 12 years when the investment increases 5.95% each year. Write a formula to describe the business investment. Use n for number of years, P for the original amount of money invested, and A for the amount of money after n years. Compare your formula to the formula describing wavelengths of notes in a musical scale.

ANSWERS

1. See student work. The frequencies approximate a logarithmic scale.

2. Correct to two decimal places the wavelengths are $1.06w_0$, $1.12w_0$, $1.19w_0$, $1.26w_0$, $1.34w_0$, $1.41w_0$, $1.50w_0$, $1.59w_0$, $1.68w_0$, $1.78w_0$, $1.89w_0$, and $2.00w_0$. Student graphs should approximate a logarithmic scale.

3. The graphs are alike.

4. In business $A = P(1.0595)^n$. In music, $w_n = w_0(1.0595)^n$. Although the formulas describe totally different situations, they are alike; only the letters used to represent the values are different.

5. In general, for an investment with interest compounded annually, C dollars will approximately double in n years at p percent whenever p is an integral divisor of 72. This situation is often called the *Rule of 72.* Complete the following table based on the Rule of 72.

C dollars	p percent	n years	return on investment
100	6	12	200
100	8		200
100	3		200

6. According to the Rule of 72, at what rate of interest will an investment of $1000 double (approximately) when the interest is compounded annually and the money is invested for two years?

5. The values are 9 and 24.

6. 36%; in fact, at 36% the return on investment will be $849.60; at 40% the return on investment will be $1960 and at 41.42% the return will be $1999.9616. (These results can be obtained by solving $2000 = 1000(1 + r)^2$ for r.)

THE PIPE ORGAN

MATHEMATICS CONTENT:
Inverse Variation,
Hyperbola

Vibrating strings create sound. Vibrating columns of air create sound too. Short columns of air produce high sounds. Long columns of air produce low sounds. This principle is used in a pipe organ. The length of organ pipes varies inversely as the pitch (measured by frequency in cycles per second). If you double the length of pipe, you halve the pitch. If you halve the length of pipe, you double the pitch. The exact relationship between length, L, and pitch, p, can be described by the following equation.

$$L = \frac{512}{p}$$

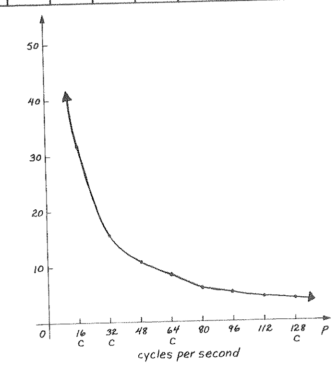
Using this equation, you can graph the relationship between length of pipe and pitch. Complete the table of values on page 1 and sketch the graph in the first quadrant. On what kind of curve do the points lie? (It's a hyperbola.)

THE PIPE ORGAN 1

$L = \frac{512}{p}$ where L = length of pipe in feet and p = pitch of pipe in cycles per second

Complete the table of values and sketch the graph.

p	16	32	48	64	80	96	112	128
L	32	16	$10.\bar{6}$	8	6.4	$5.\bar{3}$	≈ 4.6	4



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EXERCISES

1. The graph of $L = 512/p$ is a hyperbola. In what two quadrants of the coordinate plane does the graph lie?

2. Why does the graph of $L = 512/p$ representing the relationship between the length of organ pipes to their pitch lie only in the first quadrant of the coordinate plane?

3. What is the pitch (in cycles per second) of an organ pipe that is 10.7 feet long?

4. What length of organ pipe will give a pitch of 96 cycles per second?

ANSWERS

1. First and third

2. Measurements are positive numbers.

3. About 48 cycles per second

4. 5-1/3 feet

MATHEMATICS CONTENT:
Polar Coordinates,
Geometric Progression,
Exponential Function,
Logarithmic Spiral, Golden Rectangle,
Fibonacci Numbers, Golden Triangle

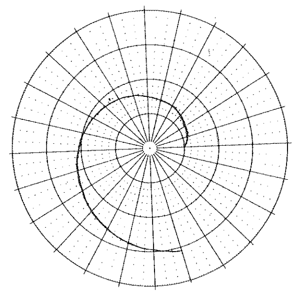
A MUSICAL SPIRAL

You have seen that the powers of $\sqrt[12]{2}$ (about 1.0595) are intimately related to the notes of the musical scale (by frequency and wavelength). You have used powers of $\sqrt[12]{2}$ to obtain some very interesting and useful graphs. There is another way to graph these values that gives a beautiful and fascinating picture.

