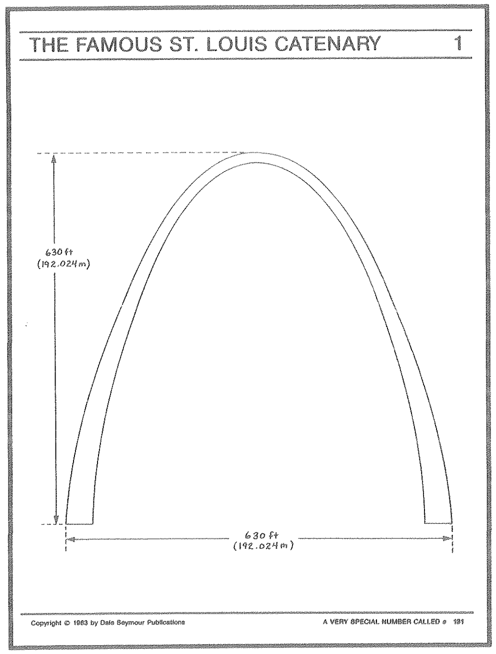
THE FAMOUS ST. LOUIS CATENARY

MATHEMATICS CONTENT:
Exponential Functions,
e

Historically, St. Louis, Missouri was an important starting point for pioneers on their way West. To commemorate its role as the gateway to the West, the city of St. Louis built a 630-foot stainless steel structure called the Gateway Arch. The design and construction of this imposing arch was an incredible feat of engineering.

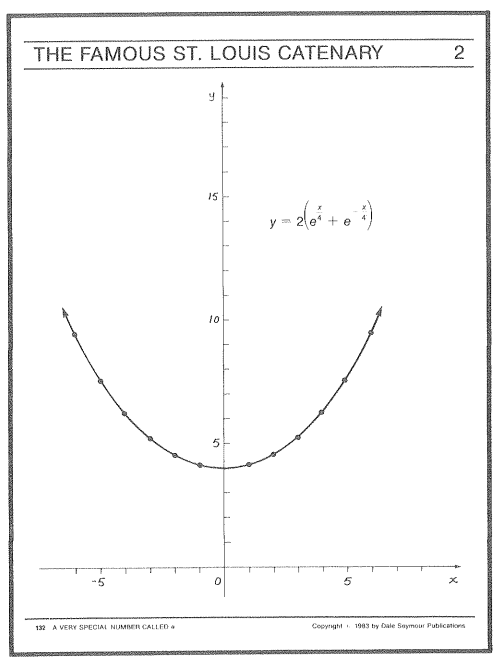


The Gateway Arch is famous not only for its symbolic role as the gateway to the West, but also for what it represents mathematically. It takes the shape of a *catenary* curve. (If you hold a chain at its two ends, letting the chain hang freely, you'll form another catenary curve.) In architecture, the catenary shape makes an extremely stable arch because the force of the arch's weight acts along its legs directly into the ground.

A general catenary curve can be described by the following equation.

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\left(\frac{x}{a}\right)}\right)$$ where a is a real nonzero constant and x is real

To see how values of the constant a affect the shape of the curve, sketch several different catenary curves on the same set of axes. Try $a = 2$, $a = 4$, and $a = 6$. The graph on page 2 shows a catenary curve for $a = 4$. The curve was determined by finding points for all integer values of x from -7 to 7 and using 2.71828 as an approximation for e.



Notice how the graphs differ for different values of a. The values of a determine the y-intercepts of the curves. Do you see why? The values also affect the degree of curvature (the slope) of the curves. As the values of a increase, do the slopes decrease (become more gradual) or increase? (They decrease.)

The Gateway Arch actually is an inverted catenary; it's upside-down. Its height from the very highest point to the ground is 630 feet (192.024 meters). Its width from one leg to the other along the ground is also 630 feet (192.024 meters).

EXERCISES

1. Sketch the graph of a catenary curve for which $a = 1$. Find points for integer values of x from 0 to 4. Also find points for $x = 3.25$, $x = 3.5$, and $x = 3.75$. Use 2.718 for e.

2. Explain how the shapes of catenaries with $a = 2$ and $a = 1/2$ differ.

3. Explain how the graphs of catenaries with *positive* values of a differ from the graphs of catenaries having *negative* values of a.

4. Give an equation for the catenary curve that has the same shape as $y = e^{x/2} + e^{-(x/2)}$ and opens in the same direction but has a y-intercept of 4.

ANSWERS

1. The curve opens up and has a y-intercept of 1. The following points are on the curve: (0, 1), (1, 1.543), (2, 3.762), (3, 10.068), (3.25, 12.915), (3.5, 16.573), (3.75, 21.27), (4, 27.308)

2. The catenary for which $a = 1/2$ has a minimum point (at the y-intercept) closer to zero than the catenary for which $a = 2$. Also, when $a = 1/2$, the curve opens very slowly (has a steeper slope).

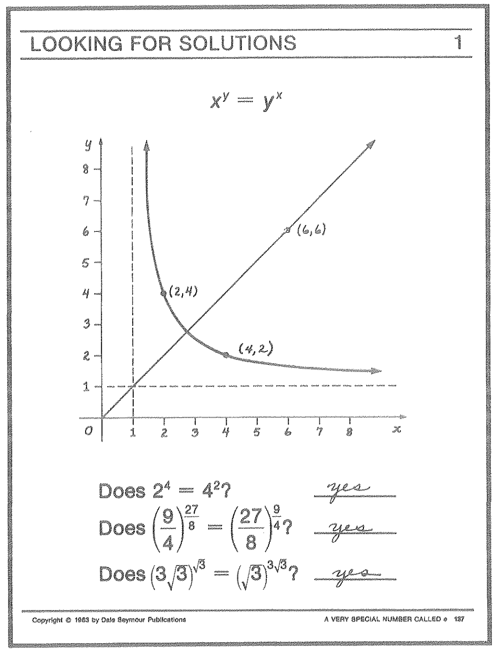
3. Catenaries having *positive* values for a open up; those having *negative* values for a open down.

4. $y = e^{x/2} + e^{-(x/2)} + 2$

LOOKING FOR SOLUTIONS

MATHEMATICS CONTENT:
Exponents, Exponential Relations,
Integers, Rational Numbers,
Irrational Numbers,
Parametric Equations

Although you may never have asked yourself, "When does $x^y = y^x$," a lot of mathematicians have. Can *you* find the integer solutions to the equation? Any ordered pairs for which $x = y$ are solutions. However, $4^4 = 4^4$ and $6^6 = 6^6$ are not very exciting results. A little trial-and-error will give you two, more interesting solutions—(2, 4) and (4, 2). Are there *negative* integer solutions to the equation for which x and y are *not* equal? Yes. (The only negative integer solutions under these conditions are $(-2, -4)$ and $(-4, -2)$.)



Once you've found a positive solution to the equation, you can find a related negative solution. For example, $(-2, -4)$ is related to (2, 4). From now on, think about *positive* solutions only. The negative solutions will follow.

So far, you've only looked for the integer solutions to $x^y = y^x$. Are there other kinds of solutions and, if so, how do you find them? The graph on the first page shows the positive solutions for $x^y = y^x$. There are two parts—a straight line representing all solutions for which $x = y$, and a curve representing those solutions for which $x \neq y$. As the values of x increase, the curve gets closer and closer to $y = 1$. As the values of y increase, the curve gets closer and closer to $x = 1$. The number e makes an appearance, too. The curve crosses the line at (e, e)!

The graph should give you an idea of the kinds of solutions you can find for $x^y = y^x$, but it doesn't really give you specifics. Does $(9/4)^{27/8} = (27/8)^{9/4}$? Does $(3\sqrt{3})^{\sqrt{3}} = (\sqrt{3})^{3\sqrt{3}}$? How do you find the coordinates of specific points on the curve?

First, what about rational number solutions? According to Martin Gardner, you can find the positive rational number solutions using the following pair of equations.

$$x = \left(1 + \frac{1}{n}\right)^{n+1} \qquad y = \left(1 + \frac{1}{n}\right)^{n}$$

where $x > y$ and n is a positive integer

The constant n is called a *parameter,* a constant that can be varied. (Parameters are also used in the equation of a line $y = mx + b$. In this case, the constants m and b are parameters.) If you complete the table on the top of page 2, you will obtain three different solutions to $x^y = y^x$. Where are these solutions graphed?