A MUSICAL SPIRAL 1

Complete the table of values. Graph the points you obtain and connect them with a smooth curve.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$(1.0595)^n$	1.0595	1.1225	1.1893	1.2601	1.3351	1.4145	1.4987	1.5878	1.6823	1.7824	1.8885	2.0008	2.1199	2.2460	2.3797	2.5213	2.6712	2.8302	2.9986	3.1770
$n(15°)$	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°	195°	210°	225°	240°	255°	270°	285°	300°



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The graph you will draw can be easily described using *polar coordinates*. (Recall that polar coordinates are of the form (r, θ) where r represents the distance, either positive or negative, of the point from the pole and θ represents the measure of the angle of rotation or angle formed between the polar axis and a ray connecting the point to the pole.) The polar coordinates of each point on your musical graph take the form $((\sqrt[12]{2})^n, n(15°))$. Complete the table of values on page 1 for n from 0 to 20. Graph the points you obtain and connect them with a smooth curve. The result should be a musical spiral. (In case you are wondering, the choice of 15° increments is arbitrary; 10° increments will generate more points; 30° increments will generate fewer points.)

If you look carefully at your graph, you will discover that each radius intersects the graph at an angle of approximately 77.5°. The radii intersect the graph at equal angles. For this reason, the spiral you have sketched is often called an *equiangular spiral*. Another name, and probably the most common, is *logarithmic spiral*. The curve has this name because the angles the radii form with the polar axis are proportional to the logarithms of the lengths of the corresponding radii. For example, when the angle is $2 \times 15°$, the corresponding log is $2 \times \log(1.0595)$ where 1.0595 approximates $\sqrt[12]{2}$; when the angle is $3 \times 15°$, the corresponding log is $3 \times \log(1.0595)$; and so on.

Logarithmic spirals are not only musical, they're golden. You can use a Golden Rectangle to generate a logarithmic spiral. You simply follow these steps.

1. Draw a Golden Rectangle *ABCD*.
2. Draw diagonal $\overline{DB}$.

3. Locate E on $\overline{DC}$ and F on $\overline{AB}$ so that figure $AFED$ is a square. Notice that the small rectangle formed, rectangle $FBCE$, is a Golden Rectangle. (Can you verify this fact?)
4. Draw diagonal $\overline{CF}$.
5. Locate G on $\overline{CB}$ and H on $\overline{EF}$ so that figure $ECGH$ is a square. This creates a new, even smaller Golden Rectangle, rectangle $HFBG$.
6. Continue creating squares and Golden Rectangles as long as space allows.
7. Starting with point F, draw an arc of a circle with radius equal to the side of square $AFED$. The arc should start at A and end at E. Continue the process for H, J, and so on.

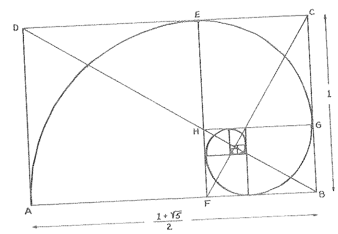
Page 2 shows rectangle $ABCD$ with $\overline{EF}$ drawn. Complete steps 4–7 for this diagram. The completed spiral is an approximation of a logarithmic spiral. (The real logarithmic spiral is not made up of circular arcs, but its shape is very close to what you've drawn. The spiral does pass through A, E, B, and so on. In other words, the locus of these points *is* a logarithmic spiral when rectangle $ABCD$ is a Golden Rectangle.)

Another way to approximate a logarithmic spiral is by using a sequence of numbers closely related to the Golden Ratio—the Fibonacci numbers. Consider the ratio of two consecutive Fibonacci numbers such as 89/55. You will obtain a decimal approximation for the Golden Ratio that is correct to three decimal places—1.618. A rectangle measuring 89 units by 55 units is, then, a fair approximation to a Golden Rectangle. If, within this rectangle, you cut off a 55-unit-by-55-unit square, you obtain a smaller rectangle that is 55 units by 34 units. The value of 55/34 rounded to three decimal places is 1.618, so this 55 by 34 rectangle again is a fair approximation to the Golden Rectangle.