Some of the solutions to $x^y = y^x$ have irrational values. Warren Manhard and his high school students developed the following parametric equations. In these equations, k is a positive real number and $k \neq 1$.

$$x = k^{\frac{1}{(k-1)}}$$

$$y = k^{\frac{k}{(k-1)}}$$

These equations give many solutions to the equation $x^y = y^x$, including many irrational number solutions. Completing the table of values on page 3 using Manhard's equations will give you a solution you've seen before and several new ones. Once you've found the solutions, try placing them on the graph.

You may want to find other points on the curve using the formulas. You may also want to read the article about Manhard's work. It's an update on the progress of finding the real and complex solutions to $x^y = y^x$. The work people have done with $x^y = y^x$ shows how a problem that is quite simple (and possibly trivial) within one set of numbers becomes more complex (and often more interesting) within other sets of numbers.

LOOKING FOR SOLUTIONS 2

These equations give positive *rational* solutions to $x^y = y^x$.

$$x = \left(1 + \frac{1}{n}\right)^{n+1}$$

$$y = \left(1 + \frac{1}{n}\right)^{n}$$

where $x > y$ and n is a positive integer.

n	x	y
1	4	2
2	$27/8$	$9/4$
3	$256/81$	$64/27$
4	$3125/1024$	$625/256$

These equations give positive solutions to $x^y = y^x$, some of which are *irrational.*

$$x = k^{\frac{1}{(k-1)}}$$

$$y = k^{\frac{k}{(k-1)}}$$

where k is a positive real number and $k \neq 1$.

k	x	y
$\frac{4}{3}$	$64/27$	$256/81$
2	2	4
3	$\sqrt{3}$	$3\sqrt{3}$
4	$\sqrt[3]{4}$	$4\sqrt[3]{4}$
5	$\sqrt[4]{5}$	$5\sqrt[4]{5}$

Compare the values for $n = 3$ to those for $k = \frac{4}{3}$.

138 A VERY SPECIAL NUMBER CALLED *e*

EXERCISES

1. Verify that $(-2, -4)$ and $(-4, -2)$ are solutions to the equation $x^y = y^x$.

2. Show that $(9/4, 27/8)$ is a solution to $x^y = y^x$.

ANSWERS

1. $(-2)^{-4} = 1/(-2)^4 = 1/16$; $(-4)^{-2} = 1/(-4)^2 = 1/16$

2. $(9/4)^{27/8} = ((3/2)^2)^{27/8} = (3/2)^{27/4}$; $(27/8)^{9/4} = ((3/2)^3)^{9/4} = (3/2)^{27/4}$

MATHEMATICS CONTENT:
Cosine and Sine Functions,
π, e, and i, Exponential Function,
Series, Sequence of Partial Sums,
Complex Number, Complex Plane

EULER'S FORMULA

What do the two most famous transcendental numbers, π and e, have in common with the imaginary unit i, and the additive inverse of the multiplicative identity for the real numbers? How are $\lim_{x \to \infty} (1 + (1/x))^x$, the ratio of the circumference of a circle to its diameter, and a solution to $x^2 + 1 = 0$ related? What is one of the most amazing and useful coincidences in mathematics?

The answer to all these questions is Euler's Formula, discovered by Leonard Euler in the 16th century and considered by many to be the most elegant formula in all of mathematics. Stated in the following way, the formula provides a link between exponential functions and circular functions.

$$e^{ix} = \cos x + i \sin x \qquad \text{where } x \text{ is a real number}$$

By substituting π for x and evaluating terms in the equation, you will obtain an important relationship among e, π, i, and 1.

$$e^{i\pi} = \cos \pi + i \sin \pi$$
$$e^{i\pi} = (-1) + i(0)$$
$$e^{i\pi} = -1$$

There is another, more visual way of obtaining this relationship. It involves graphing a sequence of approximations for $e^{i\pi}$ in the complex plane. One of e's appearances is in the following mathematical statement.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

where x is any complex number

The statement means that, for any given value of x, the value of e^x can be approximated more and more closely by adding more and more terms to the sum given. For example, the sum $1 + x + (x^2/2!)$ gives a rough approximation of e^x.

EULER'S FORMULA 1

$$e^{ix} = \cos x + i \sin x$$

where x is a real number

Use Euler's formula to find the value of $e^{i\pi}$.

$$e^{i\pi} = \cos \pi + i \sin \pi$$
$$= (-1) + i(0)$$
$$= -1$$

Euler's formula establishes the relationship among the four important numbers that are defined below. Which is which?

e $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

π $\frac{\text{circumference}}{\text{diameter}}$

-1 additive inverse of the multiplicative identity for real numbers

i a solution to $x^2 + 1 = 0$

EULER'S FORMULA 2

Approximate each partial sum. Use 3.14 for π.

Term of Sequence	Value
1	$1 + 0.00i$
$1 + \pi i$	$1 + 3.14i$
$1 + \pi i + \frac{(\pi i)^2}{2!}$	$-3.93 + 3.14i$
$1 + \pi i + \frac{(\pi i)^2}{2!} + \frac{(\pi i)^3}{3!}$	$-3.93 - 2.03i$
$1 + \pi i + \frac{(\pi i)^2}{2!} + \frac{(\pi i)^3}{3!} + \frac{(\pi i)^4}{4!}$	$0.12 - 2.03i$
$1 + \pi i + \frac{(\pi i)^2}{2!} + \frac{(\pi i)^3}{3!} + \frac{(\pi i)^4}{4!} + \frac{(\pi i)^5}{5!}$	$0.12 + 0.52i$
$1 + \pi i + \frac{(\pi i)^2}{2!} + \cdots + \frac{(\pi i)^6}{6!}$	$-1.21 + 0.52i$
$1 + \pi i + \frac{(\pi i)^2}{2!} + \cdots + \frac{(\pi i)^7}{7!}$	$-1.21 - 0.08i$
$1 + \pi i + \frac{(\pi i)^2}{2!} + \cdots + \frac{(\pi i)^8}{8!}$	$-0.98 - 0.08i$
$1 + \pi i + \frac{(\pi i)^2}{2!} + \cdots + \frac{(\pi i)^9}{9!}$	$-0.98 + 0.01i$
$1 + \pi i + \frac{(\pi i)^2}{2!} + \cdots + \frac{(\pi i)^{10}}{10!}$	$-0.95 + 0.01i$

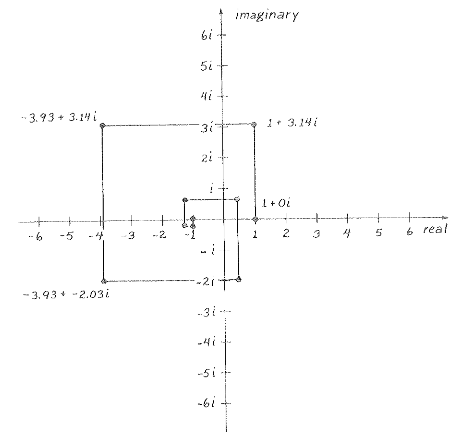
The sum $1 + x + (x^2/2!) + (x^3/3!) + (x^4/4!) + \ldots + (x^{100}/100!)$ gives a much better approximation for e^x.

Suppose x is $i\pi$. The equation becomes

$$e^{i\pi} = 1 + \pi i + \frac{(\pi i)^2}{2!} + \frac{(\pi i)^3}{3!} + \cdots$$

By graphing the sequence of *partial sums* of the series in the equation, you will get a visual interpretation of the approximations for $e^{i\pi}$. (A sequence of partial sums of a series consists of: the first term, the sum of the first two terms, the sum of the first three terms, and so on.) The table on page 2 is laid out to help you determine the first ten partial sums of the sequence.

EULER'S FORMULA 3



When connected in sequence, the terms form a counterclockwise spiral of line segments that approach _____ on the _____ axis.

 A VERY SPECIAL NUMBER CALLED e 145

EXERCISES

1. Express each of the following exponential expressions as a complex number in simplest form by using Euler's Formula.

a. $e^{(\pi i)/2}$ **b.** $e^{2\pi i}$
c. $e^{(\pi/6)(i)}$ **d.** $e^{(\pi/3)(i)}$
e. $e^{(5/6)(\pi i)}$ **f.** $e^{(4/3)(\pi i)}$
g. $e^{(11/6)(\pi i)}$ **h.** $e^{(9/4)(\pi i)}$
i. $e^{3\pi i}$

2. Express each of the following sums as a power of e by using Euler's Formula.

a. $\cos(2/3)\pi + i \sin(2/3)\pi$
b. $\cos(\pi/4) + i \sin(\pi/4)$
c. $\cos(3/2)\pi + i \sin(3/2)\pi$
d. $\cos 4\pi + i \sin 4\pi$
e. $(\sqrt{3}/2) + i(1/2)$ (Use the least value of x.)