Cutting off a 34-unit-by-34-unit square of the small Golden Rectangle leaves a second Golden Rectangle measuring 34 units by 21 units, for which the ratio 34/21 rounded to three decimal places is 1.619—a fair approximation to a third Golden Rectangle. Repeating this process about six more times gives you five more Golden Rectangle approximations. Drawing quarter-circle arcs within each square will give you a fair approximation to a logarithmic spiral. Page 3 shows the first few rectangles. Complete the picture to see the approximation.

A MUSICAL SPIRAL 2

1. Draw golden rectangle *ABCD*.
2. Draw diagonal $\overline{DB}$.
3. Locate E on $\overline{DC}$ and F on $\overline{AB}$ so that figure *AFED* is a square. Notice that the small rectangle formed, rectangle *FBCE*, is a golden rectangle.
4. Draw diagonal $\overline{CF}$.
5. Locate G on $\overline{CB}$ and H on $\overline{EF}$ so that figure *ECGH* is a square.
6. Continue creating squares and golden rectangles as long as space allows.
7. Starting with point F, draw an arc of a circle with radius equal to the side of square *AFED*. The arc should start at A and end at E. Continue the process for H, J, and so on.



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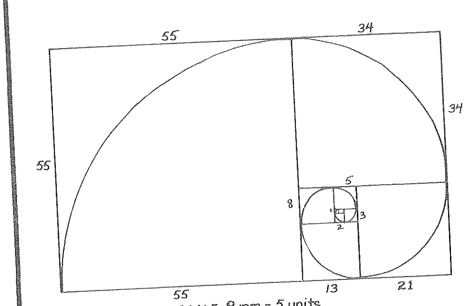
A MUSICAL SPIRAL 3



Continue the process of creating rectangles with successive Fibonacci measurements. Then draw quarter-circle arcs in each square.

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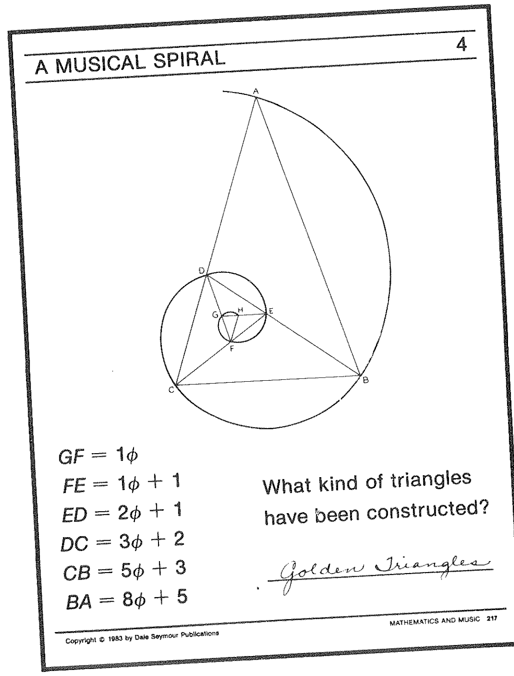
Lest you think the possibilities for constructing logarithmic spirals are exhausted, take a look at the diagram on page 4. The points A, B, C, D, E, and so on are the locus of a logarithmic spiral. The isosceles triangles have been constructed so that $GF = 1\phi$, $FE = 1\phi + 1$, $ED = 2\phi + 1$, $DC = 3\phi + 2$, $CB = 5\phi + 3$, and so on. What kind of triangles have been constructed? (They are Golden Triangles because, for example, $DC/ED = (3\phi + 2)/(2\phi + 1) = \phi^4/\phi^3 = \phi$.)