ANSWERS

1a. $0 + i$
1b. $1 + 0i$
1c. $(\sqrt{3}/2) + (1/2)i$
1d. $(1/2) + (\sqrt{3}/2)i$
1e. $-(\sqrt{3}/2) + (1/2)i$
1f. $-(1/2) - (\sqrt{3}/2)i$
1g. $(\sqrt{3}/2) - (1/2)i$
1h. $(\sqrt{2}/2) + (\sqrt{2}/2)i$
1i. $-1 + 0i$

2a. $e^{(2/3)(\pi i)}$
2b. $e^{(\pi/4)(i)}$
2c. $e^{(3/2)(\pi i)}$
2d. $e^{4\pi i}$
2e. $e^{(\pi/6)(i)}$

PUTTING EULER'S FORMULA TO WORK

MATHEMATICS CONTENT:
Exponents,
π, e, and i,
Cosine and Sine Functions,
Periodicity

Before Euler's time, mathematicians knew that the product of any two pure imaginary numbers is a real number. It was Euler, though, who discovered that a pure imaginary number raised to a pure imaginary power is a real number. Specifically, Euler found that i^i is a real number and, using his formula, found its value.

$$i^i = e^{-\left(\frac{\pi}{2}\right)}$$

Using this relationship, you can compute the decimal value of i^i to any degree of accuracy you choose. Page 1 gives the decimal value of i^i correct to ten decimal places. Using a scientific calculator and 2.7182818 for e, what approximation will you get? (The value is 0.2078796.) There appears to be no pattern to the decimal digits for i^i and, in fact, that is the case. In other words, i^i is irrational.

So far, you have only computed *one* value for i^i. There are more. As a matter of fact, i^i has an infinity of real number values. You can find the values by first taking another look at Euler's Formula and generalizing it. Remember that $e^{ix} = \cos x + i \sin x$ and that both cosine and sine are *periodic* functions with periods of 2π. (A function f is said to be periodic and have period p if $x + p$ is in the domain of the function whenever x is in the domain and $f(x + p) = f(x)$.) Since the periods of cosine and sine are 2π, Euler's Formula tells you that e^{ix} is equal to $e^{ix + 2k\pi}$ where k is any integer. What does this fact tell you about the values of i^i? It tells you this:

$$i^i = e^{-\left(\frac{\pi}{2}\right) + 2k\pi} \qquad \text{where } k \text{ is an integer}$$

The chart on page 1 shows some decimal approximations for values of i^i. Try to find the others yourself. Use 2.71828 for e.

PUTTING EULER'S FUNCTION TO WORK 1

$$i^i = e^{-\left(\frac{\pi}{2}\right)} \approx 0.2078795763\ldots$$

	i^i or $e^{-\left(\frac{\pi}{2}\right)+2k\pi}$		$\frac{1}{i^i}$ or ____	
k	value	decimal approximation	value	decimal approximation
0	$e^{-\left(\frac{\pi}{2}\right)}$	0.2078798	$e^{\frac{\pi}{2}}$	4.8104774
1	$e^{\frac{3\pi}{2}}$	111.31743	$e^{\frac{5\pi}{2}}$	2575.9703
−1	$e^{-\left(\frac{5\pi}{2}\right)}$	0.0003882	$e^{-\left(\frac{3\pi}{2}\right)}$	0.0089833
2	$e^{\frac{7\pi}{2}}$	59609.732	$e^{\frac{9\pi}{2}}$	1379410.4
−2	$e^{-\left(\frac{9\pi}{2}\right)}$	0.0000007	$e^{-\left(\frac{7\pi}{2}\right)}$	0.0000168
3	$e^{\frac{11\pi}{2}}$	31920512	$e^{\frac{13\pi}{2}}$	7.3866×10^{8}
−3	$e^{-\left(\frac{13\pi}{2}\right)}$	1.3538×10^{-9}	$e^{-\left(\frac{11\pi}{2}\right)}$	3.1328×10^{-8}