Because logarithmic spirals maintain their shapes as they grow in size, they are a favorite of nature. Some spiders spin logarithmic spirals. The spider builds an outer frame and a center region. Then she spins radii from the center to the frame. Finally, she spirals from the center, working her way out to the frame. The shell of a chambered nautilus is a beautiful and extremely popular logarithmic spiral. The cutaway view on page 5 shows the spiraling shell and the chambers in which the nautilus lives. At any time in the creature's life, it inhabits only the largest outermost chamber. As it grows, it creates ever larger chambers, abandoning its old "room" for its newer bigger one. Throughout the natural world—from galaxies to ocean waves and from pine cones to canaries—there are logarithmic spirals to be seen; you only need to look.


A MUSICAL SPIRAL 4

$GF = 1\phi$
$FE = 1\phi + 1$
$ED = 2\phi + 1$
$DC = 3\phi + 2$
$CB = 5\phi + 3$
$BA = 8\phi + 5$

What kind of triangles have been constructed?

Golden Triangles

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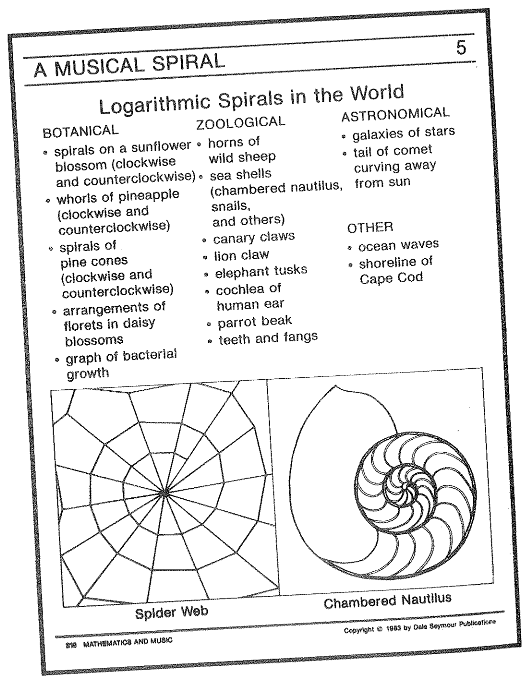
A MUSICAL SPIRAL 5

Logarithmic Spirals in the World

BOTANICAL
- spirals on a sunflower blossom (clockwise and counterclockwise)
- whorls of pineapple (clockwise and counterclockwise)
- spirals of pine cones (clockwise and counterclockwise)
- arrangements of florets in daisy blossoms
- graph of bacterial growth

ZOOLOGICAL
- horns of wild sheep
- sea shells (chambered nautilus, snails, and others)
- canary claws
- lion claw
- elephant tusks
- cochlea of human ear
- parrot beak
- teeth and fangs

ASTRONOMICAL
- galaxies of stars
- tail of comet curving away from sun

OTHER
- ocean waves
- shoreline of Cape Cod

Spider Web

Chambered Nautilus

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EXERCISES

1. Find the polar coordinates of a point on the musical spiral for $n = 40$.

2. Find the polar coordinates of a point on the musical spiral for $n = 100$.

ANSWERS

1. $(10.08, 600°)$, which is the same as $(10.09, 240°)$.

2. $(322.5, 1500°)$, which is the same as $(322.5, 60°)$.

3. Theoretically, does the musical spiral ever end?

3. no

4. How can you extend the logarithmic spiral drawn in the Golden Rectangles to make it larger?

4. Construct a square on the bottom of rectangle $ABCD$ with one side $\overline{AB}$. Then, draw a quarter-circle from B to the opposite vertex.

5. How can you extend the logarithmic spiral drawn in the Fibonacci rectangles to make it larger?

5. Add an 89-by-89 square to the bottom of the largest rectangle.

6. Can you continue the process described for Exercise 5 indefinitely?

6. Yes, assuming you have an unlimited size to your sheet of paper.

7. If you continue the process described for Exercise 5, where would the very next square be added to the figure?

7. The square should be added along the rightmost edge of the figure. The sides of the square are 144 units long.

8. The general equation for a point (r, θ) in polar coordinates on a logarithmic spiral is $r = e^{\theta \cot K}$ where K is the measure of the angle that any radius makes with the spiral curve. Approximate K for the point $(1.0595, \pi/12)$.

8. 77.5°