EXERCISES

1. Use a calculator to evaluate $(1 + (1/x))^x$ for $x = 90$.
2. Evaluate $(1 + (1/x))^x$ for $x = 500$.
3. Evaluate $(1 + (1/x))^x$ for $x = 25{,}000$.
4. Evaluate $(1 + (1/x))^x$ for $x = 60{,}000$.
5. Evaluate $(1 + (1/x))^x$ for $x = 75{,}000$.
6. Evaluate $(1 + (1/x))^x$ for $x = 79{,}000$.
7. How accurate an approximation for e do you obtain if you use 10,000 for x?

ANSWERS

1. 2.7033324
2. 2.7155682
3. 2.7182302
4. 2.718222
5. 2.7182139
6. 2.7182329
7. The approximation is correct to 3 decimal places.

The number $i^{1/i}$ has an infinite number of real number values, too. By using Euler's Formula and the fact that cosine and sine are periodic functions, you can find an exponential expression for the real number values. Try to discover what that expression is and, then, compute the various decimal approximations asked for in the table on page 1.

As you can see, Euler's Formula is a very useful mathematical tool. Not only does it help you determine important relationships between numbers, but it helps you determine the specific character of given numbers and actually helps you calculate decimal approximations of those numbers to any degree of accuracy you might need.

EXERCISES

1. Use Euler's Formula to show that $i^i = e^{-(\pi/2)}$.
   HINT: $e^{-(\pi/2)} = e^{(\pi/2)i^2} = e^{((\pi/2)i)i}$.

2. Use Euler's Formula to show that $i^{1/i} = e^{\pi/2}$.
   HINT: $e^{\pi/2} = e^{(\pi/2)(i/i)} = e^{((\pi/2)i)(1/i)}$.

ANSWERS

1. $e^{-(\pi/2)}$
$$= (e^{(\pi/2)i})^i$$
$$= (\cos(\pi/2) + i\sin(\pi/2))^i$$
$$= (0 + i(1))^i$$
$$= i^i$$

2. $e^{\pi/2}$
$$= (e^{(\pi/2)i})^{1/i}$$
$$= (\cos(\pi/2) + i\sin(\pi/2))^{1/i}$$
$$= (0 + i(1))^{1/i}$$
$$= i^{1/i}$$

MATHEMATICS CONTENT:
Sequence,
Function, Domain, Range,
Limit

APPROXIMATING *e*

Using the formula $f(x) = (1 + (1/x))^x$ you can generate a set of ordered pairs called a *sequence function.* First, let the domain be the set of positive integers. Starting with 1 for x, substitute consecutive positive integers into the expression $(1 + (1/x))^x$ to determine the corresponding elements of the range of the sequence function. The first five pairs of values are given in the table on page 1. Use a calculator to find the remaining range values in the table.

The complete set of ordered pairs obtained in the manner described and listed in order is called a sequence function. The formula $f(x) = (1 + (1/x))^x$ is called a *sequence function generator.*

As you continue to build the sequence function in this lesson, you will make a very interesting discovery. The sequence of values in the range get closer and closer together. The values approach a number that should be very familiar to you by now. That number is *e*. In fact, mathematicians often use this sequence to define *e*. They say,

$$e = \lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^x \qquad \text{where } x \text{ is a positive integer}$$

This statement means that by choosing a sufficiently large value for x, you can get a value as close to *e* as you wish.

Look at the table of values you completed. How great a value of x must you choose to obtain a value for *e* that is correct to 7 decimal digits? (80,000.)

Using today's computers, it is possible to obtain very large values for x. Consequently, it is possible to obtain very exact approximations for *e*.

APPROXIMATING *e* 1

x	$\left(1 + \frac{1}{x}\right)^x$
1	2.0000000
2	2.2500000
3	2.3703704
4	2.4414063
5	2.4883200
⋮	⋮
8	2.5657845
⋮	⋮
20	2.6532977
80	2.6991164
100	2.7048138
1000	2.7169239
50,000	2.7182547
70,000	2.7182597
80,000	2.7182818

$$e = \lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^x$$

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1. Use a calculator to evaluate $(1 + (1/x))^x$ for $x = 90$.
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